Lesson 11: Method of parameter variation

In each of problems 1 through 6 use the method of variation of parameters to find a particular solution of the given differential equation.

1. $y'' - 5y' + 6y = 2e^x$
2. $y'' - y' - 2y = 2e^{-x}$
3. $y'' + 2y' + y = 3e^{-x}$
4. $4y'' - 4y' + y = 16e^{x/2}$
5. $y'' + y = \tan x, \quad 0 < x < \pi/2$
6. $y'' + 4y' + 4y = x - 2e^{-2x}, \quad x > 0$

Solution of 1. Let us first solve the homogeneous equation

$$y'' - 5y' + 6y = 0.$$ 

The associate algebraic equation is

$$m^2 - 5m + 6 = 0,$$

with roots $m_1 = 3$, $m_2 = 2$. Therefore, the solution of homogeneous equation is $c_1e^{3x} + c_2e^{2x}$.

Let us now find the solution of the non-homogeneous equation as

$$y = f(x) = u_1(x)e^{3x} + u_2(x)e^{2x},$$

where

$$u_1(x) = c_1 - \int \frac{y_2(x)g(x)}{W[y_1(x), y_2(x)]} dx,$$

$$u_2(x) = c_2 + \int \frac{y_1(x)g(x)}{W[y_1(x), y_2(x)]} dx.$$

In our case $y_1(x) = e^{3x}$, $y_2(x) = e^{2x}$, $g(x) = 2e^x$, and

$$W[y_1(x), y_2(x)] = e^{3x}2e^{2x} - e^{2x}3e^{3x} = e^{5x}.$$ 

Therefore

$$u_1(x) = c_1 - \int \frac{e^{2x}2e^x}{e^{5x}} dx = c_1 + 2 \int e^{-2x} dx = c_1 - e^{-2x},$$

$$u_2(x) = c_2 + \int \frac{e^{3x}2e^x}{e^{5x}} dx = c_2 - 2 \int e^{-x} dx = c_2 + 2e^{-x}.$$ 

Substituting it for $y(x)$, we finally obtain

$$y(x) = c_1e^{3x} + c_2e^{2x} + e^x.$$ 

Solution of 4. We start from the homogeneous equation

$$y'' - y' + \frac{1}{4}y = 0$$
The associate algebraic equation is
\[ m^2 - m + \frac{1}{4} = 0, \]
with equal roots \( m_{1,2} = 1/2 \). Therefore, the solution of homogeneous equation is
\[ c_1e^{x/2} + c_2xe^{x/2}. \]
Let us now find the solution of the non-homogeneous equation as
\[ y = f(x) = u_1(x)e^{x/2} + u_2(x)xe^{x/2}, \]
where
\[ u_1(x) = c_1 - \int \frac{y_2(x)g(x)}{W[y_1(x), y_2(x)]} \, dx, \]
\[ u_2(x) = c_2 + \int \frac{y_1(x)g(x)}{W[y_1(x), y_2(x)]} \, dx. \]
In our case \( y_1(x) = e^{x/2}, \) \( y_2(x) = xe^{x/2}, \) \( g(x) = 16e^{x/2}, \) and
\[ W[y_1(x), y_2(x)] = e^{x/2}(e^{x/2} + \frac{1}{2}xe^{x/2}) - xe^{x/2}\frac{1}{2}e^{x/2} = e^x. \]
Therefore
\[ u_1(x) = c_1 - \int \frac{xe^{x/2}16e^{x/2}}{e^x} \, dx = c_1 - 8x^2, \]
\[ u_2(x) = c_2 + \int \frac{e^{x/2}16e^{x/2}}{e^x} \, dx = c_2 + 16x. \]
The final solution is
\[ y(x) = c_1e^{x/2} + c_2xe^{x/2} + 8x^2e^{x/2}. \]

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