Transient adjustment of high-order sliding modes

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Abstract— **Any transient dynamics can be prescribed to high-order sliding-modes. The resulting controller is capable to control the output of any smooth uncertain SISO system of a known permanent relative degree and is robust with respect to measurement errors. The control smoothness can be deliberately increased without loss of convergence completely removing the chattering effect.**

I. Introduction

CONTROL under heavy uncertainty conditions remains one of the main subjects of the modern control theory.

While a number of advanced methods like adaptation, absolute stability methods or the back-stepping procedure are based on relatively detailed knowledge of the controlled system, the sliding-mode control approach requirements are more moderate. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Sliding-mode implementation is based on its insensitivity to external and internal disturbances and high accuracy [3], [16]. The main drawback of the standard sliding modes is mostly related to the so-called chattering effect caused by the high-frequency control switching [4].

Let the constraint be given by the equation $\sigma = s - w(t) = 0$, where s is some available output variable of an uncertain single-input-single-output (SISO) dynamic system and $w(t)$ is an unknown-in-advance smooth input to be tracked in real time. Then the standard sliding-mode control $u = -k \operatorname{sign} \sigma$ may be considered as a universal output controller applicable if the relative degree is 1, i.e. if $\dot\sigma$ explicitly depends on the control u and $\dot\sigma'_u > 0$.

Higher-order sliding mode [1], [9], [12] is applicable to control SISO uncertain systems with arbitrary relative degree r. The correspondent finite-time-convergent controllers (r-sliding controllers) [9], [1], [12]-[15] require actually only the knowledge of the system relative degree. The produced control is a discontinuous function of the tracking deviation σ and of its real-time-calculated successive derivatives $\dot\sigma$, $\ddot\sigma$, ..., $\sigma^{(r-1)}$. It establishes in finite time and keeps the equalities $\sigma = \dot\sigma = ... = \sigma^{(r)} = 0$ (the r-sliding mode). The lacking derivatives can be produced by recently proposed robust exact finite-time convergent differentiators [10], [2], [7], [12], [16] generating output-feedback controllers [12]-[14]. The approach provides also for higher accuracy with discrete sampling and, properly used, totally removes the chattering effect. In order to remove the chattering, the control derivative is to be treated as a new control.

Some realization problems of high-order sliding modes are due to the complicated structure of the transient process which is difficult to monitor. Indeed, while the number of 2-sliding controllers is easily increased, providing for needed transient features [11], higher-order sliding mode design is very complicated, and the number of known controllers is very small. The difficulty comes of the high dimension of the problem. Actually, the only design idea is to decrease gradually the dimension applying some induction based logic [12], [13].

A specific problem concerns the artificial increase of the relative degree which is needed to eliminate the chattering effect. The main idea of the r-sliding control is to depress the influence of the system dynamics on $\sigma^{(r)}$ by means of the sufficiently powerful control u. When the control derivative $\dot u$ is considered as a new control, $\dot u$ has to dominate in the equation for $\sigma^{(r+1)}$. Regretfully, in general, the expression for $\sigma^{(r+1)}$ contains terms with u. Thus, $\dot u$ is to dominate over u itself, which looks like the prominent trick by Baron Munchausen. Fortunately, in the vicinity of the $(r + 1)$-sliding mode u is close to the so-called equivalent control $u_{eq}(t, x)$ [17] which is independent of $\dot u$. Hence, any standard $(r + 1)$-sliding controller will work in some vicinity of the $(r + 1)$-sliding mode $\sigma = \dot\sigma = ... = \sigma^{(r)} = 0$. That means that the initial value of $u - u_{eq}(t, x)$ is to be small enough. The global convergence is so far provided only for the transfer from $r = 1$ to $r = 2$ by a suitable controller modification [9].

The above issues could be resolved by exclusion of the transient process. The correspondent technique is developed for the standard (first order) sliding mode and is called integral sliding mode [18], [19]. It has found a lot of successful applications [19]. The idea is to construct such a

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smooth function $\Sigma(t, x)$ which equals zero at the initial point and within a finite time transforms into $\sigma(t, x)$. Thus, keeping $\Sigma(t, x) \equiv 0$ in sliding mode solves the original problem. The convergence of Σ to σ is chosen so as to provide for the needed transient features.

On the contrary to the standard integral-sliding-mode approach its r-sliding generalization proposed in the paper introduces the function Σ of the relative degree r and the equality $\Sigma(t, x) \equiv 0$ is to be kept in the r-sliding mode. In particular, with the chattering removal procedure u is automatically kept at u_{eq}, and the convergence is assured. Simulation demonstrates the practical applicability of the proposed scheme.

II. THE PROBLEM STATEMENT

Consider a smooth dynamic system with a smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback, differential equations being understood in the Filippov sense [4]. Then, provided that the successive total time derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables; and the set $\sigma = ... = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is called r-sliding (rth order sliding) mode [9], [12].

The standard sliding mode used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

Here $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \rightarrow \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$, n is also uncertain. The relative degree r of the system is assumed to be constant and known. Extend system (2) by introduction of a fictitious variable $x_{n+1} = t$, $\dot{x}_{n+1} = 1$. Denote $a_e = (a,1)^t$, $b_e = (b,0)^t$, where the last component corresponds to x_{n+1}. The equality of the relative degree to r means that the Lie derivatives $L_{b_e}\sigma$, $L_{b_e}L_{a_e}\sigma$, ..., $L_{b_e}L_{a_e}^{r-2}\sigma$ equal zero identically in a vicinity of a given point and $L_{b_e}L_{a_e}^{r-1}\sigma$ is not zero at the point [6]. It is easy to check [6] that the control first time appears explicitly in the rth total time derivative of σ and

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{u=0} = L_{a_e}^{r}\sigma$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)} = L_{b_e}L_{a_e}^{r-1}\sigma$ are some unknown functions. It is supposed that

$$0 < K_{\mathrm{m}} \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_{\mathrm{M}}, \ |\sigma^{(r)}|_{u=0}| \leq C \tag{3}$$

for some K_{m}, K_{M}, $C > 0$. Note that conditions (3) are formulated in terms of input-output relations.

It is also assumed that trajectories of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$. In practice it means that the system be weakly minimum phase.

III. INTEGRAL R-SLIDING MODE

The above problem statement is standard and is solved by known r-sliding controllers [12], [13]. Suppose that there are some additional requirements to the transient process like some transient time restrictions or some additional smoothness of the entrance into the mode $\sigma \equiv 0$, i.e. $\sigma = \dot{\sigma} = ... = \sigma^{(k)} = 0$, $k \geq r$, at the entrance moment.

Let the above requirements be fulfilled if $\sigma(t, x(t)) = \varphi(t)$, which means, in particular, that

$$\varphi(t_0) = \sigma(t_0),\ \dot{\varphi}(t_0) = \dot{\sigma}(t_0), ..., \varphi^{(r-1)}(t_0) = \sigma^{(r-1)}(t_0) \tag{4}$$

at the initial moment t_0 and

$$\varphi(t) = 0 \text{ with } t \geq t_f. \tag{5}$$

after the entrance moment $t_f > t_0$. Here and further, for the sake of brevity, $\sigma(t)$ is written instead of $\sigma(t, x(t))$ whenever the ambiguity is avoided.

Let $\varphi^{(r-1)}(t)$ be a Lipshitz function, then almost everywhere it has a globally bounded derivative $\varphi^{(r)}(t)$, and the function $\Sigma(t, x) = \sigma(t, x) - \varphi(t)$ satisfies conditions (2), (3) with some changed constants K_{m}, K_{M}, $C > 0$. Thus $\Sigma(t, x) \equiv 0$ can be kept by any known r-sliding controller. That solves the considered problem.

Integral r-sliding mode. Reformulate the above reasoning in the integral form. Consider the auxiliary dynamic system

$$z^{(r)} = v, \tag{6}$$

with the initial state

$$z(t_0) = \sigma(t_0),\ \dot{z}(t_0) = \dot{\sigma}(t_0), ..., z^{(r-1)}(t_0) = \sigma^{(r-1)}(t_0) \tag{7}$$

and define the auxiliary function v and the new constraint function Σ as follows

$$v(t) = \varphi^{(r)}(t), \quad \Sigma(t, x, z) = \begin{cases} \sigma(t,x) - z, & t_0 \leq t \leq t_f \\ \sigma(t,x), & t \geq t_f \end{cases}. \tag{8}$$

Let the bounded feedback control

$$u = \alpha\Psi_r(\Sigma, \dot{\Sigma}, ..., \Sigma^{(r-1)}) \tag{9}$$

be any finite-time convergent r-sliding controller [12], [13], $\alpha > 0$ being the only parameter to be adjusted. The following Theorem is obvious.

Theorem 1. *Let the function* φ *satisfy* (4), (5), *then with sufficiently large* α *the controller* (6) - (9) *provides for the r-sliding mode* $\sigma \equiv 0$ *starting from the moment* t_f. *During the transient process* $\sigma(t, x(t)) = \varphi(t)$.

Note that during the transient the system maintains the open-loop dynamics $\sigma^{(r)} = \varphi^{(r)}(t)$. In the case of noisy measurements the equality $\Sigma = 0$ is only approximately kept from the very beginning, and the trajectory will miss the r-sliding manifold $\sigma = ... = \sigma^{(r-1)} = 0$. Hence, the controller is robust with respect to noisy measurements, but some additional fast finite-time transient will take place after the time t_f [14]. The derivatives can be calculated by means of the robust exact differentiators [10], [7], [12] with finite-time convergence.

Model-tracking integral r-sliding mode. Consider v as some auxiliary control, $|v| \le V_M$. Let

$$v = V(z, \dot{z}, ..., z^{(r-1)}), \qquad (10)$$

be any controller providing for the global finite-time stability of (6) at the origin.

Theorem 2. *With sufficiently large* α *and* $\Sigma = \sigma - z$ *controller* (6), (7), (10) *keeps* $\sigma(t, x(t)) = z(t)$ *and provides for the globally finite-time stable r-sliding mode* $\sigma \equiv 0$, $z \equiv 0$.

The resulting controller features the standard maximal r-sliding accuracy [9], [12]. In particular, with discrete measurements with the step τ, in the absence of noises it provides for the inequalities $|\sigma^{(i)}| \le \mu_i \tau^{r-i}$, $i = 0, ..., r-1$, with some positive constants μ_i.

Real-time estimations of $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ being provided by the finite-time convergent robust differentiator [12], a robust output-feedback controller is produced with the same features. With the sampling noise of the magnitude ε the inequalities of the form $|\sigma^{(i)}| \le \mu_i \varepsilon^{(r-i)/r}$ are assured.

IV. Example of Integral *R*-Sliding Mode Design

One of the natural ways to choose a smooth function $\varphi(t)$ satisfying (4), (5) is to choose a control for (6), (7) which connects the initial point with the origin by a trajectory optimal in some sense [8] and to take $\varphi(t) = z(t)$. Note that due to the uncertainty of the original system the optimality will take place only for the auxiliary dynamic system (6).

Consider another rather simplistic way to demonstrate the approach. Any (r-1)-smooth function $\varphi(t)$ satisfying (5) can be represented in the form

$$\varphi(t) = (t - t_f)^r f(t) \qquad (11)$$

where $f(t)$ is an (r-1)-smooth function. Let $f(t)$ have the polynomial form

$$f(t) = c_0 + c_1(t - t_0) + ...+ c_{r-1}(t - t_0)^{r-1}. \qquad (12)$$

Obviously, any constant value of the transient time $t_f - t_0$ requires unacceptably large control values in order to steer the trajectory to the r-sliding mode from far distanced initial values. To avoid it require that $t_f - t_0$ be a homogeneous function of the initial conditions $\vec{\sigma}(t_0) = (\sigma(t_0), \dot{\sigma}(t_0), ..., \sigma^{(r-1)}(t_0))$ of the degree 1 with the homogeneity weights of σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ being r, r - 1, ..., 1 respectively (r-sliding homogeneity [11], [14]). Any homogeneous continuous positively-definite function of the degree 1 will fit. In particular, let

$$t_f - t_0 = \lambda\, (|\sigma(t_0)|^{p/r} + |\dot{\sigma}(t_0)|^{p/(r-1)} + ... + |\sigma^{(r-1)}(t_0)|^{p})^{1/p}, \qquad (13)$$

where p is the least common multiple of 1, 2, ..., r, and $\lambda > 0$.

Theorem 3. *Conditions* (4), (11) - (13) *define a unique function* $\varphi(t, \vec{\sigma}(t_0))$. *With sufficiently large* α *the controller* (8), (9) *establishes the r-sliding mode* $\sigma \equiv 0$ *with the transient time* (13). *The equality* $\sigma(t, x(t)) = \varphi(t, \vec{\sigma}(t_0))$ *is kept during the transient process.*

Thus, the parameter α does not depend on initial conditions. The proof follows from the following Lemma.

Lemma 4. *Conditions* (4), (11) - (13) *define a unique function* $\varphi(t, \vec{\sigma}(t_0))$ *whose rth derivative* $\varphi^{(r)}$ *is bounded within the segment* $t_0 \le t \le t_f(\vec{\sigma}(t_0))$ *by a constant which is independent of the initial conditions* $\sigma(t_0)$, $\dot{\sigma}(t_0)$, ..., $\sigma^{(r-1)}(t_0)$.

Proof of the Lemma. Differentiating (11) obtain

$$\varphi^{(i)} = \sum_{j=0}^{i} \binom{j}{i} \frac{d^{i-j}}{dt^{i-j}} (t - t_f)^r f^{(j)}, \ i = 0, ..., r.$$

Let $T(\vec{\sigma}(t_0))$ be the right-hand side of (13), and $\vec{\sigma}(t_0) \neq 0$. Taking $t = t_0$ obtain from (4)

$$\sigma^{(i)}(t_0) = \sum_{j=0}^{i} \binom{j}{i} \frac{r!}{(r-i+j)!} (-T)^{r-i+j} f^{(j)}(t_0).$$

Thus, $F_i = f^{(i)}(t_0)$ can be found recursively as

$$F_0 = (-T)^{-r}\sigma(t_0),$$

$$F_i = (-T)^{-r}\left[\sigma^{(i)}(t_0) - \sum_{j=0}^{i-1} \binom{j}{i} \frac{r!}{(r-i+j)!} (-T)^{r-i+j} F_j\right].$$

Therefore, with $i = 0, ..., r-1$

$$f^{(i)}(t) = F_i + F_{i+1}\frac{t-t_0}{1!} + ... + F_{r-1}\frac{(t-t_0)^{r-i-1}}{(r-i-1)!},\ f^{(r)}(t) = 0;$$

$$\varphi^{(i)}(t) = \sum_{j=0}^{i} \binom{j}{i} \frac{r!}{(r-i+j)!} (-T)^{r-i+j} f^{(j)}(t), \ i = 0, ..., r.$$

The statement of the Lemma follows now from the simple inequalities

$$|\sigma^{(i)}(t_0)|/T^{r-i} \le 1/\lambda^{r-i}, \quad |t - t_0|/T \le 1, \quad |t - t_f|/T \le 1$$

kept with $t \in [t_0, t_f]$ and non-zero initial conditions. ■

The above calculations are easily performed by computer in real time at the moment t_0. When λ is multiplied by $\mu > 1$, the maximal value of $|\varphi^{(r)}|$ is reduced at least by μ^r times.

Chattering removal. The same construction removes chattering. Choose some integer $k > r$ and consider $u^{(k-r)}$ as a new control. The new relative degree is k. Introduce the function φ satisfying the conditions

$$\varphi(t_0) = \sigma(t_0),\ \dot{\varphi}(t_0) = \dot{\sigma}(t_0), ..., \varphi^{(k-1)}(t_0) = \sigma^{(k-1)}(t_0); \qquad (14)$$

$$\varphi(t) = (t - t_f)^k f(t); \qquad (15)$$

$$f(t) = c_0 + c_1(t - t_0) + ...+ c_{k-1}(t - t_0)^{k-1}; \qquad (16)$$

$$t_f - t_0 = \lambda\, (|\sigma(t_0)|^{p/k} + |\dot{\sigma}(t_0)|^{p/(k-1)} + ... + |\sigma^{(k-1)}(t_0)|^{p})^{1/p}, \qquad (17)$$

where p is the least common multiple of 1, 2, ..., k, and $\lambda > 0$. Let the new constraint function be

$$\Sigma = \begin{cases} \sigma(t,x) - \varphi(t), & t_0 \le t \le t_f \\ \sigma(t,x), & t > t_f \end{cases}, \tag{18}$$

and the bounded feedback control be defined by

$$u^{(k-r)} = \alpha\Psi_k(\Sigma, \dot{\Sigma}, ..., \Sigma^{(k-1)}), \tag{19}$$

with arbitrary initial values $u(t_0)$, ..., $u^{(k-r-1)}(t_0)$.

Define the smooth function $u_{eq}(t, x) = - h(t, x)/g(t, x)$ from (2) and the condition $\sigma^{(r)} = 0$. Denote by $\zeta(t, x)$ the *(k-r)*th total derivative of $u_{eq}(t, x)$ with respect to

$$\dot{x} = a(t,x) + b(t,x)\, u_{eq}(t, x).$$

Theorem 4. *Let the initial conditions* t_0, $x(t_0)$, $u(t_0)$, ..., $u^{(k-r-1)}(t_0)$ *belong to some compact set in* $\mathbf{R}^{n+k-r+1}$ *and* $\zeta(t, x)$ *be uniformly bounded. Then with sufficiently large* α *controller* (19) *establishes the k-sliding mode* $\sigma \equiv 0$ *with the transient time* (15). *The equality* $\sigma(t, x(t)) = \varphi(t, \vec{\sigma}(t_0))$ *is kept during the transient process. The controller is robust with respect to small measurement errors.*

Proof. Differentiating (2) obtain

$$\Sigma^{(k)} = \tilde{h}(t, x, u, ..., u^{(k-r-1)}) + g(t, x)\, u^{(k-r)} \tag{20}$$

where $\tilde{h}$ can be expressed through the Lie derivatives and is, therefore, a smooth function with the subtraction of the uniformly bounded function $\varphi^{(k)}(t, \vec{\sigma}(t_0))$ when $t_0 \le t \le t_f(\vec{\sigma}(t_0))$. The motions in the *k*-sliding mode $\Sigma \equiv 0$ are described by the replacement of the new control $\xi = u^{(k-r)}$ by its *equivalent* value

$$u^{(k-r)} = \xi_{eq} = - \tilde{h}(t, x, u, ..., u^{(k-r-1)})/g(t, x).$$

From the equation $\Sigma^{(r)} = 0$ obtain that the *k*-sliding motion $\Sigma = 0$ implies the equality $u = \tilde{u}_{eq}(t, x)$, where

$$\tilde{u}_{eq}(t, x) = \begin{cases} u_{eq}(t,x) + \varphi^{(r)}(t, \vec{\sigma}(t_0)), & t_0 \le t \le t_f(\vec{\sigma}(t_0)) \\ u_{eq}(t,x), & t > t_f(\vec{\sigma}(t_0)) \end{cases}.$$

Differentiating $k - r - 1$ times the equation $\Sigma^{(r)} = 0$ obtain that

$$\xi_{eq} = \tilde{u}_{eq}^{(k-r)}(t, x),$$

where $\tilde{u}_{eq}^{(k-r)}$ is the (*k-r*)th total derivative of $\tilde{u}_{eq}$ with respect to the equation

$$\dot{x} = a(t,x) + b(t,x)\, \tilde{u}_{eq}(t, x). \tag{21}$$

The points of the trajectories of

$$\dot{x} = a(t,x) + b(t,x)u, \qquad u^{(k-r)} = \tilde{u}_{eq}^{(k-r)}(t, x)$$

starting from the given compact set Ω of initial values with $\min_{\Omega} t_0 \le t \le \max_{\Omega} t_f(\vec{\sigma}(t_0))$ form a compact set Φ in the space $t, x, u, ..., u^{(k-r-1)}$ [4]. Let $\tilde{\Phi}$ be some compact set containing Φ in its interior, $\tilde{C} = K_M \max \{\sup |\zeta(t, x)|+1, \max_{\tilde{\Phi}} |\xi_{eq}|\}$. Then, due to (21), ineqalities $|\tilde{h}| \le \tilde{C}$ and $K_m \le g \le K_M$ hold on the trajectories of (1), (19), which provides for the local *k*-sliding convergence [14] with α large enough. ■

As follows from the Theorem, the problem of the interaction between the control and its derivatives is solved here. Indeed, due to the transient absence, control and its derivatives track the smooth function $\tilde{u}_{eq} = u_{eq}(t, x(t)) + \varphi^{(r)}(t)/g(t,x(t))$ and its successive total time derivatives calculated with respect to (21).

V. Simulation Example

Consider a variable-length pendulum control problem. Friction is assumed absent. All motions are restricted to some vertical plane. A load of known mass *m* is moving along the pendulum rod (Fig. 1). Its distance from *O* equals $R(t)$ and is not measured. An engine transmits a torque *w* which is considered as control. The task is to track some function x_c given in real time by the angular coordinate *x* of the rod.

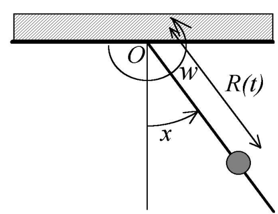

Fig. 1: Illustrative example

The system is described by the equation

$$\ddot{x} = -2\, \frac{\dot{R}}{R}\, \dot{x} - g\frac{1}{R} \sin x + \frac{1}{mR^2}\, w, \tag{20}$$

where $g = 9.81$ is the gravitational constant, $m = 1$ was taken. Let $0 < R_m \le R \le R_M$, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, $\sigma = x - x_c$ be available. The initial conditions are $x(0) = \dot{x}(0) = 0$. The relative degree of the system is 2, but the chattering should obviously be avoided here. Extend the system, artificially increasing its relative degree to 4:

$$\ddot{w} = u. \tag{21}$$

Let $w(0) = \dot{w}(0) = 0$. Since $\ddot{\sigma}|_{u=0}$ linearly depends on $\dot{x}$, it is not uniformly bounded. Nevertheless all requirements of the Theorems are satisfied in any bounded vicinity of the origin $x = \dot{x} = 0$, which provides for the local application of the method. Note that the values of w, $\dot{w}$ do not matter. Following are the functions R and x_c considered in the simulation:

$$R = 1 + 0.25 \sin 4t + 0.5 \cos t,$$
$$x_c = 0.5 \sin 0.5t + 0.5 \cos t.$$

While parameters of the controllers demonstrated further may be evaluated with respect to the above-mentioned restrictions on unknown functions $R(t)$, $x_c(t)$, their derivatives and some chosen bound on $\dot{x}$, they are usually excessively large in this case. The better way is to tune the parameters during simulation. Surely, the controlled class is somewhat smaller, but it still allows significant disturbances of the considered realizations of R and x_c.

The transient dynamics is chosen according to (14) - (17):

$$\varphi(t_0) = s_0(t_0),\ \dot{\varphi}(t_0) = s_1(t_0),\ \ddot{\varphi}(t_0) = s_2(t_0),\ \dddot{\varphi}(t_0) = s_3(t_0);$$

$$\varphi(t) = (t - t_f)^4 (c_0 + c_1(t - t_0) + \ldots + c_3(t - t_0)^3),$$

$$T = t_f - t_0 = \lambda\, (|s_0(t_0)|^3 + s_1(t_0)^4 + |s_2(t_0)|^6 + s_3(t_0)^{12})^{1/12},$$

$$c_0 = s(t_0)\, T^{-4},\quad c_1 = s_1(t_0)\, T^{-4} + 4\, s(t_0)\, T^{-5},$$

$$c_2 = [s_2(t_0)\, T^{-4} + 8\, s_1(t_0)\, T^{-5} + 20\, s(t_0)\, T^{-6}]\,/2,$$

$$c_3 = [s_3(t_0) T^{-4} + 12\, s_2(t_0) T^{-5} + 60\, s_1(t_0)\, T^{-6} + 120\, s(t_0) T^{-7}]\,/6,$$

The output-feedback controller takes now the form

$$\xi(t) = \begin{cases} 0 & \text{with } t \notin [t_0, t_f] \\ \varphi(t) & \text{with } t \in [t_0, t_f] \end{cases};$$

$$u = \begin{cases} 0 & \text{with } t < t_0 \\ \alpha\Psi_4(s_0 - \xi, s_1 - \dot{\xi}, s_2 - \ddot{\xi}, s_3 - \dddot{\xi}) & \text{with } t \in [t_0, t_f] \end{cases},$$

where s_0, s_1, s_2, s_3 are the outputs of the 3rd-order differentiator estimating σ, $\dot{\sigma}$, $\ddot{\sigma}$, $\dddot{\sigma}$ respectively:

$$\dot{s}_0 = v_0,\ \ v_0 = -\,3\, L^{1/4}\, |s_0 - \sigma|^{3/4}\, \mathrm{sign}(s_0 - \sigma) + s_1,$$

$$\dot{s}_1 = v_1,\quad v_1 = -\,2\, L^{1/3}\, |s_1 - v_0|^{2/3}\, \mathrm{sign}(s_1 - v_0) + s_2,$$

$$\dot{s}_2 = v_2,\quad v_2 = -\,1.5\, L^{1/2}\, |s_2 - v_1|^{1/2}\, \mathrm{sign}(s_2 - v_1) + s_3,$$

$$\dot{s}_3 = -\,1.1\, L\, \mathrm{sign}(s_3 - v_2).$$

Here L is to be larger than $\sup|\sigma^{(4)}|$, which exists due to Theorem 4. The time t_0 is needed to ensure that the differentiator has already converged. The initial values of the differentiator are taken $s_0(0) = \sigma(0)$, $s_1(0) = s_2(0) = s_3(0) = 0$.

Two controllers were considered ($z_i = s_i - \xi^{(i)}$): the "standard" controller [12], [14]

$$\Psi_4(z_0, z_1, z_2, z_3) = -\,\mathrm{sign}\{z_3 + 3\,(z_2^6 + z_1^4 + |z_0|^3)^{1/12}\,\mathrm{sign}[z_2 + (z_1^4 + |z_0|^3)^{1/6}\,\mathrm{sign}(z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0)]\}$$

and the "quasi-continuous" controller [13]

$$H_1 = z_3 + 3[|z_2| + (|z_1| + 0.5|z_0|^{3/4})^{-1/3}|z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0|]^{-1/2}\,[z_2 + (|z_1| + 0.5|z_0|^{3/4})^{-1/3}(z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0)],$$

$$H_2 = |z_3| + 3[|z_2| + (|z_1| + 0.5|z_0|^{3/4})^{-1/3}|z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0|]^{-1/2}\,|z_2 + (|z_1| + 0.5|z_0|^{3/4})^{-1/3}(z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0)|,$$

$$\Psi_4(z_0, z_1, z_2, z_3) = -\,H_1 / H_2\,.$$

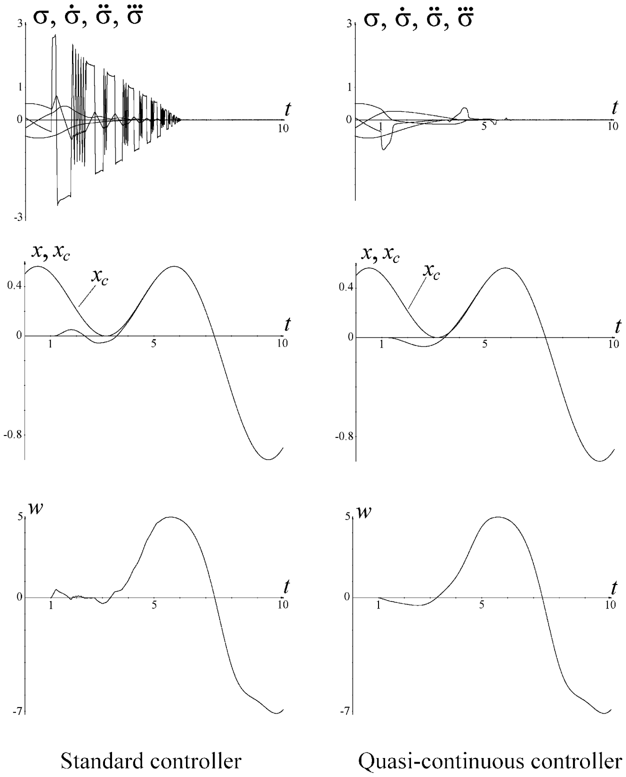

Fig. 2: Straight-forward 4-sliding pendulum control

The parameters which are still lacking are taken as follows: $\alpha = 70$, $L = 150$, $t_0 = 1$. The convergence-time parameter λ takes on values 2, 4, 6.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-5}$. The 4-sliding deviations corresponding to the standard and quasi-continuous controllers ($\xi = 0$) are demonstrated in Fig. 2. The deviations corresponding to the adjusted transient are shown in Fig. 3. The graphs practically do not depend on the choice of algorithm, only the parameter λ matters, and the sliding accuracy varies.

In the absence of output noises the tracking accuracies $|\sigma| \le 6.8{\cdot}10^{-13}$, $|\dot{\sigma}| \le 5.8{\cdot}10^{-10}$, $|\ddot{\sigma}| \le 1.9{\cdot}10^{-6}$, $|\dddot{\sigma}| \le 0.013$ were attained after application of the quasi-sliding controller with $\lambda = 2$ (Fig. 3) and $\tau = 10^{-5}$. Similar accuracies were obtained in all cases. The standard controller is slightly less precise. In particular, the tracking accuracies $|\sigma| \le 2.0{\cdot}10^{-12}$, $|\dot{\sigma}| \le 1.3{\cdot}10^{-9}$, $|\ddot{\sigma}| \le 2.3{\cdot}10^{-6}$, $|\dddot{\sigma}| \le 0.016$ were attained after application of the standard controller with $\lambda = 2$ and $\tau = 10^{-5}$.

The corresponding tracking performance, the torque and the differentiator convergence are shown for $\lambda = 2$ (Fig. 3). It is seen from Fig. 3 that the embedded third-order differentiator provides for exact estimations of output derivatives.

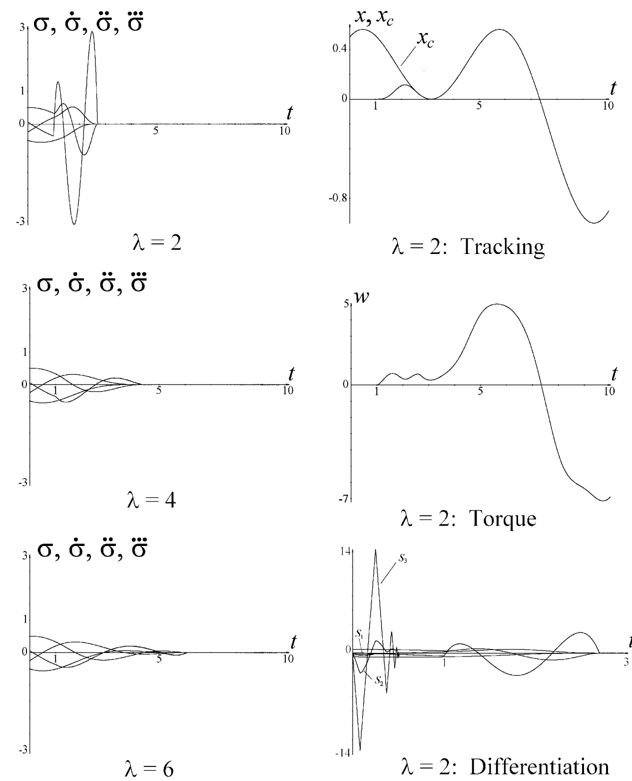

Fig. 3: Transient adjustment

VI. CONCLUSIONS

The integral sliding mode approach allows to prescribe the needed transient dynamics to high-order sliding-mode systems. In particular, any positive definite r-sliding homogeneous function of initial conditions can be realized as the settling-time function.

The same approach allows to solve the long-lasting problem of the control interaction in the chattering removal based on the artificial relative degree increase. The control smoothness can be deliberately increased without loss of convergence completely removing the chattering effect.

The resulting controller is capable to control the output of any smooth uncertain SISO system of a known permanent relative degree r and is robust with respect to measurement errors. A robust output-feedback controller is obtained when combined with recently proposed robust exact differentiator of the order r - 1.

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