Finite Differences in Homogeneous Discontinuous Control

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Abstract— **Finite differences are shown to be applicable to the on-line estimation of arbitrary-order derivatives in homogeneous discontinuous control. An output-feedback controller is produced from any finite-time-stable *r*-sliding homogeneous controller, capable to control the output of any smooth uncertain single-input-single-output system of a known permanent relative degree *r*. Variable sampling step feedback is proposed, providing for the utmost *r*-sliding accuracy corresponding to the minimal possible sampling interval in the absence of noises. In the presence of noises the tracking accuracy is proportional to the unknown noise magnitude. Theoretical results are confirmed by computer simulation.**

Index Terms **— high-order sliding mode, homogeneity, robustness, output feedback control**

I. Introduction

Differentiation problem is often encountered in control practice. Unfortunately, the well-known differentiation sensitivity to small high-frequency noises makes the problem difficult. The simplest approach to the problem is to use finite differences. With the noise magnitude being smaller than ε and the smooth signal being σ(*t*), obtain that

$$\Delta \hat{\sigma}(t) = \hat{\sigma}(t) - \hat{\sigma}(t-\tau) = \dot{\sigma}(t)\tau + o(\tau) + O(\varepsilon),$$

where $\hat{\sigma}$ is the measured value of the signal, and τ is the sampling interval. Thus, $\dot{\sigma}(t)$ can be evaluated, provided ε is much less than τ (the entities are assumed dimensionless). The same idea being applied to the estimation of the *k*th order derivative yields [6]

$$\Delta^k \hat{\sigma}(t) = \sigma^{(k)}(t)\tau^k + o(\tau^k) + O(\varepsilon),$$

where $\Delta^k \hat{\sigma}(t)$ is the *k*th-order backward finite difference. That expression contains some valuable information on $\sigma^{(k)}(t)$ only with ε being small compared with τ^k. Since also τ is obviously assumed to be small, the condition is very restrictive. The above reasoning might easily convince that high-order finite differences are of no use in feedback control. Nevertheless, it is shown in this paper that such differences can still be successfully implemented, if the control is discontinuous and homogeneous. The reason is that such homogeneous controllers are less sensitive to the errors in the estimation of higher derivatives.

Sliding-mode control is based on keeping properly chosen constraints by means of high-frequency control switching. Sliding modes are accurate and insensitive to disturbances [7, 32]. Their main drawbacks are mostly related to the so-called chattering effect [2, 5, 11, 12, 13].

Let the chosen constraint be given by the equation σ = *s* - *w*(*t*) = 0, where *s* is the output of an uncertain single-input-single-output (SISO) dynamic system and *w*(*t*) is an unknown-in-advance smooth signal to be tracked in real time. The standard sliding-mode control *u* = -α sign σ, α > 0, solves the problem if the relative degree is 1, i.e. if $\dot{\sigma}$ explicitly depends on the control *u* and $\frac{\partial}{\partial u}\dot{\sigma} > 0$. High-order sliding modes [17, 20, 4] are applicable to controlling SISO uncertain systems of arbitrary relative degrees. Corresponding finite-time-convergent controllers (*r*-sliding controllers) [2, 4, 10, 15, 20, 23] require actually only the knowledge of the system relative degree *r*. The produced control is a discontinuous function of σ and its real-time-calculated successive derivatives $\dot{\sigma}, \ddot{\sigma}, ..., \sigma^{(r-1)}$. The controllers provide also for higher accuracy with discrete sampling and, when properly used, practically avoid the chattering effect [5, 24]. For this aim the control derivative is treated as a new control, artificially increasing the relative degree. While higher-order controllers are still mostly theoretically studied, 2-sliding controllers have already found numerous applications [3, 4, 9, 15, 24 - 30].

Recall that ε is the uncertain measurement noise magnitude, and τ is the sampling time interval. The lacking derivatives can be produced by the recently proposed (*r* - 1)th order robust exact finite-time-convergent differentiators [3, 16, 18, 20, 30, 31] providing for the estimation error of $\sigma^{(i)}$ proportional to τ^{r-i} with ε = 0, *i* = 0, 1, ..., *r* - 1 or to $\varepsilon^{(r-i)/r}$ with τ << ε [18, 20]. The same tracking accuracy is maintained by the resulting output-feedback controller [20, 22, 23]. This accuracy cannot be improved [17, 18, 20]. Unfortunately, the good performance requires here not-always-available high sampling rates.

It is known that the 2-sliding sub-optimal and twisting controllers can be realized based only on the first-order finite differences [17, 2 - 4]. In that case the above-mentioned optimal asymptotic accuracy σ ~ ε (sup|σ| proportional to ε) is obtained in the steady state with a sampling step proportional to $\varepsilon^{1/2}$. Sampling-interval reduction can cause a system

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disaster. Thus, the sampling interval is always taken redundantly large, for ε is usually uncertain. Another approach was developed in the case of the twisting controller [17, 19]: with a feedback-defined variable sampling interval, a robust controller is obtained, not requiring the knowledge of ε. In that case the sampling step is taken proportional to $|\hat{\sigma}|^{1/2}$, but not smaller than the least possible sampling interval τ_m, $\hat{\sigma}$ being the result of the noisy σ measurement. The above-mentioned utmost accuracy $\sigma \sim \tau_m^{\ 2}$ is provided in the absence of noise, otherwise $\sigma \sim \varepsilon$ is obtained with $\varepsilon >> \tau_m$.

The above result for the twisting controller is generalized in this paper to the whole class of homogeneous finite-time-stable r-sliding controllers [22], $r \geq 1$, i.e. for almost all of about a dozen known high-order sliding controller families [2-4, 15, 17, 20, 22 - 24], excluding only non-homogeneous [15, 25]. Finite differences of the orders 1, ..., r - 1 are used. The constant sampling step is to be taken now proportional to $\varepsilon^{1/r}$, while the variable sampling step is to be proportional to $|\hat{\sigma}|^{1/r}$. Both methods provide for the accuracy $\sigma \sim \varepsilon$ in the presence of the noises, only the latter does not require the knowledge of ε. While the first method has always more or less the same accuracy independently of the noise existence, the second method provides for the accuracy $\sigma \sim \tau_m^{\ r}$ in the absence of noises, with τ_m being the least possible sampling interval or the discretization time step. The result is new already with $r = 2$, since it is proved here for almost all known controllers. It seems also to be the first known robust finite-differences-based output-feedback controller for non-linear systems with high relative degrees. Simulation demonstrates the practical applicability of the proposed scheme.

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II. THE PROBLEM STATEMENT

Consider a smooth dynamic system with a smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback and understood in the Filippov sense [5]. Then, provided the successive total time derivatives σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma = \ldots = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is said to be in the r-sliding (rth order sliding) mode [17, 20]. The standard sliding mode, used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous). Such systems often feature also asymptotically stable higher-order sliding modes. In particular, such modes are deliberately introduced in the systems with dynamical sliding modes [27].

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \rightarrow \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$, n can be also uncertain. The relative degree r of the system is assumed to be constant and known. That means that for the first time the control appears explicitly in the rth total time derivative of σ [14]. The task is to provide in finite time for keeping $\sigma \equiv 0$.

Extend system (1) by introduction of a fictitious variable $x_{n+1} = t$, $\dot{x}_{n+1} = 1$. Denote $a_e = (a,1)^{\mathrm{t}}$, $b_e = (b,0)^{\mathrm{t}}$, where the last component corresponds to x_{n+1}. It is known [14] that

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{u=0} = L_{a_e}^{\ r}\sigma$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)}$ $h(t,x) = L_{b_e}L_{a_e}^{\ r\text{-}1}\sigma$ are some unknown smooth functions. It is supposed that

$$0 < K_{\mathrm{m}} \leq \tfrac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}, \ |\sigma^{(r)}|_{u=0}| \leq C \tag{3}$$

for some K_{m}, K_{M}, $C > 0$. Note that conditions (3) are formulated in terms of input-output relations. It is also assumed that trajectories of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$. The system is often required in practice to be weakly minimum phase.

It is supposed also that the output σ is measured at the time moments t_0, t_1, ..., $t_{i+1} - t_i = \tau_i \geq \tau_m > 0$, and is assumed that τ_i can be assigned any value. Another important case is when τ_i is to be integer multiple of τ_m. It is supposed that the measurement noise magnitude does not exceed some uncertain $\varepsilon \geq 0$. The task is to keep σ as small as possible.

Obviously, (2), (3) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u. \tag{4}$$

The problem is solved in two steps. First a bounded feedback Lebesgue-measurable control

$$u = \varphi(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}), \tag{5}$$

is constructed, such that all trajectories of (4), (5) converge in finite time to the origin $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$ of the r-sliding phase space σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$. It is easily shown that such a control is inevitably discontinuous at least at the origin, and, therefore, r-sliding mode $\sigma = 0$ is to be established [22]. Since the differential inclusion (4), (5) does not "remember" the original dynamic system, the controller is effective for the whole class of systems (1), (3). That step is assumed already done in this paper. At the next step the lacking derivatives are real-time evaluated, producing an output-feedback controller. Some further important properties of the controllers (5) are postulated below.

III. HOMOGENEOUS DISCONTINUOUS CONTROL

A differential inclusion $\dot{x} \in F(x)$ is called further a *Filippov differential inclusion* if the vector set $F(x)$ is non-empty, closed, convex, locally bounded and upper-semicontinuous [8]. The last condition means that the maximal distance of the points of $F(y)$ from the set $F(x)$ vanishes when $y \rightarrow x$. Solutions are defined as absolutely-continuous functions of time satisfying the inclusion almost everywhere. Such solutions always exist and have most of the well-known standard properties except the uniqueness [8].

A differential equation $\dot{x} = f(x)$ with a locally-bounded Lebesgue-measurable right-hand side is said to be understood in the Filippov sense [8], if its solutions are defined as solutions of a specially built Filippov differential inclusion $\dot{x} \in F(x)$. In the most usual case, when f is continuous almost everywhere, the procedure is to take $F(x)$ being the convex closure of the set of all possible limit values of f at a given point x, obtained when its continuity point y tends to x.

A similar procedure is applied to the differential inclusion (4), (5). For this end the above Filippov procedure is applied to the function φ and the obtained Filippov set is substituted for u in (5), producing a Filippov inclusion to replace (4), (5). Any solution of (4), (5) is defined in this paper as a solution of the built Filippov inclusion.

A function f: $\mathbf{R}^n \to \mathbf{R}$ is called *homogeneous* [1] *of the degree* (*weight*) $q \in \mathbf{R}$ *with the dilation* d_κ: $(x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n)$, where $m_1, ..., m_n > 0$, if for any x and $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds. Numbers $m_1, ..., m_n$ are called the homogeneity degrees (weights) of $x_1, ..., x_n$.

A differential equation $\dot{x} = f(x)$, $x \in \mathbf{R}^n$, (respectively a differential inclusion $\dot{x} \in F(x)$) is called *homogeneous of the degree* $q \in \mathbf{R}$ *with the dilation* d_κ, if for any x and any $\kappa > 0$ the identity $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$ (respectively $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ [22]) holds.

The definition is easily understood prescribing the weight $p = -q$ to the time variable. Then the homogeneity weight of a coordinate derivative is the result of the subtraction of p from the weight of the coordinate. Thus, the homogeneity of the differential equation $\dot{x} = f(x)$ means [1] that the ith component $f_i(x)$ of the vector field $f(x)$ is a homogeneous function of the weight $m_i - p$.

The homogeneity degree of a function (differential equation or inclusion) and the coordinate weights $m_1, ..., m_n$ can be always simultaneously proportionally changed. In particular, the non-zero system (inclusion) homogeneity degree $q = -p$ can be always scaled to ±1.

The homogeneity of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) can be equivalently defined as the invariance of the equation (inclusion) with respect to the combined time-coordinate transformation $G_\kappa: (t, x) \mapsto (\kappa^{-q} t, d_\kappa x)$.

1°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is called further *globally uniformly finite-time stable* at 0, if it is Lyapunov stable at 0, and for any $R > 0$ exists $T > 0$, $T = T(R)$, such that any trajectory starting within the disk $\|x\| < R$ stabilizes at zero in the time T.

2°. A differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is called further *globally uniformly asymptotically stable* at 0, if it is Lyapunov stable at 0, and for any $R > 0$ and $\varepsilon > 0$ exists $T > 0$, $T = T(R, \varepsilon)$, such that any trajectory starting within the disk $\|x\| < R$ enters the disk $\|x\| < \varepsilon$ in the time T to stay there forever.

A set D is called *dilation retractable* if $d_\kappa D \subset D$ for any $\kappa \in [0, 1]$. In particular, any disk $x_1^2 + ... + x_n^2 < R^2$ is dilation retractable. Obviously, with any point P retractable sets contain the whole curve $x(\kappa) = d_\kappa P$, $\kappa \in [0, 1]$.

3°. A homogeneous differential inclusion $\dot{x} \in F(x)$ (equation $\dot{x} = f(x)$) is further called *contractive* if there are 2 compact sets D_1, D_2 and $T > 0$, such that D_2 lies in the interior of D_1 and contains the origin; D_1 is dilation-retractable; and all trajectories starting at the time 0 within D_1 are localized in D_2 at the time moment T.

Theorem 1 [22]. *Let* $\dot{x} \in F(x)$ ($\dot{x} = f(x)$) *be a homogeneous Filippov differential inclusion (equation) with a negative homogeneous degree -p, then properties* 1°, 2° *and* 3° *are equivalent and the maximal settling time is a continuous homogeneous function of the initial conditions of the degree p*.

Equivalence of 1° and 2° is proved also in [26], see also [1] for similar results on continuous differential equations and references therein. Obviously, local asymptotic (finite-time) stability is equivalent to the global one due to 3°. Let $\dot{x} \in F(x)$ be a homogeneous Filippov differential inclusion. Consider the case of "noisy measurements" of x_i with the magnitude $\beta_i \tau^{m_i}$

$$\dot{x} \in F(x_1 + [-\beta_1, \beta_1]\tau^{m_1}, ..., x_n + [-\beta_n, \beta_n]\tau^{m_n}), \quad \tau > 0 .$$

Applying successively the closure of the right-hand-side graph and the convex closure at each point x, obtain some new Filippov differential inclusion $\dot{x} \in F_\tau(x)$.

Theorem 2 [22]. *Let* $\dot{x} \in F(x)$ *be a globally uniformly finite-time stable homogeneous Filippov differential inclusion (equation) with the homogeneity weights* $m_1, ..., m_n$ *and the degree* $-p < 0$, *and let* $\tau > 0$. *Suppose that a continuous function* $x(t)$ *be defined for any* $t \geq -\tau^p$ *and satisfy some initial conditions* $x(t) = \phi(t)$, $t \in [-\tau^p, 0]$. *Then if* $x(t)$ *is a solution of the disturbed inclusion*

$$\dot{x}(t) \in F_\tau(x(t + [-\tau^p, 0])), \quad t > 0 , \tag{6}$$

the inequalities $|x_i| < \gamma_i \tau^{m_i}$ *are established in finite time with some positive constants* γ_i *independent of* τ *and* ϕ.

Theorem 2 covers the cases of retarded or discrete noisy measurements of all or some of the coordinates. Only infinite extendibility of solutions in time is required.

Let the homogeneity weights of $t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ be $1, r, r-1, ..., 1$ respectively ($p = 1$, therefore the homogeneity degree is -1). This homogeneity is called further the *r-sliding homogeneity* [22]. It can be shown [22] that it is the only homogeneity possible for the differential inclusion (4), (5). In other words, the inclusion (4), (5) and controller (5) are called r-sliding homogeneous, if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: (t, \Sigma) \mapsto (\kappa t, d_\kappa \Sigma),$$
$$\Sigma = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \; d_\kappa \Sigma = (\kappa^r \sigma, \kappa^{r-1} \dot{\sigma}, ..., \kappa \sigma^{(r-1)}) \tag{7}$$

preserves the closed-loop Filippov inclusion corresponding to (4), (5) and its solutions. Note that the 2-sliding sub-optimal controller [2, 4] does not exactly satisfy the described feedback form (5), but its trajectories are invariant with respect to (7). It is natural to say that it is 2-sliding homogeneous in the broad sense.

Obviously, (4) (5) is r-sliding homogeneous if the equality

$$\varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}) \qquad (8)$$

holds identically. Such controllers are naturally to be called *strictly r-sliding homogeneous*. Recall that the values of φ on any zero-measure set do not influence the corresponding Filippov differential inclusion and its homogeneity. Any strictly r-sliding-homogeneous controller is uniformly bounded, since it is locally bounded and takes on all its values in any vicinity of the origin. It is inevitably discontinuous at the origin (0, ..., 0), if φ is not a constant almost everywhere.

Following are some examples of 2-sliding homogeneous controllers. Let α, β be positive parameters. The twisting controller [17] is given by the homogeneous formula

$$u = -\alpha \operatorname{sign}\sigma - \beta \operatorname{sign}\dot\sigma \equiv -\alpha \operatorname{sign}(\kappa^2\sigma) - \beta \operatorname{sign}(\kappa\dot\sigma).$$

Its finite-time stability conditions are $\alpha > \beta$, $(\alpha + \beta)K_m - C > (\alpha - \beta)K_M + C$, $(\alpha - \beta) K_m > C$. The homogeneous form of the controller with prescribed convergence law [17] is defined as

$$u = -\alpha \operatorname{sign}(\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma) \equiv -\alpha \operatorname{sign}(\kappa\dot\sigma + \beta|\kappa^2\sigma|^{1/2}\operatorname{sign}(\kappa^2\sigma)).$$

Its finite-time stability condition is $\alpha K_m - C > \beta^2/2$. This controller is a 2-sliding homogeneous analogue of the terminal sliding mode controller [25]. The recently published quasi-continuous 2-sliding controller [23] is defined as

$$u = -\alpha\frac{\dot\sigma + \beta|\sigma|^{1/2}\operatorname{sign}\sigma}{|\dot\sigma| + \beta|\sigma|^{1/2}} \equiv -\alpha\frac{\kappa\dot\sigma + \beta|\kappa^2\sigma|^{1/2}\operatorname{sign}\kappa^2\sigma}{|\kappa\dot\sigma| + \beta|\kappa^2\sigma|^{1/2}}.$$

It is finite-time stable with any sufficiently large α and is continuous everywhere except $\sigma = \dot\sigma = 0$. The sub-optimal controller [2] is defined by the formula

$$u = -\alpha \operatorname{sign}(\sigma - \sigma^*/2) + \beta \operatorname{sign}\sigma^* \equiv -\alpha \operatorname{sign}(\kappa^2\sigma - \kappa^2\sigma^*/2) + \beta \operatorname{sign}\kappa^2\sigma^*,$$

where σ^* is the value of σ detected at the closest time when $\dot\sigma$ was 0. The initial value of σ^* is 0, and the convergence conditions are $\alpha > \beta > 0$, $2[(\alpha + \beta)K_m - C] > (\alpha - \beta)K_M + C$, $(\alpha - \beta)K_m > C$. The control u depends actually on the whole history of $\dot\sigma$ and σ measurements, i.e. on $\dot\sigma(\cdot)$ and $\sigma(\cdot)$, and does not satisfy the feedback form (5). The results of this paper are true also for this controller, but the proofs are to be modified, since the powerful Filippov results cannot be directly applied here.

It is further supposed that the controller (5) *is finite-time stable and strictly r-sliding homogeneous* (*i.e.* (8) *is supposed to hold identically*).

IV. Implementation of Finite Differences

Controller (5) requires availability of $\dot\sigma$, ..., $\sigma^{(r-1)}$. That information demand can be lowered using finite differences. In the following the usage of differences with constant and feedback-defined sampling interval is considered, and the influence of sampling noises is studied.

Let σ, $\dot\sigma$, ..., $\sigma^{(k)}$, $0 \le k \le r$-1 be available. Consider first the case, when the measurements are carried out at times t_i with constant time step $\tau > 0$. Denote $\sigma_i^{(s)} = \sigma^{(s)}(t_i, x(t_i))$. Let Δ be the backward difference operator, $\Delta\sigma_i^{(s)} = \sigma_i^{(s)} - \sigma_{i-1}^{(s)}$, $t \in [t_i, t_{i+1})$, s = 1, 2, ..., r-1. Define

$$u = \varphi(\tau^{r(r-k-1)}\sigma_i, ..., \tau^{(r-k+1)(r-k-1)}\sigma_i^{(k-1)}, \tau^{(r-k)(r-k-1)}\sigma_i^{(k)}, \tau^{(r-k)(r-k-2)}\Delta\sigma_i^{(k)}, ..., \tau^{r-k}\Delta^{r-k-2}\sigma_i^{(k)}, \Delta^{r-k-1}\sigma_i^{(k)}), \qquad (9)$$

where $\Delta^s\sigma_i^{(k)}$ is the sth-order finite difference. In particular, with $k = r$-1 (full measurements), and with $k = r - 2$ achieve respectively

$$u = \varphi(\sigma_i, ..., \sigma_i^{(r-2)}, \sigma_i^{(r-1)}), \quad u = \varphi(\tau^r\sigma_i, ..., \tau^2\sigma_i^{(r-2)}, \Delta\sigma_i^{(r-2)});$$

and with k = 0 achieve

$$u = \varphi(\tau^{r(r-1)}\sigma_i, \tau^{r(r-2)}\Delta\sigma_i, ..., \tau^r\Delta^{r-2}\sigma_i, \Delta^{r-1}\sigma_i). \qquad (10)$$

The idea of (9) is to avoid division by small numbers. Indeed, (8) implies that (9) is equivalent to

$$u = \varphi(\sigma_i, ..., \sigma_i^{(k-1)}, \sigma_i^{(k)}, \Delta\sigma_i^{(k)}/\tau, ..., \Delta^{r-k-2}\sigma_i^{(k)}/\tau^{r-k-2}, \Delta^{r-k-1}\sigma_i^{(k)}/\tau^{r-k-1}). \qquad (11)$$

Theorem 3. *Suppose that controller* (5) *be strictly r-sliding homogeneous and finite-time stable,* $0 \le k \le r - 1$*, then in the absence of noises, with discrete measurements controller* (9) *provides in finite time for the establishment of the inequalities* $|\sigma| < \gamma_0\tau^r$, $|\dot\sigma| < \gamma_1\tau^{r-1}$, ..., $|\sigma^{(r-1)}| < \gamma_{r-1}\tau$ *with some positive constants* $\gamma_0, \gamma_1, ..., \gamma_{r-1}$.

Here and further the proofs are placed in Appendix. The accuracy provided by the above Theorem is the best possible with discontinuous $\sigma^{(r)}$ and discrete sampling [17].

Let now measurement noises be present, and assume them to be any bounded functions of time. It is obvious that with τ sufficiently large controller (9) does not "feel" noises and performs according to Theorem 3. On the other hand the finite differences do not contain any useful information when τ is too small. The boundary between these two cases is revealed in the next Theorem .

Theorem 4. *Under the conditions of Theorem* 3 *let* σ, $\dot\sigma$, ..., $\sigma^{(k)}$, $k < r$, *be measured with measurement noises of the magnitudes* $\beta_0\varepsilon$, $\beta_1\varepsilon^{(r-1)/r}$, ..., $\beta_k\varepsilon^{(r-k)/r}$ *respectively, with* $\beta_0, \beta_1, ..., \beta_k > 0$ *and the measurement step* $\tau = \eta\varepsilon^{1/r}$, $\eta > 0$. *Then there are such positive constants* $\gamma_0, \gamma_1, ..., \gamma_{r-1}$ *that for any* $\varepsilon > 0$ *controller* (9) *provides in finite time for keeping the inequalities*

$$|\sigma| \le \gamma_0\varepsilon, |\dot\sigma| \le \gamma_1\varepsilon^{(r-1)/r}, ..., |\sigma^{(r-1)}| \le \gamma_{r-1}\varepsilon^{1/r}.$$

Note that there are no restrictions on η and β_i. Since noise magnitudes are often unknown, a reasonable value is directly assigned to τ, which results in large τ and unnecessarily poor performance with small noises.

Clearly, a feedback-defined variable sampling step would provide for the robustness of the controller, if τ grew with the distance from the r-sliding mode. If only σ is available, such distance is not available. Define the variable measurement step

$$\tau_{i+1} = t_{i+1} - t_i = \begin{cases} \lambda\,|\hat{\sigma}_i|^{1/r}, & \lambda\,|\hat{\sigma}_i|^{1/r} > \tau_m \\ \tau_m, & \lambda\,|\hat{\sigma}_i|^{1/r} \le \tau_m \end{cases} \qquad (12)$$

where $\lambda > 0$, $\hat{\sigma}_i$ is a noisy estimation of σ at the moment t_i. Denote $\delta^1\sigma_i = \delta\sigma_i = (\sigma_i - \sigma_{i-1})/\tau_i$, $\delta^s\sigma_i = (\delta^{s-1}\sigma_i - \delta^{s-1}\sigma_{i-1})/(\tau_i + \tau_{i-1} + \ldots + \tau_{i-s+1})$ (divided differences [6]), and consider the controller

$$u = \varphi(\hat{\sigma}_i, 1!\cdot\delta\hat{\sigma}_i, \ldots, (r-2)!\cdot\delta^{r-2}\hat{\sigma}_i, (r-1)!\cdot\delta^{r-1}\hat{\sigma}_i). \qquad (13)$$

With constant sampling steps (13) is equivalent to (10). The approach is based on the mean value formula $s!\,\delta^s\sigma_i = \sigma^{(s)}(\xi)$, which holds for some $\xi \in [t_i, t_{i-s}]$, $s = 1, 2, \ldots, r-1$ [6]. The controller

$$u = \varphi(\hat{\sigma}_i, \delta\hat{\sigma}_i, (\delta\hat{\sigma}_i - \delta\hat{\sigma}_{i-1})/\tau_i) \qquad (14)$$

can be applied instead of (13) with $r = 3$.

Theorem 5. *Let σ be measured with sampling interval* (12) *and a noise of the magnitude ε. Then for any sufficiently small $\lambda > 0$ there are such positive constants $\mu, \gamma_0, \gamma_1, \ldots, \gamma_{r-1}$ that for any $\varepsilon \ge 0$ controller* (12), (13) (*or* (14) *with $r = 3$*) *provides in finite time for keeping the inequalities*

$$|\sigma| \le \gamma_0\varepsilon,\ |\dot\sigma| \le \gamma_1\,\varepsilon^{(r-1)/r}, \ldots, |\sigma^{(r-1)}| \le \gamma_{r-1}\,\varepsilon^{1/r} \text{ with } \varepsilon > \mu\tau_m^{\,r}$$

and

$$|\sigma| \le \gamma_0\mu\tau_m^{\,r},\ |\dot\sigma| \le \gamma_1\mu^{(r-1)/r}\,\tau_m^{\,r-1}, \ldots, |\sigma^{(r-1)}| \le \gamma_{r-1}\,\mu^{1/r}\tau_m$$

with $\varepsilon \le \mu\tau_m^{\,r}$.

The parameter μ roughly defines the regions, where one of the parameters ε or τ_m is negligible. Consider a sampling law with the bounded variable sampling interval

$$t_{i+1} - t_i = \tau_i = \begin{cases} \tau_M, & \lambda\,|\hat{\sigma}_i|^{1/r} > \tau_M \\ \lambda\,|\hat{\sigma}_i|^{1/r}, & \tau_m < \lambda\,|\hat{\sigma}_i|^{1/r} \le \tau_M \\ \tau_m, & \lambda\,|\hat{\sigma}_i|^{1/r} \le \tau_m \end{cases}. \qquad (15)$$

Theorem 6. *Let σ be measured with sampling interval* (15) *and a noise of the magnitude $\varepsilon \le \varepsilon_0$, the maximal measurement step being chosen in the form $\tau_M = \max(\beta\varepsilon_0^{\,1/r}, \tau_m)$, $\beta > 0$. Then for any sufficiently small $\lambda > 0$ there are such positive constants $\mu, \gamma_0, \gamma_1, \ldots, \gamma_{r-1}, \rho_0, \rho_1, \ldots, \rho_{r-1}$ that for any $\varepsilon_0 \ge 0$ controller* (13) (*or* (14) *with $r = 3$*), (15) *provides in finite time for keeping the inequalities*

$$|\sigma| \le \rho_0\varepsilon_0,\ |\dot\sigma| \le \rho_1\,\varepsilon_0^{(r-1)/r}, \ldots, |\sigma^{(r-1)}| \le \rho_{r-1}\,\varepsilon_0^{1/r}.$$

With sufficiently small ε and τ_m the asymptotics of Theorem 5 is established:

$$|\sigma| \le \gamma_0\varepsilon,\ |\dot\sigma| \le \gamma_1\,\varepsilon^{(r-1)/r}, \ldots, |\sigma^{(r-1)}| \le \gamma_{r-1}\,\varepsilon^{1/r} \text{ with } \varepsilon > \mu\tau_m^{\,r},$$
$$|\sigma| \le \gamma_0\mu\tau_m^{\,r},\ |\dot\sigma| \le \gamma_1\mu^{(r-1)/r}\tau_m^{\,r-1}, \ldots, |\sigma^{(r-1)}| \le \gamma_{r-1}\mu^{1/r}\tau_m \text{ with } \varepsilon \le \mu\tau_m^{\,r}.$$

As follows from the Theorem, sampling step (15) features asymptotic properties of the constant step law and of (12) as well. It is convenient to choose the maximal step τ_M reasonably large, since ε_0 is mostly unknown. Though the convergence is destroyed with large noises, simulation shows that the sampling law (15) might be a better strategy, for it can provide for better accuracy with relatively large noises (see the simulation results, Table 1). It is especially useful if assumption (3) only locally holds. Note that Theorems 3, 5, 6 provide for the best possible asymptotic accuracy in the absence of noises [17].

V. Simulation Example: Car Control

Consider a simple kinematic model of car control

$$\dot x = v\cos\varphi,\quad \dot y = v\sin\varphi,$$
$$\dot\varphi = v/l\,\tan\theta,$$
$$\dot\theta = u,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while $g(x)$ and y are assumed to be measured in real time.

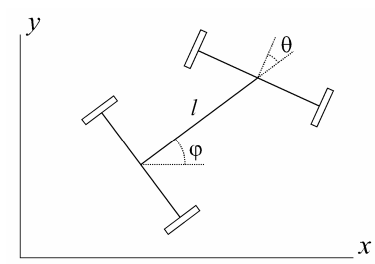

Fig. 1. Kinematic car model

Let $v = const = 10$ m/s, $l = 5$ m, $g(x) = 10\sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$. Define $\sigma = y - g(x)$. The relative degree of the system is 3 and the 3-sliding homogeneous quasi-continuous controller [23]

$$u = -\alpha\,[\ddot\sigma + 2(|\dot\sigma| + |\sigma|^{2/3})^{-1/2}(\dot\sigma + |\sigma|^{2/3}\operatorname{sign}\sigma)]\,/\,[|\ddot\sigma| + 2(|\dot\sigma| + |\sigma|^{2/3})^{1/2}]$$

can be applied here with $\alpha = 1$. Substituting estimations z_0, z_1, z_2 of $\sigma, \dot\sigma, \ddot\sigma$ respectively, obtain

$$u = -[z_2 + 2(|z_1| + |z_0|^{2/3})^{-1/2}(z_1 + |z_0|^{2/3}\operatorname{sign} z_0)]\,/\,[|z_2| + 2(|z_1| + |z_0|^{2/3})^{1/2}]. \qquad (16)$$

Consider two possibilities:

$$z_0 = \sigma_i\tau^6,\ z_1 = (\sigma_i - \sigma_{i-1})\tau^3,\ z_2 = \sigma_i - 2\sigma_{i-1} + \sigma_{i-2}, \qquad (17)$$

with a constant measurement step τ (form (9)) and

$$z_0 = \sigma_i,\ z_1 = \frac{\sigma_i - \sigma_{i-1}}{\tau_i},\ z_2 = \left(\frac{\sigma_i - \sigma_{i-1}}{\tau_i} - \frac{\sigma_{i-1} - \sigma_{i-2}}{\tau_{i-1}}\right)/\tau_i \qquad (18)$$

with the variable step (12), $\lambda = 0.15$ and $\tau_m = 10^{-4}$ (form (14)).

The control was applied only from $t = 0.5$ providing some time for the calculation of the finite differences. The integration was carried out according to the Euler method (the only one reliable with discontinuous dynamics) with the integration step 10^{-5} on the time interval of 20 seconds in the absence of noises and on the time interval of 30 seconds otherwise. The tracking accuracy was calculated as maximal absolute values of $|\sigma|$, $|\dot{\sigma}|$, $|\ddot{\sigma}|$ during the last 25% of the simulation time. The results are summarized in Table 1.

The system performance with $\tau = 10^{-4}$ in the absence of noises is shown in Fig. 2. It cannot be distinguished from the performance with full exact measurements of all derivatives [23]. The system is fully destroyed already with the noise magnitude $\varepsilon = 0.0001$ m. The system performance with the noise magnitude 0.1 m is practically the same as in the absence of noises with $\tau = 0.2$ s (Fig. 3). Note that the magnitude of the actual control θ is about 16° and the vibration frequency is about 0.5s^{-1}, which is quite feasible. Mark that $\tau = 0.2$ s is close to the typical human reaction time. Note also that the real performance might be measured by the maximal steady-state distance of the car trajectory from the desired one, which is much smaller than $\sup|\sigma|$ (Fig. 3a).

Performance with the variable measurement step in the absence of noises with $\lambda = 0.15$ and $\tau_m = 10^{-4}$ is shown in Fig. 4. The accuracy $|\sigma| \le 4.0$ is obtained with $\varepsilon = 0.05$ (Fig. 5a, b). The performance of the controller is slightly improved by the restriction (15) of the measurement step from above. With large noises the restriction $\tau \le \tau_M = 0.2$ actually provides for the same performance of the controller as with the constant sampling interval $\tau = 0.2$ (Fig. 5c,d). The demonstrated performance of the controllers does not significantly change when the noise frequency varies in the range from 10 to 100000.

Table 1. Summary of the simulation results

Controller type	Sampling parameters	Noise magnitude	$\sup\|\sigma\|$	$\sup\|\dot{\sigma}\|$	$\sup\|\ddot{\sigma}\|$
constant sampling step (16), (17)	$\tau = 10^{-4}$	0	$2.8\cdot10^{-10}$	$1.3\cdot10^{-5}$	$4.6\cdot10^{-3}$
constant sampling step (16), (17)	$\tau = 10^{-3}$	0	$2.5\cdot10^{-7}$	$2.0\cdot10^{-4}$	$4.5\cdot10^{-2}$
constant sampling step (16), (17)	$\tau = 0.2$	0	1.7	2.9	8.9
constant sampling step (16), (17)	$\tau = 0.2$	0.05	1.2	2.6	8.7
constant sampling step (16), (17)	$\tau = 0.2$	0.1	1.2	2.9	8.9
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	0	$2.7\cdot10^{-10}$	$1.3\cdot10^{-5}$	$4.6\cdot10^{-3}$
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-3}$	0	$2.7\cdot10^{-7}$	$2.0\cdot10^{-4}$	$4.5\cdot10^{-2}$
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	10^{-4}	0.0060	0.058	1.0
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	10^{-3}	0.049	0.26	2.2
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	10^{-2}	0.68	1.2	4.9
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	0.1	5.1	5.5	11
variable sampling step (16), (18), (12)	$\lambda = 0.15$, $\tau_m = 10^{-4}$	0.05	4.0	3.8	8.5
variable sampling step (16), (18), (15)	$\lambda = 0.15$, $\tau_m = 10^{-4}$, $\tau_M = 0.2$	0.05	1.4	2.0	5.0

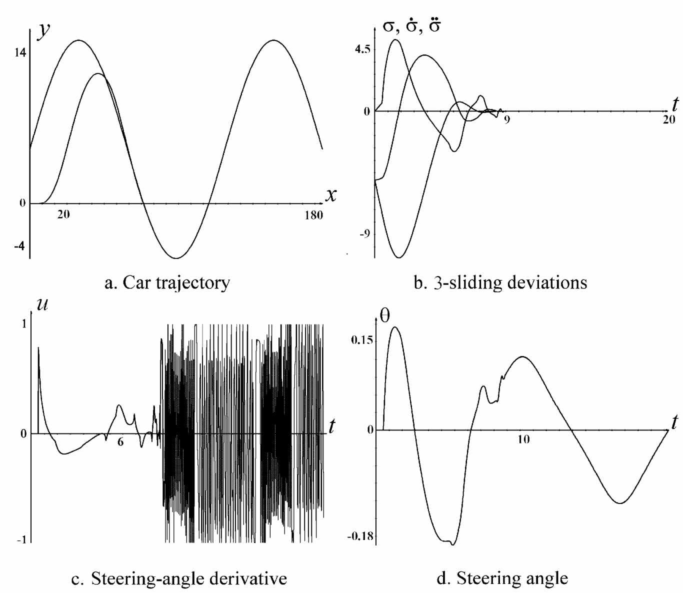

Fig. 2. Constant sampling step $\tau = 10^{-4}$, noise magnitude $\varepsilon = 0$

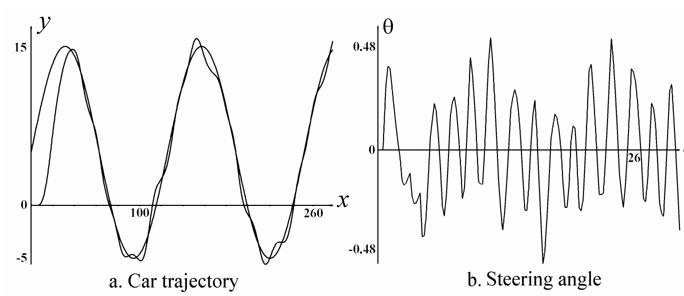

Fig. 3. Constant sampling step $\tau = 0.2$s , noise magnitude $\varepsilon = 0.1$m

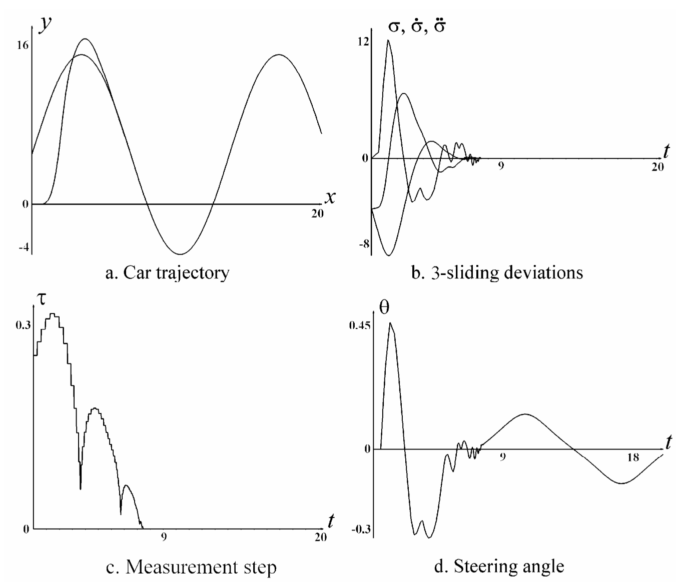

Fig. 4. Variable measurement interval, noise magnitude $\varepsilon = 0$

The simulation data listed in Table 1 confirm the asymptotics claimed in Theorems 3 - 6. It is seen that both variable-sampling-interval methods (16), (18), (12) and (16), (18), (15) provide for the ideal performance in the absence of noises, which does not differ from the results obtained with the sampling step fixed at the minimal step value. The constant step method (16), (17) actually identically performs with all noise magnitudes not exceeding some sensitivity threshold. After the noise exceeds the threshold the performance deteriorates. Variable sampling step provides for good performance in a large range of noise magnitudes, while the version with the sampling interval bounded from above looks preferable, if the maximal step is chosen sufficiently large.

VI. Conclusions

High-order finite differences are shown to provide for robust output-feedback control when applied with homogeneous sliding-mode controllers. While the on-line robust differentiation is still the preferable way [20, 22, 23], the finite differences are to be considered as a real alternative in the case, when the sampling rate is too low to provide for the good differentiation accuracy. For example, simple calculation based on the simulation results and the asymptotic accuracy from [23] shows that in the absence of noises, in the considered simulation example, a second-order differentiator [20] in the feedback would provide for the accuracy of about $|\sigma| \leq 6$ with the sampling interval $\tau = 0.02$. That is much worse than the above-obtained performance with the constant measurement interval $\tau = 0.2$, when the finite differences are used. Due to the demonstrated robustness properties of the approach, it remains competitive, or may turn out to be even the only choice, when only low sampling rates are available. In particular, high-gain observer feedback implementation is impossible with the considered low sampling rates.

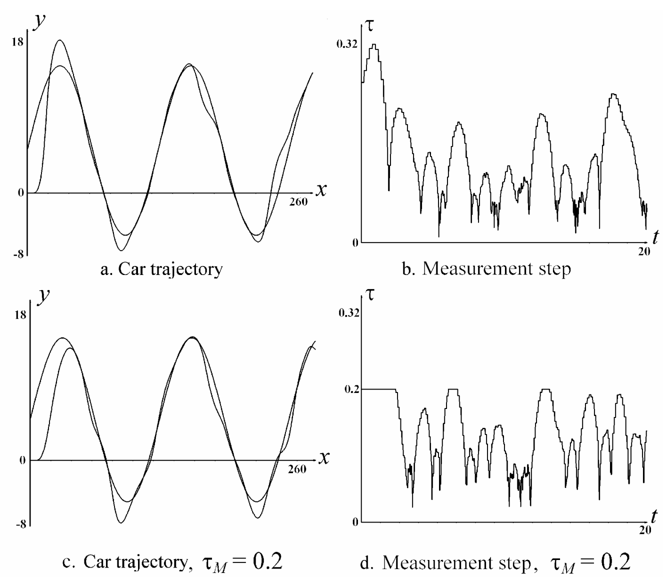

Fig. 5. Variable measurement step with noise magnitude ε = 0.05m; a, b: unbounded τ; c,d: $\tau_M = 0.2$

Finite differences can be implemented in two ways. In the case when the maximal measurement noise magnitude is known, the best way is to take a sufficiently large constant sampling interval providing for the best performance. If the noise magnitude cannot be estimated from above, a reasonable and the simplest way is still to choose the largest interval, which provides for acceptable performance in the absence of noises, for the performance is not sensitive to small enough noises. The other way is to apply the variable measurement step control with unbounded ((11), (12)) or bounded ((12), (13)) step. In that case an excellent accuracy is provided with infinitesimally small measurement noises, and the controller is still robust. The best performance seems to be obtained with the bounded variable step (13), if some reasonable estimation of the maximal noise magnitude is available, and the maximal step is appropriately chosen.

Thus, any strictly r-sliding-homogeneous finite-time-stable controller provides for the full SISO control based on the input measurements only, when the only information on the controlled uncertain process is actually its relative degree. The proposed technique is globally applicable if the relative degree is constant and few boundedness restrictions hold globally; it is also locally applicable to general-case weakly-minimum-phase SISO systems. In the absence of noises the variable-measurement-step strategy provides for the proportionality of the resulting accuracy to τ_m^r, τ_m being the minimal sampling period and r being the relative degree. That is the best possible asymptotics with discrete sampling and discontinuous control [17]. In the presence of noises the tracking accuracy is proportional to the unknown noise magnitude. Only boundedness of the measurement noise is needed, no frequency considerations are relevant.

Appendix

Proof of Theorem 3. Identity (8) implies equivalence of (9) and (11). It is known [6] that

$$|\Delta^{j-k}\sigma_i^{(k)}/\tau^{j-k} - \sigma^{(j)}| \leq (j - k)\, \tau \sup|\sigma^{(j+1)}|, \; j = k + 1, ..., r - 1,$$

where $\sup|\sigma^{(j+1)}|$ is calculated over the time interval containing the involved sampling points. Recall that $\sigma^{(r)}$ is bounded. Denote

$$\varepsilon_j(\Omega) = (j - k)\, \tau \sup_{\Sigma\in\Omega} |\sigma^{(j+1)}|, \quad j = k + 1, ..., r - 1;$$

$\varepsilon_j(\Omega) = 0$, $j = 1, ..., k$. Consider the difference $\Delta^{j-k}\sigma_i^{(k)}/\tau^{j-k} - \sigma^{(j)}$ as some noise of the magnitude ε_j which is infinitesimally small within any bounded area Ω when $\tau \to 0$. As follows from Theorem 2, with some small τ_0 all trajectories starting from some disk D centered at 0 concentrate in its subset $W = \{\Sigma \in \mathbf{R}^r |\, |\sigma| \leq a_0,\, |\dot\sigma| \leq a_1, ..., |\sigma^{(r-1)}| \leq a_{r-1}\}$ to stay there forever.

Apply transformation $(t, \Sigma, \varepsilon) \mapsto (\kappa t, d_\kappa\Sigma, d_\kappa\varepsilon)$ with such $\kappa > 1$ that $d_\kappa W \subset D$. Since the transformation transfers trajectories into trajectories, achieve that with the enlarged noise magnitudes $\tilde\varepsilon_j = \kappa^{r-j}\varepsilon_j$ the trajectories starting from $d_\kappa D$ enter $d_\kappa W \subset D$ in finite time to stay there. Taking into account that $\tilde\varepsilon_j = 0$ with $j \leq k$ and $\tilde\varepsilon_j = \kappa^{r-j}\varepsilon_j(D) = \kappa^{r-j}\tau\,(j - k) \sup_{\Sigma\in D} |\sigma^{(j+1)}| = \kappa\, \varepsilon_j(d_\kappa D) > \varepsilon_j(d_\kappa D)$ otherwise, achieve that the solutions of (4), (9) also enter $d_\kappa W \subset D$ to stay there. Thus, building infinite number of embedded sets $d_\kappa^s D$, $s \in \boldsymbol{N}$, achieve the global finite-time convergence to the set W.

It is easy to see that the transformation

$$\widetilde{G}_\kappa : (t, \tau, \Sigma) \mapsto (\kappa t, \kappa \tau, d_\kappa \Sigma)$$

preserves the discrete sampling and transfers the solutions of (4), (9) into the solutions of the same inclusion, but with a different τ. Let the set $|\sigma| \le a_0$, $|\dot{\sigma}| \le a_1$, ..., $|\sigma^{(r-1)}| \le a_{r-1}$ be attracting invariant set with some fixed sampling step τ_0. Applying now $\widetilde{G}_\kappa$ with $\kappa = \tau/\tau_0$ achieve the needed attracting-set asymptotics.■

Proof of Theorem 4. Denote by $\hat{\sigma}_i^{(j)}$ the noisy measurement of $\sigma^{(j)}_i$, $|\hat{\sigma}_i^{(j)} - \sigma_i^{(j)}| \le \beta_k \varepsilon^{(r-j)/r} = \beta_k \eta^{-(r-j)} \tau^{r-j}$, $j = 1, ..., k$. Then

$$|\Delta^{j-k} \hat{\sigma}_i^{(k)} / \tau^{j-k} - \sigma^{(j)}| \le \varepsilon_j(D) = (j - k)\tau \sup_{\Sigma\in\Omega} |\sigma^{(j+1)}| + N_{j-k}\tau^{r-j},$$

$j = k + 1, ..., r - 1$, $N_{j-k} > 0$. The rest of the proof is the same as of Theorem 3.■

Proof of Theorem 5. The idea is to show by comparison with a suitable differential inclusion that the system has a global invariant attracting set. The needed asymptotics follows then immediately from the system homogeneity. With small ε and τ_m the sampling step $\tau \approx \lambda\sigma^{1/r}$ is close to zero near the plane $\sigma = 0$, which means that the controller is highly sensitive to any noise. In order to remove the singularity enlarge the right-hand side of the inclusion (4), (5) taking

$$\sigma^{(r)} \in \begin{cases} [-C, C] + [K_m, K_M]\varphi(\Sigma), & |\sigma| > 2\varepsilon_*, \\ [-C - K_M \sup|\varphi(\Sigma)|, C + K_M \sup|\varphi(\Sigma)|], & |\sigma| \le 2\varepsilon_*. \end{cases} \quad (19)$$

Recall that φ is globally bounded. Note also that even with $\varepsilon_* = 0$ solutions of (19) might be different from the solutions of (4), (5). As always, the inclusions are replaced here by the corresponding minimal Filippov differential inclusions. The idea is to show that solutions of (4), (12), (13) approximate solutions of (19), which in its turn approximate solutions of (4), (5).

Controller (5) is finite-time stable, thus, the trajectories of the inclusion (4), (5) which start from a disk D_0 centered at the origin terminate in finite time T in some smaller disk D_1, being confined in some larger disk B during the time T . Call this the *contraction* property.

Lemma 1. *The contraction property of* (4), (5) *is preserved for* (19) *with somewhat enlarged* D_1, B, *if* $\varepsilon_* > 0$ *is chosen small enough*.

Proof. Indeed, it follows from the Lagrange Theorem that the only possible limit point of the zeros of any solution $\sigma(t)$ of (4), (5), or of (19), is the point where $\sigma = ... = \sigma^{(r-1)} = 0$, i.e. the origin $\Sigma = 0$. Therefore, the trajectories of (4), (5) and of (19) cross the hyperplane $\sigma = 0$ not stopping on it. Hence, with $\varepsilon_* = 0$ the solutions of inclusion (19) are the same as of (4), (5). The Lemma follows now from the continuous dependence of the Filippov solutions on the right-hand side graph [8]. ■

Lemma 2. *With* λ, ε, τ_m *small enough and* $|\sigma| \ge 2\varepsilon_*$ *the difference operators* δ^i, $i = 1, 2, ..., r - 1$, *are based on sampling outside of the layer* $|\sigma| \le \varepsilon_*$.

Proof. Taking into account the boundedness of $|\dot{\sigma}|$ in B, require that $\lambda \max |\dot{\sigma}|(3\varepsilon_*)^{1/r} < \varepsilon_*/r$. That means that the maximal increment of $|\sigma|$ with the sampling taken inside the layer $|\sigma| \le 2\varepsilon_*$ and the noise magnitude $\varepsilon < \varepsilon_*$, is less than ε_*/r. Thus, more than r sampling steps are needed for any point outside of the layer $|\sigma| \le 2\varepsilon_*$ to be reached from the layer $|\sigma| \le \varepsilon_*$. ■

Lemma 3. *Let* ε_* *be defined from Lemma* 1. *Then with sufficiently small* λ *the contraction property holds for* (4), (12), (13) *with somewhat enlarged* D_1, B *and* ε, τ_m *small enough*.

Proof. Show that outside of the layer $|\sigma| \le 2\varepsilon_*$ controller (12), (13) can be considered as (5) with small measurement noises, which means that (12), (13) approximates (19) in the whole region. Let λ be smaller than the value required in Lemma 2 and let τ_m and ε be so small with respect to ε_* that only the first line of (12) is actual outside of $|\sigma| \ge \varepsilon_*$ and the identity $\tau_k = \lambda|\hat{\sigma}_k|^{1/r}$ holds. According to Lemma 2 each measurement point of a trajectory outside of the layer $|\sigma| \le 2\varepsilon_*$ is preceded by at least r measurements outside of $|\sigma| \le \varepsilon_*$. Thus, with sufficiently small fixed λ and correspondingly smaller ε and τ_m the involved sampling steps are separated from zero.

When λ is small enough the feedback (12), (13) can be considered as sampling Σ in $B_1 = B \cap \{|\sigma| \ge \varepsilon_*\}$ for (19) with small measurement errors. Indeed, it follows from the mean value formula $s!\,\delta^s\sigma_i = \sigma^{(s)}(\xi)$ holding for some $\xi \in [t_i, t_{i-s}]$ that

$$s!\,|\delta^s\sigma_i - \sigma^{(s)}(t_i)| \le \lambda r \sup_{\Sigma\in B} |\sigma^{(s+1)}| \cdot \sup_{\Sigma\in B} |\sigma+\varepsilon|^{1/r}.$$

The resulting motion satisfies "noisy" (19) and is described by the inclusion

$$\sigma^{(r)} \in \begin{cases} [-C, C] + [K_m, K_M]u, & |\sigma| > 2\varepsilon_*, \\ [-C - K_M \sup|\varphi(\Sigma)|, C + K_M \sup|\varphi(\Sigma)|], & |\sigma| \le 2\varepsilon_*, \end{cases} \quad (20)$$

with u defined from (12), (13). Due to the continuous dependence of the solutions on the right-hand-side graph, enlarging D_1, B in an appropriate way, obtain the contraction property for "noisy" (19), and therefore for (4), (12), (13). Usage of (14) is justified by the asymptotic equivalence of $2\,\delta^2\sigma_i = (\delta\hat{\sigma}_i - \delta\hat{\sigma}_{i-1})/(\tau_i + \tau_{i-1})$.and $(\delta\hat{\sigma}_i - \delta\hat{\sigma}_{i-1})/\tau_i$ with small λ. ■

Let now $\varepsilon_* = \varepsilon_{*1}$ and λ be fixed so that Lemmas 1-3 hold for sufficiently small ε and τ_m.

Lemma 4. *There are a compact set* Ω *including the origin, and positive constants* τ_{m1} *and* ε_1, *such that* Ω *is a global finite-time-attracting invariant set for* (4), (12), (13) *with any* $\varepsilon \le \varepsilon_1$ *and* $\tau_m \le \tau_{m1}$.

Proof. Fix some values ε and τ_m for which Lemma 3 holds. Let also τ_m and ε be small enough with respect to ε_{*1}, so that the identity $\tau_k = \lambda|\sigma_k|^{1/r}$ holds outside of $|\sigma| \ge \varepsilon_{*1}$. Apply the transformation

$$(t, \tau_m, \varepsilon, \Sigma) \mapsto (\kappa t, \kappa\tau_m, \kappa^r\varepsilon, d_\kappa\Sigma) \quad (21)$$

with $\kappa > 1$. It preserves the trajectories of (4), (12), (13), which implies the contraction property with the set triplet

$$d_\kappa B \supset d_\kappa D_0 \supset d_\kappa D_1$$

for (4), (12), (13), and ε and τ_m changed to the *enlarged* values $\kappa^r\varepsilon$ and $\kappa\tau_m$. Inside $d_\kappa B$ these trajectories satisfy (12), (13), (20) with correspondingly changed $\varepsilon_* = \kappa^r\varepsilon_{*1}$, also (12), (13), (20) featuring the same contraction property (note that the restriction on λ from the Lemma 2 proof is invariant with respect to transformation (21)).

According to the above choice of τ_m its reduction does not influence trajectories of (12), (13), (20), therefore the reduction of $\kappa\tau_m$ back to τ_m does not violate the contraction property of (4), (12), (13) with the sets $d_\kappa D_0$, $d_\kappa D_1$, $d_\kappa B$. In its turn the reduction of $\kappa^r\varepsilon$ back to ε means just restriction to a subset of trajectories of (4), (12), (13) with smaller noises. Thus (4), (12), (13) with the original values of ε and τ_m features the contraction property with the sets $d_\kappa D_0$, $d_\kappa D_1$, $d_\kappa B$ for any $\kappa > 1$. The same reasoning implies that the property is robust with respect to any reduction of ε and τ_m.

Choosing now $\kappa > 1$ so that $d_\kappa D_1$ still lies in the interior of D_0 obtain that trajectories of (4), (12), (13) which start in $d_{\kappa^{l+1}} D_0$ terminate in $d_{\kappa^l} D_0$ in the time $\kappa^{l+1} T$ without leaving $d_{\kappa^{l+1}} B$, $l = 0, 1, ...$ Covering the whole space with the sets $d_{\kappa^l} D_0$, obtain the global finite-time convergence of the trajectories to the set D_0. The point set of all trajectory segments starting in D_0 and having the time-length T constitute the global attracting invariant set. ■

Apply Lemma 4. Define $\mu > 0$ from the equality $\varepsilon_1 = \mu\tau_{m1}^r$. Consider first the case when $\mu\tau_m^r \le \varepsilon$. Then applying transformation (21) with $\kappa = (\varepsilon_1/\varepsilon)^{1/r}$ obtain (4), (12), (13) with $\varepsilon = \varepsilon_1$, $\tau_m \le \tau_{m1}$ and the invariant set Ω. Now applying the inverse transformation and calculating the bounds of the attracting invariant set $d_\kappa\Omega$ obtain the needed asymptotics. Let now $\mu\tau_m^r > \varepsilon$. Applying transformation (21) with $\kappa = \tau_{m1}/\tau_m$ obtain (4), (12), (13) with $\varepsilon < \varepsilon_1$ and $\tau_m = \tau_{m1}$. After the inverse transformation obtain the other needed asymptotics, which ends the proof of Theorem 5. ■

Proof of Theorem 6. Choose some fixed sufficiently small value of λ as in the proof of Theorem 5. Similarly to the previous proof obtain the contraction property and global invariant set attracting in finite time with any sufficiently small τ_m, ε_0, $\varepsilon = \varepsilon_0$. Taking into account that (4), (13), (15) with the proposed in the Theorem choice of τ_M is invariant with respect to the transformation

$$(t, \tau_m, \varepsilon_0, \Sigma) \mapsto (\kappa\, t, \kappa\, \tau_m, \kappa^r \varepsilon_0, d_\kappa\Sigma)$$

readily obtain the first needed asymptotics.

Consider now a compact region D_0, including the origin and invariant with respect to (4), (5), such that all trajectories starting in it enter in finite time a smaller invariant compact subregion D_1. Consider the "noisy" control

$$u \in \begin{cases} \overline{\mathrm{co}}\,\varphi(\sigma + [-\delta,\delta], \dot\sigma + [-\delta,\delta], ..., \sigma^{(r-1)} + [-\delta,\delta]), & |\sigma| > 2\varepsilon_*, \\ [-\sup|\varphi(\Sigma)|, \sup|\varphi(\Sigma)|], & |\sigma| \le 2\varepsilon_*, \end{cases} \qquad (22)$$

where $\overline{\mathrm{co}}$ denotes the convex closure operation, meaning here that the minimal segment is taken containing all the set. With sufficiently small δ the above contraction property is preserved for (4) (22) with somewhat changed invariant regions D_0, D_1 due to the continuous dependence on the right-hand-side-graph [8]. With sufficiently small λ, τ_m, and ε control (13), (15) satisfies (22) in D_0, which means that the contraction property is true also for (4), (13), (15). Apply now the transformation

$$(t, \tau_m, \varepsilon_0, \varepsilon, \varepsilon_*, \delta, \Sigma) \mapsto (\kappa t, \kappa\tau_m, \kappa^r\varepsilon_0, \kappa^r\varepsilon_*, \kappa^r\varepsilon, \kappa^r\delta, d_\kappa\Sigma)$$

with $\kappa > 1$. Similarly to the proof of Theorem 5 obtain the contraction property of (4), (22) with changed parameters for $d_\kappa D_0$, $d_\kappa D_1$. But also (13), (15) with *unchanged* parameters satisfies (22) with $d_\kappa D_0$, $d_\kappa D_1$. The rest of the proof is the same as of Theorem 5. ■

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