Robust exact differentiation

Let a signal $f(t)$ be a function defined on $[0, \infty)$, which is a result of real-time noisy measurements of some unknown n-smooth signal $f_0(t)$ with the nth derivative $f_0^{(n)}(t)$ having a known Lipschitz constant $L > 0$. The function $f(t)$ is assumed to be a Lebesgue-measurable function, the unknown sampling noise $f(t) - f_0(t)$ is assumed bounded. The task is to find real-time estimations of f_0, $\dot{f}_0$, $\ddot{f}_0$, ..., $f_0^{(n)}$ using only values of f and the number L. The estimations are to be exact in the absence of noises when $f(t) = f_0(t)$.

Denote by $D_{n-1}(f(\cdot),L)$ the $(n-1)$th-order differentiator producing outputs $D_{n-1}^{i}(f(\cdot),L)$, $i = 0$, 1, ..., n-1, being estimations of f_0, $\dot{f}_0$, $\ddot{f}_0$, ..., $f_0^{(n-1)}$ for any input $f(t)$ with $f_0^{(n-1)}$ having Lipschitz constant $L > 0$. Then the nth-order differentiator has the outputs $z_i = D_n^i(f(\cdot),L)$, $i = 0, 1, ..., n$, defined recursively as follows:

$$\dot{z}_0 = v, \qquad v = -\lambda_n L^{1/(n+1)} | z_0 - f(t)|^{n/(n+1)} \operatorname{sign}(z_0 - f(t)) + z_1,$$

$$z_1 = D_{n-1}^{0}(v(\cdot), L), \ \ldots, z_n = D_{n-1}^{n-1}(v(\cdot), L).$$

Here $D_0(f(\cdot), L)$ is a simple nonlinear filter

$$D_0: \quad \dot{z} = -\lambda_0 L \operatorname{sign}(z - f(t)), \qquad \lambda_0 > 1.$$

Thus, the nth-order differentiator [3] has the form

$$\begin{aligned}
&\dot{z}_0 = v_0, && v_0 = -\lambda_n L^{1/(n+1)} | z_0 - f(t)|^{n/(n+1)} \operatorname{sign}(z_0 - f(t)) + z_1,\\
&\dot{z}_1 = v_1, && v_1 = -\lambda_{n-1} L^{1/n} | z_1 - v_0|^{(n-1)/n} \operatorname{sign}(z_1 - v_0) + z_2,\\
& && \ldots \\
&\dot{z}_{n-1} = v_{n-1}, && v_{n-1} = -\lambda_1 L^{1/2} | z_{n-1} - v_{n-2}|^{1/2} \operatorname{sign}(z_{n-1} - v_{n-2}) + z_n,\\
&\dot{z}_n = -\lambda_0 L \operatorname{sign}(z_n - v_{n-1}),
\end{aligned} \tag{1}$$

where $\lambda_i > 0$ are chosen sufficiently large in the list order. The solution is understood in the Filippov sense [1]. Note that it contains actually all the lower-order differentiators and each recursive step requires tuning one parameter only. With $n = 1$ the first-order differentiator from [2] is obtained. The Theorem [3] actually states that there exists a sequence $\{\lambda_i\}$ suitable for all differentiators.

There is a simple, though rather conservative, algebraic criterion for the parameter choice when $n = 1$, a practically-exact simply-verified integral criterion is also available in that case [2]. Unfortunately, such criteria lack with $n > 1$. The author found a number of computer-checked parameter combinations for $n = 5$ and less. One of the combinations is $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$, another one is $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$. Following relations are established in finite time with properly chosen parameters [2, 3]:

1. if $f(t) = f_0(t)$ (there is no noise) then

$$z_0 = f_0(t); \qquad z_i = v_{i-1} = f_0^{(i)}(t), \quad i = 1, ..., n;$$

2. if $|f(t) - f_0(t)| \leq \varepsilon$, then for some positive constants μ_i, ν_i depending exclusively on the parameters of differentiator (1)

$$|z_i - f_0^{(i)}(t)| \leq \mu_i\, \varepsilon^{(n-i+1)/(n+1)}, \ i = 0, ..., n,$$
$$|v_i - f_0^{(i+1)}(t)| \leq \nu_i\, \varepsilon^{(n-i)/(n+1)}, \ i = 0, ..., n\text{-}1;$$

3. if $f(t) = f_0(t)$, but $f(t)$ is sampled with a constant time period $\tau > 0$, then for some $\underline{\mu}_i$, $\underline{\nu}_i$

$$|z_i - f_0^{(i)}(t)| \leq \underline{\mu}_i\, \tau^{n-i+1}, \ i = 0, ..., n,$$
$$|v_i - f_0^{(i+1)}(t)| \leq \underline{\nu}_i\, \tau^{n-i}, \ i = 0, ..., n - 1.$$

It is easy to see that the nth order differentiator provides for a much better accuracy of the lth derivative, $l < n$, than the lth order differentiator. Thus, the additional smoothness of the unknown input signal $f_0(t)$ can be used to improve its derivative estimation based on the noisy measurement $f(t)$.

Results of the 5th-order numerical differentiation of the signal $f_0(t) = 0.5 \sin 0.5t + 0.5 \cos t$ with $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$, $L = 1$ are shown in Fig. 1.

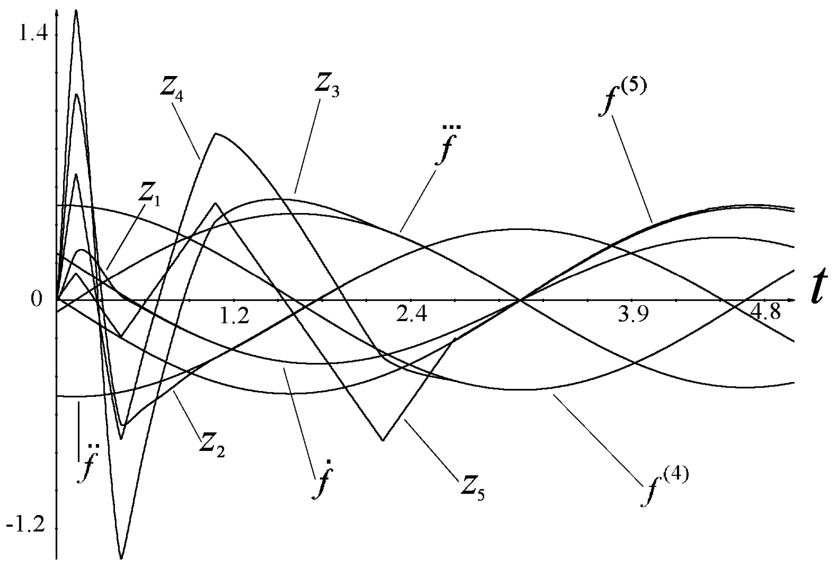

Fig. 1: 5th-order numerical differentiation

Numeric differentiation: instructions

In case $|f^{(n+1)}| \leq L$ the nth order differentiator parameters are changed with respect to the rule $\lambda_i = \lambda_{0i} L^{1/(n-i+1)}$. Following are equations of the 5th-order differentiator with simulation-tested coefficients. As mentioned, it contains also differentiators of the lower orders:

$$\dot{z}_0 = v_0, \quad v_0 = -8 L^{1/6} |z_0 - f(t)|^{5/6} \text{sign}(z_0 - f(t)) + z_1,$$

$$\dot{z}_1 = v_1, \quad v_1 = -5 L^{1/5} |z_1 - v_0|^{4/5} \text{sign}(z_1 - v_0) + z_2,$$

$$\dot{z}_2 = v_2, \quad v_2 = -3 L^{1/4} |z_2 - v_1|^{3/4} \text{sign}(z_2 - v_1) + z_3,$$

$$\dot{z}_3 = v_3, \quad v_3 = -2 L^{1/3} |z_3 - v_2|^{2/3} \text{sign}(z_3 - v_2) + z_4,$$

$$\dot{z}_4 = v_4, \quad v_4 = -1.5 L^{1/2} |z_4 - v_3|^{1/2} \text{sign}(z_4 - v_3) + z_5,$$

$$\dot{z}_5 = -1.1 L \,\text{sign}(z_5 - v_4);$$

Here z_i are the estimations of $f^{(i)}(t)$, $i = 0, ..., 5$, $|f^{(6)}(t)| \leq L$. The differentiator parameters can be easily changed. For example, increasing the gain in the line for $\dot{z}_3$ does not influence the coefficients for $\dot{z}_4$ and $\dot{z}_5$, but may require some enlargement of the above gains. The tradeoff is as follows: the larger the parameters, the faster the convergence and the higher sensitivity to input noises and the sampling step.

For example, the second order differentiator for the input f with $|\dddot{f}| \leq 1$ is

$$\dot{z}_0 = v_0, \quad v_0 = -2 |z_0 - f|^{2/3} \text{sign}(z_0 - f) + z_1,$$

$$\dot{z}_1 = v_1, \quad v_1 = -1.5 |z_1 - v_0|^{1/2} \text{sign}(z_1 - v_0) + z_2,$$

$$\dot{z}_2 = -1.1 \,\text{sign}(z_2 - v_1).$$

In particular, with $|\dddot{f}| \leq L$ the second order differentiator is

$$\dot{z}_0 = v_0, \quad v_0 = -2 L^{1/3} |z_0 - f|^{2/3} \text{sign}(z_0 - f) + z_1,$$

$$\dot{z}_1 = v_1, \quad v_1 = -1.5 L^{1/2} |z_1 - v_0|^{1/2} \text{sign}(z_1 - v_0) + z_2,$$

$$\dot{z}_2 = -1.1 L \,\text{sign}(z_2 - v_1);$$

and the first order differentiator with $|\ddot{f}| \le L$ is

$$\dot{z}_0 = v_0, \quad v_0 = -1.5\, L^{1/2}\, |z_0 - f|^{1/2}\, \mathrm{sign}(z_0 - f) + z_1,$$

$$\dot{z}_1 = -1.1\, L\, \mathrm{sign}(z_1 - v_0) = -1.1\, L\, \mathrm{sign}(z_0 - f).$$

z_0, z_1, z_2, ... stay for the smoothed input f and its successive derivatives $\dot{f}$, $\ddot{f}$, ... respectively. The last equality in the second line is just an identity, showing the compliance with [2].

With discrete sampling the latter differentiator takes on the form

$$\dot{z}_0 = v_0, \quad v_0 = -1.5\, L^{1/2}\, |z_0 - f(t_i)|^{1/2}\, \mathrm{sign}(z_0 - f(t_i)) + z_1,$$

$$\dot{z}_1 = -1.1\, L\, \mathrm{sign}(z_1 - v_0(t_i)) = -1.1\, L\, \mathrm{sign}(z_0 - f(t_i)),$$

where the current time t satisfies $t_i \le t < t_i + \tau = t_{i+1}$.

In order to differentiate a real signal, one needs to estimate approximately the constant L and to use the smallest available integration and sampling time steps. It is important to check that the auxiliary variables $v_0, \ldots, v_{n-1}$ be calculated based on the values of z and f obtained at the last sampling time (no mixing!). At first L may be taken redundantly large, afterwards it can be gradually reduced during the simulation. The criterion for the good differentiation is that the output z_0 is to track the smoothed input, which is checked looking at the graph.

Important: integration by the Euler method!

References

[1] Filippov A.F., *Differential Equations with Discontinuous Right-Hand Side*, Kluwer, Dordrecht, the Netherlands, (1988).

[2] Levant A., Robust exact differentiation via sliding mode technique. *Automatica*, Vol. 34, No. 3, pp. 379-384, (1998).

[3] Levant, A. (2003). Higher-order sliding modes, differentiation and output-feedback control, *International Journal of Control*, **76** (9/10), 924-941.