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Brief paper

Homogeneity approach to high-order sliding mode design☆

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Abstract

It is shown that a general uncertain single-input–single-output regulation problem is solvable only by means of discontinuous control laws, giving rise to the so-called high-order sliding modes. The homogeneity properties of the corresponding controllers yield a number of practically important features. In particular, finite-time convergence is proved, and asymptotic accuracy is calculated in a very general way in the presence of input noises, discrete measurements and switching delays. A robust homogeneous differentiator is included in the control structure thus yielding robust output-feedback controllers with finite-time convergence. It is demonstrated that homogeneity features significantly simplify the design and investigation of a new family of high-order sliding-mode controllers. Finally, simulation results are presented.

Keywords: Discontinuous control; Finite-time stability; High-order sliding mode; Robustness; Output feedback

1. Introduction

Control under heavy uncertainty conditions is one of the central topics of the modern control theory. The sliding-mode control approach (Utkin, 1992; Zinober, 1994; Edwards & Spurgeon, 1998) to the problem is based on keeping a properly chosen constraint exactly by means of high-frequency control switching. Although very robust and accurate, the approach also features certain drawbacks. The standard sliding mode may be implemented only if the relative degree of the constraint is 1, i.e. control has to appear explicitly already in the first total time derivative of the constraint function. Also, high-frequency control switching may cause the so-called chattering effect (Fridman, 2001, 2003).

High-gain control with saturation is used to overcome the chattering effect approximating the sign function in a boundary layer around the switching manifold (Slotine & Li, 1991), the sliding-sector method (Furuta & Pan, 2000) is suitable to control disturbed linear time-invariant systems. The sliding-mode-order approach (Emelyanov, Korovin, & Levantovsky, 1986; Levant, 1993) considered in this paper is capable of removing both the chattering and the relative-degree restrictions preserving the sliding-mode features and improving its accuracy.

Consider a smooth dynamic system with a smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables and the set $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ is non-empty and consists locally of Filippov trajectories (Filippov, 1988), the motion on the set $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ is called *r-sliding mode* (rth *order sliding mode*) (Emelyanov et al., 1986; Levant, 1993, 2003a). The rth derivative $\sigma^{(r)}$ is mostly supposed to be discontinuous or non-existent.

The standard sliding mode is of the first order ($\dot{\sigma}$ is discontinuous). Asymptotically stable higher-order sliding modes (HOSM) appear in many systems with traditional sliding-mode control (Fridman, 2001, 2003) and are deliberately introduced in systems with dynamical sliding modes (Sira-Ramírez, 1993). While finite-time-convergent

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arbitrary-order sliding-mode controllers are still studied theoretically (Levant, 2001, 2003a; Floquet, Barbot, & Perruquetti, 2003), 2-sliding controllers are already successfully implemented for the solution of real problems (Bartolini, Ferrara, & Punta, 2000; Bartolini, Pisano, Punta, & Usai, 2003; Ferrara & Giacomini, 2000; Levant, Pridor, Gitizadeh, Yaesh, & Ben-Asher, 2000; Sira-Ramírez, 2002;Orlov, Aguilar, & Cadiou, 2003; Spurgeon, Goh, & Jones, 2002;Shkolnikov & Shtessel, 2002;Shkolnikov, Sht essel,Lianos, & Thies, 2000; Shtessel & Shkolnikov, 2003).

The construction of r-sliding controllers, $r \geqslant 3$, is extremely difficult due to the high dimension of the problem. Thus, only one family of such controllers (Levant, 2001) was known until recently (Levant, 2003b). Almost all known HOSM controllers possess specific homogeneity properties. The corresponding homogeneity of r-sliding controllers is called in the paper the rth-order sliding homogeneity. This paper proposes the homogeneity-based approach to the construction of new finite-time convergent HOSM controllers. The homogeneity makes the convergence proofs of the HOSM controllers standard and provides for the highest possible asymptotic accuracy in the presence of measurement noises, delays and discrete measurements (Levant, 1993). An output-feedback controller with the same asymptotical features is obtained, when a recently developed robust exact homogeneous differentiator of the order $r-1$ (Levant, 1998, 2003a) is included as a standard part of the homogeneous r-sliding controller.

It is shown in this paper that the finite-time stability and the asymptotic accuracy are robust with respect to any small homogeneous controller perturbations. In particular, this allows the standard controllers (Levant, 2001, 2003a) to be easily regularized, drastically improving their performance. Computer simulation results are presented illustrating the main results of the paper.

2. Statement of the problem

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t,x), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is control, $\sigma \in \mathbf{R}$ is a measured output, smooth functions a, b, σ are assumed unknown, the dimension n can also be uncertain. On the contrary, the relative degree r (Isidori, 1989) of the system is constant and known. The solutions are understood in the Filippov sense (Filippov, 1988; also see below), and system trajectories are supposed to be infinitely extendible in time for any bounded Lebesgue-measurable input. Although it is formally not needed, the weakly minimum-phase property is often required in practice. The task is to make the output σ vanish in finite time and to keep $\sigma \equiv 0$.

In a simplified way, the constancy of the relative degree r means that for the first time u appears explicitly only in the rth total derivative of σ (Isidori, 1989). In that case the output σ satisfies an equation of the form

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g = \frac{\partial}{\partial u}\sigma^{(r)} \neq 0,$$
$$h = \sigma^{(r)}|_{u=0}. \tag{2}$$

Here h and g are unknown smooth functions. The uncertainty prevents immediate reduction of (1) to the standard form (2). Suppose that the inequalities

$$0 < K_{\mathrm{m}} \leqslant \frac{\partial}{\partial u}\sigma^{(r)} \leqslant K_{\mathrm{M}}, \quad |\sigma^{(r)}|_{u=0}| \leqslant C \tag{3}$$

hold for some $K_{\mathrm{m}}, K_{\mathrm{M}}, C > 0$. Note that (3) is formulated in input–output terms. These conditions are satisfied at least locally for any smooth system (1) having a well-defined relative degree at a given point with $\sigma = \cdots = \sigma^{(r-1)} = 0$. Assume that (3) holds globally. Then (2), (3) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u. \tag{4}$$

The problem is solved in two steps. First, a bounded feedback control

$$u = \varphi(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}), \tag{5}$$

is constructed such that all trajectories of (4), (5) converge in finite time to the origin $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ of the r-sliding phase space $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$. At the next step the lacking derivatives are real-time evaluated, producing an output-feedback controller. The function φ is assumed to be a Borel-measurable function, which provides for the Lebesgue measurability of composite functions to be obtained further. Actually all functions used in the sliding-mode control theory satisfy this restriction. In particular, any superposition of piecewise continuous functions is Borel measurable.

Note that the function φ has to be discontinuous at the origin. Indeed, otherwise u is close to the constant $\varphi(0,0,\ldots,0)$ in a small vicinity of the origin, and, taking $c \in [-C, C]$ and $k \in [K_{\mathrm{m}}, K_{\mathrm{M}}]$ so that $c + k\varphi(0,0,\ldots,0) \neq 0$, achieve that (5) cannot stabilize the dynamic system $\sigma^{(r)} = c + ku$. Thus, $\sigma^{(r)}$ is to be discontinuous when calculated with respect to the original system (1), (5), which means that the r-sliding mode $\sigma \equiv 0$ is to be established. All known r-sliding controllers (Bartolini, Ferrara, & Usai, 1998; Bartolini et al., 2003; Levant, 1993, 2002, 2003a,b) may be considered as controllers for (4) steering $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ to 0 in finite time. Inclusion (4) does not "remember" the original system (1). Thus, such controllers are obviously robust with respect to any perturbations preserving the system relative degree and (3).

3. Homogeneous differential inclusions

A differential inclusion $\dot{x} \in F(x)$ is further called a *Filippov differential inclusion* if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous (Filippov, 1988). The latter condition means that the maximal distance of the points of $F(x)$ from the set $F(y)$ vanishes when $x \to y$. Solutions are defined as absolutely continuous functions of time satisfying the inclusion almost everywhere. Such solutions always exist and have most of the well-known standard properties except uniqueness (Filippov, 1988).

It is said that a differential equation $\dot{x} = f(x)$ with a locally-bounded Lebesgue-measurable right-hand side is understood in the Filippov sense, if it is replaced by a special Filippov differential inclusion $\dot{x} \in F(x)$. In the most usual case, when f is continuous almost everywhere, the procedure is to take $F(x)$ as the convex closure of the set of all possible limit values of f at a given point x, obtained when its continuity point y tends to x. In the general case approximate continuity (Saks, 1964) points y are taken (one of the equivalent definitions by Filippov, 1988). A solution of $\dot{x} = f(x)$ is defined as a solution of $\dot{x} \in F(x)$. Values of f on any set of the measure 0 do not influence the Filippov solutions. Note that with continuous f the standard definition is obtained.

A similar procedure is applied to the differential inclusion (4), (5). To this end the above Filippov procedure is applied to the function φ and the obtained Filippov set is substituted for u in (5), producing a Filippov inclusion to replace (4), (5). Any solution of (4), (5) is defined as a solution of the built Filippov inclusion. Every time when a differential inclusion is considered in this paper, an appropriate Filippov inclusion replaces it, and the corresponding procedure is clarified.

A function $f : \mathbf{R}^n \to \mathbf{R}$ (respectively, a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field $f : \mathbf{R}^n \to \mathbf{R}^n$) is called *homogeneous of the degree $q \in \mathbf{R}$ with the dilation* $d_\kappa : (x_1, x_2, \ldots, x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, \ldots, \kappa^{m_n} x_n)$ (Bacciotti & Rosier, 2001), where $m_1, \ldots, m_n$ are some positive numbers (*weights*), if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds (respectively, $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$, or $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$). The non-zero homogeneity degree q of a vector field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, \ldots, m_n$.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) with respect to the combined time-coordinate transformation $G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x)$, $p = -q$.

1°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly finite-time stable* at 0, if it is Lyapunov stable and for any $R > 0$ exists $T > 0$, such that any trajectory starting within the disk $\|x\| < R$ stabilizes at zero in the time T.

2°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *globally uniformly asymptotically stable* at 0, if it is Lyapunov stable and for any $R > 0$, $\varepsilon > 0$, $T > 0$ exists such that any trajectory starting within the disk $\|x\| < R$ enters the disk $\|x\| < \varepsilon$ in the time T to stay there forever.

A set D is called *dilation retractable* if $d_\kappa D \subset D$ for any $\kappa < 1$.

3°. A homogeneous differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *contractive* if there are 2 compact sets D_1, D_2 and $T > 0$ such that D_2 lies in the interior of D_1 and contains the origin, D_1 is dilation-retractable, and all trajectories starting at the time 0 within D_1 are localized in D_2 at the time moment T.

Theorem 1. *Let $\dot{x} \in F(x)$ be a homogeneous Filippov inclusion with a negative homogeneous degree $-p$. Then properties 1°, 2° and 3° are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p.*

Proof. Obviously, both 1° and 2° imply 3°, and 1° implies 2°. Prove that 3° implies 1°. There is such a number $0 < \kappa < 1$ that $D_2 \subset d_\kappa D_1 \subset D_1$. Indeed, this follows from the continuity of the distance between D_2 and the boundary of $d_\kappa D_1$ with respect to κ in the Hausdorff metrics. Thus, trajectories starting in D_1 enter $W_1 = d_\kappa D_1$ in time T. Denote $W_j = d_\kappa^j D_1$, $j \in \mathbf{Z}$, $W_0 = D_1$, and achieve that trajectories starting in W_j finish in W_{j+1} in the time $\kappa^{jp} T$, and

$$\cdots \supset W_{-1} \supset W_0 \supset W_1 \supset \cdots, \cup W_j = \mathbf{R}^n, \cap W_j = \{O\},$$

where O is the origin. Hence, any trajectory starting in W_j converges in finite time to the origin, the convergence time being estimated by the expression $\kappa^{jp} T(1 + \kappa^p + \kappa^{2p} + \cdots) = \kappa^{jp} T/(1 - \kappa^p)$.

For any $R > 0$ there is such $\mu \ll 1$ that any trajectory starting in $d_\mu\ D_1$ will not be able to leave the disk $\|x\| \leqslant R$ in the time $\mu^p T$ due to the local boundedness of $F(x)$. That proves the Lyapunov stability. Applying the inverse transformation $G_{\mu^{-1}}$ achieve that the trajectories starting in D_1 are confined in some compact D_0 during the time T. Similarly, denoting $D_j = d_\kappa^j D_0$, achieve a sequence of embedded sets retracting to the origin O. Thus, any trajectory starting at O has to belong to all of these sets and cannot leave O.

The set of transient trajectories starting at a given point is compact in the C-metrics (Filippov, 1988). The maximal convergence time Θ of all solutions starting from x is some homogeneous function $\Theta(x)$. It equals 0 at the origin. Its continuity at the origin follows from the homogeneity: the maximal convergence time tends to zero when a disk $d_\kappa D$ of initial conditions retracts to the origin with $\kappa \to 0$. Any solution starting close to x comes in the time $\Theta(x)$ to a point close to the origin. The residual convergence time is small due to the continuity of the function $\Theta(x)$ at the origin. □

Due to the continuous dependence of solutions of the Filippov inclusion $\dot{x} \in F(x)$ on its graph $\Gamma = \{(x, y) | y \in F(x)\}$ (Filippov, 1988), the contraction feature 3° is obviously robust with respect to perturbations causing small changes of the graph in the Hausdorff metrics.

Corollary 1. *The global uniform finite-time stability of homogeneous differential equations (Filippov inclusions) with negative homogeneous degree is robust with respect to homogeneous perturbations causing locally small changes of the equation (inclusion) graph.*

Let $\dot{x} \in F(x)$ be a homogeneous Filippov differential inclusion. Consider the case of "noisy measurements" of x_i with the magnitude τ^{m_i}

$$\dot{x} \in F(x_1 + [-1,1]\tau^{m_1}, \ldots, x_n + [-1,1]\tau^{m_n}), \quad \tau > 0.$$

Applying successively the closure of the right-hand-side graph and the convex closure at each point x, obtain some new Filippov differential inclusion $\dot{x} \in F_\tau(x)$.

Theorem 2. *Let $\dot{x} \in F(x)$ be a globally uniformly finite-time stable homogeneous Filippov inclusion with the homogeneity weights $m_1, \ldots, m_n$ and the degree $-p < 0$, and let $\tau > 0$. Suppose that a continuous function $x(t)$ be defined for any $t \geqslant -\tau^p$ and satisfy some initial conditions $x(t) = \xi(t)$, $t \in [-\tau^p, 0]$. Then if $x(t)$ is a solution of the disturbed inclusion*

$$\dot{x}(t) \in F_\tau(x(t + [-\tau^p, 0])), \quad 0 < t < \infty, \tag{6}$$

inequalities $|x_i| < \gamma_i \tau^{m_i}$ are established in finite time with some positive constants γ_i independent of τ and ξ.

Note that Theorem 2 covers the cases of retarded or discrete noisy measurements of all or some of the coordinates and any mixed cases. In particular, infinitely extendible solutions certainly exist in the case of noisy discrete measurements of some variables or in the constant time-delay case.

Proof. The trajectories of the inclusion $\dot{x} \in F(x)$ which start from any disk D_0 centered at the origin converge in finite time to the origin, are confined in some larger disk, and their convergence time is uniformly bounded. Therefore, with some small parameter τ_0 the trajectories of (6) which start from D_0 gather in some small compact vicinity $W \subset D_0$ of the origin in some finite time T. The trajectories starting at the origin stay at the origin in the original system, thus also the trajectories of (6) starting within W do not leave some small vicinity of the origin during the time T. Let D_1 be the set of the points of all these trajectories during the time interval $[0, T]$, $W \subset D_1 \subset D_0$. Obviously, D_1 is an attracting invariant set of (6).

The transformation $\hat{G}_\kappa : (t, x, \tau) \mapsto (\kappa^p t, d_\kappa x, \kappa\tau)$ preserves the inclusion (6) changing τ. Let $0 < \kappa < 1$, $D_1 \subset d_\kappa D_0 \subset D_0$, then applying $\hat{G}_{\kappa^{-1}}$ achieve that the trajectories starting from $D_{-1} = d_{\kappa^{-1}} D_0 \supset D_0$ gather in D_0 with τ_0 changed to $\kappa^{-1}\tau_0$. Since $\tau_0 < \kappa^{-1}\tau_0$, the trajectories of (6) also satisfy the inclusion with τ_0 changed to $\kappa^{-1}\tau_0$. Thus, the trajectories of (6) starting in D_{-1} enter D_0 and proceed into D_1. Successively applying the transformation $\hat{G}_{\kappa^{-1}}$ achieve that D_1 is a global attracting set with the disturbance parameter τ_0. Let D_1 satisfy $|x_i| < a_i$. Now, applying the transformation $\hat{G}_\mu$ with $\mu = \tau/\tau_0$ and taking $\gamma_i = a_i/\tau_0^{m_i}$ achieve the needed asymptotic bounds of the attracting set for any τ. □

4. Homogeneity features of high-order sliding modes

Suppose that feedback (5) imparts homogeneity properties to the closed-loop inclusion (4), (5). Due to the term $[-C, C]$, the right-hand side of (5) can only have the homogeneity degree 0 with $C \neq 0$. Indeed, with a positive degree the right-hand side of (4), (5) approaches zero near the origin, which is not possible with $C \neq 0$. With a negative degree it is not bounded near the origin, which contradicts the local boundedness of φ. Thus, the homogeneity degree of $\sigma^{(r-1)}$ is to be opposite to the degree of the whole system.

Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of $t, \sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are $1, r, r-1, \ldots, 1$, respectively. This homogeneity is further called the *r-sliding homogeneity*. The inclusion (4), (5) and controller (5) are called *r-sliding homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa : (t, \sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}) \mapsto (\kappa t, \kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa\sigma^{(r-1)}) \tag{7}$$

preserves the closed-loop inclusion (4), (5).

Note that the Filippov differential inclusion corresponding to the closed-loop inclusion (4), (5) is also r-sliding homogeneous. Indeed, the convexity, the limiting process and the Lebesgue measurability are invariant with respect to the linear time-coordinate transformation (7). Recall that the values of φ on any zero-measure set do not affect the corresponding Filippov inclusion.

Transformation (7) transfers (4), (5) into

$$\frac{d^r(\kappa^r\sigma)}{(d\kappa t)^r} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]\varphi(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa\sigma^{(r-1)}).$$

Hence, (5) is r-sliding homogeneous iff

$$\varphi(\kappa^r\sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}). \tag{8}$$

Such a homogeneous controller is inevitably discontinuous at the origin $(0, \ldots, 0)$, unless φ is a constant function. It is also uniformly bounded, since it is locally bounded and takes on all its values in any vicinity of the origin.

A controller is called r-sliding homogeneous *in the broader sense* if (7) preserves the resulting trajectories of (4). Thus, the sub-optimal 2-sliding controller (Bartolini et al., 1998, 2003) is homogeneous, though it does not have the feedback form (5). Almost all known r-sliding controllers, $r \geqslant 2$, are r-sliding homogeneous. The only important exception is the terminal sliding mode controller $u = -\alpha\,\mathrm{sign}(\dot{\sigma} + \beta\sigma^\rho)$, where $\rho = (2k+1)/(2m+1)$, $\alpha, \beta > 0$, $k < m$, and k, m are natural numbers (Man, Paplinski, & Wu, 1994).

Indeed, the identity $\text{sign}(\kappa\dot{\sigma}+\beta(\kappa^2\sigma)^\rho)=\text{sign}(\dot{\sigma}+\beta\sigma^\rho)$ requires $\rho=\frac{1}{2}$ and $\sigma\geqslant 0$.

Asymptotic features of the known high-order sliding mode controllers (Levant, 1993, 2001, 2003a; Bartolini et al., 2003) are easily obtained from Theorem 2. Any r-sliding homogeneous controller can be complemented by an (r-1)th order differentiator (Atassi & Khalil, 2000;Bartolini, Pisano, & Usai, 2000; Krupp, Shkolnikov, & Shtessel, 2000; Levant, 1998, 2003a; Kobayashi, Suzuki, & Furuta, 2002;Yu & Xu, 1996) producing an output-feedback controller. In order to preserve the demonstrated exactness, finite-time stability and the corresponding asymptotic properties, the natural way is to calculate $\dot{\sigma},\ldots,\sigma^{(r-1)}$ in real time by means of a robust finite-time convergent exact *homogeneous* differentiator (Levant, 2003a). Its application is possible due to the boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function φ in (5). The resulting dynamical feedback has the form

$$u=\varphi(z_0,z_1,\ldots,z_{r-1}), \tag{9}$$

$$\begin{aligned}
&\dot{z}_0=v_0,\quad v_0=-\lambda_0L^{1/r}|z_0-\sigma|^{(r-1)/r}\,\text{sign}(z_0-\sigma)+z_1,\\
&\dot{z}_1=v_1,\\
&v_1=-\lambda_1L^{1/(r-1)}|z_1-v_0|^{(r-2)/(r-1)}\,\text{sign}(z_1-v_0)+z_2,\\
&\ldots\\
&\dot{z}_{r-2}=v_{r-2},\\
&v_{r-2}=-\lambda_{r-2}L^{1/2}|z_{r-2}-v_{r-3}|^{1/2}\,\text{sign}(z_{r-2}-v_{r-3})+z_{r-1},\\
&\dot{z}_{r-1}=-\lambda_{r-1}L\,\text{sign}(z_{r-1}-v_{r-2}),
\end{aligned} \tag{10}$$

where $L\geqslant C+\sup|\varphi|K_{\rm M}$, and parameters λ_i of differentiator (10) are adjusted in advance. A possible choice of the differentiator parameters with $r\leqslant 6$ is $\lambda_{r-1}=1.1$, $\lambda_{r-2}=1.5$, $\lambda_{r-3}=3$, $\lambda_{r-4}=5$, $\lambda_{r-5}=8$, $\lambda_{r-6}=12$. Adjustment of the parameters is described in detail in Levant (1998, 2003a).

Taking the homogeneity weight $r-i$ for z_i, $i=0,1,\ldots,r-1$, obtain a homogeneous differential inclusion (4), (9), (10) of the degree -1. Due to the finite-time convergence of the differentiator (Levant, 2003a) the corresponding Filippov inclusion is also globally uniformly finite-time stable. Let σ measurements be corrupted by a noise being an unknown bounded Lebesgue-measurable function of time. Then solutions of (1), (9), (10) are infinitely extendible in time under the assumptions of Section 2, and the following Theorems are simple consequences of Theorem 2.

Theorem 3. *Let controller* (5) *be r-sliding homogeneous and finite-time stable, and the parameters of the differentiator* (10) *be properly chosen with respect to the upper bound of* $|\varphi|$. *Then in the absence of measurement noises the output-feedback controller* (9), (10) *provides for the finite-time convergence of each trajectory to the r-sliding mode* $\sigma=0$, *otherwise convergence to a set defined by the inequalities* $|\sigma|<\gamma_0\delta$, $|\dot{\sigma}|<\gamma_1\delta^{(r-1)/r},\ldots,\sigma^{(r-1)}<\gamma_{r-1}\delta^{1/r}$ *is ensured, where* δ *is the unknown measurement noise magnitude and* $\gamma_0,\gamma_1,\ldots,\gamma_{r-1}$ *are some positive constants.*

In the absence of measurement noises the convergence time is bounded by a continuous function of the initial conditions in the space $\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)},z_0,z_1,\ldots,z_{r-1}$ which vanishes at the origin (Theorem 1).

Theorem 4. *Under the conditions of Theorem* 3 *the discrete-measurement version of the controller* (9), (10) *provides in the absence of measurement noises for the inequalities* $|\sigma|<\gamma_0\tau^r$, $|\dot{\sigma}|<\gamma_1\tau^{r-1},\ldots,\sigma^{(r-1)}<\gamma_{r-1}\tau$ *for some* $\gamma_0,\gamma_1,\ldots,\gamma_{r-1}>0$.

The asymptotic accuracy provided by Theorem 4 is the best possible with discontinuous $\sigma^{(r)}$ and discrete sampling (Levant, 1993). A Theorem corresponding to the case of discrete noisy sampling is also easily formulated based on Theorem 2. The results of this section are also valid for the sub-optimal controller (Bartolini et al., 1998, 2003).

5. Example of homogeneity-based sliding-mode design

Construction of new high-order sliding-mode controllers is difficult due to the high dimension of the problem. It can be significantly simplified by the homogeneity reasoning (Levant, 2002, 2003b). In particular, Corollary 1 allows new controller structures to be produced transforming known homogeneous controllers. Once a new controller is produced, its parameters are adjusted regardless of the controller prototype. Let q be the least common multiple of $1,2,\ldots,r$, and $\beta_1,\ldots,\beta_{r-1}>0$. Define

$$N_{i,r}=(|\sigma|^{q/r}+|\dot{\sigma}|^{q/(r-1)}+\cdots+|\sigma^{(i-1)}|^{q/(r-i+1)})^{(r-i)/q};$$

$$\varphi_{0,r}=\text{sign}\,\sigma,\quad \varphi_{i,r}=\text{sign}(\sigma^{(i)}+\beta_iN_{i,r}\varphi_{i-1,r}),\quad i=1,\ldots,r-1.$$

Then $u=-\alpha\varphi_{r-1,r}(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)})$ defines the standard r-sliding controller (Levant, 2001, 2003a). Its r-sliding homogeneity is easily checked. Here β_i can be chosen only once for each r, and the magnitude $\alpha>0$ is adjusted with respect to C, $K_{\rm m}$, $K_{\rm M}$ in order to stabilize (4) in finite time. In particular, the following controller is obtained with $r=3$:

$$u=-\alpha\,\text{sign}(\ddot{\sigma}+2(|\dot{\sigma}|^3+|\sigma|^2)^{1/6}\,\text{sign}(\dot{\sigma}+|\sigma|^{2/3}\,\text{sign}\,\sigma)). \tag{11}$$

Let $\kappa>0$. Then the 3-sliding homogeneity of (11) follows from the identity

$$\begin{aligned}
&\text{sign}(\kappa\ddot{\sigma}+2(|\kappa^2\dot{\sigma}|^3+|\kappa^3\sigma|^2)^{1/6}\,\text{sign}(\kappa^2\dot{\sigma}+|\kappa^3\sigma|^{2/3}\text{sign}\,\kappa^3\sigma))\\
&\quad=\text{sign}(\ddot{\sigma}+2(|\dot{\sigma}|^3+|\sigma|^2)^{1/6}\,\text{sign}(\dot{\sigma}+|\sigma|^{2/3}\,\text{sign}\,\sigma)).
\end{aligned}$$

The main drawback of these controllers is some trajectory chattering during the transient (see the simulation results in Fig. 1) caused by the complicated structure of the control discontinuity set. The output-feedback performance with noisy measurements is also problematic (coefficients γ_i from Theorem 3 are relatively large).

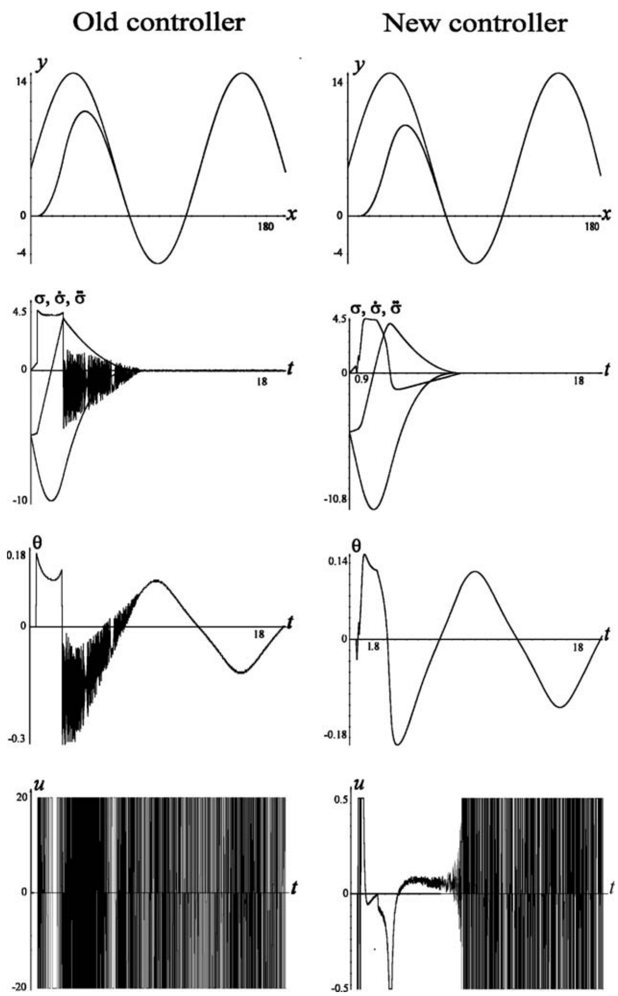

Fig. 1. Comparison of two controllers.

Recall that q is the least common multiple of $1, 2, \ldots, r$, and define the homogeneous norm and the saturation function

$$N_r = N_{r,r} = (|\sigma|^{q/r} + |\dot\sigma|^{q/(r-1)} + \cdots + |\sigma^{(r-1)}|^q)^{1/q},$$

$\mathrm{sat}(z, \varepsilon) = \min[1, \max(-1, z/\varepsilon)]$.

Let $i = 1, \ldots, r-1$. The new construction is as follows:

$$\psi_{0,r} = \mathrm{sign}\,\sigma, \quad \psi_{i,r} = \mathrm{sat}([\sigma^{(i)} + \beta_i N_{i,r}\psi_{i-1,r}]/N_r^{r-i}, \varepsilon_i).$$

Obviously $\psi_{i,r}$ turns into $\varphi_{i,r}$ with $\varepsilon_i \to 0$, $\varepsilon_i > 0$; $|\psi_{i,r}| \leqslant 1$, $\psi_{i,r}$ is homogeneous of the weight 0 and continuous everywhere except $\sigma = \dot\sigma = \cdots = \sigma^{(r-1)} = 0$. Thus, $\psi_{r-1,r}$ is a small homogeneous perturbation of $\varphi_{r-1,r}$ with small ε_i.

According to Corollary 1, the controller

$$u = -\alpha\psi_{r-1,r}(\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}) \qquad (12)$$

ensures the finite-time convergence to the r-sliding mode $\sigma \equiv 0$ with properly chosen α, β_i and small ε_i. It can be shown that β_i and ε_i can be chosen once for each r, and only $\alpha > 0$ is to be adjusted with respect to C, K_{m}, K_{M}. Controller (12) belongs to the class of quasi-continuous r-sliding controllers (Levant, 2003b) featuring continuity everywhere except the r-sliding mode itself.

Of course, Theorems 3, 4 are also valid here. The discrete-sampling and output-feedback versions of (12) are naturally constructed (Levant, 2003a). If (3) does not hold globally, the local controller application is justified as in (Levant, 2003a). Following is the list of the new r-sliding homogeneous controllers with simulation-checked $\beta_1, \ldots, \beta_{r-1}, \varepsilon_1, \ldots, \varepsilon_{r-1}$ and $r \leqslant 4$:

1. $u = -\alpha\,\mathrm{sign}\,\sigma = -\alpha\,\mathrm{sat}(\sigma/|\sigma|, 0.2)$,
2. $u = -\alpha\,\mathrm{sat}[(\dot\sigma + |\sigma|^{1/2}\,\mathrm{sign}\,\sigma)/(|\dot\sigma|^2 + |\sigma|)^{1/2}, 0.2]$,
3. $N_3 = (|\sigma|^2 + |\dot\sigma|^3 + |\ddot\sigma|^6)^{1/6}$,

$$u = -\alpha\,\mathrm{sat}\{[\ddot\sigma + 2(|\dot\sigma|^3 + |\sigma|^2)^{1/6} \times \mathrm{sat}((\dot\sigma + |\sigma|^{2/3}\,\mathrm{sign}\,\sigma)/N_3, 0.2)]/N_3, 0.2\},$$

4. $N_4 = (|\sigma|^3 + |\dot\sigma|^4 + |\ddot\sigma|^6 + |\dddot\sigma|^{12})^{1/12}$,

$$u = -\alpha\,\mathrm{sat}\{(\dddot\sigma + 3(\ddot\sigma^6 + \dot\sigma^4 + |\sigma|^3)^{1/12} \times \mathrm{sat}[(\ddot\sigma + (\dot\sigma^4 + |\sigma|^3)^{1/6} \times \mathrm{sat}((\dot\sigma + 0.5|\sigma|^{3/4}\,\mathrm{sign}\,\sigma)/N_4, 0.2))/N_4, 0.2])/N_4, 0.2\}.$$

sectionSimulation results Consider a simple kinematical model of car control

$$\dot x = v\cos\varphi,$$

$$\dot y = v\sin\varphi,$$

$$\dot\varphi = \frac{v}{l}\tan\theta,$$

$$\dot\theta = u,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 2a). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while x, $g(x)$ and y are assumed to be measured in real time.

Note that the actual control here is θ and $\dot\theta = u$ is used as a new control in order to avoid discontinuities of θ. The parameters $l = 5$ m, $v = 10$ m/s, and the initial conditions $x = y = \varphi = \theta = 0$ are taken. The function $g(x) = 10\sin(0.05x) + 5$ was taken for the simulation.

Define $\sigma = y - g(x)$. The relative degree r equals 3 and (3) holds locally. The controller parameters α and L are found by simulation. Apply the standard controller (11) with $\alpha = 20$, and the new output-feedback controller no. 3 from the list with $\alpha = 0.5$ and the differentiator parameter $L = 400$

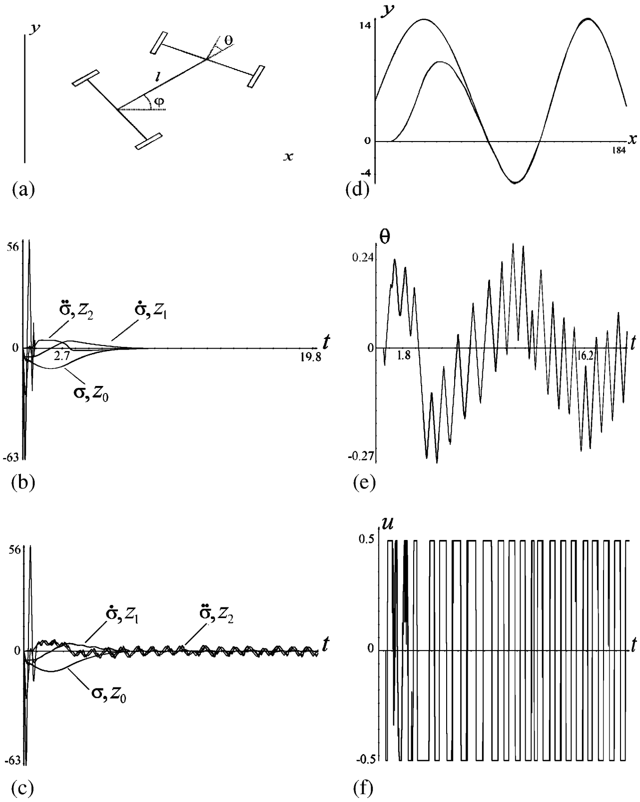

Fig. 2. Car model (a), controller performance with noisy sampling (d–f), differentiator performance with exact (b) and noisy (c) sampling.

(though already $L = 20$ suffices, the performance with noisy measurements is worse). The control is taken 0 with $t \leqslant 0.5$. The comparison of the controllers is shown in Fig. 1. The short initial chattering of u at $t=0.5$ in Fig. 1 is caused by the residual differentiator convergence (Fig. 2b). The chattering of the actual control θ is totally removed. The accuracies $|\sigma| = |x - x_c| \leqslant 3.2 \cdot 10^{-7}$, $|\dot{x} - \dot{x}_c| \leqslant 1.7 \cdot 10^{-4}$, $|\ddot{x} - \ddot{x}_c| \leqslant 1.5 \cdot 10^{-2}$ and $|x - x_c| \leqslant 3.5 \cdot 10^{-4}$, $|\dot{x} - \dot{x}_c| \leqslant 5.6 \cdot 10^{-3}$, $|\ddot{x} - \ddot{x}_c| \leqslant 1.5 \cdot 10^{-1}$ were obtained respectively with $\tau = 10^{-4}$ and $\tau = 10^{-3}$, which generally corresponds to Theorem 4.

The performance of the new controller and of the internal second order differentiator in the presence of a high-frequency measurement noise with the magnitude 0.1 m is demonstrated in Figs. 2d–f and Fig. 2c, respectively. The magnitude of the steering angle vibrations is about 7° and the frequency is about 1 Hz. The accuracy $|x - x_c| \leqslant 0.21$, $|\dot{x} - \dot{x}_c| \leqslant 0.60$, $|\ddot{x} - \ddot{x}_c| \leqslant 2.9$ is obtained. In accordance with Theorem 3, it changes to $|x - x_c| \leqslant 0.020$, $|\dot{x} - \dot{x}_c| \leqslant 0.14$, $|\ddot{x} - \ddot{x}_c| \leqslant 1.2$ with the noise magnitude 0.01. The performance does not significantly change when the noise frequency varies between 1 and 100,000 Hz.

6. Conclusions

Theorems are proved on the features of homogeneous differential inclusions, which allow simplifying and standardizing the proofs of the features of homogeneous high-order sliding mode controllers. The corresponding r-sliding homogeneity notion is introduced. It is proved that the uniform global finite-time stability is robust with respect to small homogeneous perturbations, if the homogeneity degree is negative. That fact is shown to be useful for the high-order sliding-mode controller design. A new output-feedback SISO r-sliding controller is proposed, $r = 1, 2, \ldots$, featuring control continuous everywhere except the r-sliding mode itself.

No exact model of the process is needed. The only requirements are that the relative degree of the controlled uncertain process be known and the boundedness restrictions (3) hold. Local validity of (3) provides for the local applicability of controllers.

References

Atassi, A. N., & Khalil, H. K. (2000). Separation results for the stabilization of nonlinear systems using different high-gain observer designs. *Systems & Control Letters*, *39*, 183–191.

Bacciotti, A., & Rosier, L. (2001). *Liapunov functions and stability in control theory*. Lecture Notes in Control and Information Science, Vol. 267. London: Springer.

Bartolini, G., Ferrara, A., & Punta, E. (2000). Multi-input second-order sliding-mode hybrid control of constrained manipulators. *Dynamics and Control*, *10*, 277–296.

Bartolini, G., Ferrara, A., & Usai, E. (1998). Chattering avoidance by second-order sliding mode control. *IEEE Transactions on Automatic Control*, *43*(2), 241–246.

Bartolini, G., Pisano, A., Punta, E., & Usai, E. (2003). A survey of applications of second-order sliding mode control to mechanical systems. *International Journal of Control*, *76*(9/10), 875–892.

Bartolini, G., Pisano, A., & Usai, E. (2000). First and second derivative estimation by sliding mode technique. *Journal of Signal Processing*, *4*(2), 167–176.

Edwards, C., & Spurgeon, S. K. (1998). *Sliding mode control: Theory and applications*. London: Taylor & Francis.

Emelyanov, S. V., Korovin, S. K., & Levantovsky, L. V. (1986). Higher order sliding regimes in the binary control systems. *Soviet Physics, Doklady*, *31*(4), 291–293.

Ferrara, A., & Giacomini, L. (2000). Control of a class of mechanical systems with uncertainties via a constructive adaptive/second order VSC approach. *Journal of Dynamic Systems-T ASME*, *122*(1), 33–39.

Filippov, A. F. (1988). *Differential equations with discontinuous right-hand side*. Dordrecht, The Netherlands: Kluwer.

Floquet, T., Barbot, J.-P., & Perruquetti, W. (2003). Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems. *Automatica*, *39*, 1077–1083.

Fridman, L. (2001). An averaging approach to chattering. *IEEE Transactions on Automatic Control*, *46*, 1260–1265.

Fridman, L. (2003). Chattering analysis in sliding mode systems with inertial sensors. *International Journal of Control*, *76*(9/10), 906–912.

Furuta, K., & Pan, Y. (2000). Variable structure control with sliding sector. *Automatica*, *36*, 211–228.

Isidori, A. (1989). *Nonlinear control systems* (2nd ed.), New York: Springer.

Kobayashi, S., Suzuki, S., & Furuta, K. (2002). Adaptive vs. differentiator. Advances in variable structure systems. *Proceedings of the seventh VSS Workshop*, July 2002, Sarajevo.

Krupp, D., Shkolnikov, I. A., & Shtessel, Y. B. (2000). 2-sliding mode control for nonlinear plants with parametric and dynamic uncertainties. *Proceedings of AIAA guidance, navigation, and control conference*, Denver, CO, AIAA paper No. 2000-3965.

Levant, A. (Levantovsky, L. V.) (1993). Sliding order and sliding accuracy in sliding mode control. *International Journal of Control*, *58*(6), 1247–1263.

Levant, A. (1998). Robust exact differentiation via sliding mode technique. *Automatica*, *34*(3), 379–384.

Levant, A. (2001). Universal SISO sliding-mode controllers with finite-time convergence. *IEEE Transactions on Automatic Control*, *46*(9), 1447–1451.

Levant, A. (2002). Construction principles of output-feedback 2-sliding mode design. *Proceedings of the* 42*nd IEEE conference on decision and control*, Las-Vegas, Nevada, December 10–13, 2002.

Levant, A. (2003a). Higher-order sliding modes, differentiation and output-feedback control. *International Journal of Control*, *76*(9/10), 924–941.

Levant, A. (2003b). Quasi-continuous high-order sliding mode controllers. *Proceedings of the* 43*rd IEEE conference on decision and control*, Maui, Hawaii, December 9–12, 2003.

Levant, A., Pridor, A., Gitizadeh, R., Yaesh, I., & Ben-Asher, J. Z. (2000). Aircraft pitch control via second order sliding technique. *Journal of Guidance, Control and Dynamics*, *23*(4), 586–594.

Man, Z., Paplinski, A. P., & Wu, H. R. (1994). A robust MIMO terminal sliding mode control for rigid robotic manipulators. *IEEE Transactions on Automatic Control*, *39*(12), 2464–2468.

Orlov, Y., Aguilar, L., & Cadiou, J. C. (2003). Switched chattering control vs. backlash/friction phenomena in electrical servo-motors. *International Journal of Control*, *76*(9/10), 959–967.

Saks, S. (1964). *Theory of the integral*. New York: Dover Publ. Inc.

Shkolnikov, I. A., & Shtessel, Y. B. (2002). Tracking in a class of nonminimum-phase systems with nonlinear internal dynamics via sliding mode control using method of system center. *Automatica*, *38*(5), 837–842.

Shkolnikov, I. A., Shtessel, Y. B., Lianos, D., & Thies, A. T. (2000). Robust missile autopilot design via high-order sliding mode control. *Proceedings of AIAA guidance, navigation, and control conference*, Denver, CO, AIAA paper No. 2000-3968.

Shtessel, Y. B., & Shkolnikov, I. A. (2003). Aeronautical and space vehicle control in dynamic sliding manifolds. *International Journal of Control*, *76*(9/10), 1000–1017.

Sira-Ramírez, H. (1993). On the dynamical sliding mode control of nonlinear systems. *International Journal of Control*, *57*(5), 1039–1061.

Sira-Ramírez, H. (2002). Dynamic second-order sliding mode control of the hovercraft vessel. *IEEE Transactions on Control Systems Technology*, *10*(6), 860–865.

Slotine, J.-J. E., & Li, W. (1991). *Applied nonlinear control*. London: Prentice-Hall, Inc.

Spurgeon, S., Goh, K. B., & Jones, N. B. (2002). An application of higher order sliding modes to the control of a diesel generator set (genset), Advances in Variable Structure Systems. *Proceedings of the seventh VSS Workshop*, July 2002, Sarajevo.

Utkin, V. I. (1992). *Sliding modes in optimization and control problems*. New York: Springer.

Yu, X., & Xu, J. X. (1996). An adaptive signal derivative estimator. *Electronic Letters*, *32*(16),

Zinober, A. S. I. (Ed.) (1994). *Variable structure and Lyapunov control*. Berlin: Springer.

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