Linear differential equations with constant coefficients

Method of undetermined coefficients

$$e^{u+vi} = e^{u}(\cos vx + i \sin vx),\ \ u, v \in \mathbf{R},\ i^2 = -1$$

Quasi-polynomial:

$Q_{\alpha+\beta i,k}(x) = e^{\alpha x}\,[\cos \beta x\ (f_0 + f_1 x + \ldots + f_k x^k) + \sin \beta x\ (g_0 + g_1 x + \ldots + g_k x^k)]$

$\alpha, \beta, f_0, f_1, \ldots, f_k, g_1, \ldots, g_k \in \mathbf{R}$

k is the degree, $\alpha+\beta i$ is the exponent of $Q_{\alpha+\beta i,k}(x)$

Examples:

$3 = e^{0x}\,[3 \cos 0x + \sin 0x]$ $\quad\rightarrow\quad$ $\alpha+\beta i = 0,\ k = 0$

$x^2 + 7 = e^{0x}\,[\cos 0x\,(x^2 + 7) + \sin 0x]$ $\quad\rightarrow\quad$ $\alpha+\beta i = 0,\ k = 2$

$x\,e^{2x} + e^{2x} = e^{2x}\,[\cos 0x\,(x + 1) + \sin 0x]$ $\quad\rightarrow\quad$ $\alpha+\beta i = 2,\ k = 1$

$x \cos 2x + \sin 2x = e^{0x}\,[x \cos 2x + \sin 2x]$ $\quad\rightarrow\quad$ $\alpha+\beta i = 2i,\ k = 1$

$e^{x} \cos 5x + e^{x} \sin 5x\,(x^7 + 1) = e^{x}\,[\cos 5x + (x^7 + 1) \sin 5x]$ $\quad\rightarrow\quad$ $\alpha+\beta i = 1 + 5i,\ k = 7$

$x \cos 3x + \sin x$ - not a quasi-polynomial, but a sum of two quasi-polynomials

$x^{-2} e^{x}$, $\mathrm{tg}\, x$, $\cos x / x$ - not quasi-polynomials

Homogeneous linear differential equations

Homogeneous linear differential equation of the nth order:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = 0, \qquad a_1, a_2, \ldots, a_n \in \mathbf{R} \tag{1}$$

Initial conditions:

$$y(x_0) = \xi_0, y'(x_0) = \xi_1, \ldots, y^{(n-1)}(x_0) = \xi_{n-1} \tag{2}$$

The **characteristical polynomial** and the **characteristical equation**:

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n, \quad p(\lambda) = 0$$

Let λ_1, λ_2, ..., λ_s be the different real roots of $p(\lambda)$ and m_1, m_2, ..., m_s be their multiplicities, and let μ_1, $\overline{\mu}_1$, ..., μ_q , $\overline{\mu}_q$ be the different conjugate complex pairs of roots of $p(\lambda)$ and $\underline{m}_1$, $\underline{m}_2$, ..., $\underline{m}_q$ be their multiplicities

$$m_1 + m_2 + \ldots + m_s + 2\underline{m}_1 + 2\underline{m}_2 + \ldots + 2\underline{m}_q = n.$$

The general solution of the homogeneous equation (1):

$$y = y_1 + y_2 + \ldots + y_s + \underline{y}_1 + \underline{y}_2 + \ldots + \underline{y}_q$$

Each y_j is a quasi-polynomial with the exponent $\lambda_j \in \mathbf{R}$ of the degree m_j - 1. It contains m_j coefficients. All the coefficients are arbitrary numbers.

Each $\underline{y}_j$ is a quasi-polynomial with the exponent $\mu_j \in \mathbf{C}$ of the degree $\underline{m}_j$ - 1. It contains $2m_j$ coefficients. All the coefficients are arbitrary numbers.

The total number of the free coefficients is

$m_1 + m_2 + \ldots + m_s + 2\underline{m}_1 + 2\underline{m}_2 + \ldots + 2\underline{m}_q = n$.

The concrete values of the free coefficients are determined from the initial conditions (2).

Nonhomogeneous linear differential equations

Nonhomogeneous linear differential equation of the nth order:

$$y^{(n)} + a_1 y^{(n-1)} + a_2 y^{(n-2)} + \ldots + a_n y = b(x), \qquad a_1, a_2, \ldots, a_n \in \mathbf{R} \tag{3}$$

The **general solution**

$$y = y_h + y_p$$

where y_h is the general solution of the homogeneous equation (1) and y_p is a particular solution of (2) (each one fits). If $b(x) = b_1(x) + b_2(x)$, then any particular solution is a sum of some corresponding particular solutions: $y_p = y_{p1} + y_{p2}$. The component y_h contains n arbitrary coefficients, which can be determined from the initial conditions (2) after y_p is found.

Let $b(x)$ be a quasi-polynomial of the degree k with the exponent $\alpha+\beta i$. There are two cases:

1. $\alpha+\beta i$ is not a root of the characteristical equation, i.e. $p(\alpha+\beta i) \neq 0$. Then the particular solution y_p is searched for in the form of a quasi-polynomial with the same exponent and the same degree:
$$y_p = e^{\alpha x} [\cos \beta x\, (f_0 + f_1 x + \ldots + f_k x^k) + \sin \beta x\, (g_0 + g_1 x + \ldots + g_k x^k)]. \tag{4}$$
There are k unknown coefficients with $\beta = 0$ and $2k$ coefficients with $\beta \neq 0$. All of them are to be determined from the equality obtained after the substitution of $y = y_p$ into (3).
2. $\alpha+\beta i$ is a root of the characteristical equation, i.e. $p(\alpha+\beta i) = 0$ (**resonance**). Let $m \geq 1$ be the multiplicity of the root $\alpha+\beta i$. Then the particular solution y_p is searched for in the form of a quasi-polynomial with the same exponent and the increased degree $k + m$. The terms with the degrees of x up to $m - 1$ can be canceled, for the corresponding lower degree quasi-polynomial is a solution of the homogeneous equation (1). The resulting form is
$$y_p = x^m e^{\alpha x} [\cos \beta x\, (f_0 + f_1 x + \ldots + f_k x^k) + \sin \beta x\, (g_0 + g_1 x + \ldots + g_k x^k)]. \tag{5}$$
There are k unknown coefficients with $\beta = 0$ and $2k$ coefficients with $\beta \neq 0$. All of them are to be determined from the equality obtained after the substitution of $y = y_p$ in (3).

Form (4) can be considered as a particular case of form (5) with $m = 0$.

Homogeneous systems of first-order linear differential equations

The homogeneous system

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ \dots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \ \dots + a_{2n}y_n \\ &\qquad\qquad\qquad \dots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ \dots + a_{nn}y_n \end{aligned}$$

The initial conditions:

$$y_1(x_0) = \xi_1, \ y_2(x_0) = \xi_2, \ \dots , \ y_n(x_0) = \xi_n$$

In the matrix form:

$$y' = Ay \,, \tag{6}$$

$$y(x_0) = \xi \,, \quad y, \xi \in \mathbf{R}^n \tag{7}$$

The **characteristical polynomial** and the **characteristical equation**:

$$p(\lambda) = \det(A - \lambda I) \,, \qquad p(\lambda) = 0.$$

The roots of the characteristical equation are called **eigenvalues**, any non-zero solution V of the vector equation $(A - \lambda I)V = 0$ is called an **eigenvector** with the eigenvalue λ.

The simple case: roots of the multiplicity 1

All roots of the characteristical polynomial have the multiplicity 1. Then there are n different eigenvalues and n corresponding linearly independent eigenvectors.

Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the real eigenvalues of A and $\mu_1, \overline{\mu}_1, \ldots, \mu_q, \overline{\mu}_q$ be the conjugate pairs of complex eigenvalues of A

$$s + 2q = n.$$

The general solution of the homogeneous equation (1):

$$y = y_1 + y_2 + \ldots + y_s + \underline{y}_1 + \underline{y}_2 + \ldots + \underline{y}_q ,$$

where the components corresponding to the real eigenvalues have the form

$$y_j = c_j e^{\lambda_j x} V_j ,\ j = 1, \ldots, s,$$

with V_j being the eigenvector corresponding to λ_j . The two real solutions corresponding to each conjugate pair of complex eigenvalues μ_j, $\overline{\mu}_j$ are found as the real and the imaginative parts of the complex solution of the form $e^{\mu_j x} V_j$, where V_j is the corresponding complex eigenvector:

$$\underline{y}_j = d_j \operatorname{Re}(e^{\mu_j x} V_j) + \underline{d}_j \operatorname{Im}(e^{\mu_j x} V_j),\ j = 1, \ldots, q.$$

The general solution depends on n arbitrary coefficients. These coefficients can be determined from the initial conditions.

Vector quasi-polynomial:

$Q_{\alpha+\beta i,k}(x) = e^{\alpha x}$ [$\cos \beta x$ ($F_0 + F_1 x + \ldots + F_k x^k$) + $\sin \beta x$ ($G_0 + G_1 x + \ldots + G_k x^k$)]

$F_0 ,\ F_1, \ldots, F_k, G_1, \ldots, G_k \in \mathbf{R}^n$

k is the degree, $\alpha+\beta i$ is the exponent of $Q_{\alpha+\beta i,k}(x)$

Examples:

$$\begin{pmatrix} 2 \\ x^2 \end{pmatrix} = e^{0x}\left\{ \cos 0x \left[\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x^2 \right] + \sin 0x \right\} \quad \rightarrow \quad \alpha+\beta i = 0,\ k = 2$$

$$\begin{pmatrix} e^{2x} \cos x \\ xe^{2x} \sin x \end{pmatrix} = e^{2x}\left\{ \cos x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin x \begin{pmatrix} 0 \\ 1 \end{pmatrix} x \right\} \quad \rightarrow \quad \alpha+\beta i = 2 + i,\ k = 1$$

General homogeneous case.

Let $\lambda_1, \lambda_2, ..., \lambda_s$ be the different real roots of $p(\lambda)$ (eigenvalues of A) and $m_1, m_2, ..., m_s$ be their multiplicities, and let $\mu_1, \overline{\mu}_1, ..., \mu_q, \overline{\mu}_q$ be the different conjugate complex pairs of roots of $p(\lambda)$ (eigenvalues of A) and $\underline{m}_1, \underline{m}_2, ..., \underline{m}_q$ be their multiplicities

$$m_1 + m_2 + ... + m_s + 2\underline{m}_1 + 2\underline{m}_2 + ... + 2\underline{m}_q = n.$$

The general solution of the homogeneous equation (1):

$$y = y_1 + y_2 + ... + y_s + \underline{y}_1 + \underline{y}_2 + ... + \underline{y}_q$$

Each y_j is a vector quasi-polynomial with the exponent $\lambda_j \in \mathbf{R}$ of the degree m_j - 1. It contains nm_j coefficients. Only m_j coefficients are independent and can be taken arbitrary, all the others are to be expressed through them. The independent coefficients are identified by the substitution of the general vector quasi-polynomial instead of y into (6).

Each $\underline{y}_j$ is a vector quasi-polynomial with the exponent $\mu_j \in \mathbf{C}$ of the degree $\underline{m}_j$ - 1. It contains $2n\underline{m}_j$ coefficients. Only $2\underline{m}_j$ coefficients are independent and can be taken arbitrary, all the others are to be expressed through them. The independent coefficients are identified by the substitution of the general vector quasi-polynomial instead of y into (6).

The total number of the independent free coefficients is n. The concrete values of the free coefficients are determined from the initial conditions (7).

Nonhomogeneous systems of first-order linear differential equations

Nonhomogeneous linear system:

$$y' = Ay + B(x), \qquad B(x) = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \ldots \\ b_n(x) \end{pmatrix} \tag{8}$$

The **general solution**

$$y = y_h + y_p$$

where y_h is the general solution of the homogeneous system (6) and y_p is a particular solution of (8) (each one fits). The component y_h contains n arbitrary coefficients, which can be determined from the initial conditions (7) after y_p is found.

If $B(x) = B_1(x) + B_2(x)$, then any particular solution is a sum of some corresponding particular solutions: $y_p = y_{p1} + y_{p2}$.

Let $B(x)$ be a vector quasi-polynomial of the degree k with the exponent $\alpha+\beta i$. Let m be the multiplicity of $\alpha+\beta i$ as of a root of the characteristical polynomial, i.e. $m = 0$ if $p(\alpha+\beta i) \neq 0$ and $m \geq 1$ if $p(\alpha+\beta i) = 0$ (the resonance case). Then the particular solution y_p is searched for in the form of a vector quasi-polynomial with the same exponent and the degree $k + m$:

$y_p = e^{\alpha x}$ [cos βx ($F_0 + F_1 x + \ldots + F_{k+m} x^{k+m}$) + sin βx ($G_0 + G_1 x + \ldots + G_{k+m} x^{k+m}$)].

There are $n(k + m)$ unknown coefficients with $\beta = 0$ and 2 $n(k + m)$ coefficients with $\beta \neq 0$. All of them are to be determined from the equalities obtained after the substitution of $y = y_p$ into (8). In the resonance case the number of the coefficient choices is infinite. Then some of them are defined arbitrarily (as zero, for example).

Remark. The low degrees of x cannot be canceled like in the scalar case!