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Adjustment of high-order sliding-mode controllers

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SUMMARY

Long-lasting problems of high-order sliding-mode (HOSM) design are solved. Only local uncertainty suppression was previously obtained in the case when the dynamic system uncertainties are unbounded. This restriction is removed in this paper. A universal method is proposed for the proper controller parameter adjustment based on the homogeneity approach. The method allows making the finite-time convergence arbitrarily fast or slow. In addition, a HOSM regularization procedure is proposed diminishing chattering. Computer simulation confirms the theoretical results. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Sliding modes remain among the most effective control tools under uncertainty conditions. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Sliding mode is accurate and insensitive to disturbances [1, 2]. The main drawback of the standard sliding modes is mostly related to the so-called chattering effect [3, 4], and a lot of efforts were made to diminish it [5, 6].

Let the constraint be given by the equation $\sigma = s - w(t) = 0$, where s is some available output variable of an uncertain single-input–single-output (SISO) dynamic system, and $w(t)$ is an unknown-in-advance smooth input to be tracked in real time. Then the standard sliding-mode controller $u = -k \operatorname{sign} \sigma$ may be considered as a universal controller applicable if the relative degree is one, i.e. if $\dot{\sigma}$ explicitly depends on the control u and $\dot{\sigma}'_u > 0$. Higher-order sliding mode (HOSM) [7, 8] is applicable for controlling SISO uncertain systems with arbitrary relative degree r. The corresponding finite-time-convergent controllers (r-sliding controllers) [7–10] require actually only the knowledge of the system relative degree. The produced control is a discontinuous function of the tracking deviation σ and of its real-time-calculated successive derivatives $\dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$.

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The controllers provide also for higher accuracy with discrete sampling and, when properly used, practically remove the chattering effect [11]. For this sake the control derivative is to be treated as a new control [7, 12].

Although the second-order sliding-mode controllers are already widely used [10, 13–16], the higher-order controllers are mostly theoretically studied [8, 9, 17, 18]. One of the main problems is the parameter adjustment. Indeed, no algebraic criterion was published for the parameter assignment, though it could be developed based on the constructive proofs [8, 9]. Such calculations would be carried out separately for each relative degree and would produce highly conservative conditions on the parameters. Thus, the authors consider such estimations practically useless. The proposed solution was to find such parameters by simulation. Tested parameter sets were published for the main practical cases $r=2,3,4$. Although theoretically already one set is sufficient for any relative degree, in practice one needs to adjust these parameters, in order to hasten or slow down the finite-time transient process. A simple algorithm is presented in this paper producing infinite number of valid parameter sets from a given one. The convergence can be made arbitrarily fast or slow. The corresponding formulas (Proposition 2) for parameter adjustment were developed by Michael.

A long-lasting problem of HOSM application is the uncertainty boundedness restriction. While with the second relative degree, in the presence of globally unbounded uncertainties, the problem can be solved by a sophisticated time-variant parameter adjustment of the sub-optimal controller [19], only the local convergence to the sliding mode is provided with higher relative degrees. It is shown in this paper that the main known arbitrary-order sliding-mode controllers preserve their convergence, when their constant magnitude gains are replaced by sufficiently large functions, ensuring global treatment of unbounded uncertainties. A simple criterion is developed to check (still by computer simulation) that the chosen parameter values indeed allow such implementation.

The result allows not only to increase the gain function, but also to reduce it when possible. This logically leads to the idea of diminishing or even zeroing the control, when, $\dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$ become small, in order to avoid chattering. In the latter case the ideal accuracy is inevitably lost, but approximate sliding mode persists. The corresponding regularization procedure and its accuracy are studied.

Some results of this paper were presented at the IFAC Congress [20]. Computer simulation demonstrates the applicability of the proposed schemes.

2. THE PROBLEM STATEMENT

Consider a smooth dynamic system with a smooth output function σ. Let the system be closed by some possibly dynamical discontinuous feedback and be understood in the Filippov sense [21]. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma=\cdots=\sigma^{(r-1)}=0$ is a non-empty integral set, the motion on the set is called r-sliding (rth-order sliding) mode [7, 8]. The standard sliding mode used in the most variable structure systems is of the first order (σ is continuous and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x}=a(t,x)+b(t,x)u, \quad \sigma=\sigma(t,x) \tag{1}$$

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where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$, a, b and $\sigma : \mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, n is also uncertain. The relative degree r of the system is assumed to be constant and known, which means that the control for the first time appears explicitly in the rth total time derivative of σ [22]. The output σ and its $r-1$ successive derivatives are supposed to be measured in real time. It is easy to check that

$$\sigma^{(r)} = h(t,x) + g(t,x)u \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = (\partial/\partial u)\sigma^{(r)}$ are some uncertain functions, $g(t,x) \neq 0$. The current values of a locally bounded Lebesgue-measurable non-zero function $\Phi(t,x)$ are supposed to be available, such that for any positive d, the inequality

$$\alpha g(t,x)\Phi(t,x) > d + |h(t,x)| \tag{3}$$

holds with sufficiently large α. The task is to provide in finite time for the identity $\sigma \equiv 0$.

It is also assumed that, if σ remains bounded, trajectories of (1) are infinitely extendible in time for any Lebesgue-measurable control $u(t,x)$ with bounded quotient u/Φ. Actually, the proposed method works for much larger class of systems, and this assumption is needed only to avoid finite-time escape. In practice the system is often required to be weakly minimum phase. Note also that in practice actuator presence might prevent effectiveness of any global control due to saturation effects.

The traditional assumption [8, 10] is that

$$0 < K_{\mathrm{m}} \leqslant \frac{\partial}{\partial u}\sigma^{(r)} \leqslant K_{\mathrm{M}}, \quad |\sigma^{(r)}|_{u=0}| \leqslant C \tag{4}$$

holds for some $K_{\mathrm{m}}, K_{\mathrm{M}}, C > 0$. It corresponds to $\Phi = 1$. The both problem statements (1)–(3) and (1), (2), (4) are further considered.

The information availability requirements can be significantly reduced, if the differentiator [8] was applied, producing exact robust estimations of the derivatives of σ. In particular, in the second problem setting (1), (4) the global output-feedback control would be produced [8, 18]. In the first setting only semi-global solution would be obtained, since its application requires the boundedness of $\sigma^{(r)}$. Therefore, it was decided not to use the differentiator in that case in order to simplify the presentation.

3. CONTROL MAGNITUDE ADJUSTMENT

Two known families of high-order sliding controllers are defined by recursive procedures. It is taken below $\beta_1, \ldots, \beta_{r-1} > 0$ and $i = 1, \ldots, r-1$.

1. The following procedure defines the so-called *nested r-sliding controller* [8]. The controller can be given an intuitive inexact explanation based on recursively nested standard sliding modes. The proper explanation is more complicated [8], since no sliding mode is possible on discontinuous surfaces, and a complicated motion arises around the control discontinuity set. Let p be the least common multiple of $1, 2, \ldots, r$. Define functions

$$N_{i,r} = (|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} + \cdots + |\sigma^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p}$$

$$\Psi_{0,r} = \operatorname{sign}\sigma, \quad \Psi_{i,r} = \operatorname{sign}(\sigma^{(i)} + \beta_i N_{i,r} \Psi_{i-1,r})$$

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2. Another procedure corresponds to the so-called *quasi-continuous controller* [9]. Denote

$$\varphi_{0,r}=\sigma, \quad N_{0,r}=|\sigma|, \quad \Psi_{0,r}=\varphi_{0,r}/N_{0,r}=\text{sign}\ \sigma$$

$$\varphi_{i,r}=\sigma^{(i)}+\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}, \quad N_{i,r}=|\sigma^{(i)}|+\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \quad \Psi_{i,r}=\varphi_{i,r}/N_{i,r}$$

In the both cases the controller takes of the form

$$u=-\alpha\Phi(t,x)\Psi_{r-1,r}(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)}) \tag{5}$$

where $\alpha>0$ and $\Phi\equiv 1$ in [8, 9]. Note that in the case of the quasi-continuous controller the function $\Psi_{r-1,r}$ can be redefined according to the continuity everywhere except the r-sliding set $\sigma=\dot{\sigma}=\cdots=\sigma^{(r-1)}=0$. Recall in this context that values of the control on any set of the zero Lebesgue measure do not influence the solutions [21].

Theorem 1
Let $\beta_1,\ldots,\beta_{r-1},\alpha>0$ be chosen sufficiently large in the list order, then controller (5) provides for the finite-time establishment of the identity $\sigma\equiv 0$ for any initial conditions. Moreover, any increase of the gain function Φ does not interfere with the convergence.

The theorem proofs are listed in the next section. In other words, the finite-time stable r-sliding mode $\sigma\equiv 0$ is established in the system (1), (5). Note that the theorem does not claim that *all* parametric combinations providing for the finite-time convergence to the r-sliding mode allow the arbitrary increase of α and Φ. The corresponding parameter combinations are further called *gain-function robust*. The following theorem provides for some method of finding gain-function robust parameter sets.

Theorem 2
Finite-time stability of the differential equations $\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}=0$ and $\sigma^{(r-1)}+\beta_{r-1}N_{r-1,r}\Psi_{r-2,r}=0$ for the quasi-continuous and the nested controller, respectively, provides for the gain-function robustness of the parameter combination $\beta_1,\ldots,\beta_{r-1}>0$. The number of such parameter sets is infinite.

Some other HOSM controllers also satisfy Theorem 1. Such controllers and parameter combinations are also called gain-function robust. The popular sub-optimal and twisting controllers are not gain-function robust and require special efforts to deal with unbounded uncertainties [7, 19].

Consider another implementation of Theorem 1. In any practical application the operation region is bounded. The standard application of the controller (5) corresponds to $\Phi=1$, and the gain α is chosen sufficiently large to cope with the uncertainty over the whole region. According to Theorem 1, the control effort magnitude $\Phi(t,x(t))$ can be chosen in real time, varying over the region with respect to the changing uncertainty bounds. In particular, in the case of the system stabilization at the point being 'almost equilibrium', the discontinuity can be significantly diminished, also reducing the chattering. This approach can be complemented by decreasing or even zeroing the control magnitude in a small vicinity of the r-sliding set $\Sigma_r=0$, where $\Sigma_r=(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)})$. In such a case, for example, the control can be made globally continuous if the quasi-continuous controller [9] is used.

Theorem 3
Let $\zeta_\delta(\Sigma_r)$ be a locally bounded Lebesgue-measurable function, $\zeta_\delta(\Sigma_r)\equiv 1$ with $\|\Sigma_r\|\geqslant\delta$, and $\varepsilon>0$. Let the conditions of Theorem 1 be fulfilled, $\beta_1,\ldots,\beta_{r-1}$ be chosen with respect to Theorem 2

and α be sufficiently large. Then with sufficiently small δ, the controller

$$u = -\alpha\Phi(t,x)\zeta_\delta(\sigma,\ldots,\sigma^{(r-1)})\Psi_{r-1,r}(\sigma,\ldots,\sigma^{(r-1)}) \tag{6}$$

provides for the finite-time convergence into the vicinity $\|\Sigma_r\| \leqslant \varepsilon$ of the r-sliding mode.

This theorem is actually a HOSM analogue of the well-known first-order sliding-mode regularization [5]. As usual the ideal accuracy is abandoned here for the sake of less chattering. Theorem 3 can be made more precise in the standard problem setting case [7–12], when $\Phi(t,x)\equiv 1$ is taken, and $\zeta_\delta(\Sigma_r)$ is r-sliding homogeneous.

Theorem 4
Let (4) hold, and $\varepsilon > 0$. Then with any $k > 0$ and sufficiently large $\alpha > 0$, the controller

$$u = -\alpha\Psi_{r-1,r}(\sigma,\dot\sigma,\ldots,\sigma^{(r-1)})\min\left[1, \frac{1}{\varepsilon}(|\sigma| + |\dot\sigma|^{r/(r-1)} + \cdots + |\sigma^{(r-1)}|^r)^{k/r}\right] \tag{7}$$

provides for the global finite-time convergence into the vicinity $|\sigma^{(i)}| \leqslant \mu_i \varepsilon^{(r-i)/k}$, $i = 0,\ldots,r-1$ of the r-sliding mode, where μ_i are some positive constants depending only on the parameters of the controller k, α, and the constants K_m, K_M and C from (4).

Note that any other positive-definite r-sliding homogeneous function [18, 23] of the weight k can be used in the function $\zeta_\delta(\Sigma_r)$. Using the standard homogeneity reasoning [18] the asymptotics can be shown not to change, if the output σ is measured with a noise of the maximal magnitude proportional to $\varepsilon^{r/k}$, and the successive derivatives of σ are estimated by the homogeneous $(r-1)$th-order differentiator [8]. Theorem 4 also has a local analogue, which is true for any smooth system (1) with the relative degree r. In such a case, ε and the initial values are to be sufficiently small.

4. PROOFS OF THE THEOREMS

Proof of Theorem 1
The proofs are similar for the both controllers. The main idea is that with sufficiently large α any system trajectory enters some specific region in finite time to stay in it. The region is described by some homogeneous differential inequalities, which do not ‘remember’ anything on the original process. These inequalities determine the further convergence.

Consider, for example, the quasi-continuous controller. The proof is based on a number of lemmas and is a careful modification of the proof from [9], the modification being concentrated in Lemma 3, which is new. Although the whole system satisfies specific homogeneous properties in [9], the homogeneity features are revealed now only after the statement of Lemma 3 is fulfilled.

Lemma 1
$N_{i,r}$ is positive definite, $i = 0,\ldots,r-1$, i.e. $N_{i,r} = 0$ iff $\sigma = \dot\sigma = \cdots = \sigma^{(i)} = 0$. The inequality $|\Psi_{i,r}| \leqslant 1$ holds whenever $N_{i,r} > 0$. The function $\Psi_{i,r}(\sigma,\dot\sigma,\ldots,\sigma^{(i-1)})$ is continuous everywhere (i.e. it can be redefined by continuity) except the point $\sigma = \dot\sigma = \cdots = \sigma^{(i-1)} = 0$.

Assign the weights (homogeneity degrees) $r-i$ to $\sigma^{(i)}, i = 0,\ldots,r-1$ and the weight 1 (minus system homogeneity degree, [23]) to t, which corresponds to the r-sliding homogeneity [18].

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Lemma 2
The weight of $N_{i,r}$ equals $r-i, i=0,\dots,r-1$. Each homogeneous locally bounded function $\omega(\sigma,\dot\sigma,\dots,\sigma^{r-i})$ of the weight $r-i$ satisfies the inequality $|\omega|\leqslant cN_{i,r}$ for some $c>0$.

Lemma 3
For any $\gamma>0$ and $\beta_{r-1}>0$ with sufficiently large $\alpha>0$ the inequality

$$|\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}|\leqslant\gamma N_{r-2,r}^{1/2} \tag{8}$$

is established in finite time and kept afterwards.

Proof
Consider the point set $\Omega(\xi)=\{(\sigma,\dot\sigma,\dots,\sigma^{(r-1)})||\Psi_{r-1,r}|\leqslant\xi\}\cup\{0\}$ for some fixed $\xi, 0<\xi<\gamma/4$, $\xi<3/4$. Assuming $N_{r-2,r}>0$ and $|\sigma^{(r-1)}|\geqslant 3\beta_{r-1}N_{r-2,r}^{1/2}$ obtain that, since $|\Psi_{r-2,r}|\leqslant 1$,

$$|\Psi_{r-1,r}|=|\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}|/(|\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2})\geqslant 3/4$$

Thus, $|\Psi_{r-1,r}|\leqslant\xi$, implies $|\sigma^{(r-1)}|\leqslant 3\beta_{r-1}N_{r-2,r}^{1/2}$. Simple calculations show now that $\Omega(\xi)\subset\Omega_1(\xi)$, where $\Omega_1(\xi)$ is defined by the inequality

$$|\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}|\leqslant 4\xi\beta_{r-1}N_{r-2,r}^{1/2}$$

In turn the inequality can be rewritten in the form $\theta_-\leqslant\sigma^{(r-1)}\leqslant\theta_+$, where θ_-, θ_+ are the corresponding homogeneous functions of $\sigma,\dot\sigma,\dots,\sigma^{(r-2)}$ of the weight 1. Approximate them by globally smooth homogeneous functions. Restricting θ_- and θ_+ to the homogeneous sphere $\sigma^{2p/r}+\dot\sigma^{2p/(r-1)}+\dots+(\sigma^{(r-2)})^{2p}=1$, where p is the least multiple of $1,2,\dots,r-1$, achieve some continuous on the sphere functions θ_{1-} and θ_{1+}. Functions θ_{1-} and θ_{1+} can be approximated on the sphere by some smooth functions θ_{2-} and θ_{2+} from beneath and from above, respectively. Functions θ_{2-} and θ_{2+} are in their turn extended by homogeneity to the homogeneous functions Θ_- and Θ_+ of $\sigma,\dot\sigma,\dots,\sigma^{(r-2)}$ of the weight 1, smooth everywhere except 0, so that $\Omega(\xi)\subset\Omega_2=\{(\sigma,\dot\sigma,\dots,\sigma^{(r-1)})|\Theta_-\leqslant\sigma^{(r-1)}\leqslant\Theta_+\}$ [9].

Thus, the inequality $|\Psi_{r-1,r}|\geqslant\xi$ is assured outside of Ω_2. Prove now that Ω_2 is invariant and attracts the trajectories with large α. The 'upper' boundary of Ω_2 is given by the equation $\pi_+=\sigma^{(r-1)}-\Theta_+=0$. Suppose that at the initial moment $\pi_+>0$ and, therefore, $\Psi_{r-1,r}\geqslant\xi$. Recall also that $\xi<1$. Taking into account that $\dot\Theta_+(\sigma,\dot\sigma,\dots,\sigma^{(r-1)})$ is a locally bounded homogeneous function of the zero weight, obtain $|\dot\Theta_+|\leqslant\kappa$ for some $\kappa>0$. By differentiating we achieve

$$\dot\pi_+=h(t,x)-\alpha g(t,x)\Phi(t,x)\Psi_{r-1,r}-\dot\Theta_+$$

Choosing α_1 to be sufficiently large, so that $\alpha_1\ g(t,x)\Phi(t,x)>\kappa+1+|h|$, and taking $\alpha=\alpha_1/\xi$, we obtain that $\dot\pi_+\leqslant -1$.

Hence, π_+ vanishes in finite time with α large enough. Thus, the trajectory inevitably enters the region Ω_2 in finite time. Similarly, the trajectory enters Ω_2 if the initial value of π_- is negative and, therefore, $\Psi_{r-1,r}\leqslant -\xi$. Obviously, Ω_2 is invariant.

Choose Θ_- and Θ_+ sufficiently close to θ_- and θ_+ on the homogeneous sphere and α, respectively, large enough to achieve from Lemma 2 that $\Omega_2\subset\Omega_1(\gamma/4\beta_{r-1})$ and the statement of Lemma 3. □

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The fulfilment of the statement of Lemma 3 triggers a chain collapse as follows from the next lemma.

Lemma 4

Let $1\leqslant i\leqslant r-2$, then for any positive $\beta_i, \gamma_i, \gamma_{i+1}$ with sufficiently large $\beta_{i+1}>0$ the inequality

$$|\sigma^{(i+1)}+\beta_{i+1}N_{i,r}^{(r-i-1)/(r-i)}\Psi_{i,r}|\leqslant\gamma_{i+1}N_{i,r}^{(r-i-1)/(r-i)}$$

provides for the finite-time establishment and keeping of the inequality

$$|\sigma^{(i)}+\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}|\leqslant\gamma_i N_{i-1,r}^{(r-i)/(r-i+1)}$$

The proof is similar to Lemma 3. The point set $\Omega(\xi)=\{(\sigma,\dot{\sigma},\ldots,\sigma^{(i)})||\Psi_{i,r}|\leqslant\xi\}$ is considered for some fixed $\xi>0, \xi<\gamma_i/4$. The set $\Omega_1(\xi)\supset\Omega(\xi)$ is defined by the inequality

$$|\sigma^{(i)}+\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\Psi_{i-1,r}|\leqslant 4\xi\beta_i N_{i-1,r}^{(r-i)/(r-i+1)}$$

The further proof uses Lemma 2 to estimate $\dot{\Theta}_+$. Since $N_{0,r}=|\sigma|, \varphi_{0,r}=\sigma$, Lemma 4 is replaced by the next simple lemma with $i=0$.

Lemma 5

The inequality $|\dot{\sigma}+\beta_1|\sigma|^{(r-1)/r}\operatorname{sign}\sigma|\leqslant\gamma_1|\sigma|^{(r-1)/r}$ provides with $0\leqslant\gamma_1<\beta_1$ for the establishment in finite time and keeping the identity $\sigma\equiv 0$.

Thus, β_1, γ_1 are assigned arbitrary values satisfying the conditions of Lemma 5. Now the parameters β_2, γ_2 are assigned with respect to Lemma 4, then comes the turn of β_3, γ_3, etc. At the last step the parameter α is assigned a proper value according to Lemma 3. After a finite transient time, the trajectory belongs to the intersection of closed invariant regions, one of which is the origin (Lemma 5).

This finishes the proof of the theorem in the case of the quasi-continuous controller. In the case of the nested controller a homogeneous vicinity of the controller discontinuity set is shown to attract the trajectories in finite time. The proof closely follows [8]. □

Proof of Theorem 2

Consider the quasi-continuous case. As follows from the previous proof and Lemma 3, the parameters are chosen so as to provide for the finite-time stability of the r-sliding homogeneous differential inclusion

$$\sigma^{(r-1)}\in\gamma[-N_{r-2,r}^{1/2}, N_{r-2,r}^{1/2}]-\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}$$

with sufficiently small $\gamma>0$. Note that in order to satisfy the Filippov conditions it is enlarged at the discontinuity points of $\Psi_{r-2,r}$ in order to get certain convexity and semi-continuity properties [18, 21]. The needed Filippov differential equation is obtained with $\gamma=0$. The proof of Theorem 1 shows how to choose the parameters to provide for the finite-time stability of the equation. As follows from [18] the finite-time stability of a homogeneous differential inclusion with a negative homogeneity degree is always robust with respect to small homogeneous perturbations; therefore, the finite-time stability of the homogeneous differential equation obtained with $\gamma=0$ causes the finite-time stability of the inclusion with any sufficiently small γ. The case of the nested controller is similarly considered. □

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Proof of Theorem 3
Let the parameters be chosen as in the proof of Theorem 1, which in particular means that inequality (8) is established with some $\gamma>0$, corresponding to a finite-time stable differential inclusion. The theorem follows from the following lemmas:

Lemma 6
The trajectories of (1), (5) satisfying $|\sigma^{(r-1)}+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}|\leqslant\gamma N_{r-2,r}^{1/2}$ at the initial moment and starting in a small vicinity of $\Sigma_{r-1}=0$ do not leave another small vicinity of $\Sigma_{r-1}=0$.

The lemma immediately follows from the invariance of the region (8) (Lemma 3) and the finite-time stability of the r-sliding homogeneous differential inclusion corresponding to (8) [18].

Lemma 7
Projections of trajectories of (1), (5) starting in the vicinity $\|\Sigma_r\|\leqslant\delta$ of $\Sigma_r=0$ of the space Σ_r, do not leave some another vicinity Ω_δ of $\Sigma_r=0$ small with small δ.

In fact, the projection trajectories starting in a compact region uniformly converge to $\Sigma_r=0$.

Proof
As follows from the proof of Lemma 3, the time of the transient of the trajectories to the region (8) uniformly tends to zero with $\delta\to 0$. Denote by $\sigma^{(r-1)}(0)$ the initial value of $\sigma^{(r-1)}$, and consider the cases possible at the initial moment. The case $\sigma^{(r-1)}(0)+\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}=0$ is to be excluded, since then the point already satisfies (8). The remaining cases are $\sigma^{(r-1)}(0)>-\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}\geqslant 0, -\beta_{r-1}N_{r-2,r}^{1/2}\Psi_{r-2,r}>\sigma^{(r-1)}(0)\geqslant 0$, and symmetrical cases with $\sigma^{(r-1)}(0)\leqslant 0$. During the transient into the region (8) the coordinate $\sigma^{(r-1)}$ changes monotonously. Therefore, the inequality

$$|\sigma^{(r-1)}|\leqslant\max\{|\sigma^{(r-1)}(0)|,\beta_{r-1}N_{r-2,r}^{1/2}(\Sigma_r)\}$$

holds during the transient, which means that the trajectory does not leave some (small) vicinity of $\Sigma_r=0$. From the moment when (8) holds, the trajectories are confined to a small vicinity of $\Sigma_r=0$ due to Lemma 6. □

Choose sufficiently small δ, so that $\Omega_\delta\subset\{\Sigma_r,\|\Sigma_r\|\leqslant\varepsilon\}$. Outside of $\|\Sigma_r\|\leqslant\delta$ the trajectories of (1), (6) satisfy (1), (5). Thus, any trajectory of (1), (6) sooner or later enters $\|\Sigma_r\|\leqslant\delta$. From that time on it never leaves Ω_δ due to Lemma 7. □

Proof of Theorem 4
The trajectories of (1), (7) satisfy the differential inclusion

$$\sigma^{(r)}\in[-C,C]+[K_{\mathrm{m}},K_{\mathrm{M}}]u,\quad u\in\begin{cases}-\alpha\Psi_{r-1,r}(\Sigma_r), & (|\sigma|+|\dot\sigma|^{r/(r-1)}+\cdots+|\sigma^{(r-1)}|^r)^{k/r}>\varepsilon\\ [-\alpha,\alpha], & (|\sigma|+|\dot\sigma|^{r/(r-1)}+\cdots+|\sigma^{(r-1)}|^r)^{k/r}\leqslant\varepsilon\end{cases}$$

which is invariant with respect to the time-coordinate-parameter transformation

$$(t,\sigma,\dot\sigma,\ldots,\sigma^{(r-1)},\varepsilon)\mapsto(\kappa t,\kappa^r\sigma,\kappa^{r-1}\dot\sigma,\ldots,\kappa\sigma^{(r-1)},\kappa^k\varepsilon)$$

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Similarly to the proof of Theorem 3, obtain that with some ε the trajectories concentrate in a bounded region. The theorem follows now in a standard way from the homogeneity reasoning [18]. □

5. ADJUSTMENT OF THE PARAMETERS

Consider the problem (1), (4). Then the equality (3) implies the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u \tag{9}$$

The problem is now solved building a bounded feedback control of the form

$$u = \alpha \Psi(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}) \tag{10}$$

which provides for the finite-time stability of the closed inclusion (9), (10).

The solutions of (9), (10) are understood as solutions of a some-what larger Filippov differential inclusion [18, 21], in order to provide for the solution existence and other convenient solution properties. Inclusion (9), (10) and the controller (10) are called r-sliding homogeneous, if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa : (t, \Sigma_r)(\kappa t, d_\kappa \Sigma_r) \tag{11}$$

where $\Sigma_r = (\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$, $d_\kappa \Sigma_r = (\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa \sigma^{(r-1)})$ preserve the closed-loop inclusion (9), (10) and its solutions [18]. It is easy to check that (10) is r-sliding homogeneous if

$$\Psi(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa \sigma^{(r-1)}) = \Psi(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}) \tag{12}$$

Note that any changes of the controller on zero-measure sets do not influence the corresponding Filippov inclusion and its solution, also preserving the homogeneity.

Almost all known HOSM controllers are r-sliding homogeneous and satisfy (12). Note that though the sub-optimal controller [10, 12] does not exactly satisfy the described feedback form (10), its trajectories are invariant with respect to (11) with $r = 2$ and it can be considered as two-sliding homogeneous in the broad sense.

Denote by $T_{\max}(s_1, s_2, \ldots, s_r)$ and $T_{\min}(s_1, s_2, \ldots, s_r)$ the maximal and the minimal convergence times to the origin $\Sigma_r = 0$ of the solutions of (9), (10) with initial conditions $\sigma(0) = s_1, \dot{\sigma}(0) = s_2, \ldots, \sigma^{(r-1)}(0) = s_r$. These functions are well defined [21]. If (10) is r-sliding homogeneous, they are also continuous and homogeneous with the homogeneity degree 1 [18].

Let $\lambda > 0$. Consider the differential inclusion

$$\sigma^{(r)} \in \lambda^r [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u \tag{13}$$

and the controller

$$u = \lambda^r \alpha \Psi(\sigma, \dot{\sigma}/\lambda, \ldots, \sigma^{(r-1)}/\lambda^{r-1}) \tag{14}$$

Denote by Ω_R and $\bar{\Omega}_R$, the sets $|\sigma|^{1/r} + |\dot{\sigma}|^{1/(r-1)} + \cdots + |\sigma^{(r-1)}| \leqslant R$ and $|\sigma|^{1/r} + \cdots + |\sigma^{(r-1)}| \geqslant R$, and let $\tilde{T}_{\max}(\Sigma_r)$ and $\tilde{T}_{\min}(\Sigma_r)$ be the convergence-time functions for controller (14).

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Proposition 1

Let the differential inclusion (9), (10) be finite-time stable and r-sliding homogeneous, then also the inclusion (13), (14) is finite-time stable and

$$\max\{\tilde{T}_{\max}(\Sigma_r)|\Sigma_r\in\Omega_R\}\leqslant\frac{1}{\lambda}\max\{T_{\max}(\Sigma_r)|\Sigma_r\in\Omega_R\} \tag{15}$$

$$\min\{\tilde{T}_{\min}(\Sigma_r)|\Sigma_r\in\bar{\Omega}_R\}\geqslant\frac{1}{\lambda}\min\{T_{\min}(\Sigma_r)|\Sigma_r\in\bar{\Omega}_R\} \tag{16}$$

hold with $\lambda>1$ and $\lambda<1$, respectively.

Proof

Apply the time transformation $t=\lambda\tau$. Then $d/\mathrm{d}t=(1/\lambda)d/\mathrm{d}\tau$ and in the new time the closed-loop inclusion takes the form (13), (14). Obviously,

$$T_{\max}(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)})=\lambda\tilde{T}_{\max}(\sigma,\dot{\sigma}/\lambda,\ldots,\sigma^{(r-1)}/\lambda^{r-1})$$

$$T_{\min}(\sigma,\dot{\sigma},\ldots,\sigma^{(r-1)})=\lambda\tilde{T}_{\min}(\sigma,\dot{\sigma}/\lambda,\ldots,\sigma^{(r-1)}/\lambda^{r-1})$$

The proposition follows now from the fact that with $\lambda>1$ the point $(\sigma,\dot{\sigma}/\lambda,\ldots,\sigma^{(r-1)}/\lambda^{r-1})$ belongs to Ω_R, whereas with $\lambda<1$ it belongs to $\bar{\Omega}_R$. Owing to the homogeneity, the minimum of $T_{\min}$ in $\bar{\Omega}_R$ exists and takes place on the set $|\sigma|^{1/r}+\|^{1/(r-1)}+\cdots+|\sigma^{(r-1)}|=R$. □

Note that with $\lambda>1$ inequality (15) holds also for the inclusion (9), (14), which means that controller (14) provides for the convergence acceleration. In the case, when $C=0$, pure acceleration or slow down of the convergence takes place.

Obviously, if (10) is a gain-function robust r-sliding homogeneous controller, then also (14) is gain-function robust and r-sliding homogeneous. In the special case of the quasi-continuous controller, Equation (14) has the same form as the original controller (10). The following proposition lists the corresponding parameter transformation.

Proposition 2

The above-defined quasi-continuous controller preserves its form after the transformation (14) with $r>1$. Its new parameters take on the values

$$\tilde{\beta}_1=\lambda\beta_1,\quad \tilde{\beta}_2=\lambda^{r/(r-1)}\beta_2,\ldots,\tilde{\beta}_{r-1}=\lambda^{r/2}\beta_{r-1},\quad \tilde{\alpha}=\lambda^r\alpha$$

Note that since this controller is gain-function robust, each λ produces a new valid combination of β_i effective for any SISO system with the given relative degree r, provided a sufficiently large gain function is taken. Following are the resulting quasi-continuous controllers with $r\leqslant 4$, simulation-tested β_i and a general gain function Φ:

1. $u=-\alpha\Phi\,\mathrm{sign}\,\sigma$,
2. $u=-\alpha\Phi(\dot{\sigma}+\lambda|\sigma|^{1/2}\mathrm{sign}\,\sigma)/(|\dot{\sigma}|+\lambda|\sigma|^{1/2})$,
3. $u=-\alpha\Phi[\ddot{\sigma}+2\lambda^{3/2}(|\dot{\sigma}|+\lambda|\sigma|^{2/3})^{-1/2}(\dot{\sigma}+\lambda|\sigma|^{2/3}\mathrm{sign}\,\sigma)]/[|\ddot{\sigma}|+2\lambda^{3/2}(|\dot{\sigma}|+\lambda|\sigma|^{2/3})^{1/2}]$,
4. $\varphi_{3,4}=\sigma+3\lambda^2[\ddot{\sigma}+\lambda^{4/3}(|\dot{\sigma}|+0.5\lambda|\sigma|^{3/4})^{-1/3}(\dot{\sigma}+0.5\lambda|\sigma|^{3/4}\,\mathrm{sign}\,\sigma)][|\ddot{\sigma}|+\lambda^{4/3}(|\dot{\sigma}|+0.5\lambda|\sigma|^{3/4})^{2/3}]^{-1/2}$, $N_{3,4}=|\sigma|+3\lambda^2[|\ddot{\sigma}|+\lambda^{4/3}(|\dot{\sigma}|+0.5\lambda|\sigma|^{3/4})^{2/3}]^{1/2}$, $u=-\alpha\Phi\varphi_{3,4}/N_{3,4}$.

As follows from Proposition 2 one needs a valid basic set of parameters to produce sets featuring different convergence rate changing λ. The larger the λ, the faster the convergence. New valid

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basic sets can be found by simulation based on Theorem 2. While Theorem 2 and the proof of Theorem 1 allow developing some algebraic criteria for the parameter set choice, it should be done separately for each r, and seems practical only with $r\leqslant 3$.

6. SIMULATION EXAMPLE

6.1. Gain and parameter adjustment

Consider a model example of an 'exploding' process

$$\dddot{x}_1 = \cos 10t\,(e^{x_2} + x_1^2 \sin \dot{x}_1) + (2+\sin t)(x_2^2+1)u$$
$$\dot{x}_2 = x_1 - x_2 + \cos t$$

where x_1 is the output. Obviously, x_2 remains bounded, provided x_1 is bounded. The task is to track the function

$$x_{1c} = 0.08 \sin t + 0.12 \cos 0.3t$$

Respectively, $\sigma = x_1 - x_{1c}$ is taken. Global control is not available here in the standard HOSM framework. The nested three-sliding controller (5) takes the form

$$u = -\alpha\Phi\, \mathrm{sign}(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{1/6} \mathrm{sign}(\dot{\sigma} + |\sigma|^{2/3} \mathrm{sign}\, \sigma)$$

where Φ is the gain function to be specified further. In addition, the three-sliding quasi-continuous controller was applied listed in Section 4. The initial conditions $x_1 = 6, \dot{x}_1 = 1, \ddot{x}_1 = 15, x_2 = 10$ were taken at $t = 0$. The gain functions

$$\Phi = (e^{x_2} + x_1^2)/(x_2^2+1)+1 \tag{17}$$

and

$$\Phi = e^{x_2} + x_1^2 + 1 \tag{18}$$

were considered. In all the cases $\alpha = 5$ is taken. The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics) with the integration step 10^{-5}.

Both considered controllers were applied with both listed gain functions. With the smaller gain function (17) the controllers demonstrate their standard transient features (Figures 1 and 2, $\lambda = 1$). With the redundantly large gain function (18) some large but quickly decreasing chattering of the control and $\ddot{\sigma}$ arises. One cannot distinguish between the joint graphs of $\sigma, \dot{\sigma}, \ddot{\sigma}$ for the both controllers in the latter case. It is interesting to mark that the graphs of σ and $\dot{\sigma}$ do not significantly change with the change of the gain function. In addition, the transient time does not differ much (Figure 1). This is explained by the common dynamics in the 'configuration' space $\sigma, \dot{\sigma}$ (see the proof of Theorem 1).

It is seen in Figure 2 that the control magnitude drops instantly from very large values. After the sliding mode is established, i.e. the trajectory approaches the control discontinuity set $\sigma = \dot{\sigma} = \ddot{\sigma} = 0$, the character sliding-mode control chattering arises with the magnitude $\alpha\Phi(t, x(t))$.

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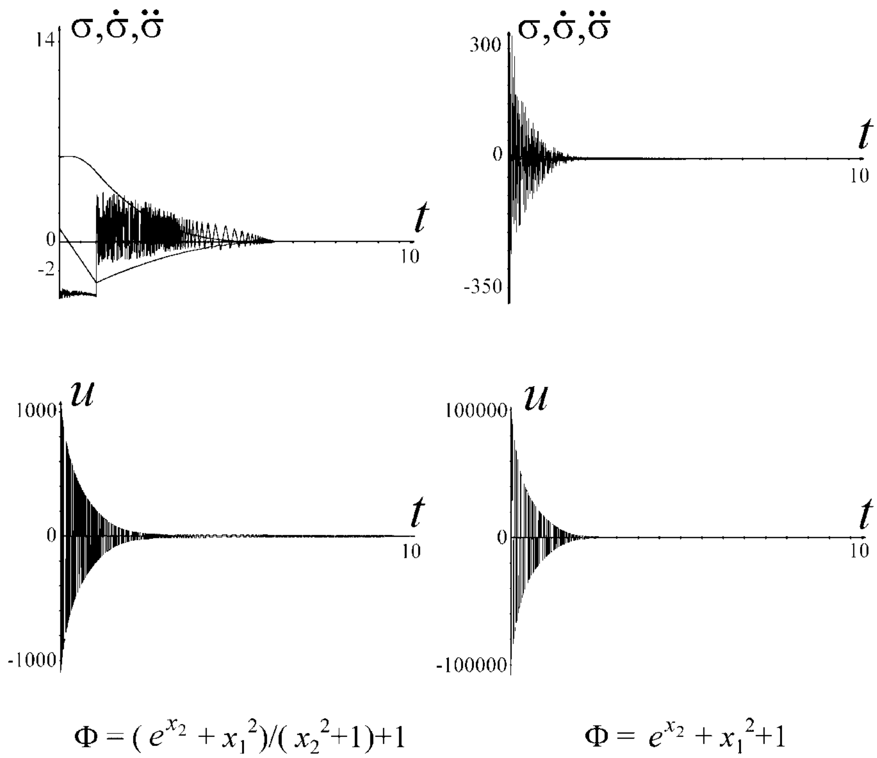

Figure 1. Nested three-sliding controller with different gain functions.

In all the cases almost the same sliding accuracy is obtained $|\sigma|\leqslant 2\times 10^{-12}$, $|\dot{\sigma}|\leqslant 3\times 10^{-8}$, $|\ddot{\sigma}|\leqslant 1\times 10^{-3}$ for the nested controller, and $|\sigma|\leqslant 6\times 10^{-13}$, $|\dot{\sigma}|\leqslant 2\times 10^{-8}$, $|\ddot{\sigma}|\leqslant 8\times 10^{-4}$ for the quasi-continuous controller with $\lambda=1$. After the integration step was changed to 10^{-6}, the accuracy of the nested controller changed to $|\sigma|\leqslant 2\times 10^{-15}$, $|\dot{\sigma}|\leqslant 4\times 10^{-10}$, $|\ddot{\sigma}|\leqslant 1\times 10^{-4}$, which corresponds to the classical 3-sliding accuracy.

The parametric adjustment is demonstrated for the quasi-continuous controller (Figure 2). It is seen that with $\lambda=0.5$ the transient is two times longer, whereas with $\lambda=2$ it is two times shorter. In the latter case, with respect to (14); hence, α was changed to the value $2^3\times 5=40$.

6.2. *Regularization procedure*

Consider the kinematical car model (Figure 3(a))

$$\dot{x}=v\cos\varphi,\quad \dot{y}=v\sin\varphi,\quad \dot{\varphi}=\frac{v}{l}\tan\theta,\quad \dot{\theta}=u$$

Here, x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle, u is the control. The task is to steer the car from a given initial position to the trajectory $y=g(x)$, while $g(x)$ and y are measured in real time. Define $\sigma=y-g(x)$. Let $v=\text{const}=10\,\text{m/s}, l=5\,\text{m}, g(x)=10\sin(0.05x)+5, x=y=\varphi=\theta=0$ at $t=0$.

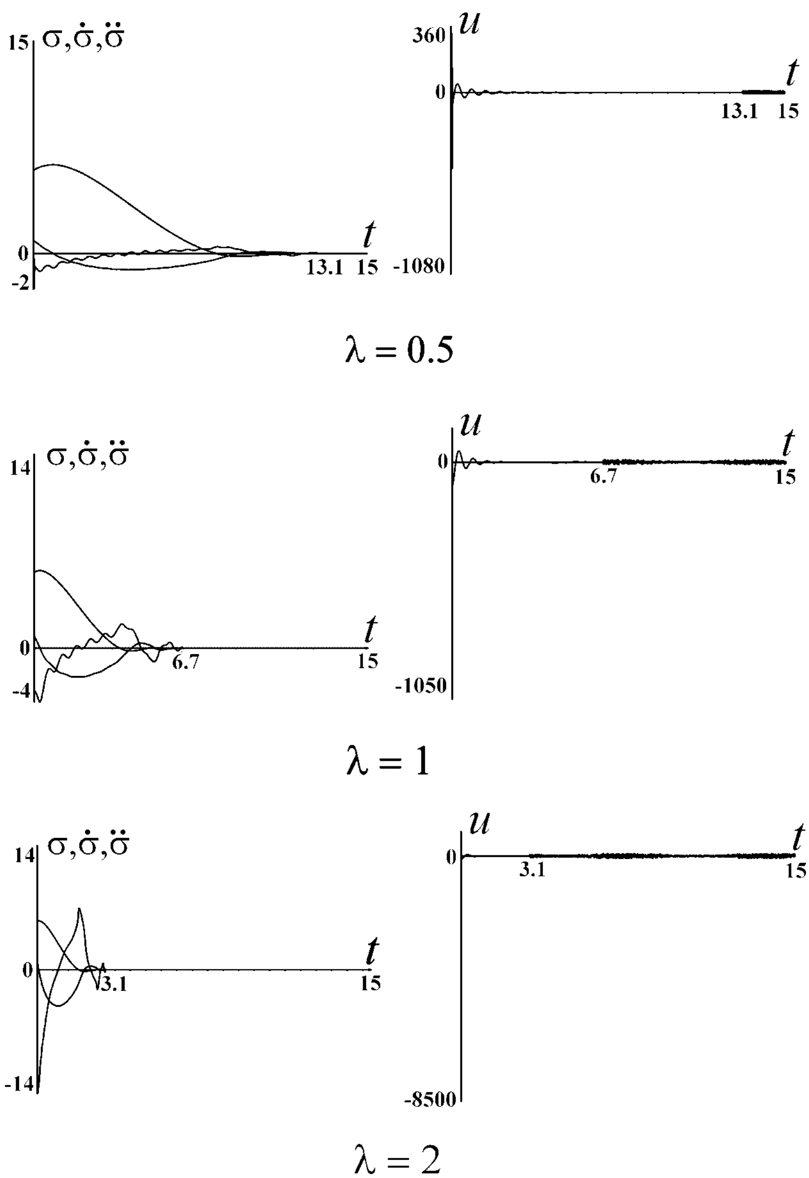

Figure 2. Adjustment of the quasi-continuous controller with $\Phi = (e^{x_2} + x_1^2)/(x_2^2 + 1) + 1$.

It is well known that human driving skills do not depend significantly on the concrete car parameters. The HOSM approach demonstrates the same features. The relative degree of the system is three. Apply the 3-sliding quasi-continuous controller

$$u = -0.5\zeta[\ddot{\sigma} + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{-1/2}(\dot{\sigma} + |\sigma|^{2/3}\operatorname{sign}\sigma)]/[|\ddot{\sigma}| + 2(|\dot{\sigma}|) + |\sigma|^{2/3})^{1/2}]$$
$$\zeta = \min[1, (|\sigma| + |\dot{\sigma}| + |\ddot{\sigma}|)^2]$$

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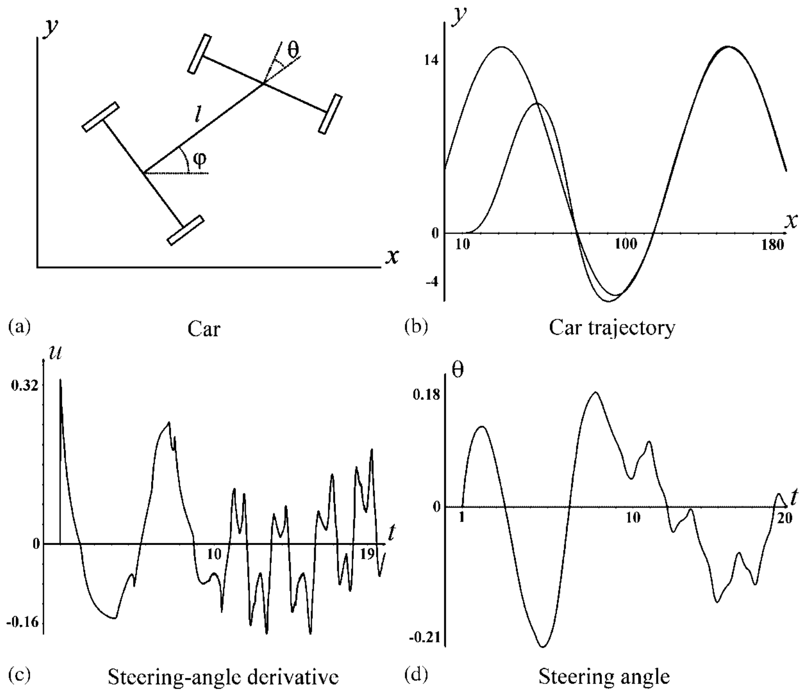

Figure 3. Regularization of three-sliding quasi-continuous car control: (a) car; (b) car trajectory; (c) steering-angle derivative; and (d) steering angle.

Only the control gain is to be chosen here and is taken as 0.5. The resulting controller is continuous, since $\zeta(0)=0$, but exactly keeping $\sigma=0$ is lost. The values of $\sigma, \dot{\sigma}, \ddot{\sigma}$ were replaced by their estimations obtained by the differentiator [8], taken exactly as in [9]. The control was applied from the moment $t=1$. The resulting performance is shown in Figure 3. Obviously, the chattering was removed. The accuracies $|\sigma|\leqslant 0.21, |\dot{\sigma}|\leqslant 0.32, |\ddot{\sigma}|\leqslant 0.97$ were obtained. The performance did not change, when noises were introduced in the measurements of σ not exceeding 0.1 m in the absolute value. Frequency properties of the noises did not show any significance.

7. CONCLUSIONS

This paper is summarized as follows. The controller parameters can be easily adjusted providing for the desired convergence rate. The discontinuity magnitude can change with respect to the uncertainty bounds, and it can even vanish in a small vicinity of the goal mode without significant accuracy loss.

Both main types of HOSM controllers allow functional magnitude gains of very general form (Theorem 1). The convergence rate is not much influenced by the functional gain. It is defined mostly by other controller parameters, which can be adjusted providing for the faster or slower convergence (Propositions 1, 2). Thus, having one valid parameter set, one obtains a whole family of parameter sets with different convergence rates. The main method of building such basic

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parameters' sets remains the computer simulation. It is sufficient to carry out such simulation for the lower-order differential equation not containing uncertainty (Theorem 2).

A list of quasi-continuous controllers is presented in Section 4 with relative degrees less or equal to four, and simulation-tested combinations of parameters, which are valid with any sufficiently large gain function. Since in the most practically important problems of output regulation, the relative degree r does not exceed four, this list constitutes a base for easy application of HOSM controllers.

It is known that the chattering effect can be diminished, provided the relative degree is artificially increased, causing smooth control to be obtained [7–19]. Unfortunately, more information or higher-order differentiation is needed in that case, inevitably increasing the system sensitivity with respect to the measurement noises. A regularization procedure is proposed in this paper generalizing the traditional procedure [8] to the HOSM. It removes the control discontinuity in a small vicinity of the set with zero successive output derivatives. That procedure reduces the chattering, while still preserving good accuracy (Theorems 3 and 4).

High-order real-time exact differentiators [8] are known to provide for the output-feedback control of the SISO systems with *bounded* uncertainties. Unfortunately, their application needs the boundedness of $\sigma^{(r)}$, which is true only locally or within the problem setting (1), (4). A differentiator with a known functional bound of $\sigma^{(r)}$ was recently reported [24] and can probably be implemented here.

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