Mathematical Black-Box Control

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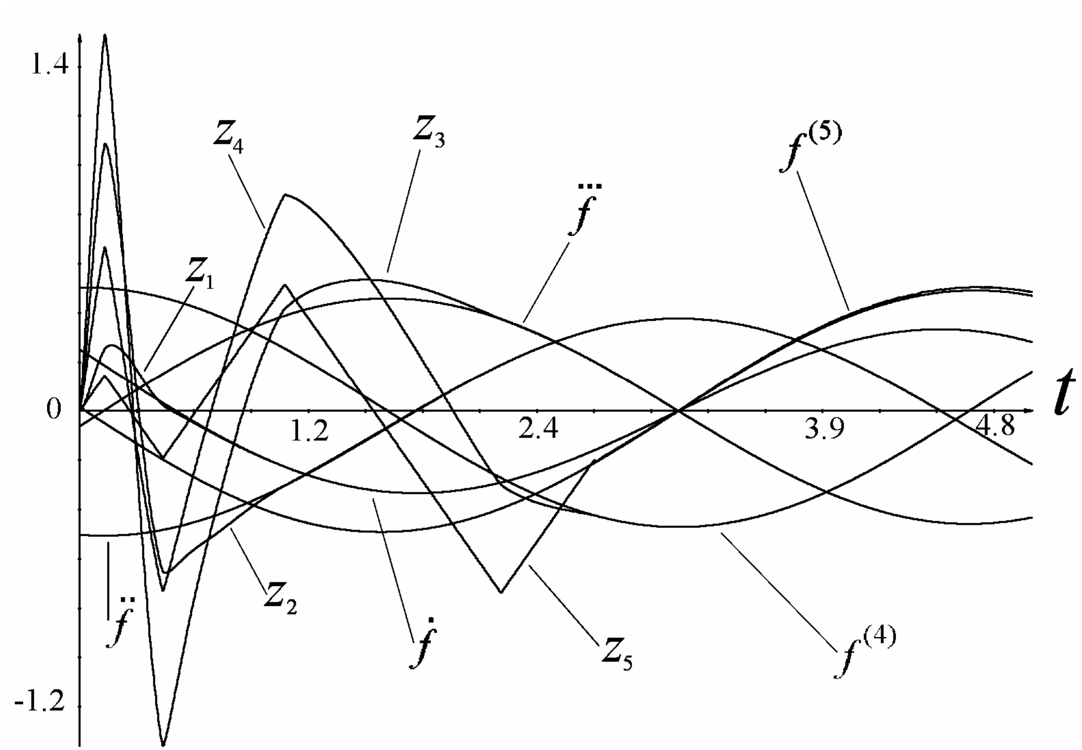


General Output-Regulation Problem

$$\dot{x} = f(t,x,u), \;\; x \in \mathbf{R}^n,$$

output $s(t,x)$, control (input) $u \in \mathbf{R}$

The task: tracking a real-time given signal $\varphi(t)$ by s

$\sigma(t,x) = s - \varphi(t)$, The goal: $\sigma = 0$

$v = \dot{u}$ is taken as a new control,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} f(t,x,u) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v$$

From now on the system is assumed affine in control.

Relative Degree

$$\dot{x} = a(t,x) + b(t,x)u,\ x \in \mathbf{R}^n,\ \sigma,\ u \in \mathbf{R}$$

Informal definition: The number r of the first total derivative where the control explicitly appears with a non-zero coefficient.

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad \tfrac{\partial}{\partial u}\sigma^{(r)} = g \neq 0$$

Formal definition: $\begin{pmatrix} \dot{x} \\ \dot{t} \end{pmatrix} = \begin{pmatrix} a(t,x) \\ 1 \end{pmatrix} + \begin{pmatrix} b(t,x) \\ 0 \end{pmatrix} u, \quad \tilde{x} = \begin{pmatrix} x \\ t \end{pmatrix}$

$\dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{b}(\tilde{x})u$, r is defined by the conditions (Isidori 1985)

$$L_{\tilde{b}}\sigma = L_{\tilde{b}}L_{\tilde{a}}\sigma = \ldots = L_{\tilde{b}}L_{\tilde{a}}^{r-2}\sigma = 0, \quad L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma = g(\tilde{x}) \neq 0$$

to be held at a point (in a region).

Output-regulation problem

System affine in control

$$\dot{x} = a(t,x) + b(t,x)u,\ x \in \mathbf{R}^n,\ \sigma,\ u \in \mathbf{R}$$

The output: $\sigma: \mathbf{R}^{n+1} \to \mathbf{R}$ **The goal**: $\sigma(t,x) = 0$

$a(t,x)$, $b(t,x)$ are smooth; a, b, n are uncertain

Assumptions. **known:** $r \in N$, $0 < K_{\mathrm{m}} \leq K_{\mathrm{M}}$, $C \geq 0$

1. The relative degree is r:

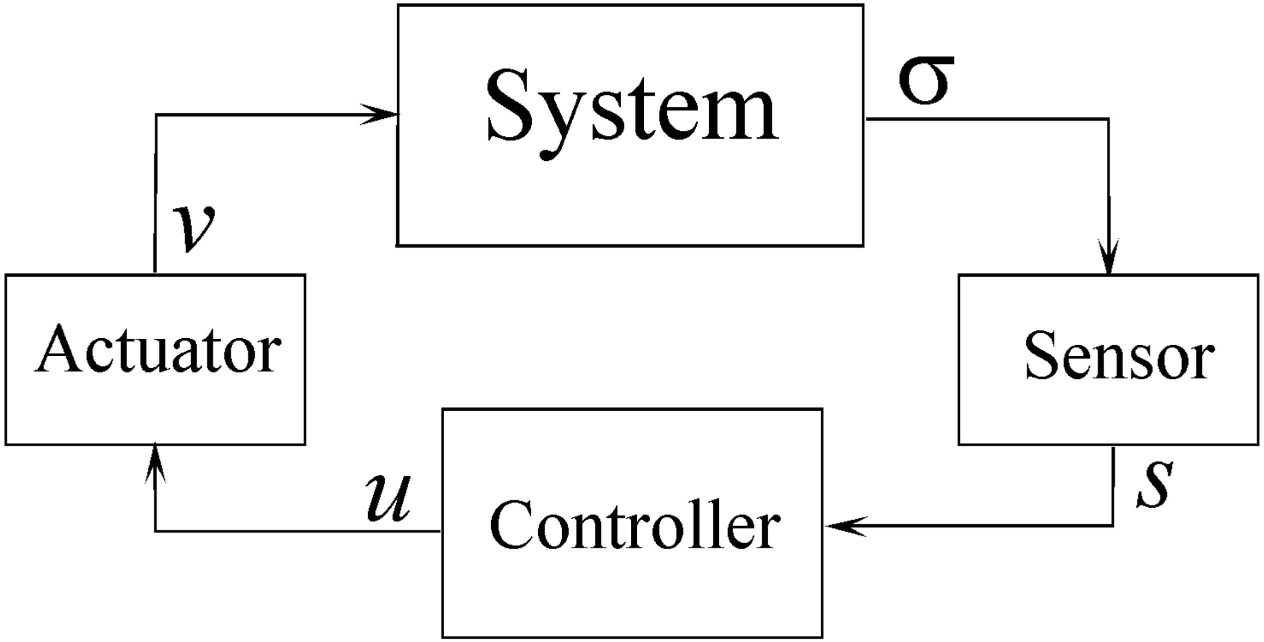
$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g \neq 0 \quad \tfrac{\partial}{\partial u}\sigma^{(r)} = g, \quad \sigma^{(r)}|_{u=0} = h,$$

2. $0 < K_{\mathrm{m}} \leq \tfrac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}$, $|\,\sigma^{(r)}|_{u=0}| \leq C$.

3. $|u(t)|$ bounded $\Rightarrow \exists\ x(t),\ t \in [t_0, \infty)$

In practice: $r = 2, 3, 4, 5$ (mechanical systems)

SISO Black-Box Control



The goal is to keep $\sigma \equiv 0$.

The uncertain system dynamics is smooth.

The very presence of an actuator and sensor is uncertain. The control is to be robust to their presence.

The main result - 1

A list of universal Single Input - Single Output controllers is developed, solving the stated problem for any given relative degree

in finite time and with ideal accuracy.

The controllers depend on $\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}$, include exact robust differentiators, are insensitive to any disturbances preserving the assumptions, and are robust with respect to

- Lebesgue-measurable bounded output noises
- **any** small smooth system disturbances
- discrete sampling, small delays
- unaccounted-for fast stable dynamics of sensors and actuators

The control signal can be made as smooth as needed.

The main result – 1a

The convergence can be made arbitrarily fast under the considered conditions.

In the case when the assumptions 1, 2 are only locally valid, also the controller will be only locally effective. This still makes sense due to the finite-time convergence to the mode $\sigma \equiv 0$.

Differentiation: main results

In the absence of noises the proposed differentiator in finite time yields **exact** successive k derivatives of any signal, provided its kth derivative is a Lipschitzian function with a known Lipschitz constant.

If the signal is corrupted by a bounded Lebesgue-measurable noise, the evaluated derivatives differ from the ideal ones. But *the error continuously depends on the noise magnitude*, and the corresponding asymptotics is optimal.

Main Restriction

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g \neq 0 \quad \tfrac{\partial}{\partial u}\sigma^{(r)} = g, \quad \sigma^{(r)}|_{u=0} = h,$$

$$0 < K_{\mathrm{m}} \leq \tfrac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}, \ |\ \sigma^{(r)}|_{u=0}| \leq C.$$

$\Rightarrow$ **The control is to be discontinuous at**

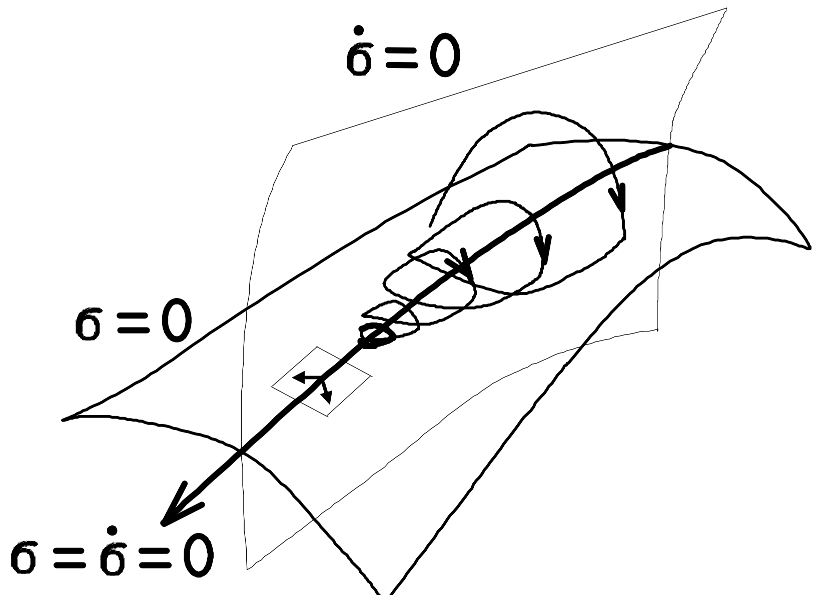
$$\sigma = \dot{\sigma} = \ldots = \sigma^{(r\,-\,1)} = 0$$

That means that an ***r*th order sliding mode** $\sigma \equiv 0$ is to be established.

Sliding mode order (informally)

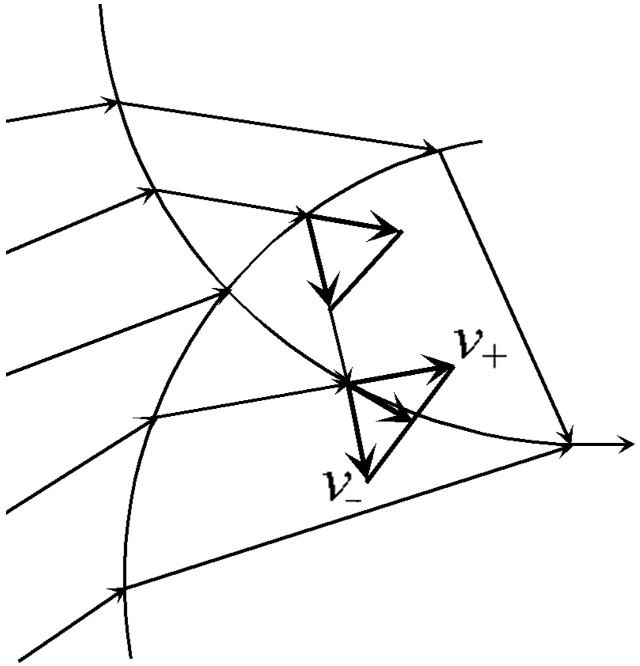
There is an ***r*th order sliding (*r*-sliding) mode** $\sigma \equiv 0$ in a system if

- $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the system coordinates and time,
- $\sigma^{(r)}$ is discontinuous.



2-sliding mode

Discontinuous Differential Equations
Filippov Definition



$$\dot{x} = v(x) \;\Leftrightarrow\; \dot{x} \in V(x)$$

$x(t)$ is an absolutely continuous function

$$V(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu N=0} \text{convex\_closure}\, v(O_\varepsilon(x) \setminus N)$$

v_+, v_- - limit values, $\dot{x} = pv_+ + (1\text{-}p)v_-$, $\quad p \in [0,1]$

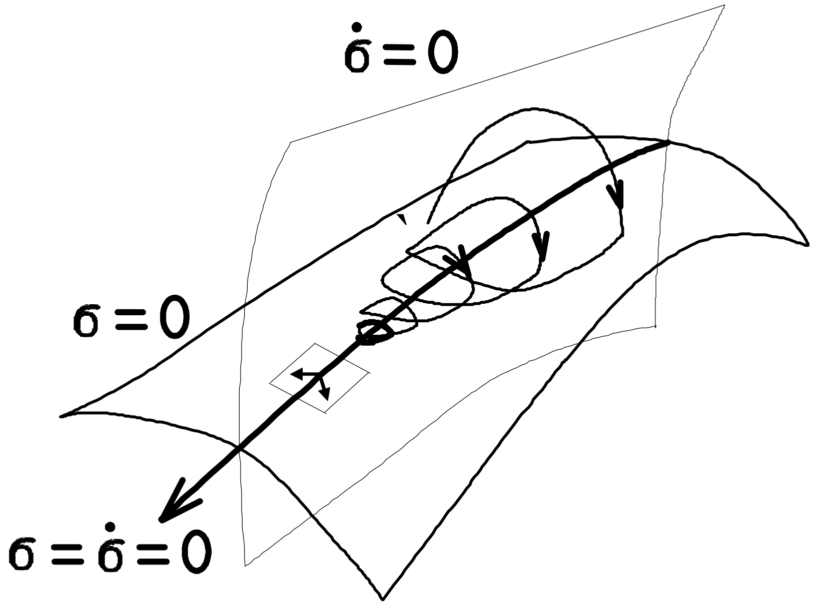
Solutions exist for any locally bounded Lebesgue-measurable $v(x)$; or for any upper-semicontinuous, convex, closed, non-empty, locally-bounded $V(x)$.

Formal sliding mode order definition

$$\dot{x} = v(x),\ \ x \in \mathbf{R}^n,\ \sigma\colon \mathbf{R}^n \to \mathbf{R}$$

The motion $\sigma \equiv 0$ is r-sliding mode with respect to constraint function σ if

1. $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r\text{-}1)}$ are continuous functions
2. $L_r = \{x \mid \ \sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r\text{-}1)} = 0\} \neq \varnothing$ - integral set
3. The Filippov set at L_r contains more than one vector



2-sliding mode

Non-autonomous case: $\dot{t} = 1$

The main idea

Relative degree is $r \Rightarrow$

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g \neq 0$$

$$h, g \text{ unknown: } 0 < K_\mathrm{m} \leq |g| \leq K_\mathrm{M}, \quad |h| \leq C.$$

$$\Rightarrow \quad \sigma^{(r)} \in [- C, C] + [K_\mathrm{m}, K_\mathrm{M}]u$$

To find

$$u = U(\sigma, \dot{\sigma}, ..., \sigma^{(r - 1)}) : \sigma, \dot{\sigma}, ..., \sigma^{(r - 1)} \to 0 \text{ in finite time}$$

1. $C > 0 \Rightarrow$ U is **discontinuous** at 0

2. Differentiator is needed!

Homogeneity of a function

$$f\colon \mathbf{R}^n \to \mathbf{R}$$

Weights (degrees) of the coordinates are chosen:

$$\deg x_i = m_i \geq 0,\ i = 1, 2, \ldots, n$$

Dilation is a linear transformation

$$d_\kappa\colon (x_1, x_2, \ldots, x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, \ldots, \kappa^{m_n} x_n)$$

$$\deg f = q \iff \forall x\ \forall \kappa > 0 \quad f(d_\kappa x) = \kappa^q f(x)$$

Homogeneity of differential inclusions

Formally: $\dot{x} \in F(x)$, $x \in \mathbf{R}^n$ (in particular $\dot{x} = f(x)$)

- p is the homogeneity degree if $\forall x\ \forall \kappa > 0 \quad F(x) = \kappa^{p} d_\kappa^{-1} F(d_\kappa x)$

Equivalent definition:

The homogeneity degree is - p, if deg $t = p$ and

$$\forall x\ \forall \kappa > 0 \qquad (t, x) \mapsto (\kappa^p t, d_\kappa x)$$

preserves the differential inclusion (equation).

The homogeneity degree can be always scaled to ± 1 or 0:

$$\kappa^p = (\kappa^s)^{p/s}, \quad \kappa^{m_i} = (\kappa^s)^{m_i/s}, \quad \kappa := \kappa^s$$

Homogeneity: informal explanation

Arithmetical operations "+" and "–"are only allowed for the operands of the same weight.

Examples:

$m_1 = 1,\ m_2 = 3.\ p = \deg t = 2.$

*H*omogeneity degree $q = -2$

$\deg (x_2{}^5) = 3{\cdot}5 = 15:\ (\kappa^3 x_2)^5 = \kappa^{15} x_2{}^5$

$\deg (x_1{}^2 x_2) = 1{\cdot}2+3 = 5$

$\deg (\dot{x}_2) = 3 - 2 = 1:\quad d(\kappa^3 x_2)/\mathrm{d}(\kappa^2 t) = \kappa \dot{x}_2$

Differential inclusion:

$$|x_2 \dot{x}_1| + \dot{x}_2{}^2 \le |x_1{}^3 + x_2|^{2/3} \qquad 3+1-2 = (3-2)2 = 3{\cdot}2/3$$

Equivalent form:

$$(\dot{x}_1, \dot{x}_2) \in \{(z_1, z_2) \in \mathbf{R}^2 \big|\ |x_2 z_1| + z_2{}^2 \le |x_1{}^3 + x_2|^{2/3}\}$$

Stability of differential inclusions

Suppose $x(t)$ exists for any $t > 0$.

Global uniform finite-time stability at 0

$\forall\delta > 0\ \exists T > 0$: $|x(0)| < \delta,\ t \geq T \Rightarrow x(t) = 0$

$\forall\Omega$ compact $\quad \exists T > 0$: $\forall x(0) \in \Omega$, $\forall t \geq T \Rightarrow\ x(t) = 0$

Global uniform asymptotic stability at 0

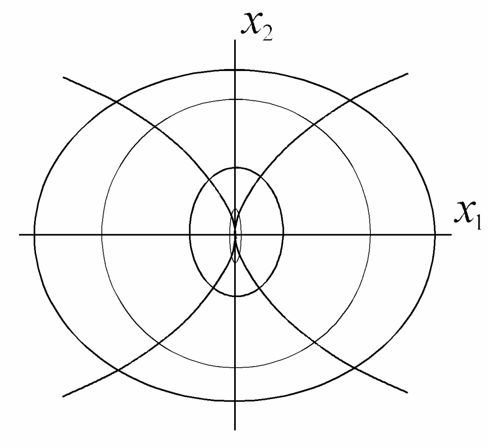
$\forall\delta > 0\ \forall\varepsilon > 0\ \exists\ t_0 > 0$: $|x(0)| < \delta,\ t \geq t_0 \Rightarrow \|x(t)\| < \varepsilon$

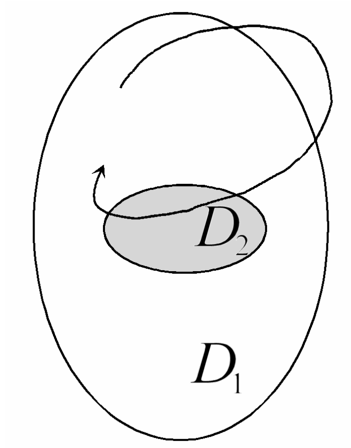
$\forall\Omega$ compact $\quad \forall\varepsilon > 0\ \exists t_0 > 0$: $\forall x(0) \in \Omega$, $\forall t \geq t_0 \Rightarrow\ \|x(t)\| < \varepsilon$

Contraction of homogeneous differential inclusions

$\dot{x} \in F(x)$ is homogeneous.

$D \subset \mathbf{R}^n$ is called *dilation-retractable* if $d_\kappa D \subset D$ for any $0 \le \kappa \le 1$.





$\dot{x} \in F(x)$ is *contractive* if $\exists\ D_1, D_2$ compact sets, $\exists\ T$:

1. $0 \in D_2 \subset \text{interior}(D_1)$,
2. D_1 is dilation-retractable
3. $\forall x(0) \in D_1 \Rightarrow x(T) \in D_2$

Finite-time stability of homogeneous differential inclusions

Theorem Levant (2005)**.**

Let the homogeneity degree be - $p < 0$.

Then the following properties are equivalent:

- Global finite-time stability
- Global uniform asymptotic stability
- Contraction property

The settling time is a continuous homogeneous function of the initial conditions of the weight p.

Finite-time stability is robust with respect to small homogeneous perturbations!

Accuracy of finite-time stable homogeneous systems (Levant, 2005)

$$d_\kappa: (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$$

$$\dot{x} \in F(x) \Leftrightarrow \frac{d}{d\kappa^p t}(d_\kappa x) \in F(d_\kappa x), \ \ p > 0$$

Let $\delta > 0$.

$\forall i$: x_i is measured with error $\leq \mu_i \delta^{m_i}$

maximal time delay $\leq \nu_i \delta^p$.

The solution can be indefinitely extended in time.

Then starting from some moment

$$\| x_i \| \leq \gamma_i \delta^{m_i}$$

$\gamma_i > 0$ do not depend on the initial conditions and δ.

r-sliding homogeneity

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]\, U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$$

Let the weight of time t be 1.
With $C > 0$ the only possible homogeneity weight of the right side is 0.

$\Rightarrow$

$\deg(\sigma^{(r-1)}) -1 = 0$

$\deg(\sigma^{(r-2)}) -1 = \deg(\sigma^{(r-1)})$

…

$\Rightarrow \deg(\sigma) = r, \ldots, \deg(\sigma^{(r-1)}) = 1, \deg(\sigma^{(r)}) = 0$

$\forall\kappa > 0 \qquad U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) = U(\kappa^{r}\sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa\sigma^{(r-1)})$

r-sliding homogeneity:

The weights:

deg t =1,

deg $\sigma = r$, deg $\dot{\sigma} = r - 1$, ..., deg $\sigma^{(r-1)} = 1$, deg $\sigma^{(r)} = 0$.

Invariance $\forall \kappa > 0$:

$(t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \longmapsto (\kappa t, \kappa^r \sigma, \kappa^{r-1} \dot{\sigma}, ..., \kappa \sigma^{(r-1)})$

In other words deg $U = 0$

$$U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) = U(\kappa^r \sigma, \kappa^{r-1} \dot{\sigma}, ..., \kappa \sigma^{(r-1)})$$

Standard known solution: *r* = 1

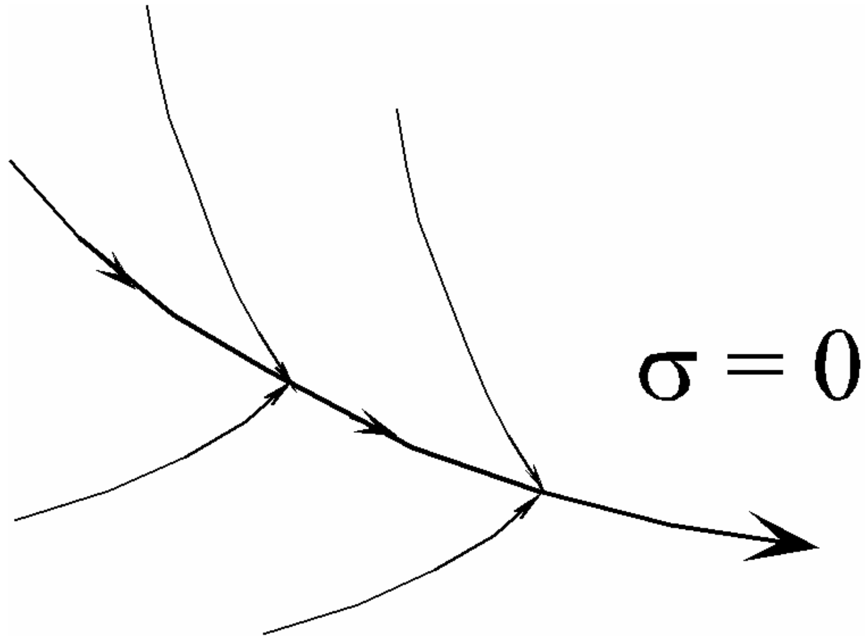
1-sliding mode (standard)

$$\dot{x} = a(t,x) + b(t,x)u$$

$$\dot{\sigma} = h(t,x) + g(t,x)u, \quad g > 0$$

Locally: $u = -\, k \operatorname{sign} \sigma \Rightarrow$

$$\sigma\dot{\sigma} < 0, \ \sigma \neq 0$$



$\forall \kappa > 0 \quad \operatorname{sign} \sigma = \operatorname{sign} (\kappa\sigma)$

Main drawback: chattering effect

r = 2: prescribed law of σ variation (1986)

(homogeneous terminal control)

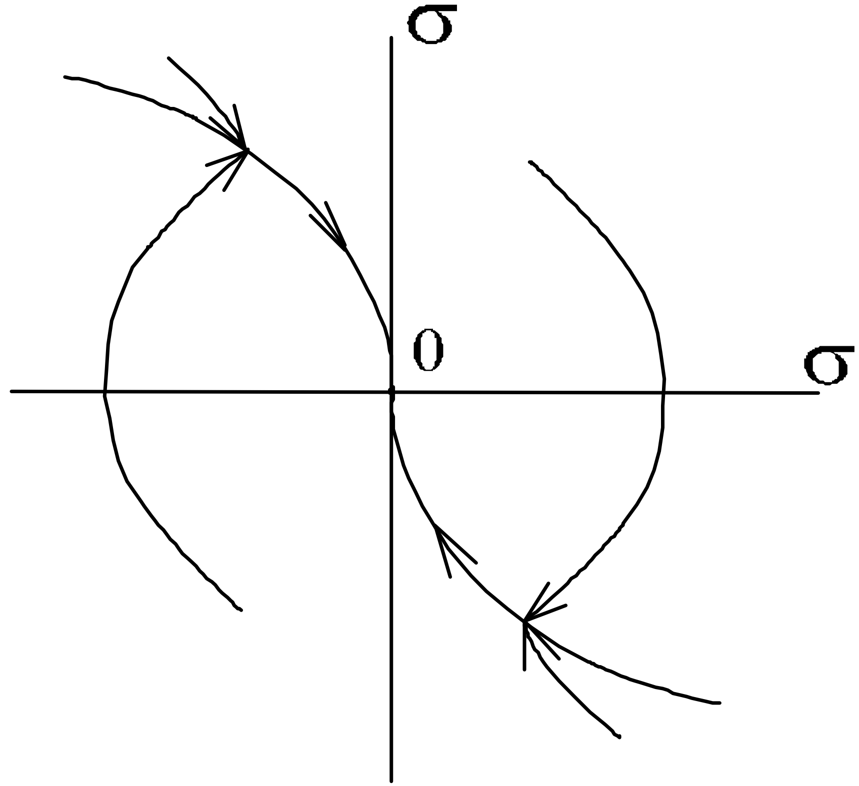
$$\ddot{\sigma} = h(t,x) + g(t,x)u, \quad g > 0$$

$$u = -\alpha\,\mathrm{sign}(\dot{\sigma} + \lambda\,|\sigma|^{1/2}\,\mathrm{sign}\,\sigma)$$

$$\alpha > 0,\ \lambda > 0,\ \alpha > (C + 0.5\,\lambda^2)\,/\,K_m.$$

$\forall\kappa > 0$

$$\mathrm{sign}(\dot{\sigma} + \lambda\,|\sigma|^{1/2}\,\mathrm{sign}\,\sigma) =$$
$$\mathrm{sign}(\kappa\dot{\sigma} + \lambda\,|\kappa^2\sigma|^{1/2}\,\mathrm{sign}\,(\kappa^2\sigma))$$



Arbitrary order sliding mode controllers

$$u = -\alpha\, \Psi_{r\text{-}1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r\text{-}1)})$$

$\alpha, \beta_1, ..., \beta_{r\text{-}1} > 0$: $i = 1, ..., r\text{-}1$;

1. **Nested *r*-sliding controller.** Let $q > 1$

$$N_{i,r} = (|\sigma|^{q/r} + |\dot{\sigma}|^{q/(r\text{-}1)} + ... + |\sigma^{(i\text{-}1)}|^{q/(r\text{-}i+1)})^{(r\text{-}i)/q} \ ;$$

$$\Psi_{0,r} = \text{sign}\, \sigma, \quad \Psi_{i,r} = \text{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i\text{-}1,r});$$

2. **Quasi-continuous** (continuous everywhere except $\sigma = \dot{\sigma} = ... = \sigma^{(r\text{-}1)} = 0$)

$$\varphi_{0,r} = \sigma, \ N_{0,r} = |\sigma|, \ \Psi_{0,r} = \varphi_{0,r} / N_{0,r} = \text{sign}\, \sigma,$$

$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)} \Psi_{i\text{-}1,r},$$

$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-1)/(r-i+1)},$$

$$\Psi_{i,r} = \varphi_{i,r} / N_{i,r}$$

List of quasi-continuous controllers ($r = 1 - 4$)

1. $u = - \alpha \operatorname{sign} \sigma$,

2. $u = - \alpha \left(|\dot\sigma| + |\sigma|^{1/2}\right)^{-1} \left(\dot\sigma + |\sigma|^{1/2} \operatorname{sign} \sigma\right)$,

3. $u = - \alpha \left[|\ddot\sigma| + 2\left(|\dot\sigma| + |\sigma|^{2/3}\right)^{-1/2} |\dot\sigma + |\sigma|^{2/3} \operatorname{sign} \sigma|\right]^{-1}$
$\left[\ddot\sigma + 2\left(|\dot\sigma| + |\sigma|^{2/3}\right)^{-1/2}\left(\dot\sigma + |\sigma|^{2/3} \operatorname{sign} \sigma\right)\right]$,

4. $\varphi_{3,4} = \dddot\sigma + 3\left[\ddot\sigma + \left(|\dot\sigma| + 0.5|\sigma|^{3/4}\right)^{-1/3} |\dot\sigma + 0.5|\sigma|^{3/4} \operatorname{sign} \sigma|\right]^{-1/2}$
$\left[\ddot\sigma + \left(|\dot\sigma| + 0.5|\sigma|^{3/4}\right)^{-1/3}\left(\dot\sigma + 0.5|\sigma|^{3/4} \operatorname{sign} \sigma\right)\right]$,

$N_{3,4} = |\dddot\sigma| + 3\left[|\ddot\sigma| + \left(|\dot\sigma| + 0.5|\sigma|^{3/4}\right)^{-1/3} |\dot\sigma + 0.5\,|\sigma|^{3/4} \operatorname{sign} \sigma|\right]^{-1/2}$
$|\ddot\sigma + \left(|\dot\sigma| + 0.5|\sigma|^{3/4}\right)^{-1/3}\left(\dot\sigma + 0.5\,|\sigma|^{3/4} \operatorname{sign} \sigma\right)|$,

$u = - \alpha\, \varphi_{3,4} / N_{3,4}$.

Quasi-continuous controller $r = 2$

$$u = -\alpha \frac{\dot\sigma + |\sigma|^{1/2}\operatorname{sign}\sigma}{|\dot\sigma| + |\sigma|^{1/2}}$$

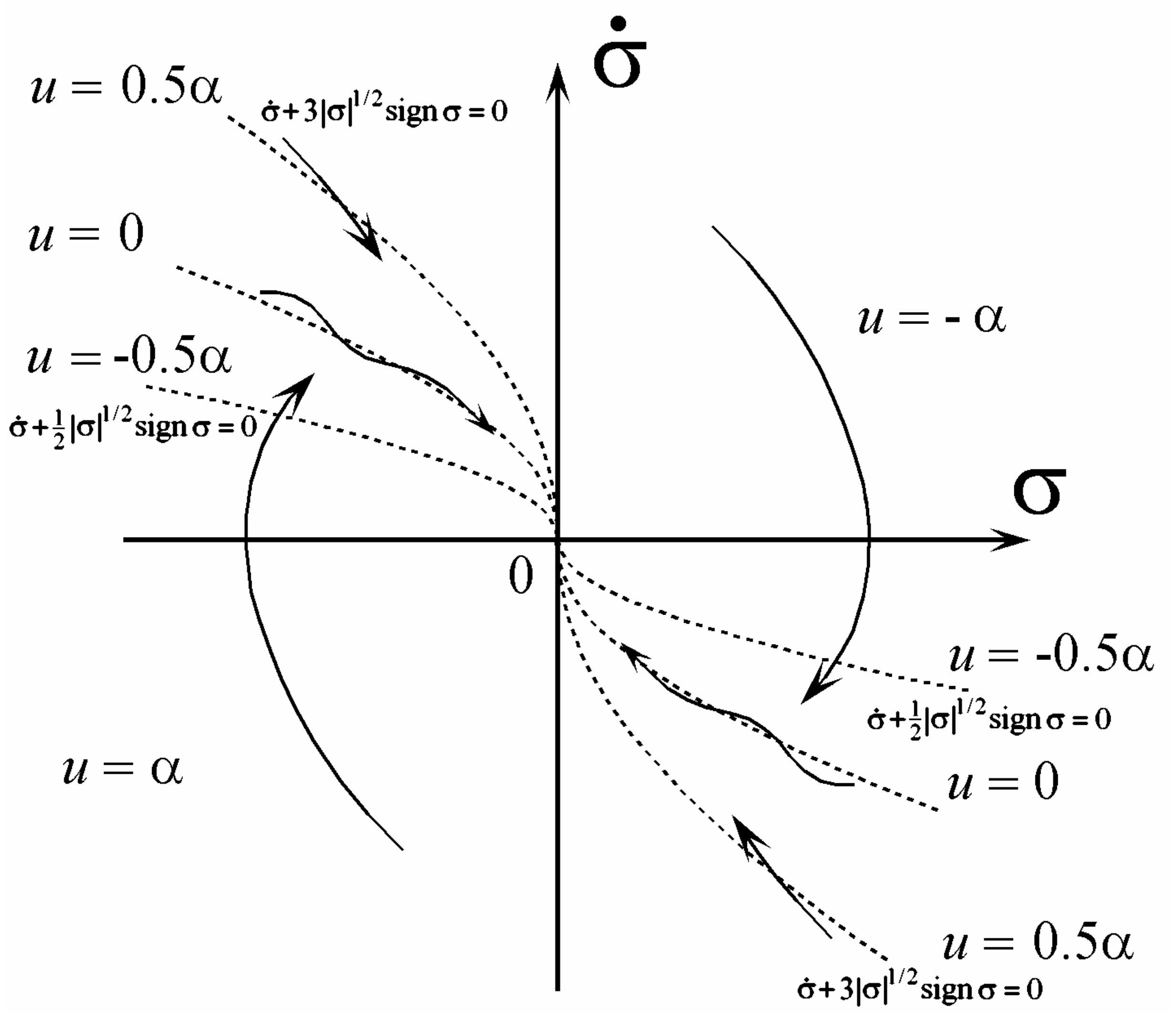
Weights: $\sigma \sim 2$, $\dot\sigma \sim 1$

Quasi-continuous controller $r = 3$

$$u = -\alpha \frac{\ddot\sigma + 2\dfrac{(\dot\sigma + |\sigma|^{2/3}\operatorname{sign}\sigma)}{(|\dot\sigma| + |\sigma|^{2/3})^{1/2}}}{|\ddot\sigma| + 2(|\dot\sigma| + |\sigma|^{2/3})^{1/2}}$$

Weights: $\sigma \sim 3$, $\dot\sigma \sim 2$, $\ddot\sigma \sim 1$

Explanation: $r = 2$: $u = -\alpha \dfrac{\dot\sigma + |\sigma|^{1/2}\,\mathrm{sign}\,\sigma}{|\dot\sigma| + |\sigma|^{1/2}}$



Theorem

$$u = -\alpha\Psi_{r\text{-}1,r}(\sigma, \dot\sigma, \dots, \sigma^{(r\text{-}1)})$$

provides for the r-sliding mode $\sigma \equiv 0$ *in finite time*.

Discrete measurements$\Rightarrow$$r$-real-sliding mode:

$$|\sigma| < a_0\tau^r,\ |\dot\sigma| < a_1\tau^{r\text{-}1},\ \dots,\ |\sigma^{(r\text{-}1)}| < a_{r\text{-}1}\tau.$$

Chattering avoidance:

$\dot u = u_1 \quad \Rightarrow \quad u_1$ is the new control, $r := r + 1$

The idea of the proof

$$\sigma^{(r)} \in [-C, C] - \alpha\, [K_{\mathrm{m}}, K_{\mathrm{M}}]\, \Psi_{r-1,r}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$$

The inclusion is *r*-sliding homogeneous.

Contraction $\Rightarrow$ Finite-time stability $\Rightarrow$ Accuracy

Differentiation problematics:

Division by zero: $f'(t) = \lim\limits_{\tau \to 0} \dfrac{f(t+\tau) - f(t)}{\tau}$

Let $f(t) = f_0(t) + \eta(t)$, $\eta(t)$ – noise

$$\frac{f(t+\tau) - f(t)}{\tau} = \frac{\Delta f}{\tau} + \frac{\Delta \eta}{\tau}, \quad \frac{\Delta \eta}{\tau} \in (-\infty, \infty)$$

Differentiation Problem

Input:

$$f(t) = f_0(t) + \eta(t), \qquad |\, \eta \,| < \varepsilon$$

$\eta(t)$ - Lebesgue-measurable function,

f_0 ,η, ε are unknown, known : $\quad |f_0^{(k+1)}(t)| \le L$

(or |Lipschitz constant of $f_0^{(k)}| \le L$)

The goal:

real-time estimation of $\dot{f}_0(t), \ddot{f}_0(t), ..., f_0^{(k)}(t)$

Differentiator (Levant 1998, 2003)

$$\dot{z}_0 = v_0\ , v_0 = -\lambda_k \,|\, z_0 - f(t)|^{k/(k+1)} \operatorname{sign}(z_0 - f(t)) + z_1,$$

$$\dot{z}_1 = v_1\ , v_1 = -\lambda_{k-1} \,|\, z_1 - v_0|^{(k-1)/k} \operatorname{sign}(z_1 - v_0) + z_2,$$

$$\ldots$$

$$\dot{z}_{k-1} = v_{k-1}\ , \quad v_{k-1} = -\lambda_1 \,|\, z_{k-1} - v_{k-2}|^{1/2} \operatorname{sign}(z_{k-1} - v_{k-2}) + z_k,$$

$$\dot{z}_k = -\lambda_0 \operatorname{sign}(z_k - v_{k-1}).$$

$|f^{(k+1)}| \le L$; z_i is the estimation of $f^{(i)}$, $i = 0, 1, \ldots, k$.

Optional choice:

$$\lambda_0 = 1.1L,\ \lambda_1 = 1.5L^{1/2},\ \lambda_2 = 2L^{1/3},\ \lambda_3 = 3L^{1/4},$$
$$\lambda_4 = 5L^{1/5},\ \lambda_5 = 8L^{1/6},\ \ldots$$

5th-order differentiator, $|f^{(6)}| \leq L$.

$$\dot{z}_0 = v_0 \,,\; v_0 = -8\, L^{1/6} |\, z_0 - f(t)|^{\,5/6} \text{sign}(z_0 - f(t)) + z_1 \,,$$

$$\dot{z}_1 = v_1 \,,\; v_1 = -5\, L^{1/5} |\, z_1 - v_0|^{\,4/5} \text{ sign}(z_1 - v_0) + z_2 \,,$$

$$\dot{z}_2 = v_2 \,,\; v_2 = -3\, L^{1/4} |\, z_2 - v_1|^{\,3/4} \text{ sign}(z_2 - v_1) + z_3 \,,$$

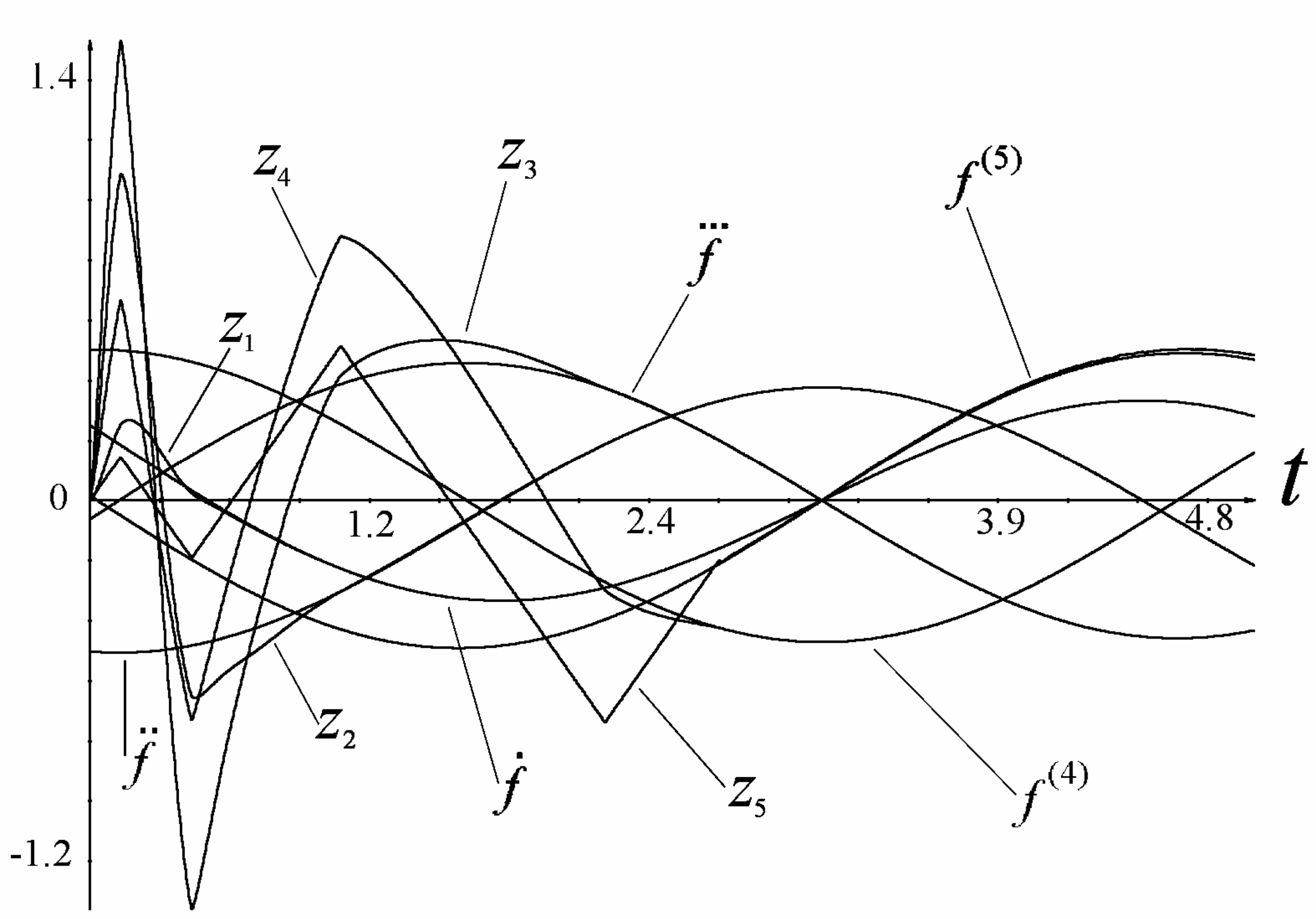
$$\dot{z}_3 = v_3 \,,\; v_3 = -2\, L^{1/3} |\, z_3 - v_2|^{\,2/3} \text{ sign}(z_3 - v_2) + z_4 \,,$$

$$\dot{z}_4 = v_4 \,,\; v_4 = -1.5\, L^{1/2} |\, z_4 - v_3|^{\,1/2} \text{ sign}(z_4 - v_3) + z_5 \,,$$

$$\dot{z}_5 = -1.1\, L \text{ sign}(z_5 - v_4);$$

5th-order differentiation

$$f(t) = \sin 0.5t + \cos 0.5t, \quad L = 1$$



The differentiation accuracy

$\varepsilon = 0$ (no noise) $\Rightarrow$ in a finite time

$$z_0 = f_0(t); \qquad z_i = v_{i\text{-}1} = f_0^{(i)}(t), \quad i = 1, ..., k.$$

In the presence of the noise with the magnitude ε

$$|z_i - f_0^{(i)}(t)| \leq \mu_i \varepsilon^{(k\text{-}i+1)/(k+1)}, \;\; |z_k - f_0^{(k)}(t)| \leq \mu_n \varepsilon^{1/(k+1)}$$

Discrete-sampling case with the sampling step τ:

$$|z_i - f_0^{(i)}(t)| \leq \nu_i \, \tau^{k - i + 1}, \;\; |z_k - f_0^{(k)}(t)| \leq \nu_k \, \tau$$

This asymptotics cannot be improved!

(Kolmogorov, ≈ 1935)

The idea of the differentiator proof - 1

Denote $s_i = z_i - f_0^{(i)}(t)$, then

$$\dot{s}_0 = -\lambda_0 \,|\, s_0 \,|^{k/(k+1)} \operatorname{sign}(s_0) + s_1 \,,$$

$$\dot{s}_1 = -\lambda_1 \,|\, s_1 - \dot{s}_0 |^{(k-1)/k} \operatorname{sign}(s_1 - \dot{s}_0) + s_2,$$

$$\dots$$

$$\dot{s}_{k-1} = -\lambda_{k-1} \,|\, s_{k-1} - \dot{s}_{k-2} |^{1/2} \operatorname{sign}(s_{k-1} - \dot{s}_{k-2}) + s_k,$$

$$\dot{s}_k \in -\lambda_k \operatorname{sign}(s_k - \dot{s}_{k-1}) + [-L, L].$$

The idea of the differentiator proof – 2

The obtained differential inclusion is homogeneous with the dilation

$$\forall \kappa > 0 \; d_\kappa : (s_0, s_1, s_2, \ldots, s_k) \mapsto (\kappa^{k+1} s_0, \kappa^{k} s_1, \kappa^{k-1} s_2, \ldots, \kappa s_k)$$

and the negative homogeneity degree –1,

i.e. there is the invariance Trajectory $\mapsto$ Trajectory

$$(t, s_0, s_1, s_2, \ldots, s_k) \mapsto (\kappa t, \kappa^{k+1} s_0, \kappa^{k} s_1, \kappa^{k-1} s_2, \ldots, \kappa s_k).$$

contraction⇒finite-time stability⇒accuracy

Universal SISO controller

$$\dot{x} = a(t,x) + b(t,x)u,$$

$$\exists r\text{: } \frac{\partial}{\partial u}\sigma^{(r)} \neq 0,\ r \text{ is known, } 0 < K_{\mathrm{m}} \le \left|\frac{\partial}{\partial u}\sigma^{(r)}\right| \le K_{\mathrm{M}},\quad |\sigma^{(r)}|_{u=0}| \le C$$

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u \Rightarrow |\sigma^{(r)}| \le C + \alpha K_{\mathrm{M}}$$

$$u = -\,\alpha\, U_r(z_0, z_1, \ldots, z_{r-1}),$$

$$\dot{z}_0 = v_0,\ \ v_0 = -\lambda_0 L^{1/r}|z_0 - \sigma|^{(r-1)/r}\,\mathrm{sign}(z_0 - \sigma) + z_1,$$

$$\dot{z}_1 = v_1,\ \ v_1 = -\lambda_1 L^{1/(r-1)}|z_1 - v_0|^{(r-2)/(r-1)}\mathrm{sign}(z_1 - v_0) + z_2,$$

$$\ldots$$

$$\dot{z}_{r-2} = v_{r-2},\qquad v_{r-2} = -\lambda_{r-2} L^{1/2}|z_{r-2} - v_{r-3}|^{1/2}\mathrm{sign}(z_{r-2} - v_{r-3}) + z_{r-1},$$

$$\dot{z}_{r-1} = -\lambda_{r-1}\, L\ \mathrm{sign}(z_{r-1} - v_{r-2}),\quad L \ge C + \alpha K_{\mathrm{M}}\,,$$

λ_i are chosen as previously.

$\sigma^{(i)} = O(\varepsilon^{(r-i)/r})$, ε is the noise magnitude; $\sigma^{(i)} = O(\tau^{r-i})$, τ is the sampling interval

Accuracy of output-feedback homogeneous sliding-modes

Theorem.

Measurements: $\tilde{\sigma} = \sigma + \eta$, noise $|\eta| < \varepsilon$

r-sliding controller with $(r$-1)-differentiator

$\Rightarrow$ **after a finite-time transient with $\varepsilon > 0$**:

$$\sigma = O(\varepsilon), \quad \sigma^{(i)} = O(\varepsilon^{(r-i)/r})$$

$\varepsilon = 0$ (no measurement noises):

$$\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0 \text{ (ideal } r\text{-sliding)}$$

with discrete measurements: $\sigma \sim O(\tau^{r}), \quad \sigma^{(i)} \sim O(\tau^{r-i})$

The idea of the proof

Denote $s_i = z_i - \sigma^{(i)}(t)$,

$$\sigma^{(r)} \in [-C, C] + [K_\mathrm{m}, K_\mathrm{M}]u$$

$$u = -\alpha\, U_r\,(s_0+\sigma,\ s_1+\dot\sigma,\ \ldots,\ s_{r-1}+\sigma^{(r-1)}),$$

$$\dot s_0 = -\lambda_0\, L^{1/r}|s_0|^{(r-1)/r}\ \mathrm{sign}(s_0) + s_1\ ,$$

$$\dot s_1 = -\lambda_1 L^{1/(r-1)}|s_1-\dot s_0|^{(r-2)/(r-1)}\mathrm{sign}(s_1-\dot s_0) + s_2,$$

$$\ldots.$$

$$\dot s_{r-2} = -\lambda_{r-1}L^{1/2}|s_{r-2}-\dot s_{r-3}|^{1/2}\mathrm{sign}(s_{r-2}-\dot s_{r-3}) + s_{r-1},$$

$$\dot s_{r-1} \in -\lambda_r\ L\ \mathrm{sign}(s_r - \dot s_{r-2}) + [-L, L].$$

weights: $t \sim 1$, $s_i, \sigma^{(i)} \sim r - i$

contraction $\Rightarrow$ finite-time stability, accuracy

Chattering Removal

$\dot{u}$ (or any higher derivative $u^{(k)}$) is considered as a new control.

$\Rightarrow$ The relative degree is increased.
(r + 1)-sliding (or (r + k)-sliding) mode is established.

Any dangerous high-frequency vibrations are removed. Only vibrations featuring infinitesimal energy may persist.

Robustness with respect to singular perturbations (Levant, ECC 2007). The presence of fast stable actuators and sensors lead to the appearance of real (approximate) sliding modes, but do not cause high-frequency vibrations of considerable energy, i.e. the chattering.

Control magnitude adjustment

$$\sigma^{(r)} = h(t,x) + g(t,x)u$$

$$\alpha\, g(t,x)\Phi(t,x) > d + |h(t,x)|,\ \ d > 0$$

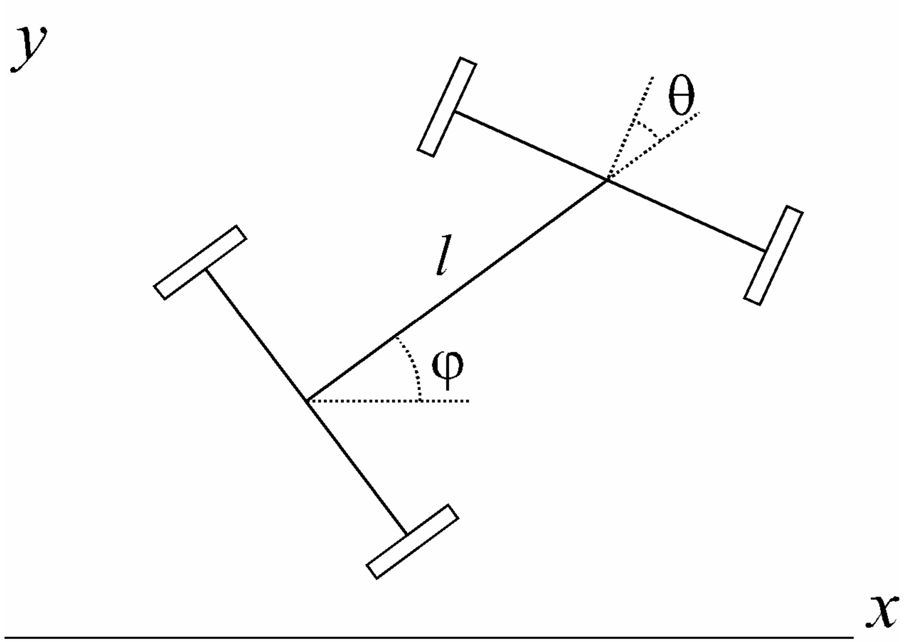
$$u = -\alpha\, \Phi(t,x)\Psi_{r-1,r}\,(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$$

Here $\alpha > 0$, and $\Psi_{r-1,r}$ is one of the introduced r-sliding homogeneous controllers.

$\Phi(t,x)$ can be arbitrarily increased preserving convergence.
Regularization of r-sliding mode is obtained if $\Phi(t,x)$ vanishes in a vicinity of the sliding mode.

Simulation

Car control



$\dot{x} = v \cos \varphi$, $\dot{y} = v \sin \varphi$,

$\dot{\varphi} = v/l \tan \theta$,

$\dot{\theta} = u$ (the relative degree is increased)

x, y are measured.

The task: real-time tracking $y = g(x)$

$v = const = 10$ m/s = 36 km/h, $l = 5$ m,

$x = y = \varphi = \theta = 0$ at $t = 0$

Solution: $\sigma = y - g(x)$, $r = 3$

3-sliding controller (N°3), $\alpha = 1$, $L = 500$

$\dot{u}$ - new control $\Rightarrow$

4-sliding controller (N°4), $\alpha = 5$, $L = 700$

3-sliding car control

$\sigma = y - g(x)$.

Simulation: $g(x) = 10 \sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$.

The controller:

$u = 0$, $0 \le t < 1$,

$u = - [z_2 + 2\,(|z_1| + |z_0|^{2/3})^{-1/2} (z_1 + |z_0|^{2/3} \operatorname{sign} z_0)]\,[|z_2| + 2\,(|z_1| + |z_0|^{2/3})^{1/2}]^{-1}$,

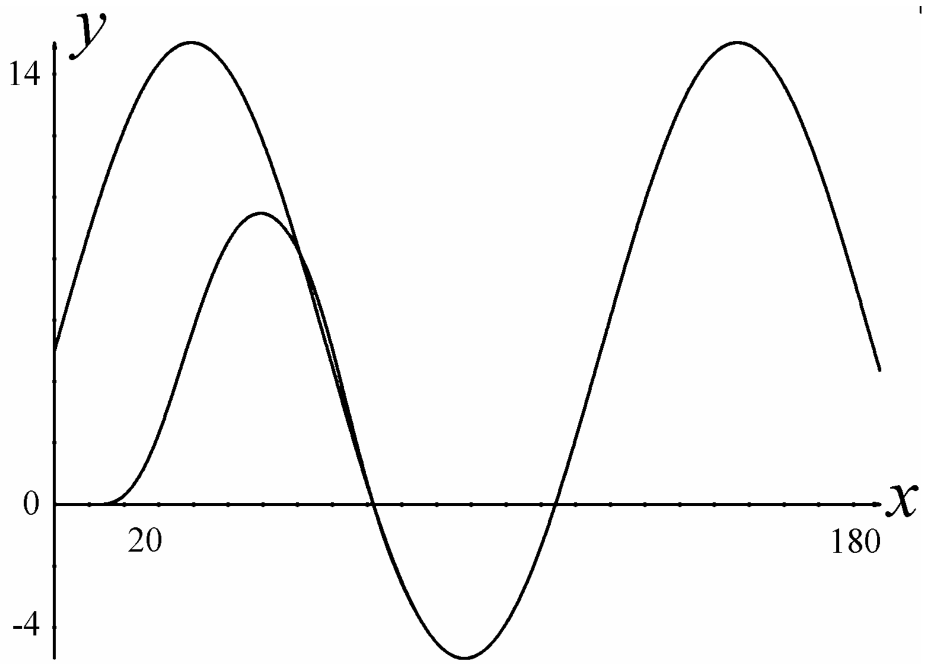
Differentiator:

$\dot z_0 = v_0$, $v_0 = - 15.9\,|z_0 - \sigma|^{2/3} \operatorname{sign}(z_0 - \sigma) + z_1$,

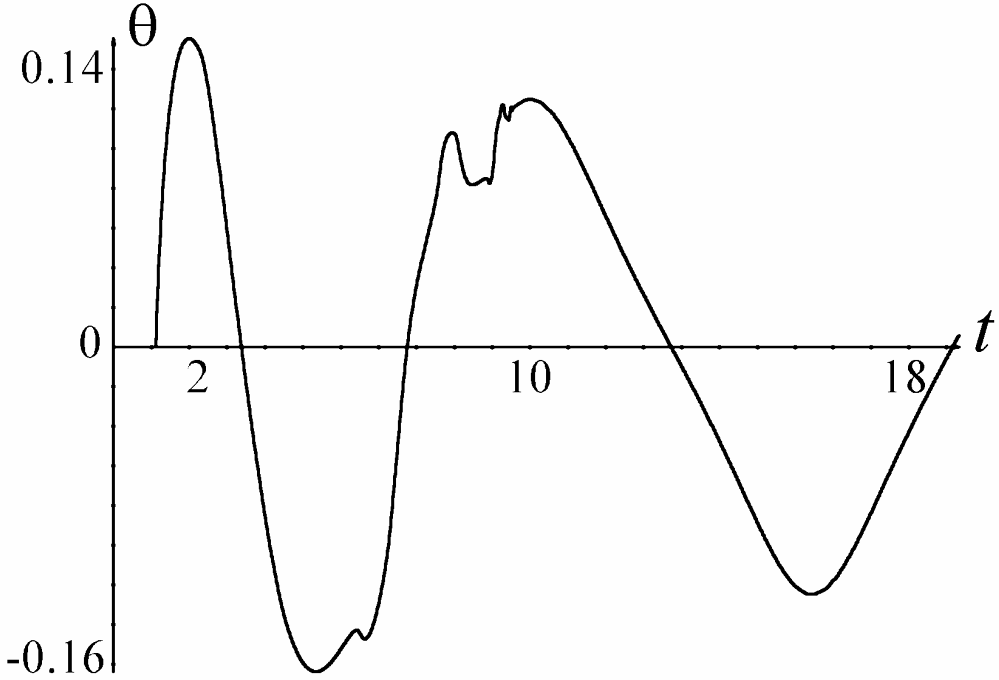
$\dot z_1 = v_1$, $v_1 = - 33.5\,|z_1 - v_0|^{1/2} \operatorname{sign}(z_1 - v_0) + z_2$,

$\dot z_2 = - 550 \operatorname{sign}(z_2 - v_1)$.

3-sliding car control - 1



Car trajectory

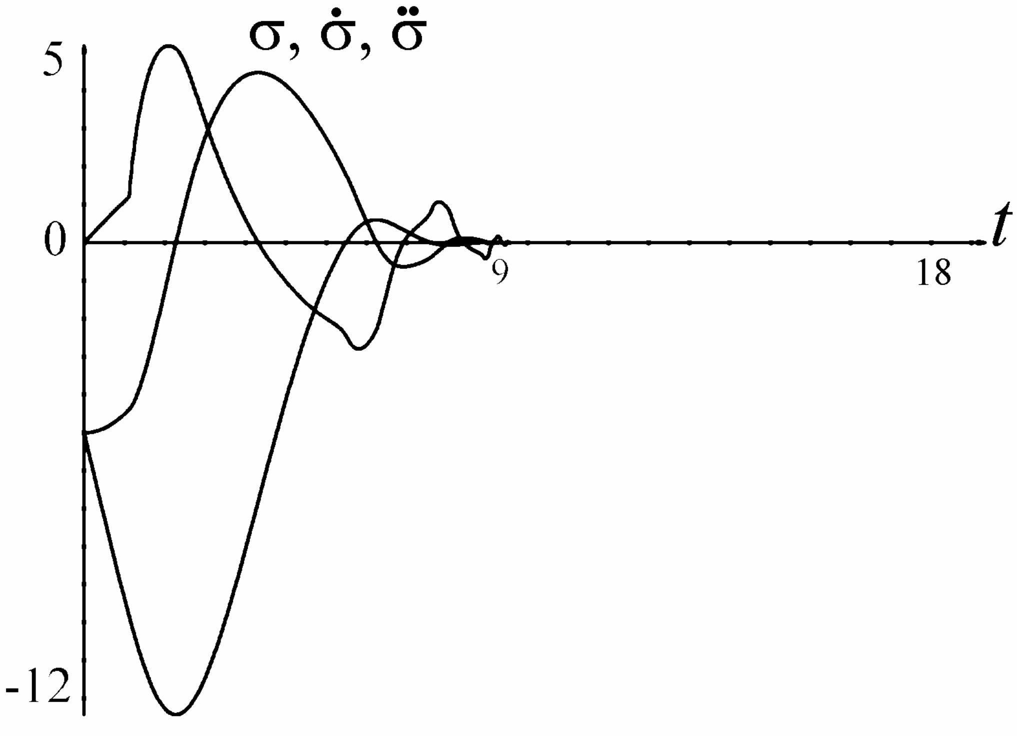


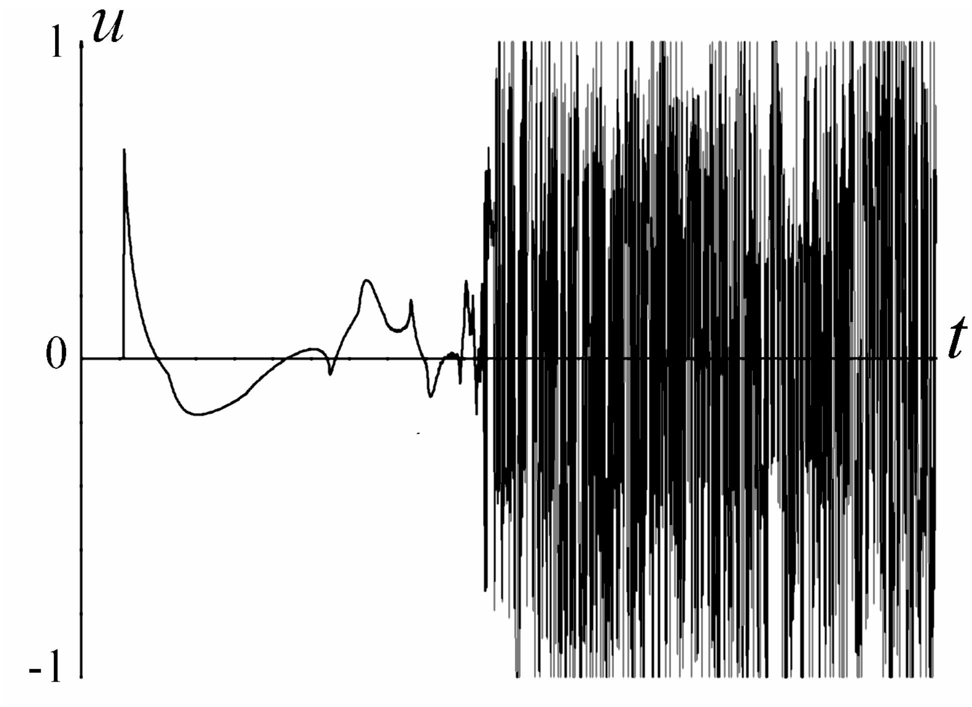
Steering angle

$\tau = 10^{-4} \Rightarrow \quad |\sigma| \le 5.4\cdot 10^{-7}, \; |\dot\sigma| \le 2.5\cdot 10^{-4}, |\ddot\sigma| \le 0.04$

$\tau = 10^{-5} \Rightarrow \quad |\sigma| \le 5.6\cdot 10^{-10}, \; |\dot\sigma| \le 1.4\cdot 10^{-5}, |\ddot\sigma| \le 0.004$

3-sliding car control – 2

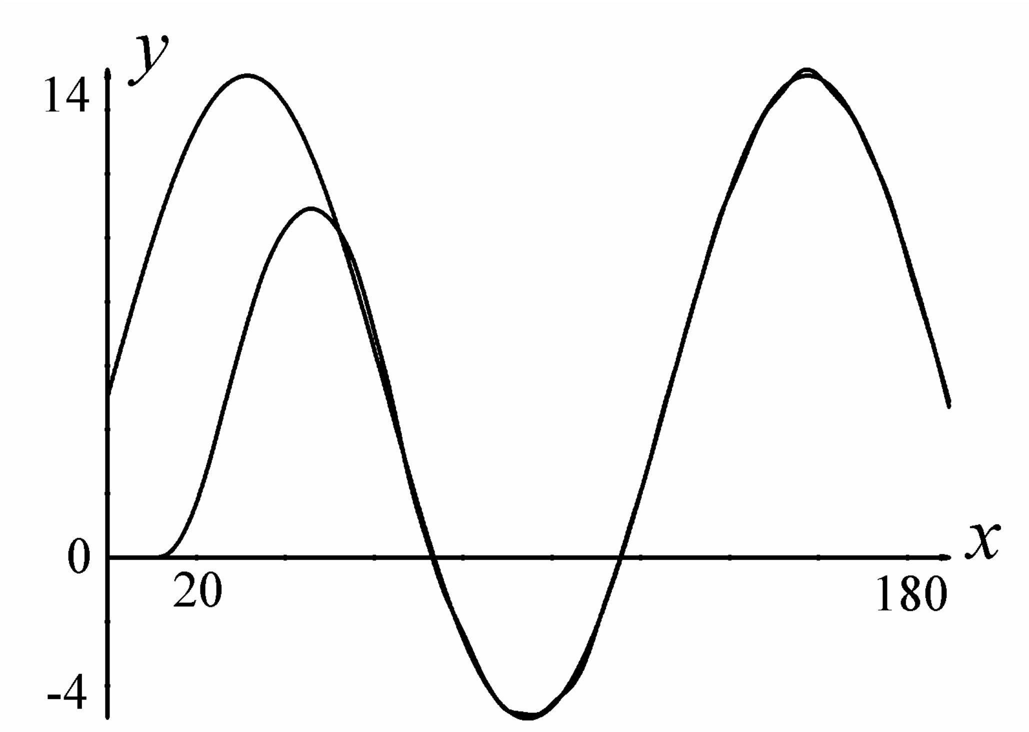




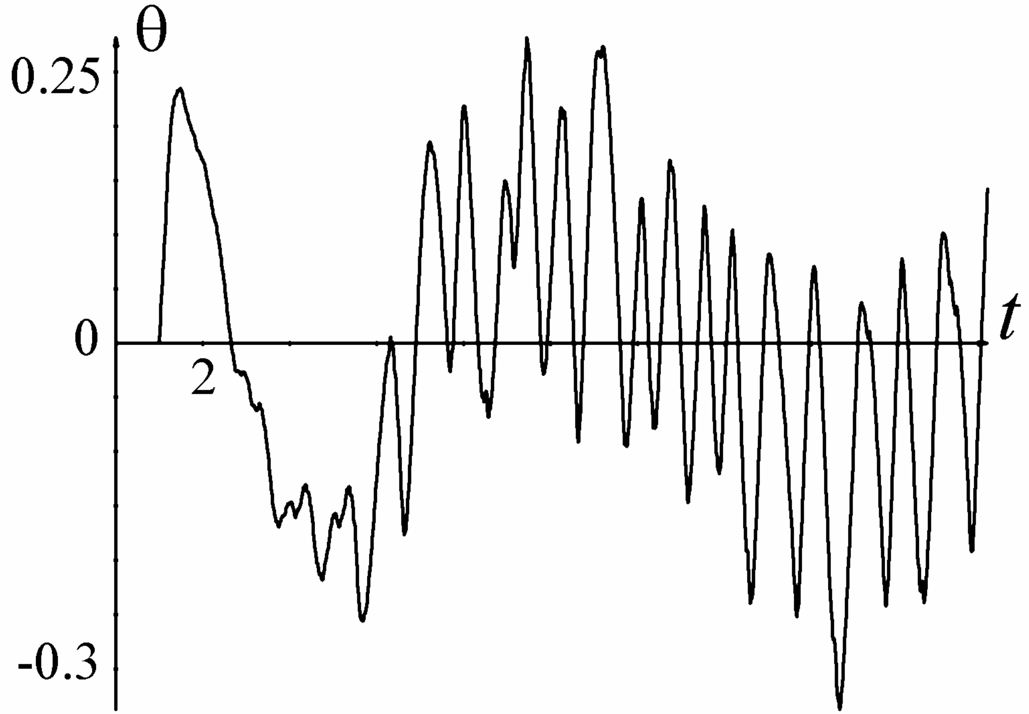
3-sliding deviations

Control

Performance with the input noise magnitude 0.1m



Car trajectory

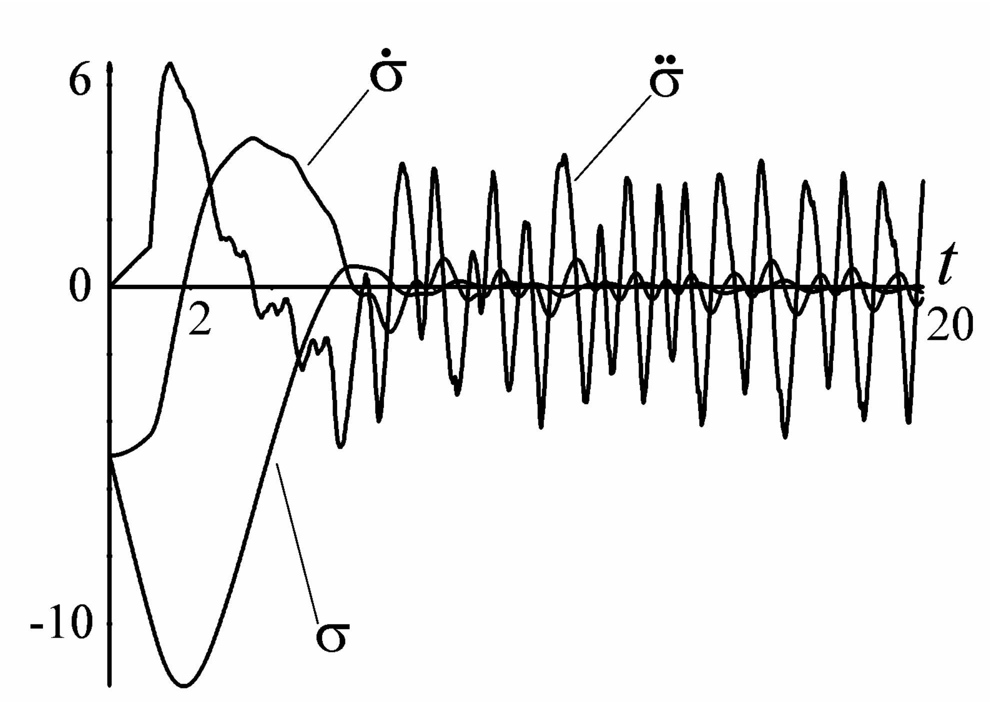


Steering angle

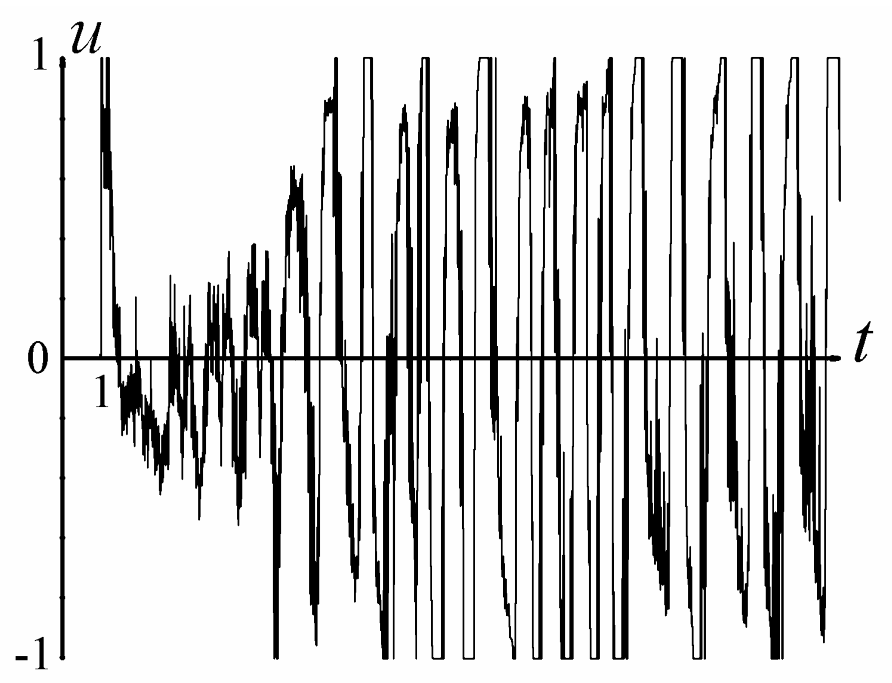
Noise of the magnitude 0.01m $\Rightarrow |\sigma| \leq 0.018,\ |\dot{\sigma}| \leq 0.15,\ |\ddot{\sigma}| \leq 1.9$

Noise of the magnitude 0.1m $\Rightarrow |\sigma| \leq 0.19,\ |\dot{\sigma}| \leq 0.81,\ |\ddot{\sigma}| \leq 4.5$

Performance with the input noise magnitude 0.1m



3-sliding deviations



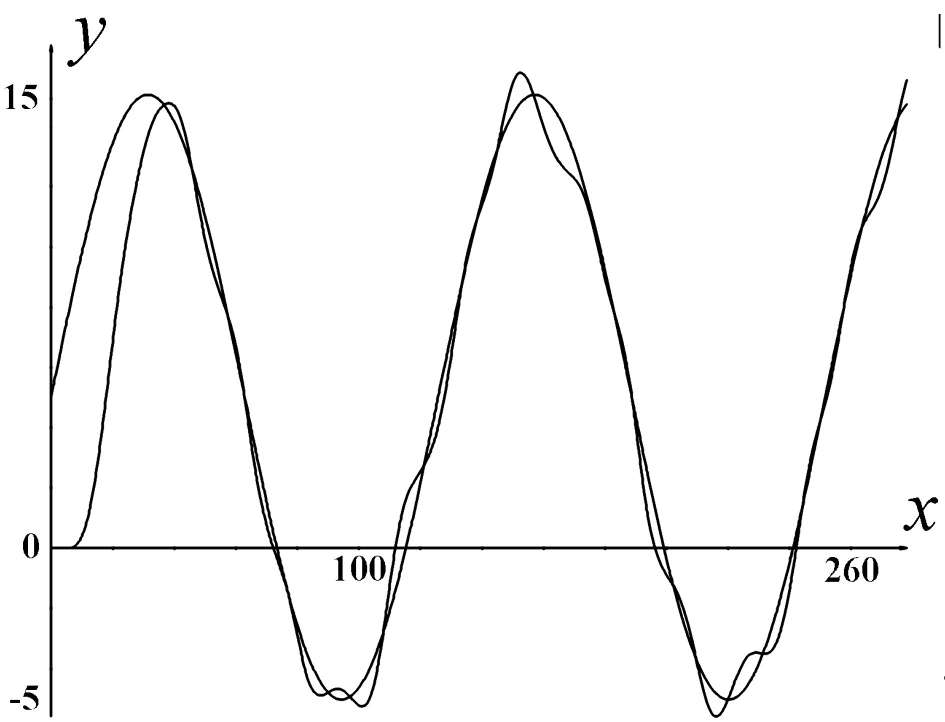
Control

Noise of the magnitude 0.01m $\Rightarrow |\sigma| \leq 0.018$, $|\dot{\sigma}| \leq 0.15$, $|\ddot{\sigma}| \leq 1.9$

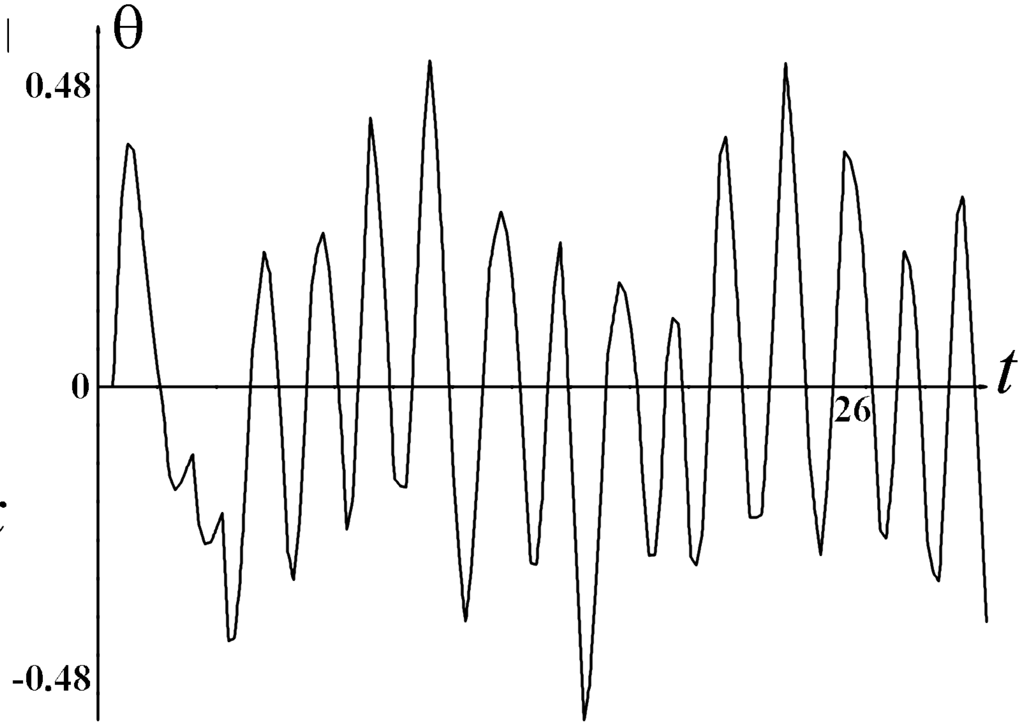
Noise of the magnitude 0.1m $\Rightarrow |\sigma| \leq 0.19$, $|\dot{\sigma}| \leq 0.81$, $|\ddot{\sigma}| \leq 4.5$

Constant sampling step $\tau = 0.2$s, $\varepsilon = 0.1$m

finite differences used, no differentiation



Car trajectory



Steering angle

$|\sigma| \leq 1.2\text{m}, \quad |\dot{\sigma}| \leq 2.9, \quad |\ddot{\sigma}| \leq 8.9$

4-sliding car control

$$\dot{x} = v\cos\varphi, \quad \dot{y} = v\sin\varphi, \quad \dot{\varphi} = \frac{v}{l}\tan\theta, \quad \dot{\theta} = v$$

Linear sensor (output s instead of $\sigma = y - g(x)$)

$$s = \hat{y} - g(x), \qquad \lambda^3 \dddot{\hat{y}} + 2\lambda^2 \ddot{\hat{y}} + 2\lambda \dot{\hat{y}} + \hat{y} = \lambda \dot{y} + y.$$

The initial values $\hat{y} = -10$, $\dot{\hat{y}} = 20$, $\ddot{\hat{y}} = -8$ were taken.

Nonlinear actuator (input u, output v)

$$\mu \dot{z}_1 = z_2, \qquad \mu \dot{z}_2 = -(z_1 - u)^3 + (z_1 - u) + z_2, \; v = z_1,$$

with zero initial conditions.

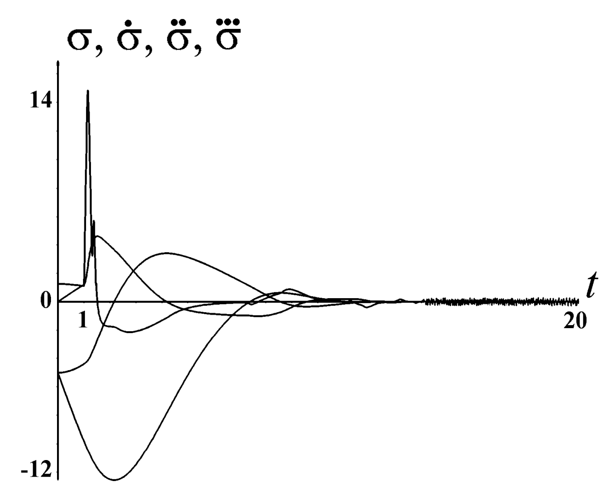
$$\dot{u} = -5\{s_3 + 3[|s_2| + (|s_1| + 0.5|s_0|^{3/4})^{-1/3}|s_1 + 0.5|s_0|^{3/4}\text{sign } s_0|\,]\,[|s_2| + (|s_1| + 0.5|s_0|^{3/4})^{2/3}]^{-1/2}\}/\{|s_3| + 3[|s_2| + (|s_1| + 0.5|s_0|^{3/4})^{2/3}]^{1/2}\}$$

s_i are the 3rd order differentiator outputs

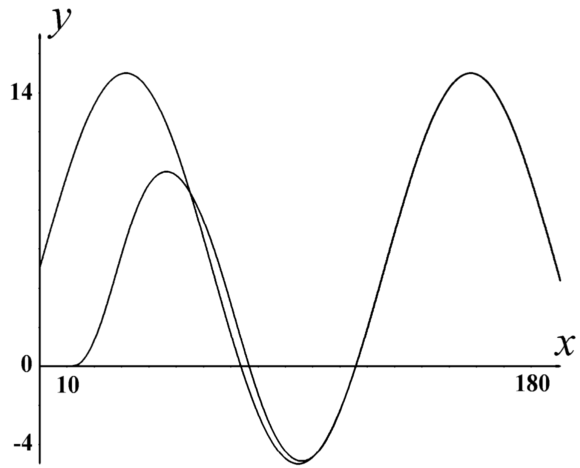
3^{rd} order differentiator

$$L = 700$$

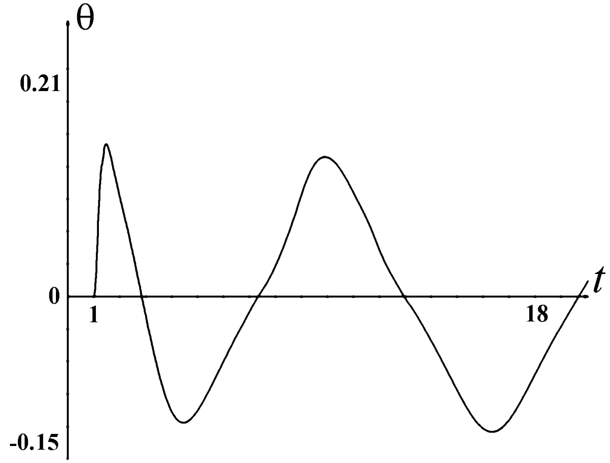
$$\dot{s}_0 = \xi_0, \qquad \xi_0 = -\ 15.3\ |s_0 - s|^{3/4}\ \mathrm{sign}(s_0 - s) + s_1,$$
$$\dot{s}_1 = \xi_1, \qquad \xi_1 = -\ 17.8\ |s_1 - \xi_0|^{2/3}\ \mathrm{sign}(s_1 - \xi_0) + s_2,$$
$$\dot{s}_2 = \xi_2, \qquad \xi_2 = -\ 39.7\ |s_2 - \xi_1|^{1/2}\ \mathrm{sign}(s_2 - \xi_1) + s_3,.$$
$$\dot{s}_3 = -\ 770\ \mathrm{sign}(s_3 - \xi_2).$$



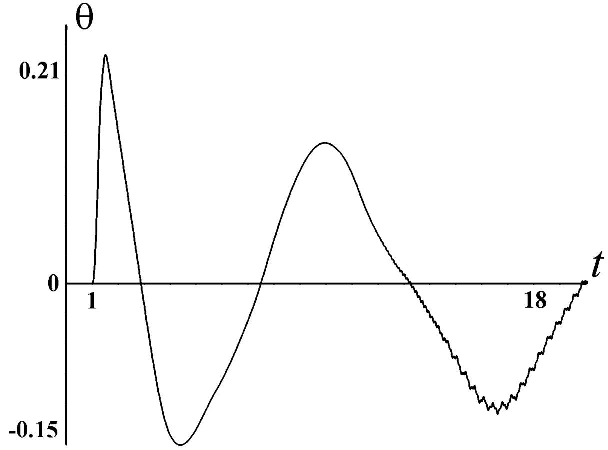
a. $\mu = \lambda = 0.001$



b. $\mu = \lambda = 0.01$



c. $\mu = \lambda = 0.001$



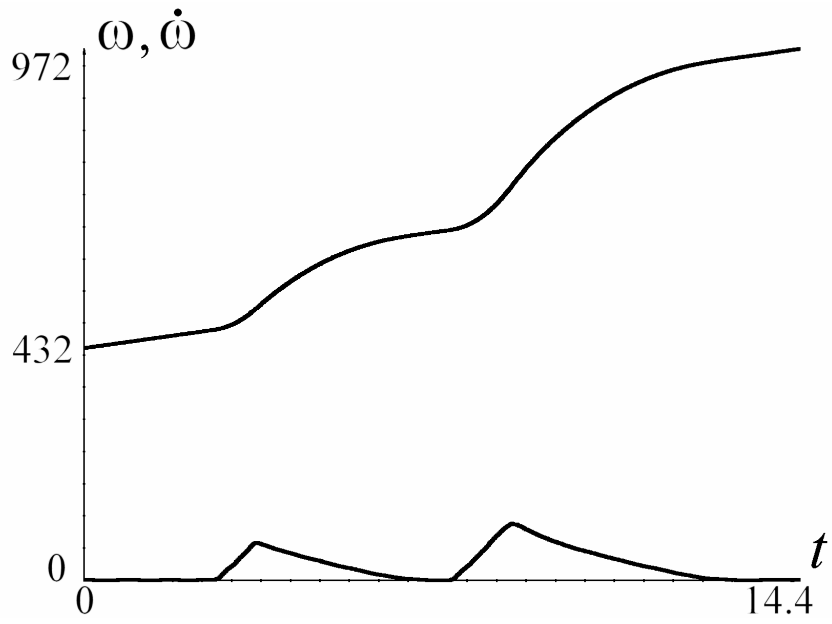
d. $\mu = \lambda = 0.01$

$\lambda = \mu = 0.001$: $|\sigma| \le 0.0045$, $|\dot{\sigma}| \le 0.0026$, $|\ddot{\sigma}| \le 0.0024$, $|\dddot{\sigma}| \le 0.34$

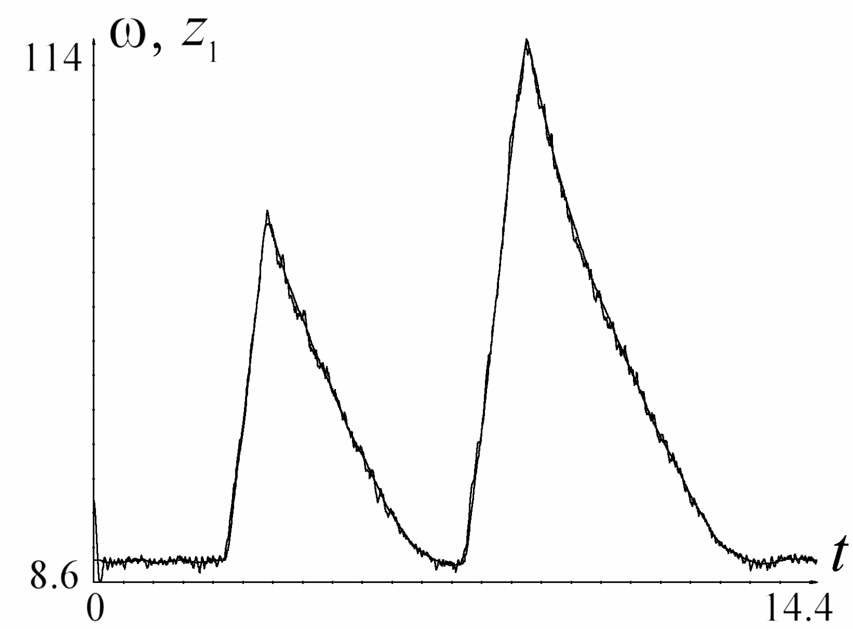
$\lambda = \mu = 0.01$: $|\sigma| \le 0.045$, $|\dot{\sigma}| \le 0.030$, $|\ddot{\sigma}| \le 0.13$, $|\dddot{\sigma}| \le 3.1$

APPLICATIONS

On-line calculation of the angular motor velocity and acceleration (data from Volvo Ltd)



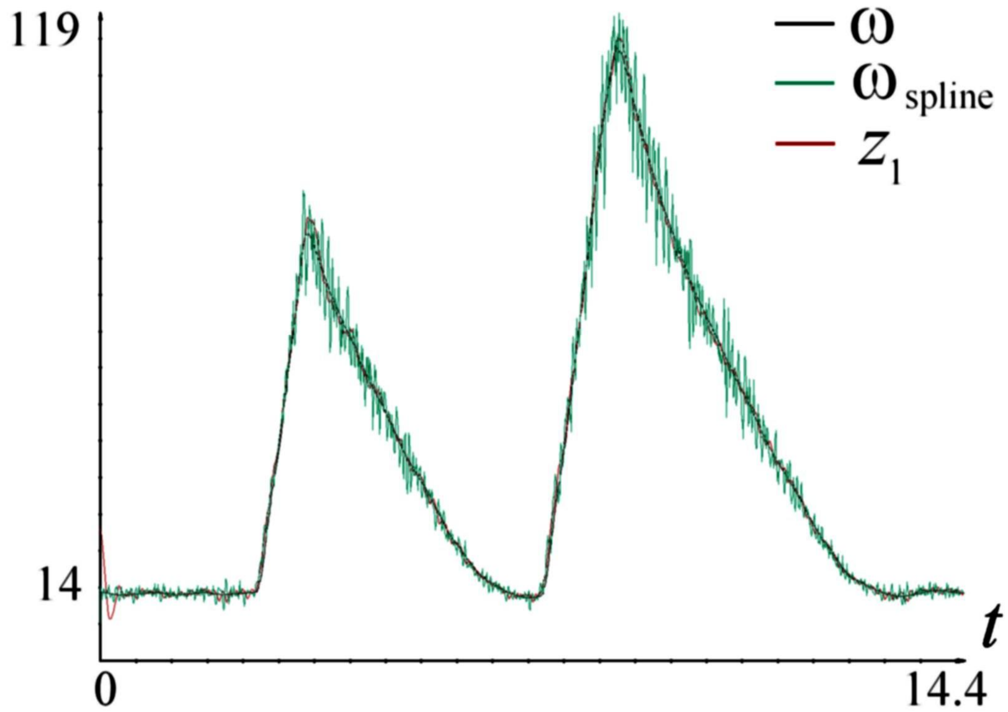
Experimental data, $\tau = 0.004$

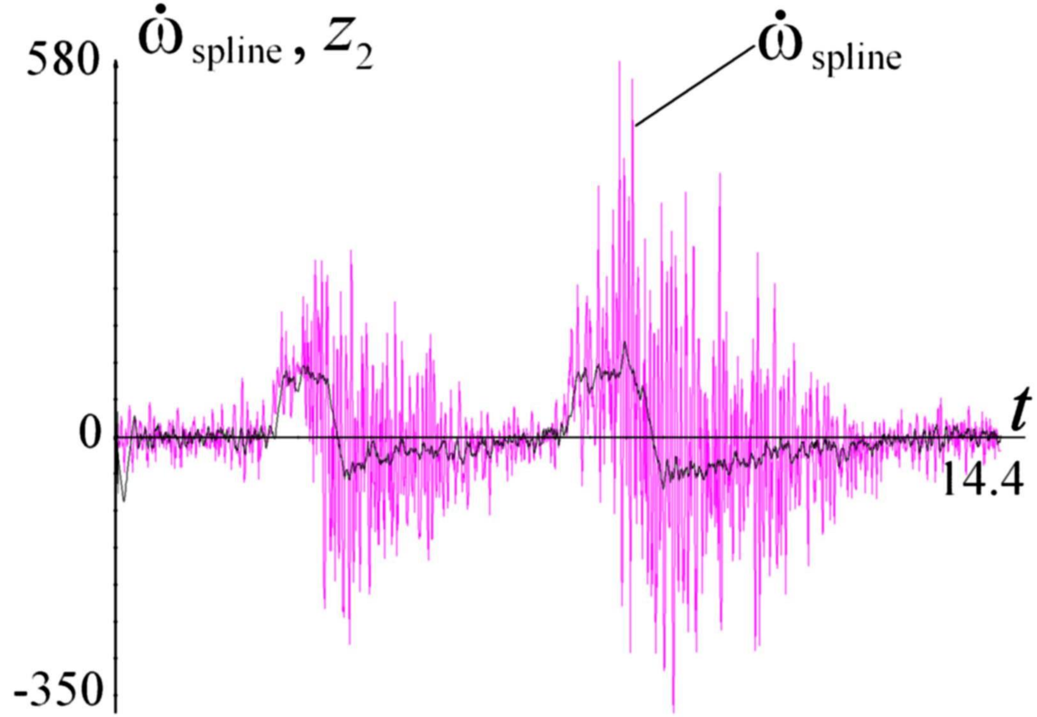


1st derivative.
2nd order differentiation, $L = 625$

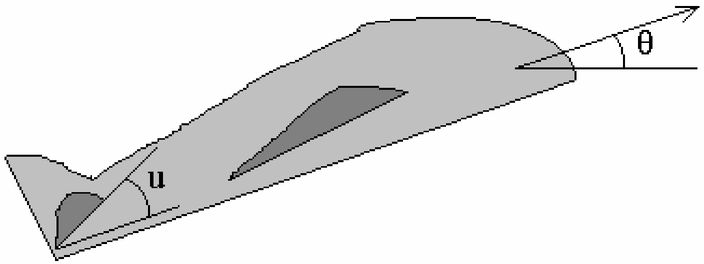
On-line 2nd order differentiation

Volvo: comparison with optimal spline approximation





Pitch Control



Problem statement. A non-linear process is given by a set of 42 linear approximations

$$\tfrac{d}{dt}(x,\theta,q)^t = G(x,\theta,q)^t + Hu,\ q = \dot{\theta},$$

$$x \in \mathbf{R}^3,\ \theta,\ q,\ u \in \mathbf{R},$$

x_1, x_2 -velocities, x_3 - altitude

The Task: $\theta \to \theta_c(t)$, $\theta_c(t)$ is given in real time.

G and H are not known properly

Sampling Frequency: 64 Hz, Measurement noises

Actuator: delay and discretization.

$d\theta/dt$ does not depend explicitly on u (relative degree 2)

Primary Statement:

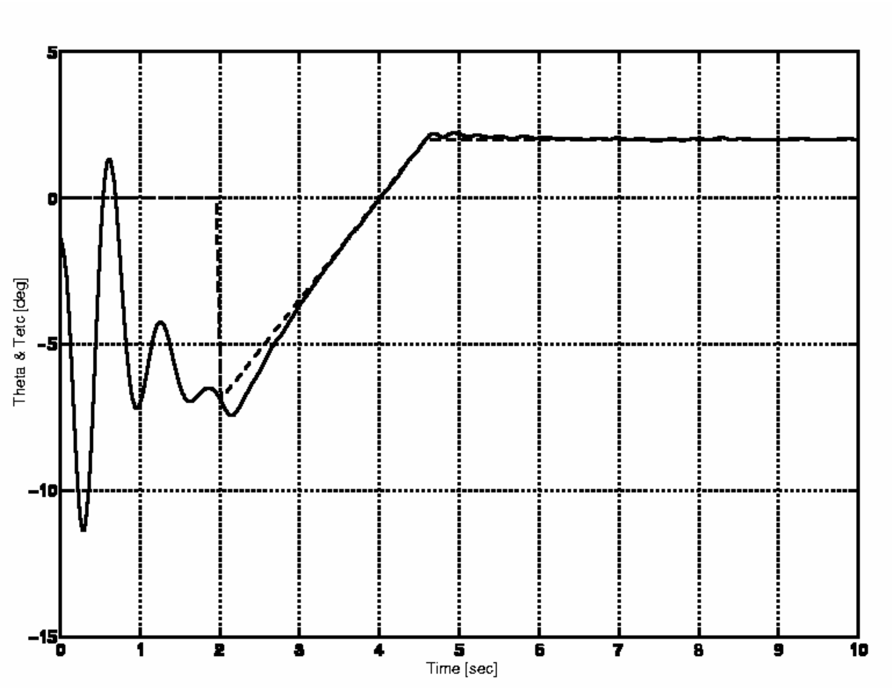
Available: θ, θ_c, Dynamic Pressure and Mach.

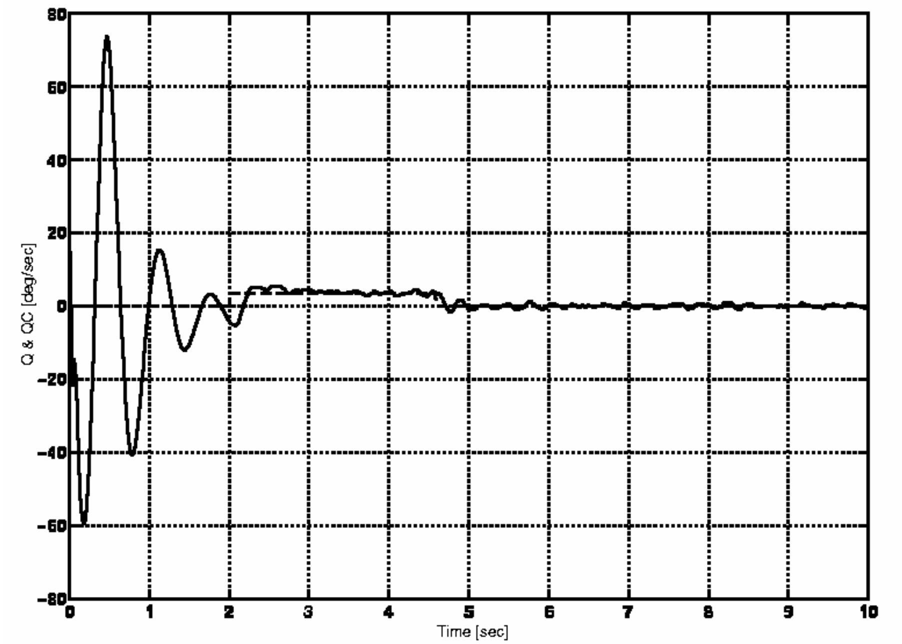
Main Statement: also $\dot{\theta}$, $\dot{\theta}_c$ are measured

The idea: keeping $5(\theta - \theta_c) + (\dot{\theta} - \dot{\theta}_c) = 0$ in 2-sliding mode

(asymptotic 3-sliding)

Flight Experiments

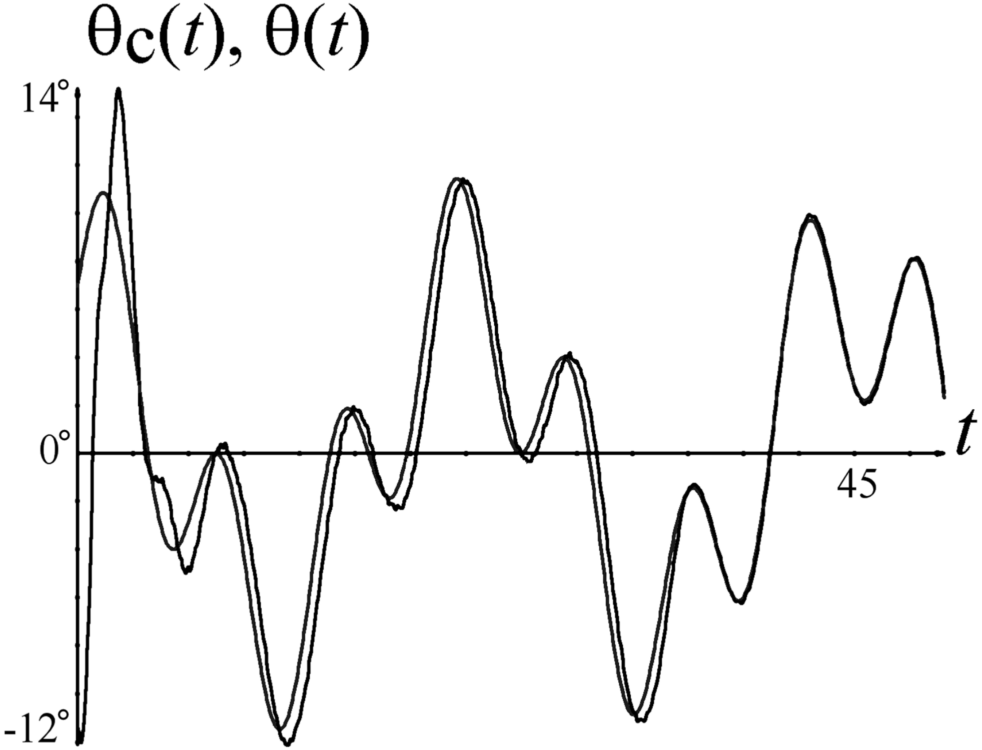


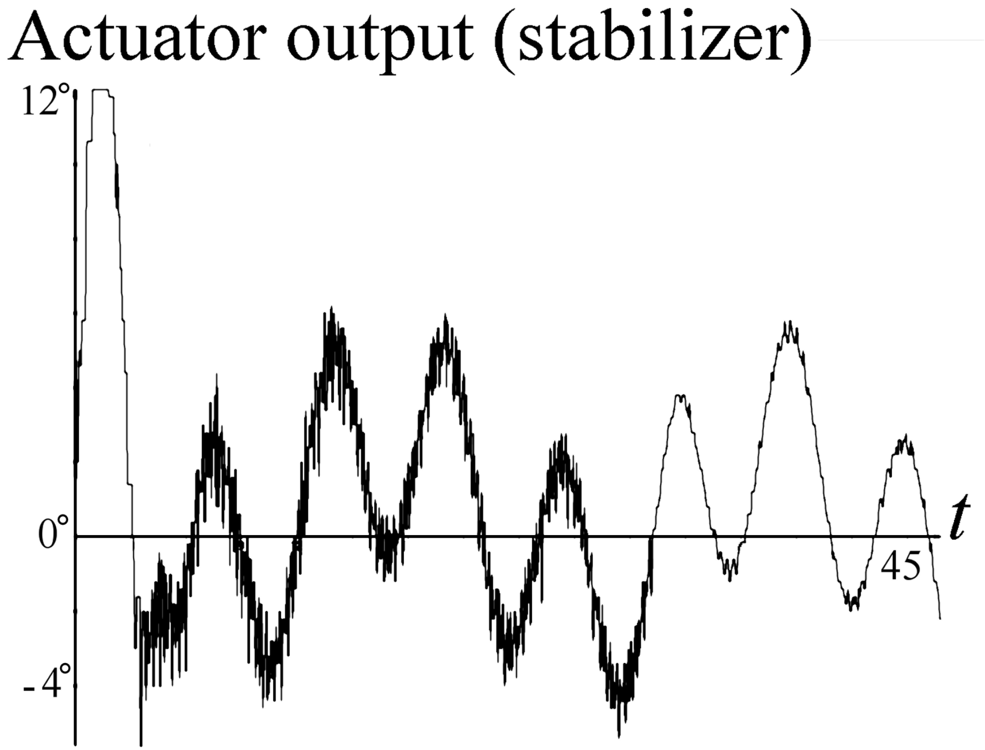


$\theta_C(t), \theta(t)$

$\dot{\theta}_c = q_c(t), \dot{\theta} = q(t)$

Actuator output:
switch from a linear to the 3-sliding control
(simulation with delays and noises)





Application: RAFAEL , October 2004

Nonlinear roll control design

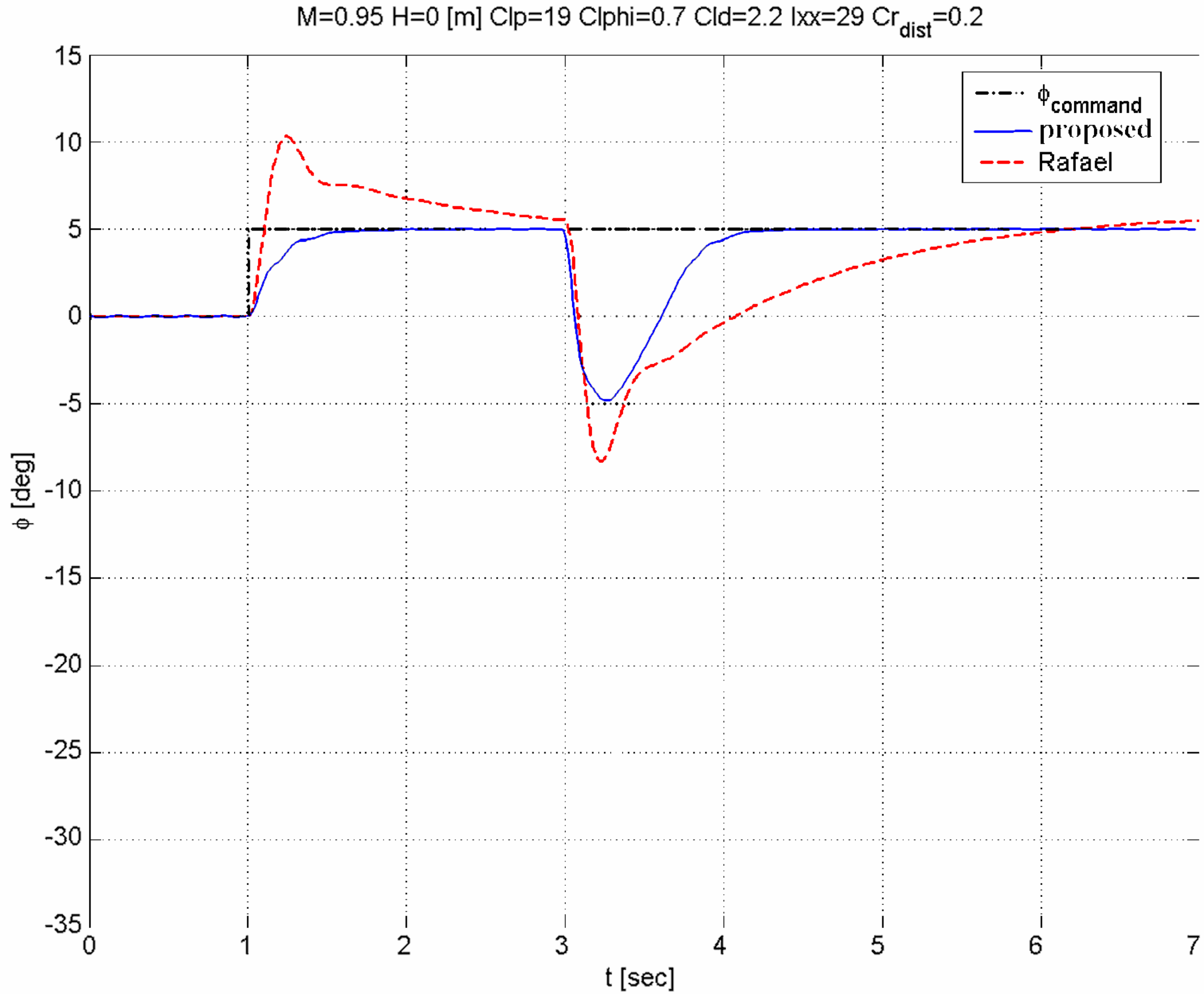
$$\ddot{\varphi} = a_1\dot{\varphi} + a_2\varphi + M + bu,$$

a_1, a_2, b are uncertain, M changes from 0 to an unknown value in a moment. The system itself is highly unstable. φ and $\dot{\varphi}$ are measured 120 times per second with different delays and not simultaneously. An uncertain fast actuator (singular disturbance) is present.

$$\sigma = \varphi - \varphi_c,\ r = 3 \text{ (}\dot{u}\text{ is a new control)}$$

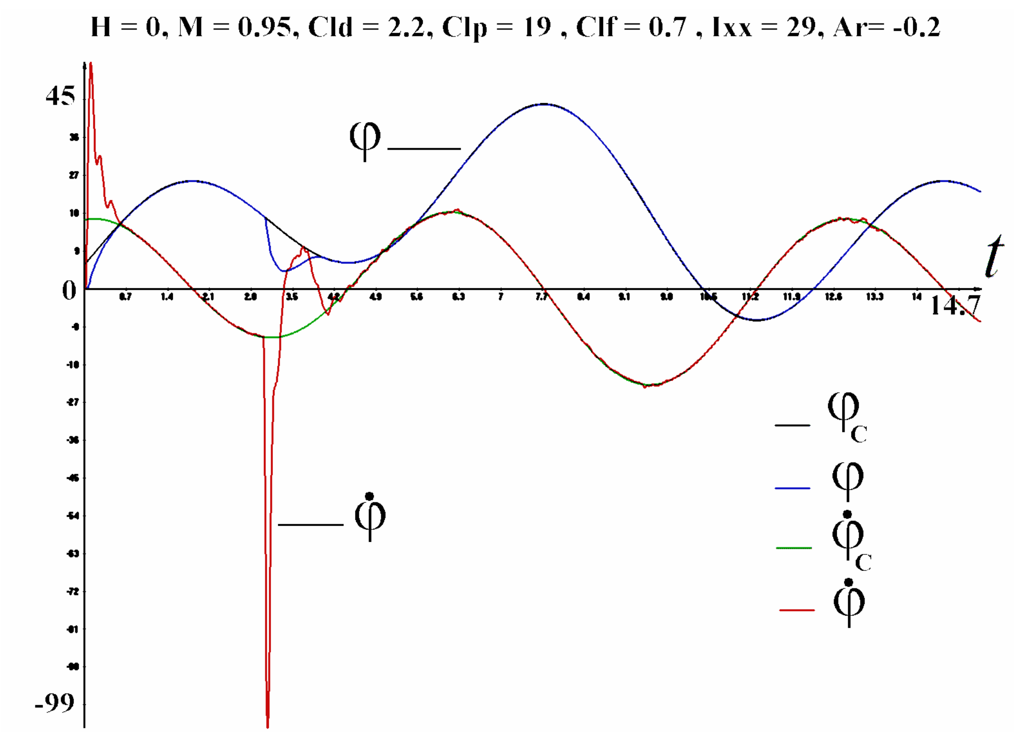
Solution: 3-sliding controller with 2-differentiator

Comparison of the controller by RAFAEL and the proposed one

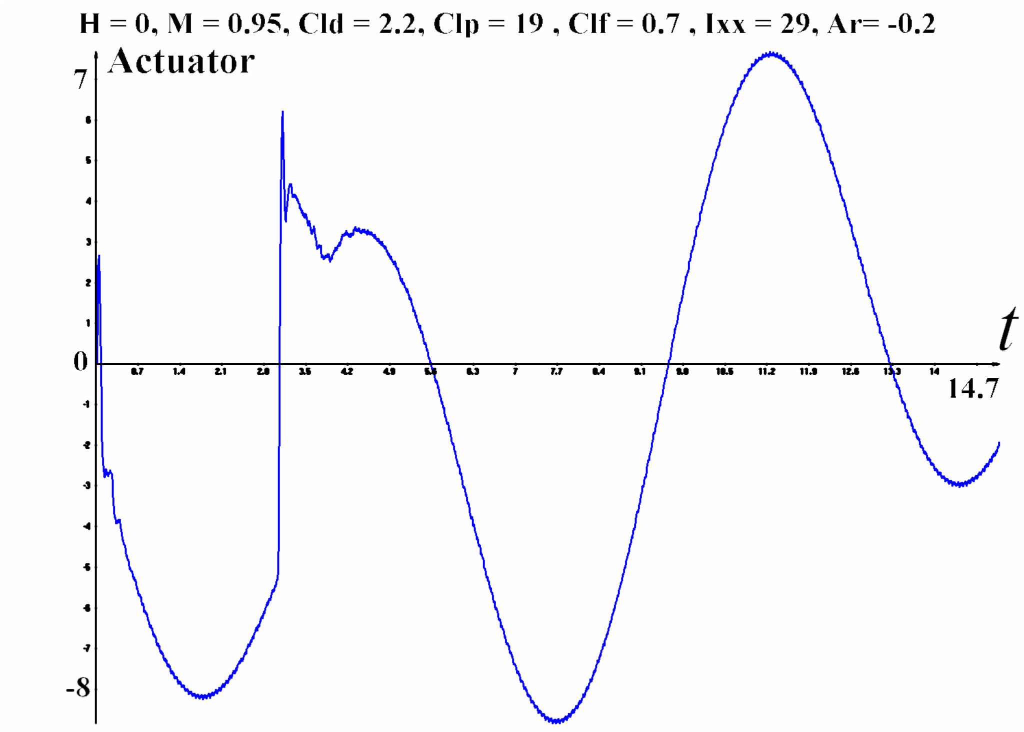


o

Tracking arbitrary smooth function

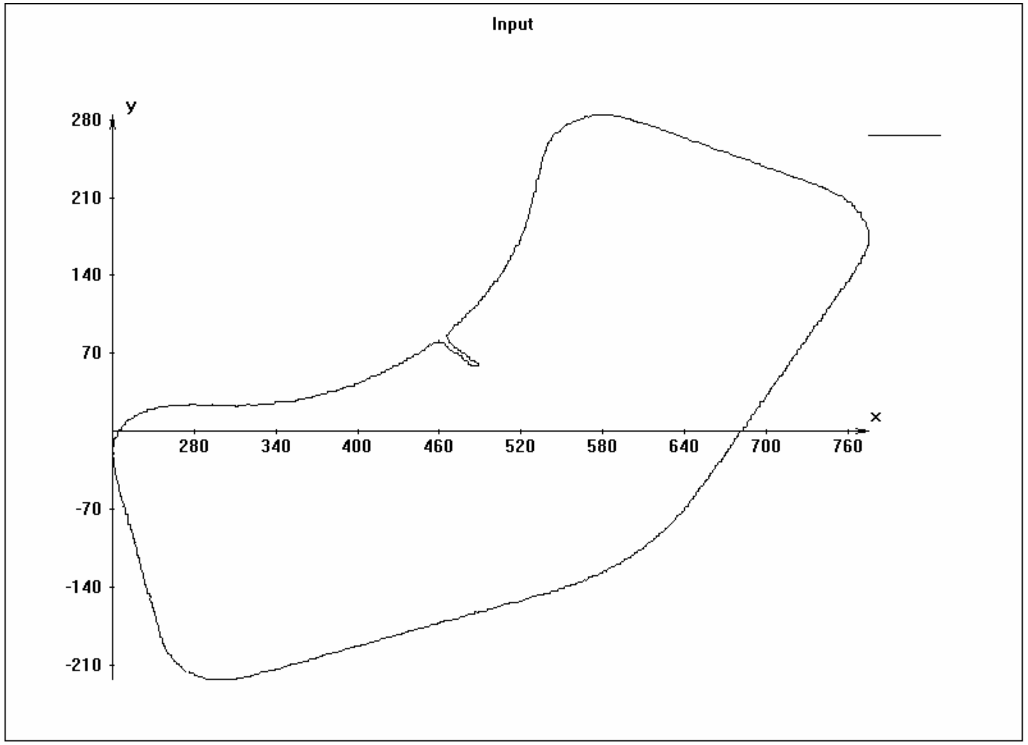


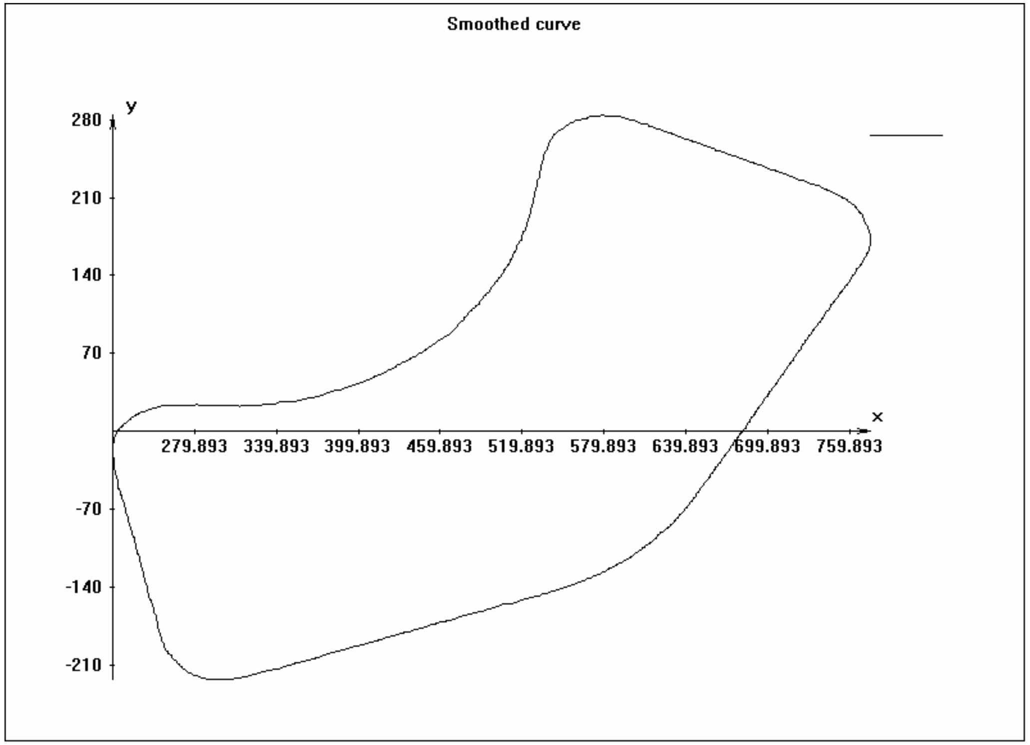
Tracking



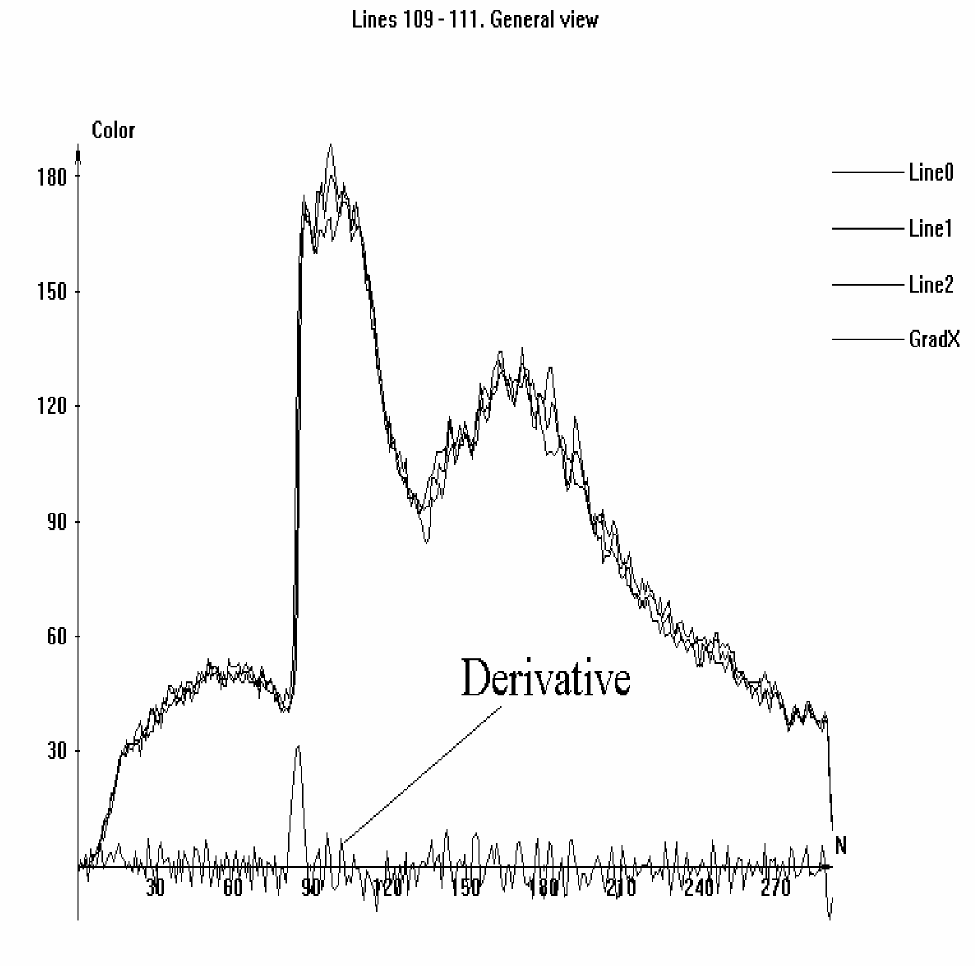
Actuator output

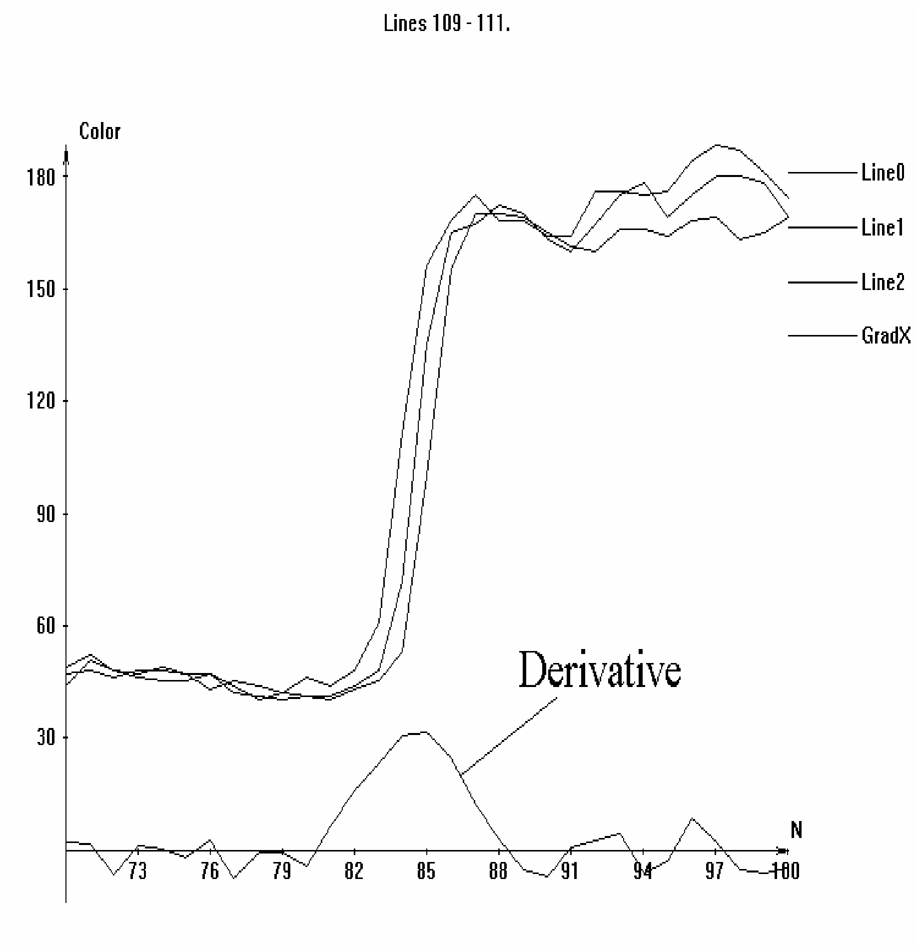
Image Processing: Crack Elimination





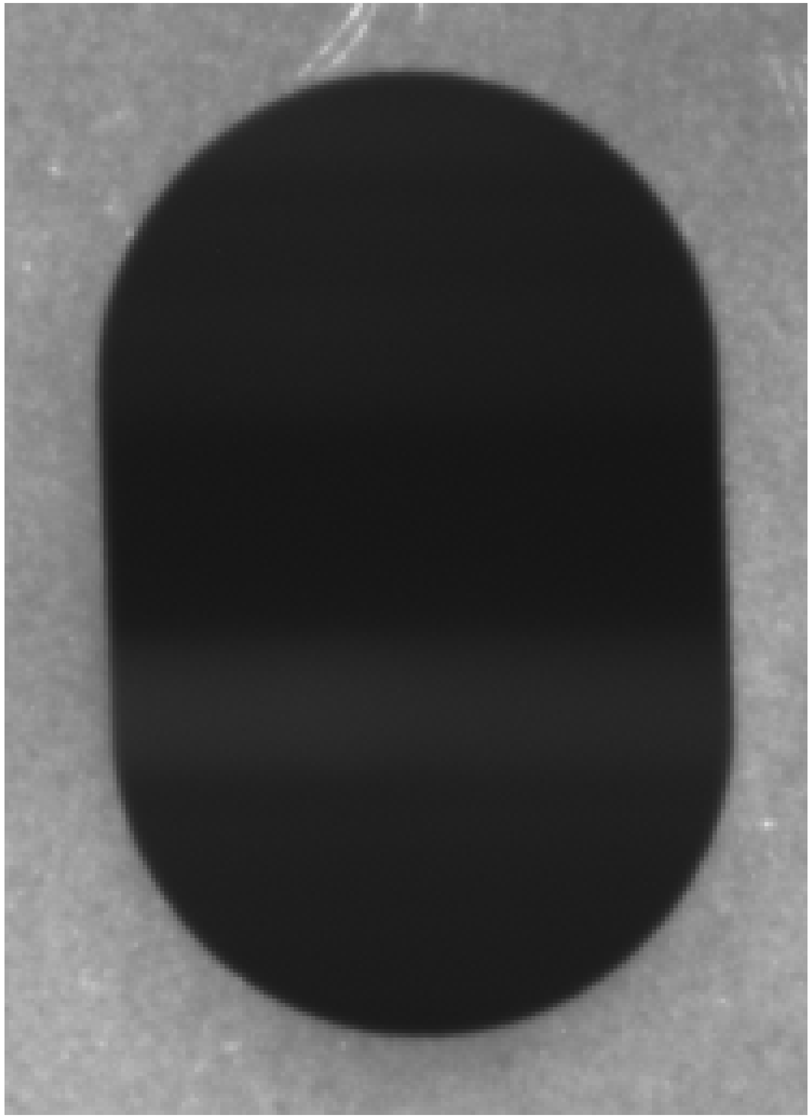
Edge Detection

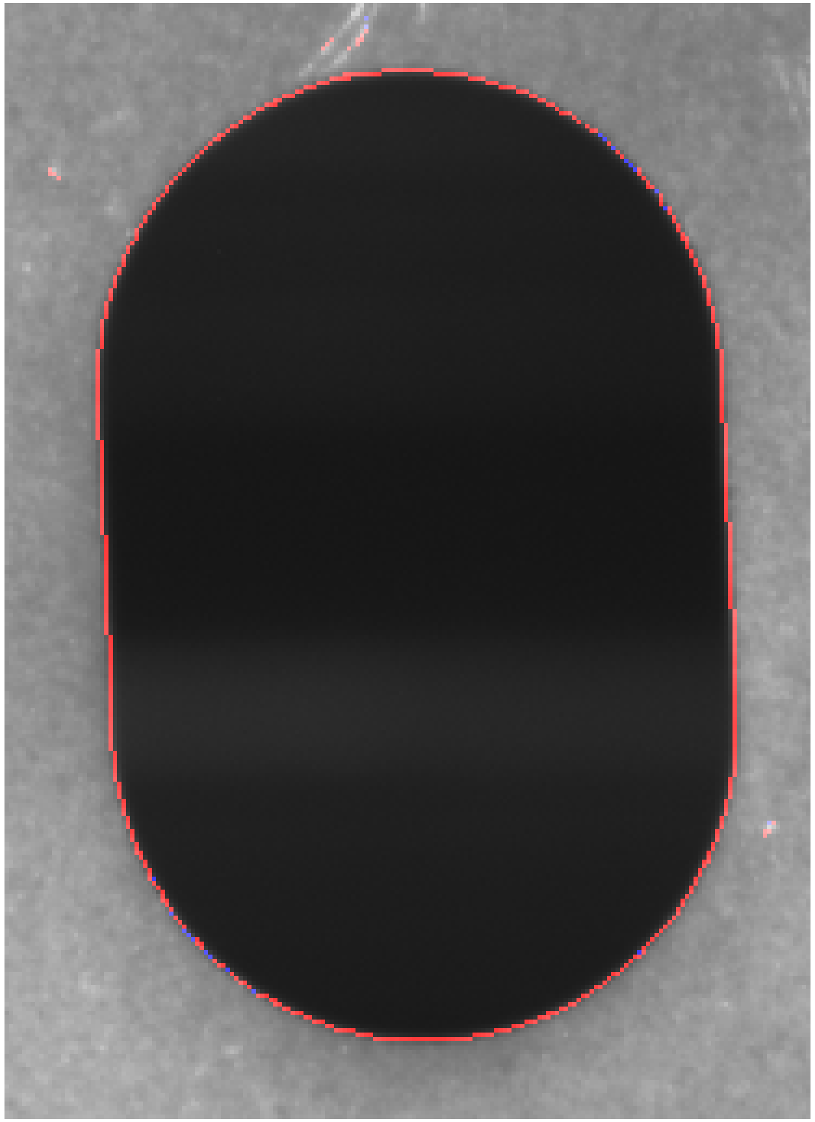




3 successive lines of a grey image   zoom

Edge Detection





Conclusions (the main result)

A list of universal Single Input - Single Output controllers is developed, solving the stated problem for any given relative degree

in finite time and with ideal accuracy

The controllers include exact robust differentiators, are insensitive to any disturbances preserving the assumptions, and robust to

- Lebesgue-measurable bounded output noises
- any small smooth system disturbances
- discrete sampling, small delays
- unaccounted-for fast stable dynamics of sensors and actuators

The control signal can be made as smooth as needed.