ARBITRARY-ORDER REAL-TIME EXACT ROBUST DIFFERENTIATION

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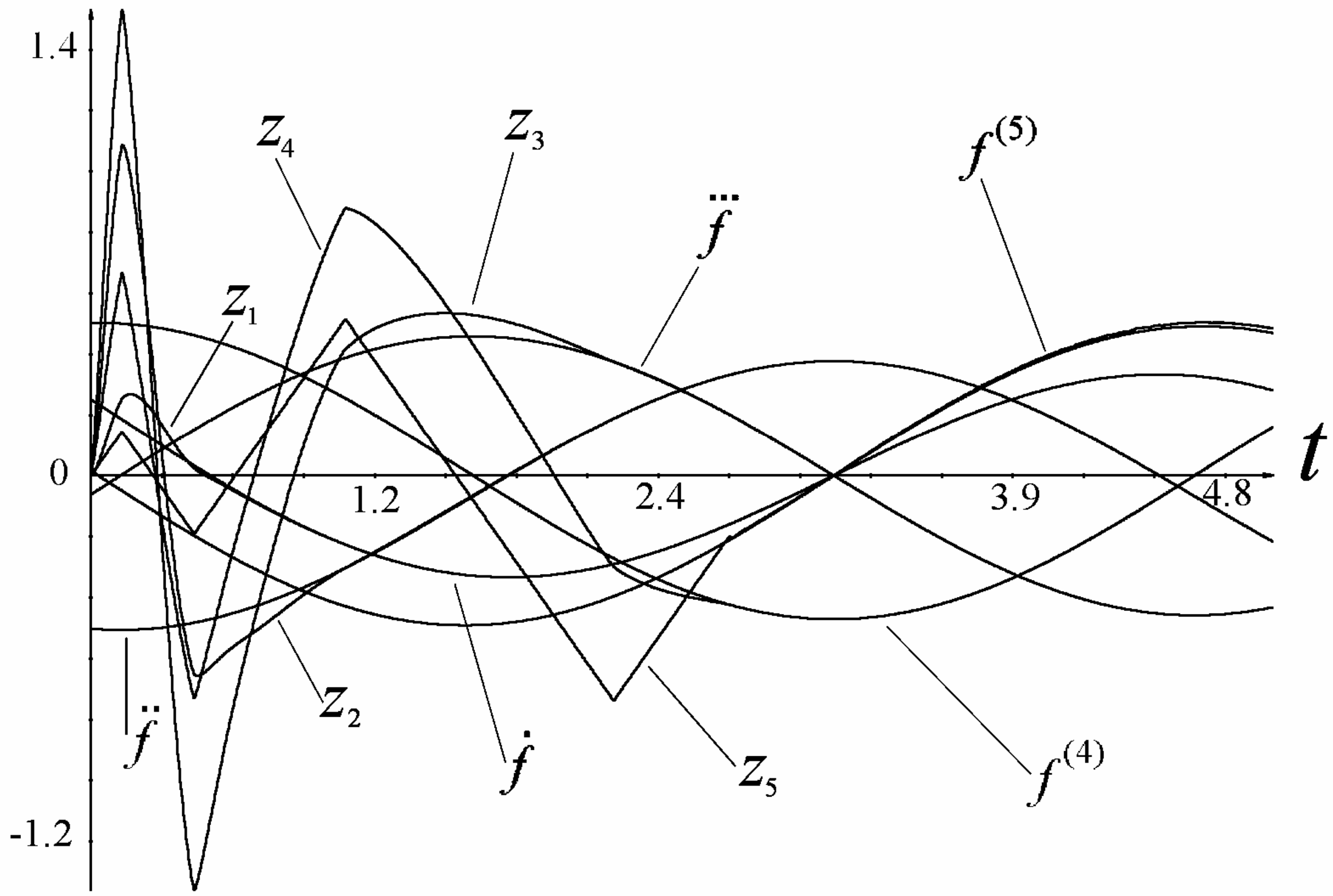


5th-order differentiation

Differentiation problematics:

Division by zero: $f'(t) = \lim_{\tau \to 0} \dfrac{f(t+\tau) - f(t)}{\tau}$

Let $f(t) = f_0(t) + \eta(t)$, $\eta(t)$ - noise

$$\frac{f(t+\tau) - f(t)}{\tau} = \frac{\Delta f}{\tau} + \frac{\Delta \eta}{\tau}, \quad \frac{\Delta \eta}{\tau} \in (-\infty, \infty)$$

1st way: $\sin(\omega t)' = \omega \cos(\omega t)$

Fourier transform, **high harmonics neglection**

$\Rightarrow$ linear filter with transfer function

$$\frac{P_m(p)}{Q_n(p)} \approx p, m \le n. \qquad \frac{p}{(0.01p+1)^2} \approx p$$

Advantages: robustness

Drawbacks: not exact, asymptotic convergence

2nd way: **numeric differentiation**

$$Df(t) = F(f(t_1), ..., f(t_n))$$

Advantages: exact on certain functions (for example polynomials $P_m(t)$)

Drawbacks: actual division by zero with $t_i \to t$

Transfer Functions

$$y^{(n)} + a_1 y^{(n\text{-}1)} + \ldots + a_n y = b_0 u^{(m)} + b_1 u^{(m\text{-}1)} + \ldots + b_m u$$

$$(D^n + a_1 D^{n\text{-}1} + \ldots + a_n)y = (b_0 D^m + b_1 D^{m\text{-}1} + \ldots + b_m)u$$

$$G(p) = \frac{b_0 p^m + b_1 p^{m-1} + \ldots + b_m}{p^n + a_1 p^{n-1} + \ldots + a_n}$$

$$u \xrightarrow{\quad G(p) \quad} y$$

$$u = \mathrm{e}^{\lambda t} \longmapsto y = G(\lambda)\, \mathrm{e}^{\lambda t}$$

in particular

$$\lambda = i\omega,\ \omega \to \infty \Rightarrow |y| = |\ G(i\omega)| \to \infty \text{ with } n < m$$

Thus, only $n \geq m$ is feasible

Realization:

$$y^{(n)} + a_1 y^{(n\text{-}1)} + \ldots + a_n y = u,\ G = \frac{1}{p^n + a_1 p^{n-1} + \ldots + a_n}$$

$$Y = (b_0 p^m + b_1 p^{m\text{-}1} + \ldots + b_m) \frac{1}{p^n + a_1 p^{n-1} + \ldots + a_n} U$$

Differentiation error = intrinsic differentiator error + noise differentiation error

The goal:

to build a differentiator with

zero intrinsic error

for a large class of functions

and with **error**

continuously dependent on

the noise magnitude;

\+ **finite-time convergence**

The main trade-of:

$$D: f_1 \mapsto \frac{d}{dt} f_1 \,,\; D: f_2 \mapsto \frac{d}{dt} f_2 \,,$$

$$\text{noise } \eta = f_2 - f_1$$

$$\Rightarrow \mathrm{D}: f_1 + \eta \mapsto \frac{d}{dt} f_1 + \frac{d}{dt}\eta$$

$\Rightarrow$ no robust differentiator is exact on all polynomials

or on all smooth functions

Differentiation Problem

Input:

$$f(t) = f_0(t) + \eta(t), \quad |\eta| < \varepsilon$$

$\eta(t)$ - Lebesgue-measurable function,

f_0, η, ε are unknown, known : $|f_0^{(n+1)}(t)| \le L$

(or |Lipschitz constant of $f_0^{(n)}| \le LC$)

The goal:

real-time estimation of $\dot{f}_0(t),\ \ddot{f}_0(t),\ \ldots,\ f_0^{(n)}(t)$

Theorem: Optimal differentiator existence

$$\exists\ \gamma_i(n) \ge 1,\ \mathrm{D}_n : f(\cdot) \mapsto D_n^i(f)(\cdot),\ i = 0, 1, \ldots, n$$

$$\sup_f \| D_n^i(f)(t) - f_0^{(i)}(t) \|_C = \gamma_i(n)\, L^{i/(n+1)} \varepsilon^{(n-i+1)/(n+1)}$$

In particular $\gamma_1(1) = 2$ (Landau, Kolmogorov)

$$\sup_f \| D_n^n(f)(t) - f_0^{(n)}(t) \| \ge L^{n/(n+1)}\ \varepsilon^{1/(n+1)}$$

$$\varepsilon = 10^{-6} \Rightarrow \exists f,\ |f_0^{(6)}(t)| \le 1:$$

$$\sup_t | D_5^5(f)(t) - f_0^{(5)}(t) | \ge 0.1$$

That asymptotics cannot be improved

First-order differentiator idea

Auxiliary dynamic system:

$$\dot{z} = u$$

Tracking problem:

$$\sigma = z - f(t) \to 0$$

Any 2-sliding controller being applied,

$$\sigma = \dot{\sigma} = 0$$

$$\Rightarrow z - f(t) = u - \dot{f}(t) = 0$$

With input noises

$$x - f(t) \approx 0, \ \ u - \dot{f}(t) \approx 0$$

A proof sketch

Consider a smooth noise

$\eta(t) = \varepsilon \sin (L/\varepsilon)^{1/(n+1)} t$

$\sup|\eta^{(i)}(t)| = L^{i/(n+1)} \varepsilon^{(n-i+1)/(n+1)}, \; i = 0, 1, ..., n+1$

$\Rightarrow D$ differentiates both 0 and $0 + \eta(t)$

Lemma. $\exists \beta_i(n) \geq 1, \beta_0(n) = \beta_n(n+1) = 1$:

$$L^{i/(n+1)} \varepsilon^{(n-i+1)/(n+1)} \leq \sup_{\text{L.const } f^{(n)} \leq L, \sup|f(t)| \leq \varepsilon} |f^{(i)}(t)| \leq \beta_i(n) L^{i/(n+1)} \varepsilon^{(n-i+1)/(n+1)}.$$

Some evaluations: $\beta_1(1) = 2\sqrt{2}$, $\beta_1(2) = 7.07$, $\beta_2(2) = 6.24$.

$W(C,n) = \{f: [a, b] \to \mathbf{R}$, L.const of $f^{(n)} \leq L\}$

Let $\Delta : g \mapsto \Delta g$, Δg is closest to g in $W(L,n)$ (not unique).

$Dg = (\Delta g)^{(i)}$.

Robust first order differentiator

Input: $f(t) = f_0(t) + \eta(t)$, $|\ddot{f}_0| < L$

$|\eta| < \varepsilon$, $\eta(t)$ - measurable (Lebesgue) function

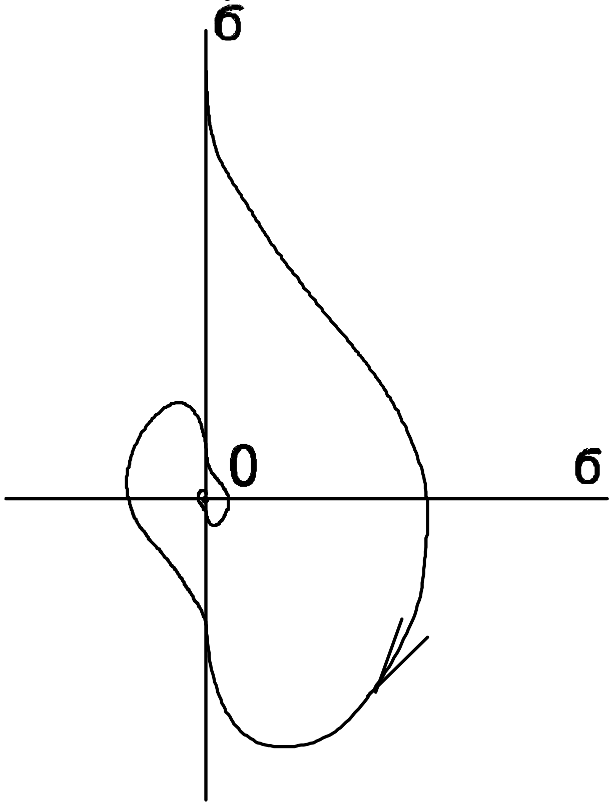
$$\dot{z}_0 = v_0,\ v_0 = -\lambda_0 |z_0 - f|^{1/2} \operatorname{sign}(z_0 - f) + z_1,$$

$$\dot{z}_1 = -\lambda_1 \operatorname{sign}(z_0 - f) \quad (= -\lambda_1 \operatorname{sign}(z_1 - v_0)).$$

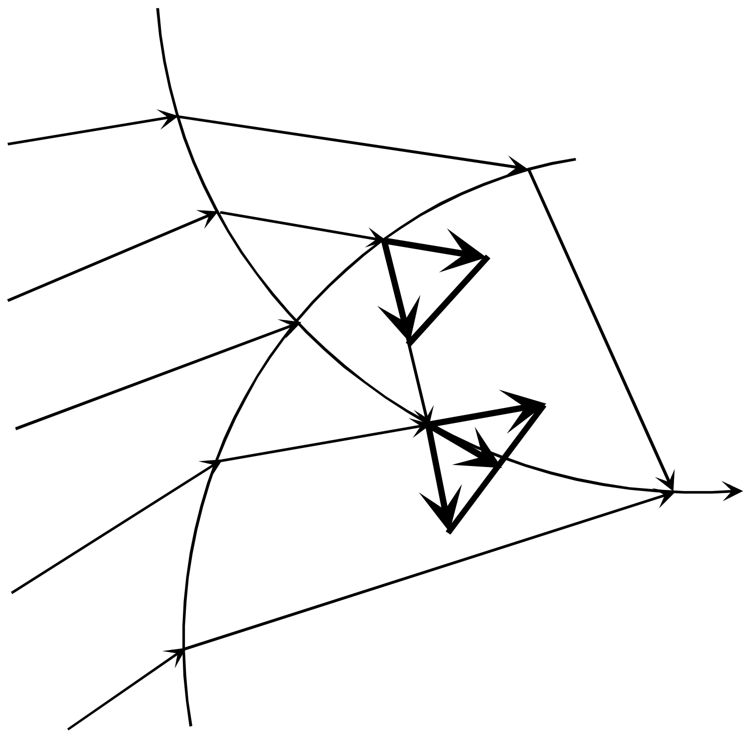
Output: $z_1(t), v_0(t)$ $\quad |z_1 - \dot{f}_0| \sim L^{1/2}\varepsilon^{1/2}$

$$\sigma = z_0 - f(t) \to 0$$

For example: $\lambda_1 = 1.1L,\ \lambda_0 = 1.5L^{1/2}$



Filippov Definition



$$\dot{x} = v(x) \Leftrightarrow \dot{x} \in V(x)$$

$x(t)$ is an absolutely continuous function

$$V(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu N=0} \overline{\text{conv}}\, v(O_\varepsilon(x) \setminus N)$$

v_+, v_- - limit values, $\dot{x} = pv_+ + (1\text{-p})v_-$, $\quad p \in [0,1]$

Solutions exist for any locally bounded Lebesgue-measurable $v(x)$;
or for any upper-semicontinuous, convex, closed, non-empty, locally-bounded $V(x)$

1-differentiator explanation

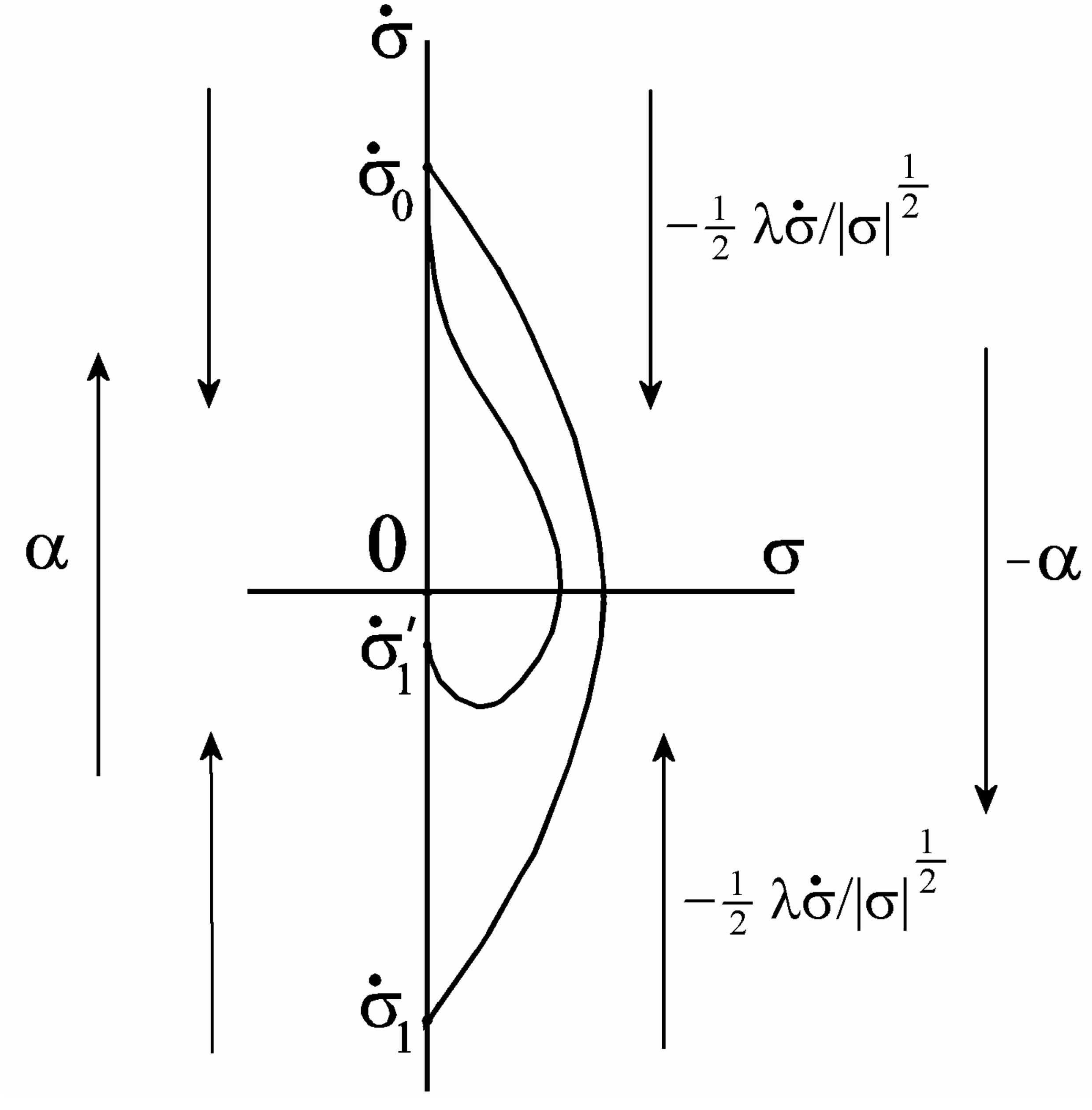
$$\text{Let } \sigma = z_0 - f_0(t),\ \varepsilon = 0$$

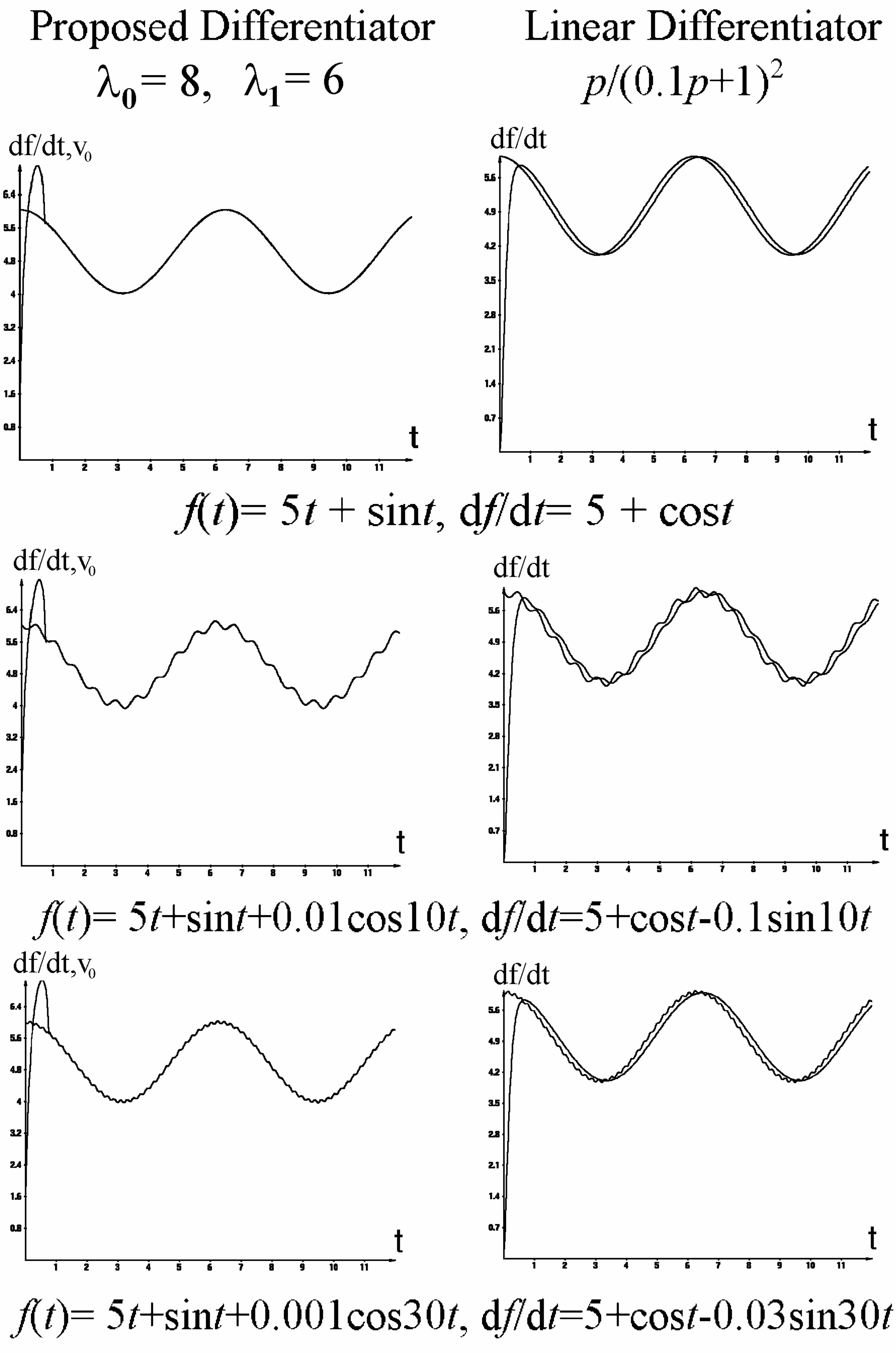
$$\Rightarrow \ddot{\sigma} = -\ddot{f}_0 - \tfrac{1}{2}\lambda\dot{\sigma}|\sigma|^{-1/2} - \alpha \operatorname{sign} \sigma.$$

$$\ddot{\sigma} \in -\tfrac{1}{2}\lambda\dot{\sigma}|\sigma|^{-1/2} - [\alpha - L,\ \alpha + L] \operatorname{sign} \sigma.$$

invariant with respect to

$$g_\nu\colon (\sigma, \dot{\sigma}, t) \mapsto (\nu^2\sigma, \nu\dot{\sigma}, \nu t)$$





Proposed Differentiator
λ0= 8, λ1= 6
Linear Differentiator
p/(0.1p+1)2
df/dt,v0
df/dt
t
f(t)= 5t + sint, df/dt= 5 + cost
f(t)= 5t+sint+0.01cos10t, df/dt=5+cost-0.1sin10t
f(t)= 5t+sint+0.001cos30t, df/dt=5+cost-0.03sin30t

Optimal *n*-differentiator

$$\dot z_0 = v_0,\ v_0 = -\lambda_0 L^{1/(n+1)}|z_0 - f(t)|^{n/(n+1)}\operatorname{sign}(z_0 - f(t)) + z_1,$$

$$\dot z_1 = v_1,\ v_1 = -\lambda_1 L^{1/n}\ |z_1 - v_0|^{(n-1)/n}\ \operatorname{sign}(z_0 - v_0) + z_2,$$

$$\ldots$$

$$\dot z_i = v_i,\ v_i = -\lambda_i L^{1/(n-i+1)}|z_i - v_{i-1}|^{(n-i)/(n-i+1)}\operatorname{sign}(z_i - v_{i-1}) + z_{i+1},$$

$$\ldots$$

$$\dot z_{n-1} = v_{n-1},\ v_{n-1} = -\lambda_{n-1} L^{1/2}|z_{n-1} - v_{n-2}|^{1/2}\operatorname{sign}(z_{n-1} - v_{n-2}) + z_n,$$

$$\dot z_n = -\lambda_n\ L \operatorname{sign}(z_n - v_{n-1})$$

D_n :

$$\dot z_0 = v\ ,\qquad v = -\mu\ |\ z_0 - f(t)|^{n/(n+1)}\ \operatorname{sign}(z_0 - f(t)) + z_1,$$

$$z_1 = D^0_{n-1}\ (v(\cdot), L),\ \ \ldots,\ z_n = D^{n-1}_{n-1}\ (v(\cdot), L).$$

$$D_0:\quad \dot z = -\mu \operatorname{sign}(z - f(t)),\qquad \mu > L.$$

Non-recursive form:

$$\dot z_0 = -\kappa_0\ |\ z_0 - f(t)|^{n/(n+1)}\ \operatorname{sign}(z_0 - f(t)) + z_1,$$

$$\ldots$$

$$\dot z_i = -\kappa_i\ |\ z_0 - f(t)|^{(n-i)/(n+1)}\operatorname{sign}(z_0 - f(t)) + z_{i+1},$$

$$i = 1, \ldots, n-1$$

$$\ldots$$

$$\dot z_n = -\kappa_n\ \operatorname{sign}(z_0 - f(t))$$

5th-order differentiator, $L = 1$

$$\dot z_0 = v_0,\ v_0 = -12|z_0 - f(t)|^{5/6}\operatorname{sign}(z_0 - f(t)) + z_1,$$

$$\dot z_1 = v_1,\ v_1 = -8\,|z_1 - v_0|^{4/5}\operatorname{sign}(z_1 - v_0) + z_2,$$

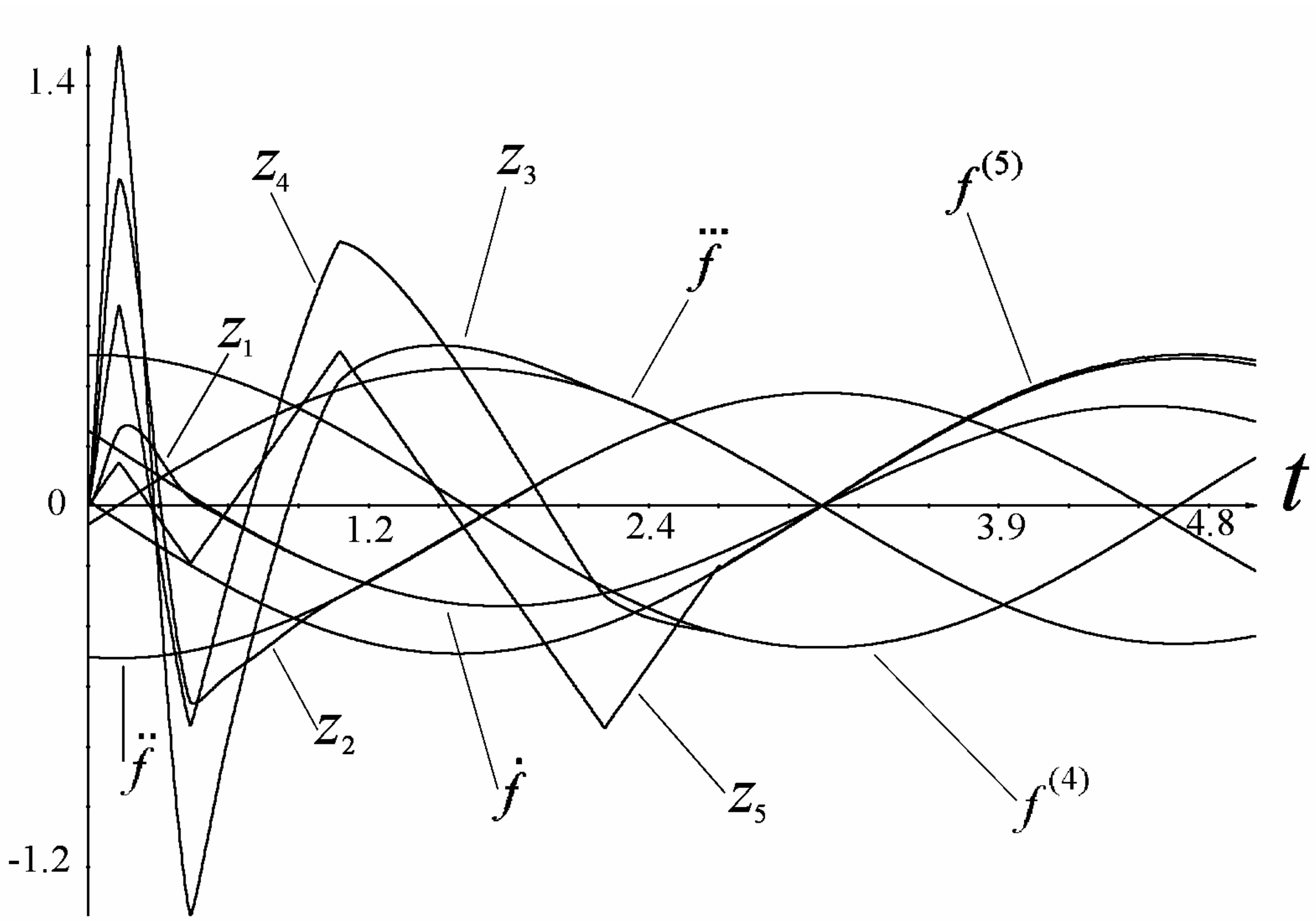
$$\dot z_2 = v_2,\ v_2 = -5\,|z_2 - v_1|^{3/4}\operatorname{sign}(z_2 - v_1) + z_3,$$

$$\dot z_3 = v_3,\ v_3 = -3\,|z_3 - v_2|^{2/3}\operatorname{sign}(z_3 - v_2) + z_4,$$

$$\dot z_4 = v_4,\ v_4 = -1.5\,|z_4 - v_3|^{1/2}\operatorname{sign}(z_4 - v_3) + z_5,$$

$$\dot z_5 = -1.1\operatorname{sign}(z_5 - v_4);$$

$$f(t) = 0.5\sin 0.5t + 0.5\cos t$$



Theorem 1. $f(t) = f_0(t) \Rightarrow$ *in finite time*

$$z_0 = f_0(t); \qquad z_i = v_{i-1} = f_0^{(i)}(t), \quad i = 1, ..., n.$$

2-sliding modes: $z_i = f_0^{(i)}(t) \quad i = 0, ..., n\text{-}1.$

Theorem 2. $|f(t) - f_0(t)| \le \varepsilon \Rightarrow$

$$|z_i - f_0^{(i)}(t)| \le \mu_i\, \varepsilon^{(n - i + 1)/(n + 1)},\ i = 0, ..., n;$$
$$|v_i - f_0^{(i+1)}(t)| \le \nu_i\, \varepsilon^{(n - i)/(n + 1)},\ i = 0, ..., n\text{-}1.$$
$$|z_n - f_0^{(n)}(t)| \le \mu_n\, \varepsilon^{1/(n + 1)}$$

Discrete-sampling case:

$z_0(t_j) - f(t_j)$ is substituted for $z_0 - f(t)$

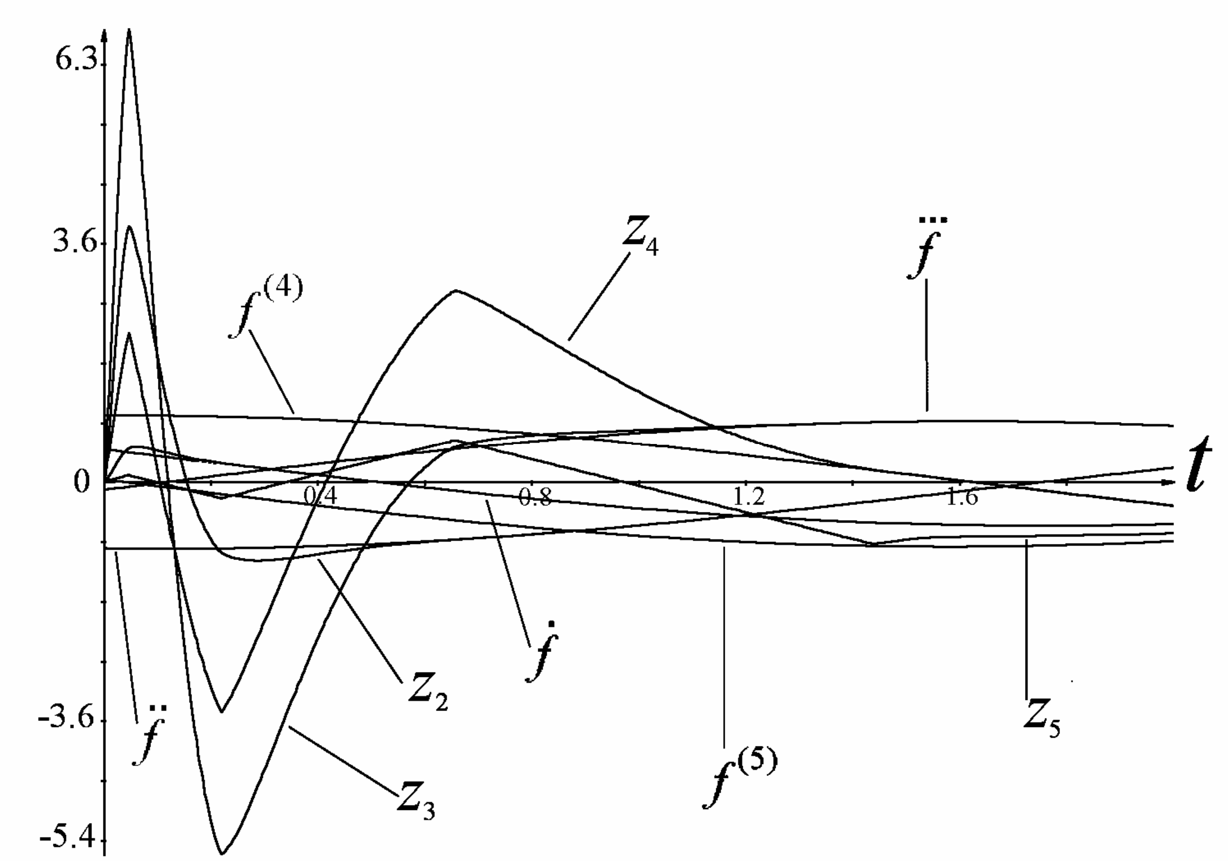
with $\quad t_j \le t < t_{j+1},\ t_{j+1} - t_j = \tau > 0.$

Theorem 3.

$$|z_i - f_0^{(i)}(t)| \le \mu_i\, \tau^{n - i + 1},\ i = 0, ..., n;$$
$$|v_i - f_0^{(i+1)}(t)| \le \nu_i\, \tau^{n - i},\ i = 0, ..., n - 1.$$
$$|z_n - f_0^{(n)}(t)| \le \mu_n\, \tau$$

Software restrictions of higher-order differentiation

5th-order differentiator



$\tau = 5\cdot 10^{-4} \Rightarrow$

$|z_0 - f_0| \leq 1.46\cdot 10^{-13}$, $|z_1 - \dot{f}_0| \leq 7.16\cdot 10^{-10}$,

$|z_2 - \ddot{f}_0| \leq 9.86\cdot 10^{-7}$, $|z_3 - \dddot{f}_0| \leq 3.76\cdot 10^{-4}$

$|z_4 - f_0^{(4)}| \leq 0.0306$, $|z_5 - f_0^{(5)}| \leq 0.449$

$\tau = 5\cdot 10^{-5} \Rightarrow$

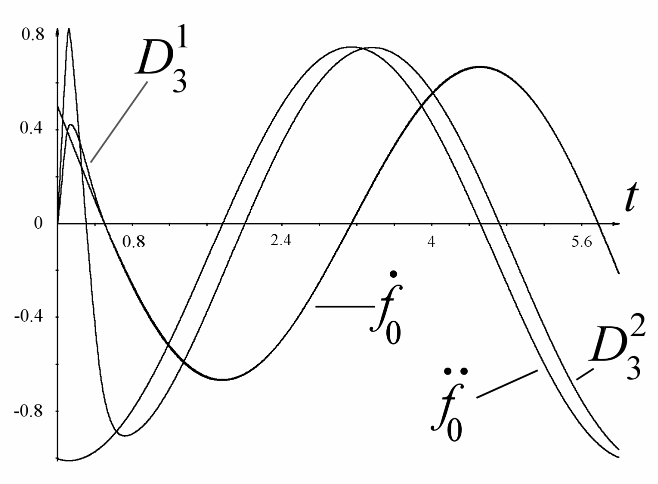
$|z_0 - f_0| \leq 4.44\cdot 10^{-16}$, $|z_1 - \dot{f}_0| \leq 6.34\cdot 10^{-12}$,

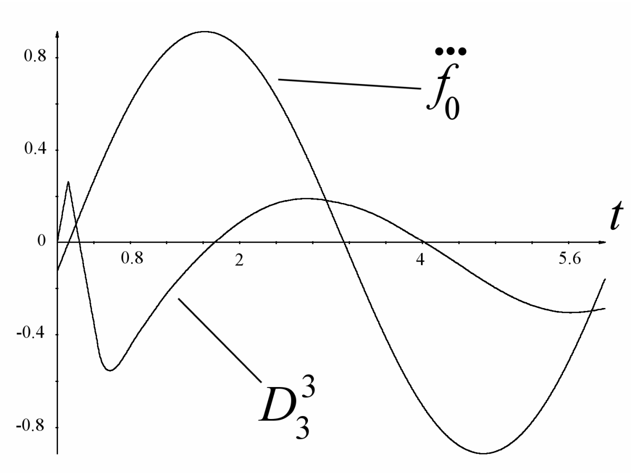
$|z_2 - \ddot{f}_0| \leq 2.24\cdot 10^{-8}$, $|z_3 - \dddot{f}_0| \leq 2.44\cdot 10^{-5}$

$|z_4 - f_0^{(4)}| \leq 0.00599$, $|z_5 - f_0^{(5)}| \leq 0.265$

Sensitivity to noises

3rd-order differentiator, $\varepsilon = 0.0001$, $\omega \approx 1000$

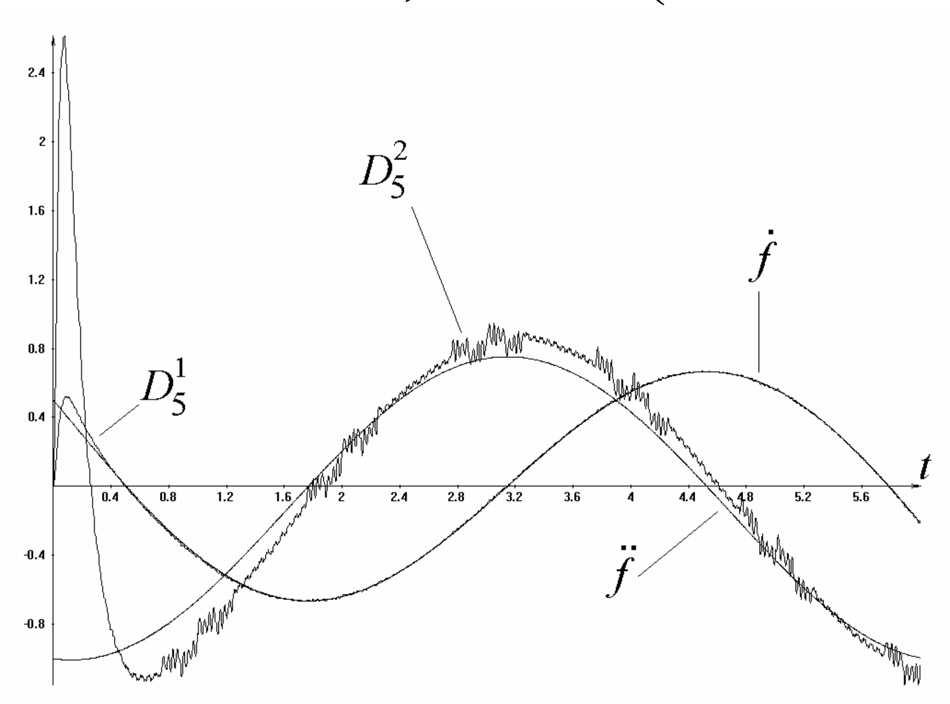




$|D_3{}^1 - \dot{f}_0| \le 2.6 \cdot 10^{-3}$, $|D_3{}^2 - \ddot{f}_0| \le 0.14$, $|D_3{}^3 - \dddot{f}_0| \le 0.48$

5th-order differentiator,

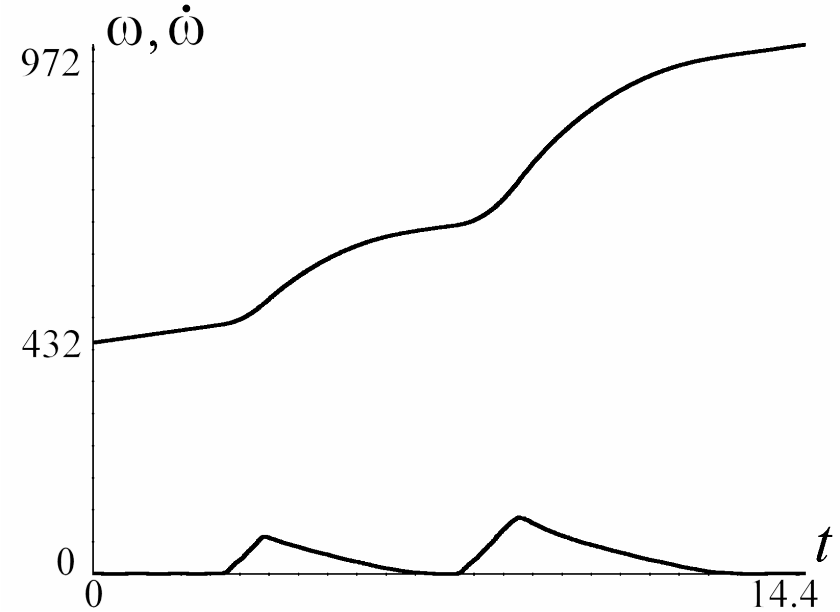
$\varepsilon = 0.01$, $\omega \approx 10$ (the worst case)



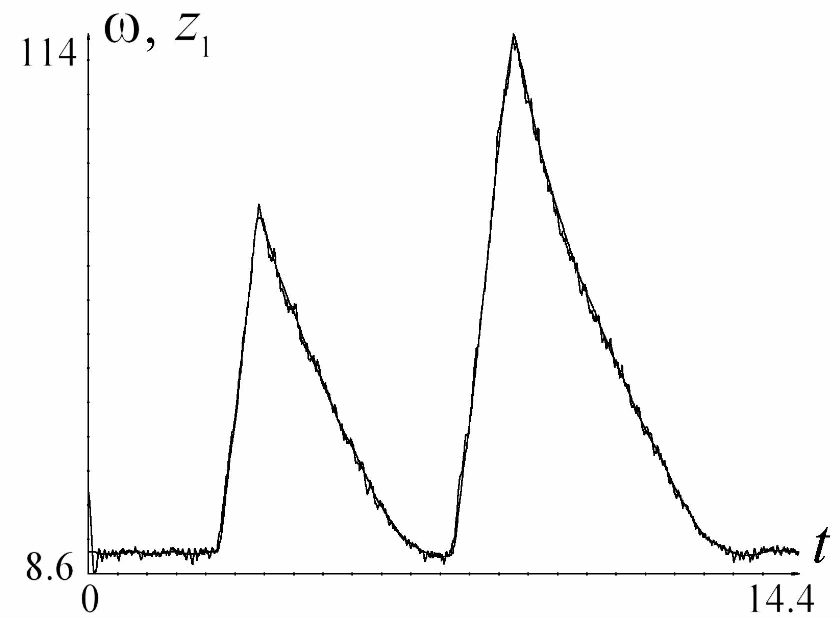
$|z_1 - \dot{f}_0| \le 0.014$, $|z_2 - \ddot{f}_0| \le 0.18$, $|z_3 - \dddot{f}_0| \le 0.65$

$|z_4 - f_0^{(4)}| \le 1.05$, $|z_5 - f_0^{(5)}| \le 0.74$

On-line calculation of the angular motor velocity and acceleration (data from Volvo Ltd)

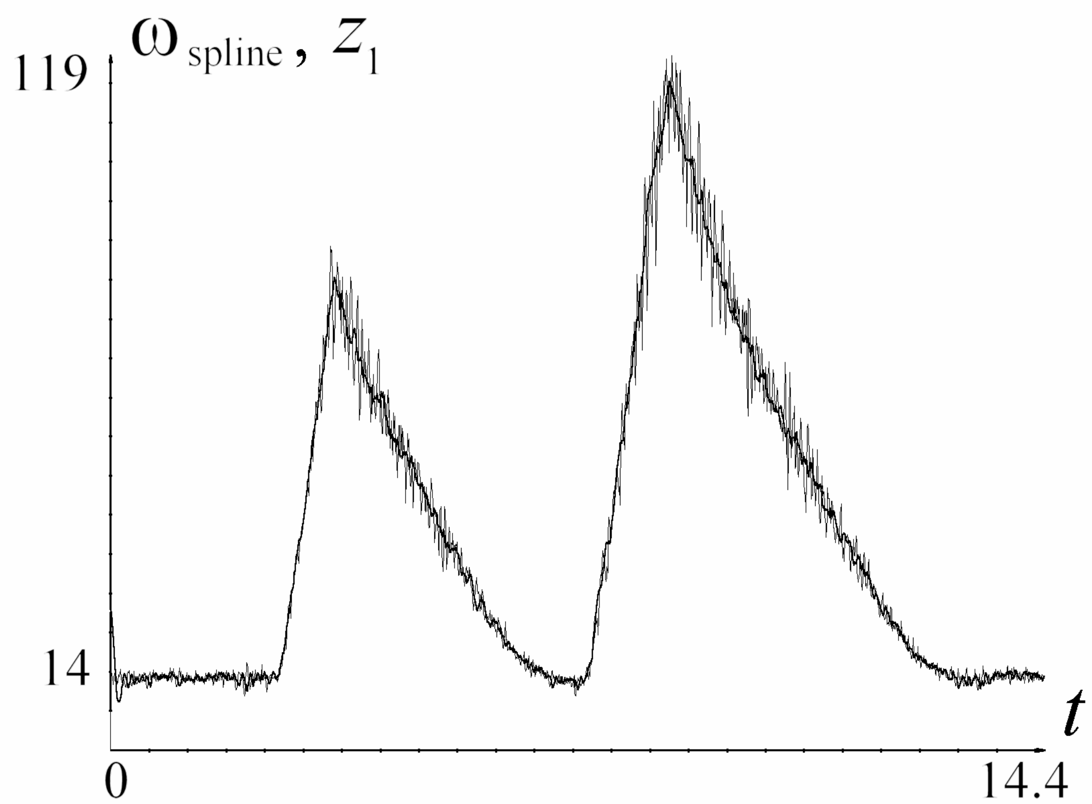


Experimental data, $\tau = 0.004$



2nd order differentiation $L = 625$

On-line 2nd order differentiation
comparison with optimal spline approximation



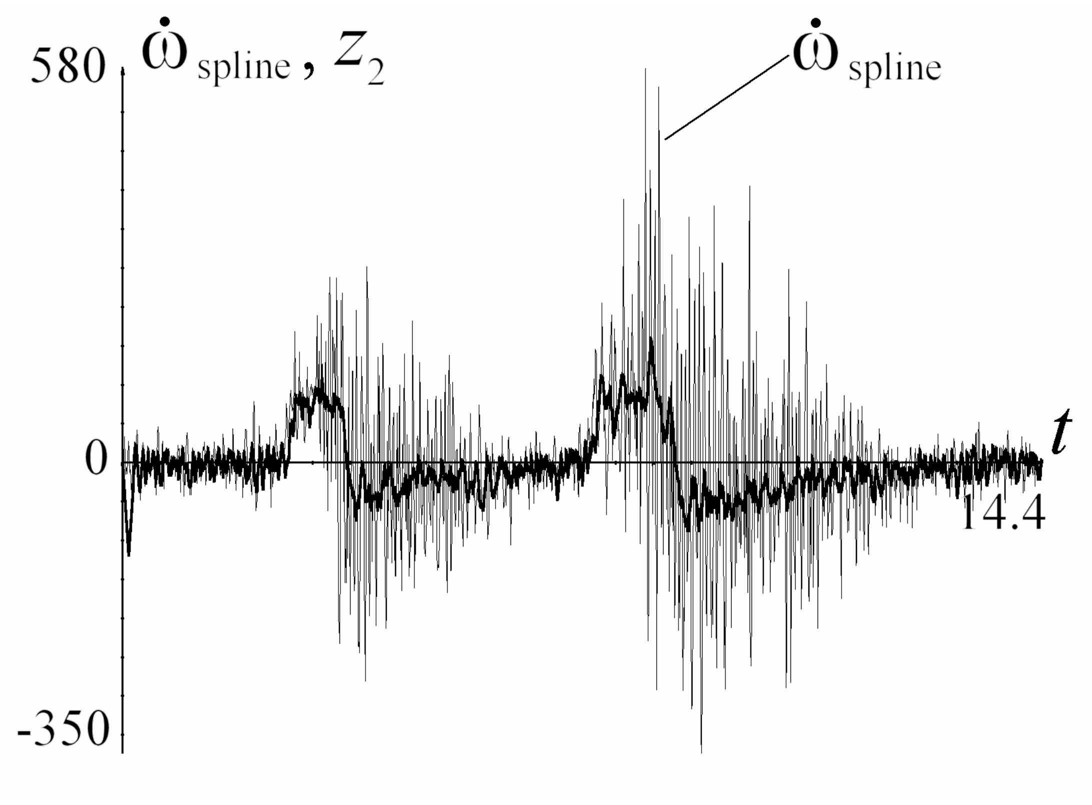
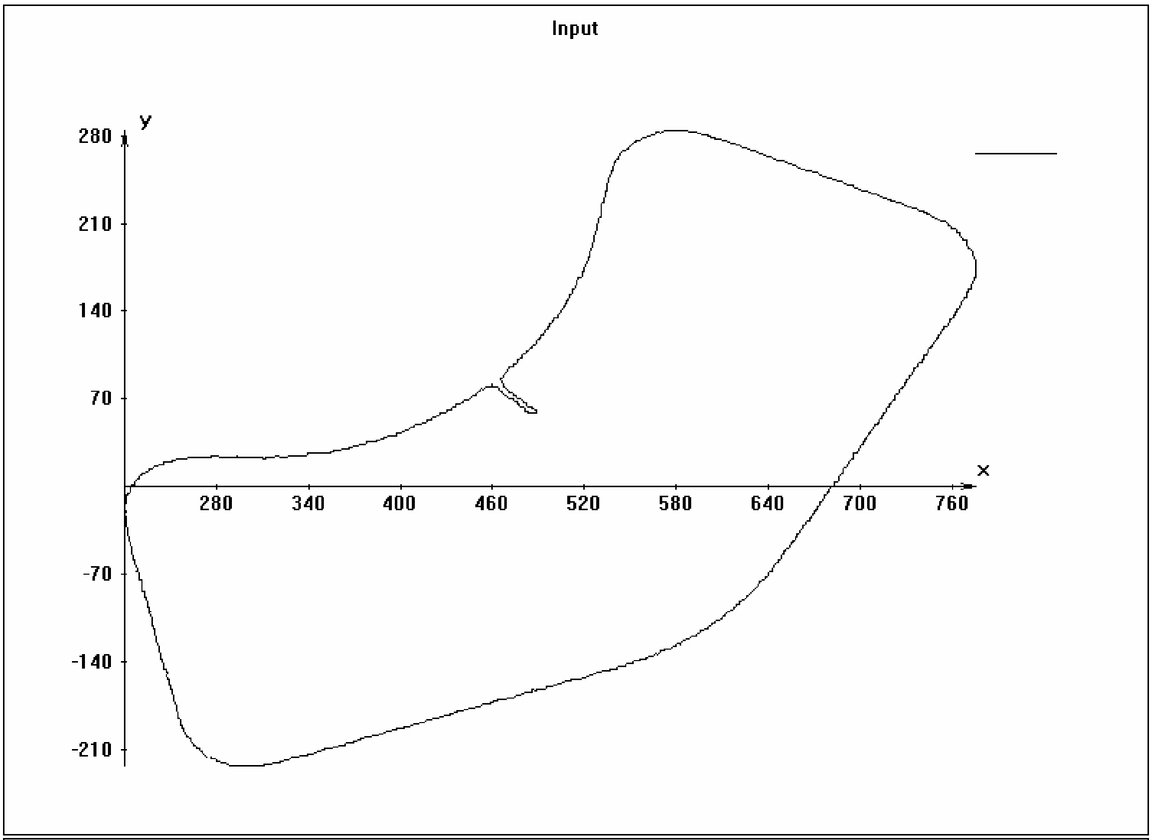
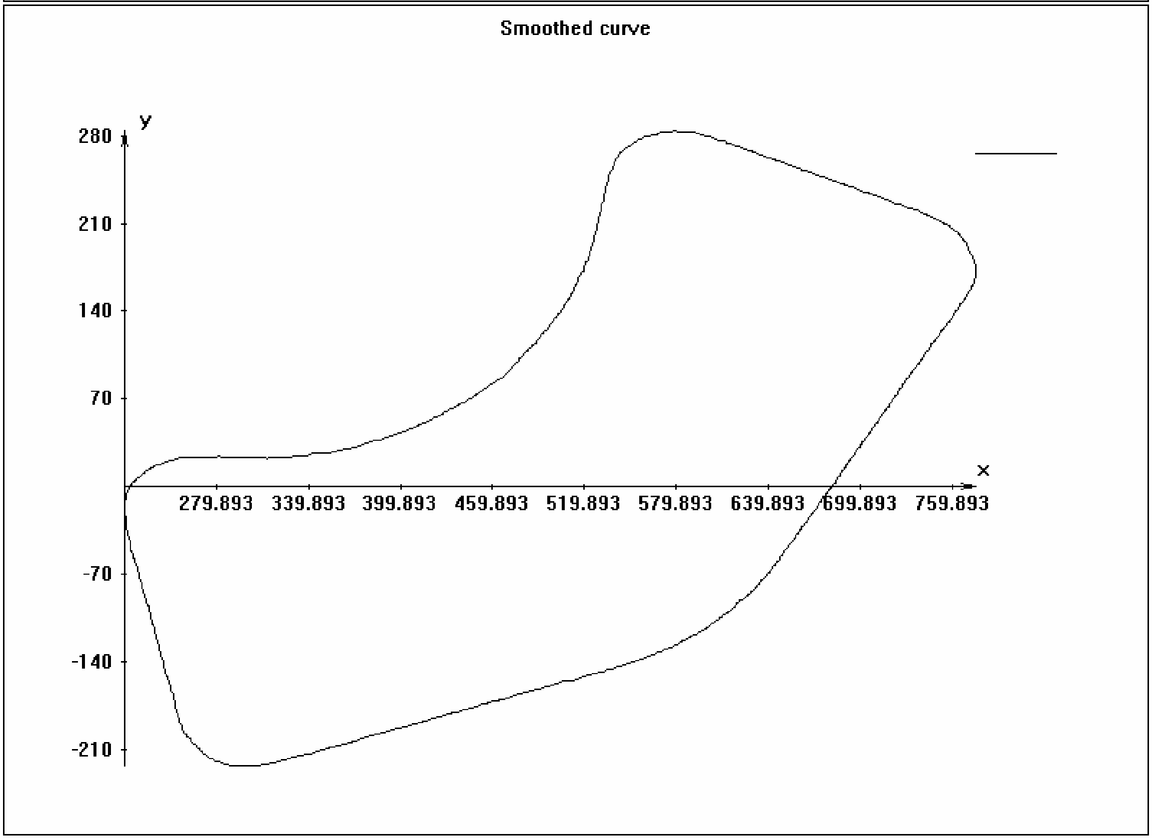


Image Processing: Crack Elimination





Edge Detection