Quasi-Continuous High-Order Sliding-Mode Controllers

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***Abstract*—A universal finite-time-convergent controller is developed capable to control the output of any uncertain single-input-single-output system with a known permanent relative degree r. The tracking error σ is steered to zero by means of a control dependent only on $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ and continuous everywhere except the set $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$. A robust output-feedback controller version provides for the tracking accuracy proportional to the sampling noise magnitude.**

***Index Terms*—Finite-time stability, high-order sliding mode, output feedback control, robustness.**

I. Introduction

Sliding mode control remains one of the most robust and effective tools to cope with heavy uncertainty conditions [22]. The main drawback of the standard sliding modes is mostly related to the so-called chattering effect caused by the high-frequency control switching [7], [8].

Let s be the output variable of an uncertain single-input–single-output (SISO) dynamic system and $w(t)$ be an unknown-in-advance smooth input, both available in real time. The task is to establish and keep $\sigma = s - w(t) = 0$. The standard sliding-mode control $u = -k \ \mathrm{sign}\sigma$ is applicable if the relative degree is 1, i.e., if $\dot{\sigma}$ explicitly depends on the control u, and $\dot{\sigma}'_u > 0$. Higher-order sliding mode [13], [16] is applicable for controlling SISO uncertain systems of arbitrary relative degrees [3], [6], [11], [14]–[16], [20]. The corresponding finite-time-convergent controllers (r-sliding controllers) [13], [16] require actually only the knowledge of the system relative degree r. The produced control is a discontinuous function of the tracking error σ and of its real-time-calculated successive derivatives $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$. The accuracy is improved in the presence of switching delays, and the chattering effect is successfully treated, provided the control derivative is used as a new control input [3], [13]. The discontinuity set of controllers [15], [16] is a stratified union of manifolds with codimension varying in the range from 1 to r, which causes certain transient chattering. To avoid the chattering one needs to increase artificially the relative degree r, inevitably complicating the controller implementation [15], [16]. The finite-time-stable exact tracking is lost with alternative controllers developed in [2] and [21] for $r = 3$ and $r = 2$, respectively.

A sliding-mode controller of a new type is proposed in this note, being a feedback function of $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$, continuous everywhere except the manifold defined by the equations

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \cdots = \sigma^{(r-1)} = 0 \tag{1}$$

of the r-sliding mode. The mode $\sigma \equiv 0$ is established after a finite-time transient. In the presence of errors in evaluation of the output σ and its derivatives, a motion in some vicinity of (1) takes place. Therefore, control is practically a continuous function of time, for the trajectory never hits the manifold (1) with $r > 1$. The controller design is based on the homogeneity reasoning [18], [19], and can be considered as a demonstration of the principles [18].

Combining with the recently proposed robust exact finite-time-convergent differentiator [16] an output-feedback controller is obtained providing for exact tracking $\sigma \equiv 0$ if the measurements of the tracking error σ are exact, and for σ proportional to the maximal measurement error otherwise. Its transient features are much better than those of the known r-sliding controllers [15], [16] (Section VI). Simulation demonstrates the practical applicability of the new controller.

II. Preliminaries and the Problem Statement

Consider a smooth dynamic system with a smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense [5]. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set (1) is a nonempty integral set, the motion (1) is called r-sliding (rth-order sliding) mode [13], [3], [16]. The standard sliding mode used in the most variable structure systems is of the first-order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u \quad \sigma = \sigma(t,x). \tag{2}$$

Here, $x \in \boldsymbol{R}^n$, a, b and $\sigma : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ are unknown smooth functions, $u \in \boldsymbol{R}$, n is also uncertain. The task is to provide in finite time for exact keeping of $\sigma \equiv 0$.

The relative degree r of the system is assumed to be constant and known. In other words [10], the control explicitly appears first time in the rth total time derivative of σ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u \tag{3}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = (\partial/\partial u)\sigma^{(r)} \neq 0$. It is supposed that for some $K_{\mathrm{m}}, K_{\mathrm{M}}, C > 0$

$$0 < K_{\mathrm{m}} \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}} \quad |\sigma^{(r)}|_{u=0}| \leq C \tag{4}$$

which is always true at least locally. Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t,x)$.

Finite time stabilization of smooth systems at an equilibrium point by means of continuous control is considered in [1]–[9]. In our case, any continuous control $u = U(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$ providing for $\sigma \equiv 0$, would satisfy the equality $U(0, 0, \ldots, 0) = -h(t,x)/g(t,x)$, whenever (1) holds. Since the problem uncertainty prevents it [18], *the control has to be discontinuous at least on the set* (1). Hence, the r-sliding mode $\sigma = 0$ is to be established. The controller which is designed in this note, is *r-sliding homogeneous* [18], which means that the identity $U(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}) \equiv U(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa\sigma^{(r-1)})$ is kept for any $\kappa > 0$.

III. Feedback Design

As follows from (3) and (4)

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u. \tag{5}$$

The closed differential inclusion is understood here in the Filippov sense [5], which means that the right-hand vector set is enlarged in

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a special way [18], in order to satisfy certain convexity and semicontinuity conditions. This inclusion does not "remember" anything on system (2) except the constants $r, C, K_{\mathrm{m}}, K_{\mathrm{M}}$. Thus, the finite-time stabilization of (5) at the origin solves the stated problem simultaneously for all systems (2) satisfying (4). Let $i = 0, \ldots, r-1$. Denote

$$\varphi_{0,r} = \sigma \quad N_{0,r} = |\sigma| \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \mathrm{sign}\sigma$$
$$\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \Psi_{i-1,r}$$
$$N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)} \quad \Psi_{i,r} = \varphi_{i,r}/N_{i,r}$$

where $\beta_1, \ldots, \beta_{r-1}$ are positive numbers, obviously $\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{-1/(r-i+1)} \varphi_{i-1,r}$. Recall that according to the Filippov definition values of the control on any set of the zero Lebesgue measure do not influence the solutions. The following proposition is easily proved by induction.

Proposition 1: Let $i = 0, \ldots, r-1$. $N_{i,r}$ is positive definite, i.e., $N_{i,r} = 0$ iff $\sigma = \dot{\sigma} = \cdots = \sigma^{(i)} = 0$. The inequality $|\Psi_{i,r}| \leq 1$ holds whenever $N_{i,r} > 0$. The function $\Psi_{i,r}(\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)})$ is continuous everywhere (i.e., it can be redefined by continuity) except the point $\sigma = \dot{\sigma} = \cdots = \sigma^{(i-1)} = 0$.

Theorem 1: Provided $\beta_1, \ldots, \beta_{r-1}, \alpha > 0$ are chosen sufficiently large in the list order, the controller

$$u = -\alpha\, \Psi_{r-1,r}\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \tag{6}$$

is r-sliding homogeneous and provides for the finite-time stability of (5), (6). The finite-time stable r-sliding mode $\sigma \equiv 0$ is established in the system (2), (6).

The proof is given in Section V. It follows from Proposition 1 that *control* (6) *is continuous everywhere except the r-sliding mode* $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$. Any time the finite-time stability is mentioned in this note it means that the maximal possible transient time is a locally bounded function of initial conditions [18].

Each choice of parameters $\beta_1, \ldots, \beta_{r-1}$ determines a controller family applicable to all systems (2) of the relative degree r. The parameter α is chosen specifically for any fixed $C, K_{\mathrm{m}}, K_{\mathrm{M}}$, most conveniently by computer simulation, avoiding redundantly large estimations of $C, K_{\mathrm{m}}, K_{\mathrm{M}}$. Obviously, α is to be negative with $(\partial/\partial u)\sigma^{(r)} < 0$. Following are controllers with $r \leq 4$ and simulation-tested β_i:

1) $u = -\alpha \mathrm{sign}\sigma$;
2) $u = -\alpha(\dot{\sigma} + |\sigma|^{1/2}\mathrm{sign}\sigma)/(|\dot{\sigma}| + |\sigma|^{1/2})$;
3) $u = -\alpha[\ddot{\sigma} + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{-1/2}(\dot{\sigma} + |\sigma|^{2/3}\mathrm{sign}\sigma)]/[|\ddot{\sigma}| + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}]$;
4) $\varphi_{3,4} = \dddot{\sigma} + 3[\ddot{\sigma} + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{-1/3}(\dot{\sigma} + 0.5|\sigma|^{3/4}\mathrm{sign}\sigma)][|\ddot{\sigma}| + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{2/3}]^{-1/2}$, $N_{3,4} = |\dddot{\sigma}| + 3[|\ddot{\sigma}| + (|\dot{\sigma}| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}$, $u = -\alpha\varphi_{3,4}/N_{3,4}$.

The control is a continuous function of time everywhere except the r-sliding set (1). It may have infinite derivatives when certain surfaces are crossed. All further theorems are standard consequences [18] of the r-sliding homogeneity of controller (6) and Theorem 1.

Theorem 2: Let the control value be updated at the moments t_i, with $t_{i+1} - t_i = \tau = \mathrm{const} > 0, t \in [t_i, t_{i+1})$ (the discrete sampling case). Then, controller (6) provides in finite time for keeping the inequalities $|\sigma| < \mu_0\tau^r, |\dot{\sigma}| < \mu_1\tau^{r-1}, \ldots, |\sigma^{(r-1)}| < \mu_{r-1}\tau$ with some positive constants $\mu_0, \mu_1, \ldots, \mu_{r-1}$.

That is the best possible accuracy attainable with discontinuous $\sigma^{(r)}$ [13]. The following result shows robustness of controller (6) with respect to measurement errors.

Theorem 3: Let $\sigma^{(i)}$ be measured with accuracy $\nu_i\varepsilon^{(r-i)/r}$ for some fixed $\nu_i > 0, i = 1, \ldots, r-1$. Then with some positive constants μ_i the inequalities $|\sigma^{(i)}| \leq \mu_i\varepsilon^{(r-i)/r}, i = 0, \ldots, r-1$, are established in finite time for any $\varepsilon > 0$.

IV. Universal Output-Feedback SISO Controller

Controller (5) requires the real-time exact calculation or direct measurement of $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$. Equality (3) implies $|\sigma^{(r)}| \leq C + \alpha K_{\mathrm{M}}$, which allows the real-time robust $(r-1)$th-order differentiation of σ [16]. Other differentiators [8] can be also used. Consider uncertain system (2), (4). Combining controller (6) and the *homogeneous* differentiator [16], [18] achieve

$$u = -\alpha\Psi_{r-1,r}(z_0, z_1, \ldots, z_{r-1}) \tag{7}$$
$$\dot{z}_0 = v_0, v_0 = -\lambda_r L^{1/r}|z_0 - \sigma|^{(r-1)/r}\mathrm{sign}(z_0 - \sigma) + z_1 \tag{8}$$
$$\dot{z}_k = v_k, v_k = -\lambda_{r-k}L^{1/(r-k)}|z_k - v_{k-1}|^{(r-k-1)/(r-k)}\mathrm{sign}(z_k - v_{k-1}) + z_{k+1}, \quad k = 1, \ldots, r-2 \tag{9}$$
$$\dot{z}_{r-1} = -\lambda_1 L \mathrm{sign}(z_{r-1} - v_{r-2}) \tag{10}$$

where the parameters of the differentiator (8)–(10) are chosen according to the condition $|\sigma^{(r)}| \leq L, L$ is to satisfy $L \geqslant C + \alpha K_{\mathrm{M}}$. The sequence λ_i is chosen in advance [16]. Hence, in the case when C and $K_{\mathrm{m}}, K_{\mathrm{M}}$ are known, only one parameter α is really needed to be tuned. Usually, both L and α are found by computer simulation.

The computer-tested values $\lambda_1 = 1.1, \lambda_2 = 1.5, \lambda_3 = 2, \lambda_4 = 3, \lambda_5 = 5, \lambda_6 = 8$ can be taken. Due to the recursive form of the differentiator, these values are sufficient for up to the fifth-order differentiation and $r \leq 6$. The lacking values need to be tuned in the unlikely case of $r > 6$.

Theorem 4: Let σ be measured with a Lebesgue-measurable noise $\eta, |\eta| \leq \varepsilon$. Then with properly chosen parameters of the controller (7)–(10) the inequalities $|\sigma^{(i)}| \leq \mu_i\varepsilon^{(r-i)/r}, i = 0, \ldots, r-1$, are established in finite time for some positive μ_i.

With exact measurements $(\varepsilon = 0)$ the r-sliding mode $\sigma \equiv 0$ is established in the closed system, globally attracting trajectories in finite time.

Theorem 5: Let $\tau > 0$ be the constant sampling interval and the noises be absent. Then inequalities $|\sigma^{(i)}| \leq \mu_i\tau^{r-i}, i = 0, \ldots, r-1$, are established in finite time for some positive constants μ_i.

V. Proof of Theorem 1

The proof is based on a few Lemmas. Only the main proof points are listed below. Assign the weights (homogeneity degrees) $r - i$ to $\sigma^{(i)}, i = 0, \ldots, r-1$ and the weight 1 (minus system homogeneity degree [1]) to t, which corresponds to the r-sliding homogeneity [15].

Lemma 1: The weight of $N_{i,r}$ equals $r - i, i = 0, \ldots, r-1$. Each homogeneous locally-bounded function $\omega(\sigma, \dot{\sigma}, \ldots, \sigma^{(i)})$ of the weight $r - i$ satisfies the inequality $|\omega| \leq cN_{i,r}$ for some $c > 0$.

Indeed, $N_{i,r}$ is a positive–definite locally bounded function (Proposition 1), which implies that $\omega/N_{i,r}$ is bounded on a unit sphere and, therefore, everywhere.

Lemma 2: Let $1 \leq i \leq r-2$, then for any positive $\beta_i, \gamma_i, \gamma_{i+1}$ with sufficiently large $\beta_{i+1} > 0$ the inequality $|\sigma^{(i+1)} + \beta_{i+1}N_{i,r}^{(r-i-1)/(r-i)}\Psi_{i,r}| \leq \gamma_{i+1}N_{i,r}^{(r-i-1)/(r-i)}$ provides for the finite-time establishment and keeping of the inequality $|\sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}| \leq \gamma_i N_{i-1,r}^{(r-i)/(r-i+1)}$.

Proof: Consider the point set $\Omega(\xi) = \{(\sigma, \dot{\sigma}, \ldots, \sigma^{(i)}) | |\Psi_{i,r}| \leq \xi\}$ for some fixed $\xi > 0, \xi < \gamma_i/(3\beta_i), \xi < 1/3$. The inequality $|\Psi_{i,r}| \leq \xi$ implies $|\sigma^{(i)}| \leq 2\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}$ and, therefore, $\Omega(\xi) \subset \Omega_1(\xi)$, where $\Omega_1(\xi)$ is defined by the inequality

$$\left|\sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}\right| \leq 3\xi\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}.$$

That is equivalent, in its turn, to $\phi_- \leq \sigma^{(i)} \leq \phi_+$, where ϕ_-, ϕ_+ are homogeneous functions of $\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)}$ of the weight $r - i$. Restricting ϕ_- and ϕ_+ to the homogeneous sphere of the radius $\rho = 1$, where $\rho^p = \sigma^{p/r} + \dot{\sigma}^{p/(r-1)}, \ldots + (\sigma^{(i-1)})^{p/(r-i+1)}, p = 2r!$, achieve some continuous on the sphere functions ϕ_{1-} and ϕ_{1+}. Functions ϕ_{1-} and ϕ_{1+} can be approximated on the sphere by some smooth functions ϕ_{2-} and ϕ_{2+} from beneath and from above, respectively.

Any function ϕ defined on the homogeneous sphere $\rho = 1$ is uniquely extended to the function Φ of the weight $w > 0$ defined in the whole space $\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)}$ by the formula $\Phi(\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)}) = \rho^w \phi(\rho^{-r}\sigma, \rho^{-r+1}\dot{\sigma}, \ldots, \rho^{(-r-i+1)}\sigma^{(i-1)})$, where the function ρ is defined above. Thus, functions ϕ_{2-} and ϕ_{2+} are extended by homogeneity to the continuous homogeneous functions Φ_- and Φ_+ of $\sigma, \dot{\sigma}, \ldots, \sigma^{(i-1)}$ of the weight $r - i$, smooth everywhere except 0, so that $\Omega(\xi) \subset \Omega_2 = \{(\sigma, \dot{\sigma} \ldots, \sigma^{(i-1)}) | \Phi_- \leq \sigma^{(i)} \leq \Phi_+\}$.

Prove now that Ω_2 is invariant and attracts the trajectories with large β_{i+1}. The "upper" boundary of Ω_2 is given by the equation $\pi_+ = \sigma^{(i)} - \Phi_+ = 0$. The inequality $|\Psi_{i,r}| \geq \xi$ is assured outside of Ω_2. Suppose that at the initial moment $\pi_+ > 0$ and, therefore, $\Psi_{i,r} \geqslant \xi$. Taking into account that $\dot{\Phi}_+$ is homogeneous of the weight $r-i-1$ and, according to Lemma 1, $|\dot{\Phi}_+| \leq \kappa N_{i,r}^{(r-i-1)/(r-i)}$, and $|\pi_+| \leq \kappa_1 N_{i,r}$ for some $\kappa, \kappa_1 > 0$, achieve differentiating that with sufficiently large β_{i+1}

$$\begin{aligned}\dot{\pi}_+ &\leq (-\beta_{i+1}\xi + \gamma_{i+1}) N_{i,r}^{(r-i-1)/(r-i)} - \dot{\Phi}_+ \\ &\leq (-\beta_{i+1}\xi + \gamma_{i+1} + \kappa) N_{i,r}^{(r-i-1)/(r-i)} \\ &\leq (-\beta_{i+1}\xi + \gamma_{i+1} + \kappa)\left(\kappa_1^{-1}\pi_+\right)^{(r-i-1)/(r-i)}.\end{aligned}$$

Hence, π_+ vanishes in finite time with β_{i+1} large enough. Thus, the trajectory inevitably enters the region Ω_2 in finite time. Similarly, the trajectory enters Ω_2 if the initial value of π_+ is negative and, therefore, $\Psi_{i,r} \leq -\xi$. Obviously, Ω_2 is invariant.

Choosing Φ_- and Φ_+ sufficiently close to ϕ_- and ϕ_+ on the homogeneous sphere and β_{i+1} large enough, achieve from Lemma 1 that $\Omega_2 \subset \Omega_1(\gamma_i/(3\beta_i))$ and the statement of Lemma 2. ■

Since $N_{0,r} = |\sigma|, \varphi_{0,r} = \sigma$, Lemma 2 is replaced by the next simple lemma with $i = 0$.

Lemma 3: The inequality $|\dot{\sigma} + \beta_1|\sigma|^{(r-1)/r}\mathrm{sign}\sigma| \leq \gamma_1|\sigma|^{(r-1\}/r}$ provides with $0 \leq \gamma_1 < \beta_1$ for the establishment in finite time and keeping of the identity $\sigma \equiv 0$.

The proof of the theorem is now finished by the similar to Lemma 2 proof that for any $\gamma > 0$ with sufficiently large α the inequality $|\sigma^{(r-1)} + \beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}| \leq \gamma N_{r-2,r}^{1/2}$ is established in finite time and kept afterwards. ■

VI. Simulation Example: Car Control

Consider a simple kinematic model of car control

$$\dot{x} = v\cos\varphi \quad \dot{y} = v\sin\varphi$$
$$\dot{\varphi} = v/l\tan\theta \quad \dot{\theta} = u$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, where $g(x)$ and y are assumed to be available in real time.

Define $\sigma = y - g(x)$. Let $v = \mathrm{const} = 10$ m/s, $l = 5$ m, $g(x) = 10\sin(0.05x) + 5, x = y = \varphi = \theta = 0$ at $t = 0$. The relative degree of the system is 3 and the 3-sliding controller [15], [16] solves the problem, but its transient features some chattering (Fig. 2). Also

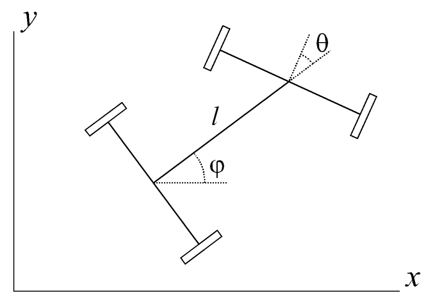

Fig. 1. Kinematic car model.

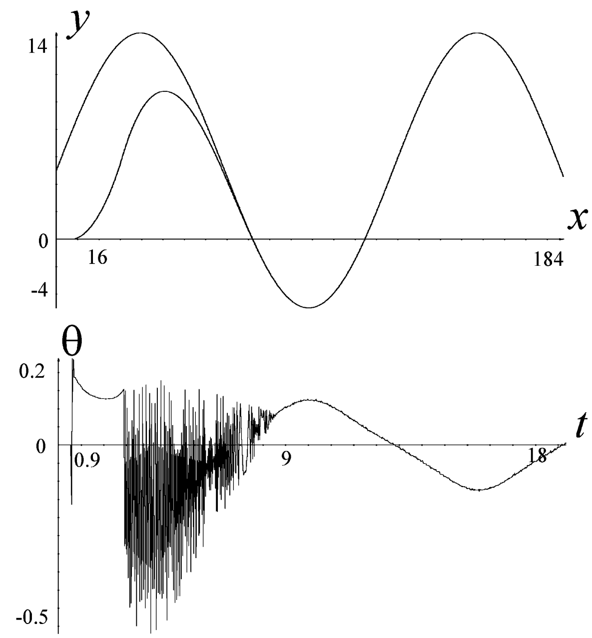

Fig. 2. 3-sliding car control [15].

3-sliding controller $N^\circ 3$ can be applied here. It was taken $\alpha = 1, L = 400$. The resulting output-feedback controller (8)–(10) is

$$\begin{aligned}u = &-[z_2 + 2(|z_1| + |z_0|^{2/3})^{-1/2}(z_1 + |z_0|^{2/3}\mathrm{sign} z_0)]/[|z_2| \\ &+ 2(|z_1| + |z_0|^{2/3})^{1/2}]\end{aligned}$$
$$\dot{z}_0 = v_0 \quad v_0 = -14.7361|z_0 - \sigma|^{2/3}\mathrm{sign}(z_0 - \sigma) + z_1$$
$$\dot{z}_1 = v_1 \quad v_1 = -30|z_1 - v_0|^{1/2}\mathrm{sign}(z_1 - v_0) + z_2$$
$$\dot{z}_2 = -440\ \mathrm{sign}(z_2 - v_1).$$

The controller parameter α is convenient to find by simulation [3], [14]. The differentiator parameter $L = 400$ is taken deliberately large, in order to provide for better performance in the presence of measurement errors ($L = 25$ is also sufficient, but is much worse with sampling noises). The control was applied only from $t = 1$, in order to provide some time for the differentiator convergence.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. In the absence of noises the tracking accuracies $|\sigma| \leq 5.4 \cdot 10^{-7}, |\dot{\sigma}| \leq 2.4 \cdot 10^{-4}, |\ddot{\sigma}| \leq 0.042$ were obtained. With $\tau = 10^{-5}$ the accuracies $|\sigma| \leq 5.6 \cdot 10^{-10}, |\dot{\sigma}| \leq 1.4 \cdot 10^{-5}, |\ddot{\sigma}| \leq 0.0042$ were attained, which mainly corresponds to the asymptotics stated in Theorem 5. The car trajectory, 3-sliding tracking errors, steering angle θ and its derivative u are shown in Fig. 3(a)–(d), respectively. It is seen from Fig. 3(c) that the control u remains continuous until the entrance into

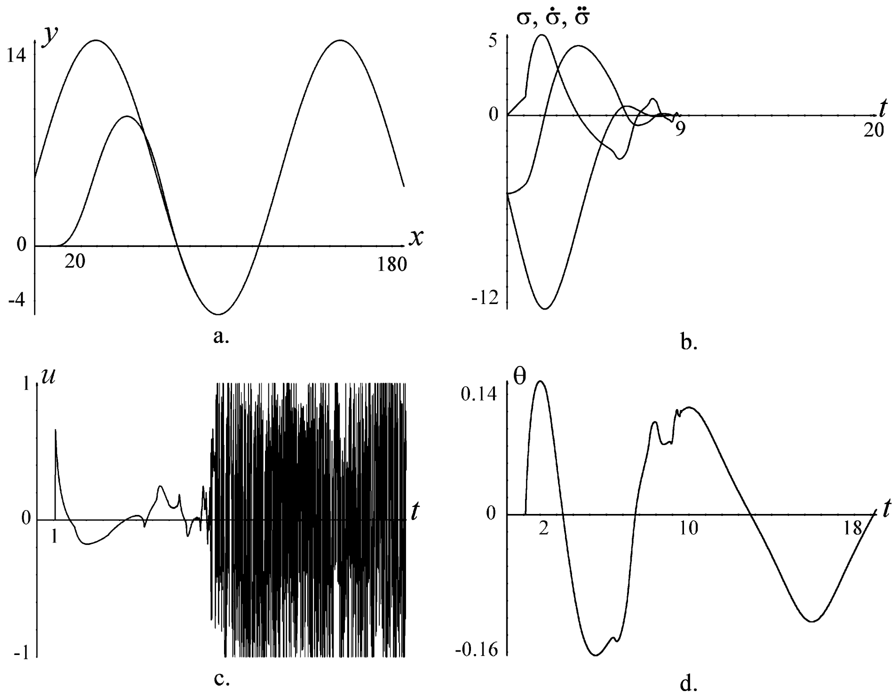

Fig. 3. New 3-sliding car control.

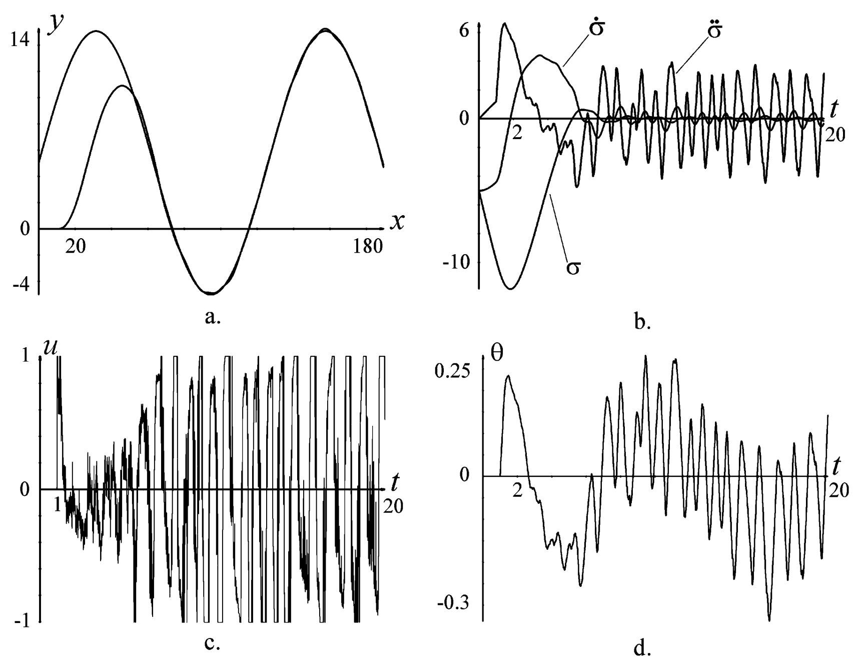

Fig. 4. Performance with the input noise magnitude 0.1 m.

the 3-sliding mode. The steering angle θ remains rather smooth and is quite feasible.

In the presence of output noise with the magnitude 0.01 m the tracking accuracies $|\sigma| \leq 0.02, |\dot{\sigma}| \leq 0.14, |\ddot{\sigma}| \leq 1.3$ were obtained. With the measurement noise of the magnitude 0.1 the accuracies changed to $|\sigma| \leq 0.20, |\dot{\sigma}| \leq 0.62, |\ddot{\sigma}| \leq 2.8$ which corresponds to the asymptotics stated by Theorem 4. The performance of the controller with the measurement error magnitude 0.1 m is shown in Fig. 4. It is seen from Fig. 4(c) that the control u is a continuous function of t. The steering angle vibrations have the magnitude of about 7° and the frequency 1, which is also quite feasible. The performance does not change, when the frequency of the noise varies in the range 100–100 000. The advantages of the new controller are obvious (compare Figs. 2 and 3). Simulation shows that the previous controller [15], [16] is also much more sensitive to the parameter choice and noises.

VII. Conclusion

A new arbitrary-order sliding mode controller is proposed. It is actually only the second known family of such controllers. It is also a sliding-mode SISO controller of a new type, for it provides for the finite-time stable sliding motion on the zero-dynamics manifold of high relative degree by means of control continuous everywhere except this manifold. As a result, the chattering effect is significantly reduced.

The real-time exact differentiator [16] of the appropriate order is combined with the proposed controller providing for the full SISO control based on the input measurements only, when the only information on the controlled uncertain process is actually its relative degree. Both the proposed controller and its output-feedback version are very robust with respect to measurement noises. Only boundedness of the noise is needed, no frequency considerations are relevant. *The simulation shows that it is probably the first practically applicable output-feedback r-sliding controller with* $r > 2$.

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Global Compensation of Unknown Sinusoidal Disturbances for a Class of Nonlinear Nonminimum Phase Systems

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***Abstract*—A class of output feedback stabilizable nonlinear systems with known output dependent nonlinearities and affected by unknown sinusoidal disturbances is considered: Nonminimum phase systems are also allowed. The problem of designing a global output feedback compensator which drives the state of the system exponentially to zero is solved when the disturbance consists of a known number of biased sinusoids with any unknown bias, magnitudes, phases, and frequencies.**

***Index Terms*—Adaptive estimation, nonlinear systems, output feedback, regulators.**

I. INTRODUCTION AND PROBLEM STATEMENT

Over the last decades, the problem of adaptive compensation of sinusoidal disturbances has attracted a considerable attention. If the frequencies of the disturbances are known, an early approach is based on the internal model principle [7]: The disturbances are viewed as outputs of known exosystems and may be rejected by incorporating the exosystem in the feedback path of the closed loop. The disturbance rejection in the case of a single sinusoidal disturbance with unknown frequency has been studied for stable linear systems in [1] and [17] and for minimum phase linear systems in [18] and [19]. Two schemes (a direct one and an indirect one) are presented and analyzed in [1]: While the direct scheme is local, the indirect one allows for larger initial conditions in the frequency estimate; on the other hand only the direct scheme guarantees exact disturbance compensation. If a positive lower bound for the frequency is known, a compensation scheme which allows for any initial frequency estimate greater than the given bound is presented in [17]. In [18], output tracking with multiple sinusoidal disturbance rejection is achieved for linear minimum phase systems and in [19] an adaptive extension is presented when the system parameters are unknown. As far as nonlinear systems are concerned, several results are available under the minimum phase assumption (MP): The semiglobal output regulation problem is addressed in [24] for systems with unknown parameters in the exosystem; in [21], a global robust state feedback control scheme is presented for systems affected by both structured unknown disturbances and an unknown noise, following earlier work in [20]. The global output tracking problem is studied in [2] for uncertain cascaded systems in lower triangular form coupled with a neutrally stable exosystem, while the output regulation problem is addressed in [27] for a class of large-scale nonlinear interconnected systems perturbed by a neutrally stable exosystem via a decentralized error feedback controller. Recently, global output feedback regulators for the same class of nonlinear systems considered in this note under the MP assumption have been proposed in [6] (an adaptive version of this strategy is presented in [5]) following [4] and [3]; semiglobal output feedback regulators have been described in [25]. Preliminary results on the semiglobal regulation of nonminimum phase (NMP) systems are given in [26] and [12], for classes of NMP systems which are more general than those considered in this note, under the assumption of sinusoidal disturbances with known frequency.

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