Chapter 3

HIGHER ORDER SLIDING MODES

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3.1 Introduction

One of the most important control problems is control under heavy uncertainty conditions. While there are a number of sophisticated methods like adaptation based on identification and observation, or absolute stability methods, the most obvious way to withstand the uncertainty is to keep some constraints by "brutal force". Indeed any strictly kept equality removes one "uncertainty dimension". The most simple way to keep a constraint is to react immediately to any deviation of the system stirring it back to the constraint by a sufficiently energetic effort. Implemented directly, the approach leads to so-called sliding modes, which became main operation modes in the variable structure systems (VSS) [55]. Having proved their high accuracy and robustness with respect to various internal and external disturbances, they also reveal their main drawback: the so-called chattering effect, i.e. dangerous high-frequency vibrations of the controlled system. Such an effect was considered as an obvious intrinsic feature of the very idea of immediate powerful reaction to a minutest deviation from the chosen constraint. Another important feature is proportionality of the maximal deviation from the constraint to the time interval between the measurements (or to the switching delay).

To avoid chattering some approaches were proposed [15, 51]. The main idea was to change the dynamics in a small vicinity of the discontinuity surface in order to avoid real discontinuity and at the same time to preserve the main properties of the whole system. However, the ultimate accuracy and robustness of the sliding mode were partially lost. Recently invented higher order sliding modes (HOSM) generalize the basic sliding mode idea acting on the higher order time derivatives of the system deviation from the constraint instead of influencing the first deviation derivative like it happens in standard sliding modes. Keeping the main advantages of the original approach, at the same time they totally remove the chattering effect and provide for even higher accuracy in realization. A number of such controllers were described in the literature [16, 34, 35, 38, 3, 5].

HOSM is actually a movement on a discontinuity set of a dynamic system understood in Filippov's sense [22]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. If the task is to provide for keeping a constraint given by equality of a smooth function s to zero, the sliding order is a number of continuous total derivatives of s (including the zero one) in the vicinity of the sliding mode. Hence, the rth order sliding mode is determined by the equalities

$$s = \dot{s} = \ddot{s} = ... = s^{(r-1)} = 0. \tag{3.1}$$

forming an r-dimensional condition on the state of the dynamic system. The words "rth order sliding" are often abridged to "r-sliding".

The standard sliding mode on which most variable structure systems (VSS) are based is of the first order ($\dot{s}$ is discontinuous). While the standard modes feature finite time convergence, convergence to HOSM may be asymptotic as well. r-sliding mode realization can provide for up to the rth order of sliding precision with respect to the measurement interval [35, 38, 41]. In that sense r-sliding modes play the same role in sliding mode control theory as Runge - Kutta methods in numerical integration. Note that such utmost accuracy is observed only for HOSM with finite-time convergence.

Trivial cases of asymptotically stable HOSM are easily found in many classic VSSs. For example there is an asymptotically stable 2-sliding mode with respect to the constraint $x = 0$ at the origin $x = \dot{x} = 0$ (at the one point only) of a 2-dimensional VSS keeping the constraint $x + \dot{x} = 0$ in a standard

1-sliding mode. Asymptotically stable or unstable HOSMs inevitably appear in VSSs with fast actuators [23, 25, 26, 27, 30]. Stable HOSM reveals itself in that case by spontaneous disappearance of the chattering effect. Thus, examples of asymptotically stable or unstable sliding modes of any order are well known [16, 14, 50, 35, 30]. On the contrary, examples of r-sliding modes attracting in finite time are known for $r = 1$ (which is trivial), for $r = 2$ [34, 16, 17, 35, 4, 5] and for $r = 3$ [30]. Arbitrary order sliding controllers with finite-time convergence were only recently presented [38, 41]. Any new type of higher-order sliding controller with finite-time convergence is unique and requires thorough investigation.

The main problem in implementation of HOSMs is increasing information demand. Generally speaking, any r-sliding controller keeping $s = 0$ needs $s, \dot{s}, ..., s^{(r-1)}$ to be available. The only known exclusion is a so-called "super-twisting" 2-sliding controller [35, 37], which needs only measurements of s. First differences of $s^{(r-2)}$ having been used, measurements of $s, \dot{s}, ..., s^{(r-2)}$ turned out to be sufficient, which solves the problem only partially. A recently published robust exact differentiator with finite-time convergence [37] allows that problem to be solved in the theoretical way. In practice, however, the differentiation error proves to be proportional to $\varepsilon^{(2^{-k})}$, where $k < r$ is the differentiation order and ε is the maximal measurement error of s. Yet the optimal one is proportional to $\varepsilon^{(r-k)/r}$ ($s^{(r)}$ is supposed to be discontinuous, but bounded [37]). Nevertheless, there is another way to approach HOSM.

It was mentioned above that r-sliding mode realization provides for up to the rth order of sliding precision with respect to the switching delay τ, but the opposite is also true [35]: keeping $|s| = O(\tau^r)$ implies $|s^{(i)}| = O(\tau^{r-i}), i = 0, 1, ..., r-1$, to be kept, if $s^{(r)}$ is bounded. Thus, keeping $|s| = O(\tau^r)$ corresponds to approximate r-sliding. An algorithm providing for fulfillment of such relation in finite time, independent on τ, is called rth order real-sliding algorithm [35]. Few second order real sliding algorithms [35, 52] differ from 2-sliding controllers with discrete measurements. Almost all rth order real sliding algorithms known to date require measurements of $s, \dot{s}, ..., s^{(r-2)}$ with $r > 2$. The only known exceptions are two real-sliding algorithms of the third order [7, 39] which require only measurements of s.

Definitions of higher order sliding modes (HOSM) and order of sliding are introduced in Section 3.2 and compared with other known control theory notions in Section 3.3. Stability of relay control systems with higher sliding orders is discussed in Section 3.4. The behavior of sliding mode systems with dynamic actuators is analyzed from the sliding-order viewpoint in Section 3.5.

A number of main 2–sliding controllers with finite_time convergence are listed in Section 3.6. A family of arbitrary-order sliding controllers with finite-time convergence is presented in Section 3.7. The main notions are illustrated by simulation results.

3.2 Definitions of higher order sliding modes

Regular sliding mode features few special properties. It is reached in finite time, which means that a number of trajectories meet at any sliding point. In other words, the shift operator along the phase trajectory exists, but is not invertible in time at any sliding point. Other important features are that the manifold of sliding motions has a nonzero codimension and that any sliding motion is performed on a system discontinuity surface and may be understood only as a limit of motions when switching imperfections vanish and switching frequency tends to infinity. Any generalization of the sliding mode notion has to inherit some of these properties.

Let us recall first what Filippov's solutions [21, 22] are of a discontinuous differential equation

$$\dot{x} = v(x),$$

where $x \in \mathbb{R}^n$, v is a locally bounded measurable (Lebesgue) vector function. In that case, the equation is replaced by an equivalent differential inclusion

$$\dot{x} \in \mathcal{V}(x).$$

In the particular case when the vector-field v is continuous almost everywhere, the set-valued function $\mathcal{V}(x)$ is the convex closure of the set of all possible limits of $v(y)$ as $y \to x$, while $\{y\}$ are continuity points of v. Any solution of the equation is defined as an absolutely continuous function $x(t)$, satisfying the differential inclusion almost everywhere.

The following Definitions are based on [34, 16, 17, 19, 35, 30]. Note that the word combinations "rth order sliding" and "r-sliding" are equivalent.

3.2.1 Sliding modes on manifolds

Let $\mathcal{S}$ be a smooth manifold. Set $\mathcal{S}$ itself is called the 1-sliding set with respect to $\mathcal{S}$. The 2-sliding set is defined as the set of points $x \in \mathcal{L}$, where $\mathcal{V}(x)$ lies entirely in tangential space $\mathbf{T}_x$ to manifold $\mathcal{S}$ at point x (Fig.3.1).

Definition 1 *It is said that there exists a first (or second) order sliding mode on manifold $\mathcal{S}$ in a vicinity of a first (or second) order sliding point x, if in this vicinity of point x the first (or second) order sliding set is an integral set, i.e. it consists of Filippov's sense trajectories.*

Let $\mathcal{S}_1 = \mathcal{S}$. Denote by $\mathcal{S}_2$ the set of 2-sliding points with respect to manifold $\mathcal{S}$. Assume that $\mathcal{S}_2$ may itself be considered as a sufficiently smooth manifold. Then the same construction may be considered with respect to $\mathcal{S}_2$. Denote by $\mathcal{S}_3$ the corresponding 2-sliding set with respect to $\mathcal{S}_2$. $\mathcal{S}_3$ is called *the* 3*-sliding set with respect to manifold* $\mathcal{S}$. Continuing the process, achieve sliding sets of any order.

Definition 2 *It is said that there exists an r-sliding mode on manifold $\mathcal{S}$ in a vicinity of an r-sliding point $x \in \mathcal{S}_r$, if in this vicinity of point x the r-sliding set $\mathcal{S}_r$ is an integral set, i.e. it consists of Filippov's sense trajectories.*

3.2.2 Sliding modes with respect to constraint functions

Let a constraint be given by an equation $s(x) = 0$, where $s : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth constraint function. It is also supposed that total time derivatives along the trajectories $s, \dot{s}, \ddot{s}, \ldots, s^{(r-1)}$ exist and are single-valued functions of x, which is not trivial for discontinuous dynamic systems. In other words, this means that discontinuity does not appear in the first $r-1$ total time derivatives of the constraint function s. Then the rth order sliding set is determined by the equalities

$$s = \dot{s} = \ddot{s} = \ldots = s^{(r-1)} = 0. \tag{3.2}$$

Here (3.2) is an r-dimensional condition on the state of the dynamic system.

Definition 3 *Let the r-sliding set* (3.2) *be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying* (3.2) *is called an r-sliding mode with respect to the constraint function s (Fig.3.1).*

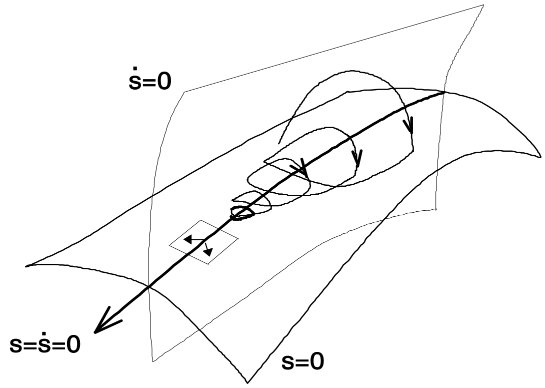

Figure 3.1: Second order sliding mode trajectory

To exhibit the relation with the previous Definitions, consider a manifold $\mathcal{S}$ given by the equation $s(x) = 0$. Suppose that $s, \dot{s}, \ddot{s}, \ldots, s^{(r-2)}$ are differentiable functions of x and that

$$\text{rank}\{\nabla s, \nabla \dot{s}, \ldots, \nabla s^{(r-2)}\} = r - 1 \tag{3.3}$$

holds locally (here rank $\mathcal{V}$ is a notation for the rank of vector set $\mathcal{V}$). Then $\mathcal{S}_r$ is determined by (3.2) and all $\mathcal{S}_i, i = 1, \ldots, r-1$ are smooth manifolds. If in its turn $\mathcal{S}_r$ is required to be a differentiable manifold, then the latter condition is extended to

$$\text{rank}\{\nabla s, \nabla \dot{s}, \ldots, \nabla s^{(r-1)}\} = r \tag{3.4}$$

Equality (3.4) together with the requirement for the corresponding derivatives of s to be differentiable functions of x will be referred to as *the sliding regularity condition*, whereas condition (3.3) will be called *the weak sliding regularity condition.*

With the weak regularity condition satisfied and $\mathcal{S}$ given by equation $s = 0$ Definition 3 is equivalent to Definition 2. If regularity condition (3.4) holds, then new local coordinates may be taken. In these coordinates the system will take the form

$$y_1 = s, \quad \dot{y}_1 = y_2; \quad \ldots; \quad \dot{y}_{r-1} = y_r;$$

$$s^{(r)} = \dot{y}_r = \Phi(y, \xi);$$

$$\dot{\xi} = \Psi(y, \xi), \quad \xi \in \mathbf{R}^{n-r}.$$

Proposition 1 *Let regularity condition* (3.4) *be fulfilled and* r*-sliding manifold* (3.2) *be non-empty. Then an* r*-sliding mode with respect to the constraint function* s *exists if and only if the intersection of the Filippov vector-set field with the tangential space to manifold* (3.2) *is not empty for any* r*-sliding point.*

Proof. The intersection of the Filippov set of admissible velocities with the tangential space to the sliding manifold (3.2), mentioned in the Proposition, induces a differential inclusion on this manifold. This inclusion satisfies all the conditions by Filippov [21, 22] for solution existence. Therefore manifold (3.2) is an integral one.□

Let now s be a smooth vector function, $s : \mathbb{R}^n \to \mathbb{R}^m, s = (s_1, \ldots, s_m)$, and also $r = (r_1, \ldots, r_m)$, where r_i are natural numbers.

Definition 4 *Assume that the first* r_i *successive full time derivatives of* s_i *are smooth functions, and a set given by the equalities*

$$s_i = \dot{s}_i = \ddot{s}_i = \ldots = s_i^{(r_i-1)} = 0, \quad i = 1, \ldots, m,$$

is locally an integral set in Filippov's sense. Then the motion mode existing on this set is called a sliding mode with vector sliding order r *with respect to the vector constraint function* s*.*

The corresponding sliding regularity condition has the form

$$\mathrm{rank}\{\nabla s_i, \ldots, \nabla s_i^{(r_i-1)} | i = 1, \ldots, m\} = r_1 + \ldots + r_m.$$

Definition 4 corresponds to Definition 2 in the case when $r_1 = \ldots = r_m$ and the appropriate weak regularity condition holds.

A sliding mode is called *stable* if the corresponding integral sliding set is stable.

Remarks

1. These definitions also include trivial cases of an integral manifold in a smooth system. To exclude them we may, for example, call a sliding mode "not trivial" if the corresponding Filippov set of admissible velocities $V(x)$ consists of more than one vector.
2. The above definitions are easily extended to include non-autonomous differential equations by introduction of the fictitious equation $\dot{t} = 1$. Note that this differs slightly from the Filippov definition considering time and space coordinates separately.

3.3 Higher order sliding modes in control systems

Single out two cases: *ideal sliding* occurring when the constraint is ideally kept and *real sliding* taking place when switching imperfections are taken into account and the constraint is kept only approximately.

3.3.1 Ideal sliding

All the previous considerations are translated literally to the case of a process controlled

$$\dot{x} = f(t, x, u), \quad s = s(t, x) \in \mathbb{R}, \ u = U(t, x) \in \mathbb{R},$$

where $x \in \mathbb{R}^n$, t is time, u is control, and f, s are smooth functions. Control u is determined here by a feedback $u = U(t, x)$, where U is a discontinuous function. For simplicity we restrict ourselves to the case when s and u are scalars. Nevertheless, all statements below may also be formulated for the case of vector sliding order.

Standard sliding modes satisfy the condition that the set of possible velocities V does not lie in tangential vector space T to the manifold $s = 0$, but intersects with it, and therefore a trajectory exists on the manifold with the velocity vector lying in T. Such modes are the main operation modes in variable structure systems [54, 55, 12, 57] and according to the above definitions they are of the first order. When a switching error is present the trajectory leaves the manifold at a certain angle. On the other hand, in the case of second order sliding all possible velocities lie in the tangential space to the manifold, and even when a switching error is present, the state trajectory is tangential to the manifold at the time of leaving.

To see connections with some well-known results of control theory, consider at first the case when

$$\dot{x} = a(x) + b(x)u, \quad s = s(x) \in \mathbb{R}, \ u \in \mathbb{R},$$

where a, b, s are smooth vector functions. Let the system have a relative degree r with respect to the output variable s [31] which means that Lie derivatives $L_b s, L_b L_a s, \ldots, L_b L_a^{r-2} s$ equal zero identically in a vicinity of a given point and $L_b L_a^{r-1} s$ is not zero at the point. The equality of the relative degree to r means, in a simplified way, that u first appears explicitly only in the rth total time derivative of s. It is known that in that case $s^{(i)} = L_a^i s$ for $i = 1, \ldots, r-1$, regularity condition (3.4) is satisfied automatically and also $\frac{\partial}{\partial u} s^{(r)} = L_b L_a^{r-1} s \neq 0$. There is a direct analogy between the relative degree notion and the sliding regularity condition. Loosely speaking, it may be said that the sliding regularity condition (3.4) means that the "relative degree with respect to discontinuity" is not less than r. Similarly, the rth order sliding mode notion is analogous to the zero-dynamics notion [31].

The relative degree notion was originally introduced for the autonomous case only. Nevertheless, we will apply this notion to the non-autonomous case as well. As was already done above, we introduce for the purpose a fictitious variable $x_{n+1} = t$, $\dot{x}_{n+1} = 1$. It has to be mentioned that some results by Isidori will not be correct in that case, but the facts listed in the previous paragraph will still be true.

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad s = s(t,x), \; u = U(t,x) \in \mathbb{R}.$$

Theorem 2 *Let the system have relative degree r with respect to the output function s at some r-sliding point (t_0, x_0). Let, also, the discontinuous function U take on values from sets $[K, \infty)$ and $(-\infty, -K]$ on some sets of non-zero measure in any vicinity of any r-sliding point near point (t_0, x_0). Then it provides, with sufficiently large K, for the existence of r-sliding mode in some vicinity of point (t_0, x_0). r-sliding motion satisfies the zero-dynamics equations.*

Proof. This Theorem is an immediate consequence of Proposition 1, nevertheless, we will detail the proof. Consider some new local coordinates $y = (y_1, \ldots, y_n)$, where $y_1 = s, y_2 = \dot{s}, \ldots, y_r = s^{(r-1)}$. In these coordinates manifold L_r is given by the equalities $y_1 = y_2 = \ldots = y_r = 0$ and the dynamics of the system is as follows:

$$\begin{aligned} &\dot{y}_1 = y_2, \quad \ldots, \quad \dot{y}_{r-1} = y_r, \\ &\dot{y}_r = h(t,y) + g(t,y)u, \quad g(t,y) \neq 0, \\ &\dot{\xi} = \Psi_1(t,y) + \Psi_2(t,y)u, \;\; \xi = (y_{r+1}, \ldots, y_n). \end{aligned} \tag{3.5}$$

Denote $U_{eq} = -h(t,y)/g(t,y)$. It is obvious that with initial conditions being on the r-th order sliding manifold $\mathcal{S}_r$ equivalent control $u = U_{eq}(t,y)$ provides for keeping the system within manifold $\mathcal{S}_r$. It is also easy to see that the substitution of all possible values from $[-K, K]$ for u gives us a subset of values from Filippov's set of the possible velocities. Let $|U_{eq}|$ be less than K_0, then with $K > K_0$ the substitution $u = U_{eq}$ determines a Filippov's solution of the discontinuous system which proves the Theorem.□

The trivial control algorithm $u = -K \operatorname{sign} s$ satisfies Theorem 2. Usually, however, such a mode will not be stable. It follows from the proof above that the equivalent control method [54] is applicable to r-sliding mode and produces equations coinciding with the zero-dynamics equations for the corresponding system.

The sliding mode order notion [11, 14] seems to be understood in a very close sense (the authors had no possibility to acquaint themselves with the work by Chang). A number of papers approach the higher order sliding mode technique in a very general way from the differential-algebraic point of view [48, 49, 50, 43]. In these papers so-called "dynamic sliding modes" are not distinguished from the algorithms generating them. Consider that approach.

Let the following equality be fulfilled identically as a consequence of the dynamic system equations [50]:

$$P(s^{(r)}, \ldots, \dot{s}, s, x, u^{(k)}, \ldots, \dot{u}, u) = 0. \qquad (3.6)$$

Equation (3.6) is supposed to be solvable with respect to $s^{(r)}$ and $u^{(k)}$. Function s may itself depend on u. The rth order sliding mode is considered as a steady state $s \equiv 0$ to be achieved by a controller satisfying (3.6). In order to achieve for s some stable dynamics

$$\sigma = s^{(r-1)} + a_1 s^{(r-2)} + \ldots + a_{r-1} s = 0$$

the discontinuous dynamics

$$\dot{\sigma} = -\operatorname{sign} \sigma \qquad (3.7)$$

is provided. For this purpose the corresponding value of $s^{(r)}$ is evaluated from (3.7) and substituted into (3.6). The obtained equation is solved for $u^{(k)}$.

Thus, a dynamic controller is constituted by the obtained differential equation for u which has a discontinuous right hand side. With this controller successive derivatives $s, \ldots, s^{(r-1)}$ will be smooth functions of closed system state space variables. The steady state of the resulting system will satisfy at least (3.2) and under some relevant conditions also the regularity requirement (3.4), and therefore Definition 3 will hold.

Hence, it may be said that the relation between our approach and the approach by Sira-Ramìrez is a classical relation between geometric and algebraic approaches in mathematics. Note that there are two different sliding modes in system (3.6), (3.7): a standard sliding mode of the first order which is kept on the manifold $\sigma = 0$, and an asymptotically stable r-sliding mode with respect to the constraint $s = 0$ which is kept in the points of the r-sliding manifold $s = \dot{s} = \ddot{s} = \ldots = s^{(r-1)} = 0$.

3.3.2 Real sliding and finite time convergence

Recall that the objective is synthesis of such a control u that the constraint $s(t, x) = 0$ holds. The quality of the control design is closely related to the sliding accuracy. In reality, no approaches to this design problem may provide for ideal keeping of the prescribed constraint. Therefore, there is a need to introduce some means in order to provide a capability for comparison of different controllers.

Any ideal sliding mode should be understood as a limit of motions when switching imperfections vanish and the switching frequency tends to infinity (Filippov [21, 22]). Let ε be some measure of these switching imperfections. Then sliding precision of any sliding mode technique may be featured by a sliding precision asymptotics with $\varepsilon \to 0$ [35]:

Definition 5 *Let $(t, x(t, \varepsilon))$ be a family of trajectories, indexed by $\varepsilon \in \mathbb{R}^\mu$, with common initial condition $(t_0, x(t_0))$, and let $t \geq t_0$ (or $t \in [t_0, T]$). Assume that there exists $t_1 \geq t_0$ (or $t_1 \in [t_0, T]$) such that on every segment $[t', t'']$, where $t' \geq t_1$, (or on $[t_1, T]$) the function $s(t, x(t, \varepsilon))$ tends uniformly to zero with ε tending to zero. In that case we call such a family a real-sliding family on the constraint $s = 0$. We call the motion on the interval $[t_0, t_1]$ a transient process, and the motion on the interval $[t_1, \infty)$ (or $[t_1, T]$) a steady state process.*

Definition 6 *A control algorithm, dependent on a parameter $\varepsilon \in \mathbb{R}^\mu$, is called a real-sliding algorithm on the constraint $s = 0$ if, with $\varepsilon \to 0$, it forms a real-sliding family for any initial condition.*

Definition 7 *Let $\gamma(\varepsilon)$ be a real-valued function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. A real-sliding algorithm on the constraint $s = 0$ is said to be of order r ($r > 0$) with respect to $\gamma(\varepsilon)$ if for any compact set of initial conditions and for any* *time interval $[T_1, T_2]$ there exists a constant C, such that the steady state process for $t \in [T_1, T_2]$ satisfies*

$$|s(t, x(t, \varepsilon))| \leq C|\gamma(\varepsilon)|^r.$$

In the particular case when $\gamma(\varepsilon)$ is the smallest time interval of control smoothness, the words "with respect to γ" may be omitted. This is the case when real sliding appears as a result of switching discretization.

As follows from [35], with the r-sliding regularity condition satisfied, in order to get the rth order of real sliding with discrete switching it is necessary to get at least the rth order in ideal sliding (provided by infinite switching frequency). Thus, the real sliding order does not exceed the corresponding sliding mode order. The standard sliding modes provide, therefore, for the first order real sliding only. The second order of real sliding was really achieved by discrete switching modifications of the second order sliding algorithms [34, 16, 17, 18, 19, 35]. Any arbitrary order of real sliding can be achieved by discretization of the same order sliding algorithms from [38, 39, 41](see section 3.7).

Real sliding may also be achieved in a way different from the discrete switching realization of sliding mode. For example, high gain feedback systems [47] constitute real sliding algorithms of the first order with respect to k^{-1}, where k is a large gain. A special discrete-switching algorithm providing for the second order real sliding were constructed in [52], another example of a second order real sliding controller is the drift algorithm [18, 35]. A third order real-sliding controller exploiting only measurements of s was recently presented [7].

It is true that in practice the final sliding accuracy is always achieved in finite time. Nevertheless, besides the pure theoretical interest there are also some practical reasons to search for sliding modes attracting in finite time. Consider a system with an r-sliding mode. Assume that with minimal switching interval τ the maximal r-th order of real sliding is provided. That means that the corresponding sliding precision $|s| \sim \tau^r$ is kept, if the r-th order sliding condition holds at the initial moment. Suppose that the r-sliding mode in the continuous switching system is asymptotically stable and does not attract the trajectories in finite time. It is reasonable to conclude in that case that with $\tau \to 0$ the transient process time for fixed general case initial conditions will tend to infinity. If, for example, the sliding mode were exponentially stable, the transient process time would be proportional to

$r\ln(\tau^{-1})$. Therefore, it is impossible to observe such an accuracy in practice, if the sliding mode is only asymptotically stable. At the same time, the time of the transient process will not change drastically if it was finite from the very beginning. It has to be mentioned, also, that the authors are not aware of a case when a higher real-sliding order is achieved with infinite-time convergence.

3.4 Higher order sliding stability in relay systems

In this section we present classical results by Tsypkin [53] (published in Russian in 1956) and Anosov (1959) [1]. They investigated the stability of relay control systems of the form

$$\begin{array}{l} \dot{y}_1 = y_2, \quad \ldots, \quad \dot{y}_{l-1} = y_l, \\ \dot{y}_l = \sum_{j=1}^{n} a_{l,j} y_j + k \operatorname{sign} y_1, \\ \dot{y}_i = \sum_{j=1}^{n} a_{i,j} y_j, \quad i = l+1, \ldots, n \end{array} \tag{3.8}$$

where $a_{i,j} = const$, $k \neq 0$, and $y_1 = y_2 = \ldots = y_l = 0$ is the lth order sliding set.

The main result is as follows:

- for stability of equilibrium point of relay control system (3.8) with second order sliding ($l = 2$) three main cases are singled out: exponentially stable, stable and unstable;
- it is shown that the equilibrium point of the system (3.8) is always unstable with $l \geq 3$. Consequently, all higher order sliding modes arriving in the relay control systems are unstable with order of sliding more than 2.

Consider the ideas of the proof.

3.4.1 2-sliding stability in relay systems

Consider a simple example of a second order dynamic system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = a y_1 + b y_2 + k \operatorname{sign} y_1. \tag{3.9}$$

The 2-sliding set is given here by $y_1 = y_2 = 0$. At first, let $k < 0$. Consider the Lyapunov function

$$E = \frac{y_2^2}{2} - a\frac{y_1^2}{2} + |k||y_1| - \frac{b}{2}y_1y_2. \tag{3.10}$$

Function E is an energy integral of system (3.9). Computing the derivative of function E achieve

$$\dot{E} = \frac{b}{2}y_2^2 + \frac{b}{2}|y_1|(|k| - a|y_1| - by_2 \operatorname{sign} y_1).$$

It is obvious that for some positive $\alpha_1 \le \alpha_2$, $\beta_1 \le \beta_2$

$$\alpha_1|y_1| + \beta_1 y_2^2 \le E \le \alpha_2|y_1| + \beta_2 y_2^2.$$

Thus, the inequalities $-\gamma_2 E \le \dot{E} \le -\gamma_1 E$ or $\gamma_1 E \le \dot{E} \le \gamma_2 E$ hold for $b < 0$ or $b > 0$ respectively, in a small vicinity of the origin with some $\gamma_2 \ge \gamma_1 > 0$.

Let now $k > 0$. It is easy to see in that case that trajectories cannot leave the set $y_1 > 0$, $y_2 = \dot{y}_1 > 0$ if $a \ge 0$. The same is true with $y_1 < k/|a|$ if $a < 0$. Starting with infinitisimally small $y_1 > 0$, $y_2 > 0$, any trajectory inevitably leaves some fixed origin vicinity.

It allows three main cases to be singled out for investigation of stability of system (3.9).

- *Exponentially stable case.* Under the conditions

$$b < 0, \quad k < 0 \tag{3.11}$$

  the equilibrium point $y_1 = y_2 = 0$ is exponentially stable.

- *Unstable case.* Under the condition

$$k > 0 \;\text{ or }\; b > 0$$

  the equilibrium point $y_1 = y_2 = 0$ is unstable.

- *Critical case.*

$$k \le 0, \quad b \le 0, \quad bk = 0.$$

  With $b = 0, k < 0$ the equilibrium point $y_1 = y_2 = 0$ is stable.

It is easy to show that if the matrix A consisting of $a_{i,j}$, $i, j > 2$ is Hurvitz and conditions $a_{2,2} < 0$, $k < 0$ are true, the equilibrium point of system (3.8) is exponentially stable.

3.4.2 Relay system instability with sliding order more than 2

Let us illustrate the idea of the proof on an example of a simple third order system

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \quad \dot{y}_3 = a_{31}y_1 + a_{32}y_2 + a_{33}y_3 - k\,\mathrm{sign}\, y_1, \; k > 0. \tag{3.12}$$

Consider the Lyapunov function[1]

$$V = y_1 y_3 - \frac{1}{2} y_2^2.$$

Thus,

$$\dot{V} = -k|y_1| + y_1(a_{31}y_1 + a_{32}y_2 + a_{33}y_3),$$

and $\dot{V}$ is negative at least in a small neighbourhood of origin $(0, 0, 0)$. That means that the zero solution of system (3.12) is unstable.

On the other hand, in relay control systems with order of sliding more than 2 a stable periodic solution can occur [46, 32].

3.5 Sliding order and dynamic actuators

Let the constraint be given by the equality of some constraint function s to zero and let the sliding mode $s \equiv 0$ be provided by a relay control. Taking into account an actuator conducting a control signal to the process controlled, we achieve more complicated dynamics. In that case the relay control u enters the actuator and continuous output variables of the actuator z are transmitted to the plant input (Fig. 3.2). As a result discontinuous switching is hidden now in the higher derivatives of the constraint function [55, 23, 24, 25, 26, 27, 9].

3.5.1 Stability of 2-sliding modes in systems with fast actuators

Condition (3.11) is used in [25, 26, 27, 9] for analysis of sliding mode systems with fast dynamic actuators. Here is a simple outline of these reasonings. One of the actuator output variables is formally replaced by $\dot{s}$ after application

[1]this function was suggested by V.I. Utkin in private communications

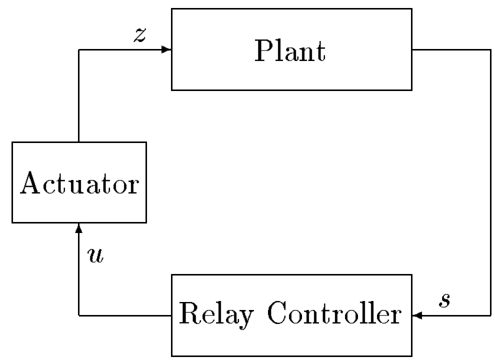

Figure 3.2: Control system with actuator

of some coordinate transformation. Let the system under consideration be rewritten in the following form:

$$\begin{aligned} \mu\dot{z} &= Az + B\eta + D_1 x, \\ \mu\dot{\eta} &= Cz + b\eta + D_2 x + k\,\mathrm{sign}\, s, \\ \dot{s} &= \eta, \\ \dot{x} &= F(z, \eta, s, x), \end{aligned} \tag{3.13}$$

where $z \in \mathbb{R}^m, x \in \mathbb{R}^n, \eta, s \in \mathbb{R}$.

With (3.11) fulfilled and $\mathrm{Re\,Spec}\, A < 0$ system (3.13) has an exponentially stable integral manifold of slow motions being a subset of the second order sliding manifold and given by the equations

$$z = H(\mu, x) = -A^{-1}D_1 x + O(\mu), \;\; s = \eta = 0.$$

Function H may be evaluated with any desired precision with respect to the small parameter μ .

Therefore, according to [25, 26, 27, 9] under the conditions

$$\mathrm{Re\,Spec}\, A < 0, \;\; b < 0, \;\; k < 0 \tag{3.14}$$

the motions in such a system with a fast actuator of relative degree 1 consist of fast oscillations, vanishing exponentially, and slow motions on a submanifold of the second order sliding manifold.

Thus, if conditions (3.14) of chattering absence hold, the presence of a fast actuator of relative degree 1 does not lead to chattering in sliding mode control systems.

Remark

The stability of the fast actuator and of the second order sliding mode in (3.13) still does not guarantee absence of chattering if $\dim z > 0$ and $\frac{\partial F}{\partial z} \neq 0$, for in that case fast oscillations may still remain in the 2-sliding mode itself. Indeed, the stability of a fast actuator corresponds to the stability of the fast actuator matrix

$$\mathrm{Re\,Spec}\begin{pmatrix} A & B \\ C & b \end{pmatrix} < 0.$$

Consider the system

$$\begin{aligned}
\mu \dot z_1 &= z_1 + z_2 + \eta + D_1 x, \\
\mu \dot z_2 &= 2z_2 + z_3 + D_2 x; \\
\mu \dot\eta &= 24 z_1 - 60 z_2 - 9\eta + D_3 x + k\,\mathrm{sign}\, s, \\
\dot s &= \eta, \\
\dot x &= F(z_1, z_2, \eta, s, x),
\end{aligned}$$

where z_1, z_2, η, s are scalars. It is easy to check that the spectrum of the matrix is $\{-1, -2, -3\}$ and condition (3.11) holds for this system. On the other hand the motions in the second order sliding mode are described by the system

$$\begin{aligned}
\mu \dot z_1 &= z_1 + z_2 + D_1 x; \\
\mu \dot z_2 &= 2z_2 + D_2 x; \\
\dot x &= F(z_1, z_2, 0, 0, x).
\end{aligned}$$

The fast motions in this system are unstable and the absence of chattering in the original system cannot be guaranteed.

Example

Without loss of generality we illustrate the approach by some simple examples. Consider, for instance, sliding mode usage for the tracking purpose. Let the process be described by the equation $\dot x = u,\ x, u \in \mathbb{R}$, and the sliding variable be

$$s = x - f(t),\ f : \mathbb{R} \to \mathbb{R},$$

so that the problem is to track a signal $f(t)$ given in real time, where $|f|, |\dot f|, |\ddot f| < 0.5$. Only values of x, f, u are available.

The problem is successfully solved by the controller $u = -\mathrm{sign}\, s$, keeping $s = 0$ in a 1-sliding mode. In practice, however, there is always some actuator between the plant and the controller, which inserts some additional dynamics and removes the discontinuity from the real system. With respect to Fig. 3.2 let the system be described by the equation

$$\dot{x} = v,$$

where $v \in \mathbb{R}$ is an output of some dynamic actuator. Assume that the actuator has some fast first order dynamics. For example

$$\mu \dot{v} = u - v$$

The input u of the actuator is the relay control

$$u = -\operatorname{sign} s.$$

where μ is a small positive number. The second order sliding manifold $\mathcal{S}_2$ is given here by the equations

$$s = x - f(t) = 0, \ \dot{s} = v - \dot{f}(t) = 0.$$

The equality

$$\ddot{s} = \frac{1}{\mu}(u - v) - \ddot{f}(t)$$

shows that the relative degree here equals 2 and, according to Theorem 2, a 2-sliding mode exists, provided $\mu < 1$. The motion in this mode is described by the equivalent control method or by zero-dynamics, which is the same: from $s = \dot{s} = \ddot{s} = 0$ achieve $u = \mu \ddot{f}(t) + v$, $v = \dot{f}(t)$ and therefore

$$x = f(t), \ v = \dot{f}(t), \ u = \mu \ddot{f}(t) + v.$$

It is easy to prove that the 2-sliding mode is stable here with μ small enough. Note that the latter equality describes the equivalent control [54, 55] and is kept actually only in the average, while the former two are kept accurately in the 2-sliding mode.

Let

$$f(t) = 0.08 \sin t + 0.12 \cos 0.3t \ , \ \ x(0) = 0, \ v(0) = 0 \ .$$

The plots of $x(t)$ and $f(t)$ with $\mu = 0.2$ are shown in Fig. 3.3, whereas the plot of $v(t)$ is demonstrated in Fig. 3.4.

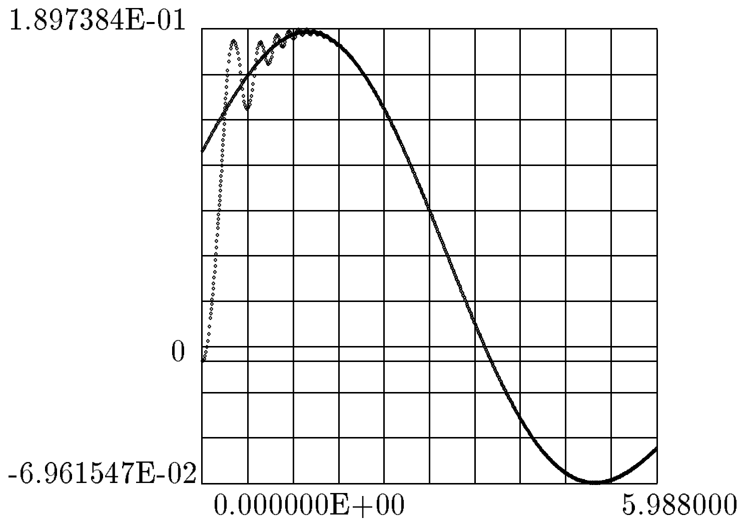

Figure 3.3: Asymptotically stable second order sliding mode in a system with a fast actuator. Tracking: $x(t)$ and $f(t)$.

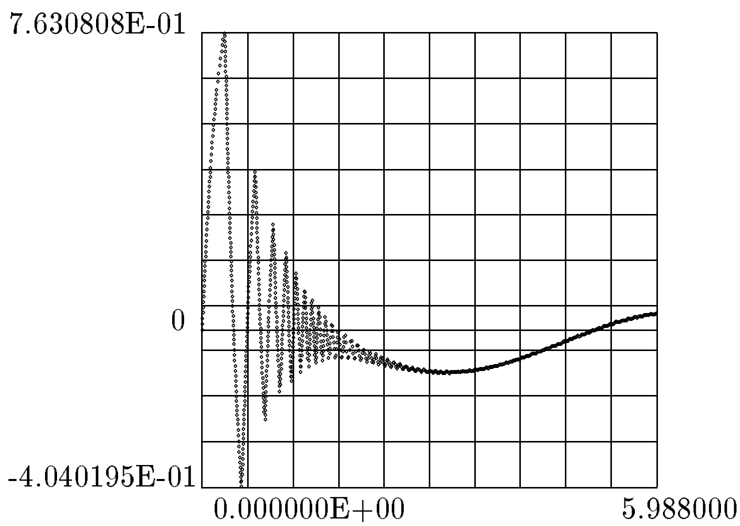

Figure 3.4: Asymptotically stable second order sliding mode in a system with a fast actuator: actuator output $v(t)$.

3.5.2 Systems with fast actuators of relative degree 3 and higher

The equilibrium point of any relay system with relative degree ≥ 3 is always unstable [1, 53] (section 3.4.2). That leads to an important conclusion: even being stable, higher order actuators do not suppress chattering in the closed-loop relay systems. For investigation of chattering phenomena in such systems the averaging technique was used [25, 29]. Higher-order actuators may give rise to high-frequency periodic solutions. The general model of sliding mode control systems with fast actuators has the form [10]

$$\dot{x} = h(x, s, \eta, z, u(s)), \quad \dot{s} = \eta,$$
$$\mu\dot{\eta} = g_2(x, s, \eta, z), \mu\dot{z} = g_1(x, s, \eta, z, u(s)), \tag{3.15}$$

where $z \in \mathbf{R}^m$, $\eta, s \in \mathbf{R}$, $x \in X \subset \mathbf{R}^n$, $u(s) = \text{sign} s$, and g_1, g_2, h are smooth functions of their arguments. Variables s, x may be considered as the state coordinates of the plant, η, z being the fast-actuator coordinates, and μ being the actuator time constant.

Suppose that following conditions are true:

1. The fast-motion system

$$\frac{ds}{d\tau} = \eta, \quad \frac{d\eta}{d\tau} = g_2(x, 0, \eta, z), \quad \frac{dz}{d\tau} = g_1(x, 0, \eta, z, u(s)), \tag{3.16}$$

has a $T(x)$-periodic solution $(s_0(\tau, x), \eta_0(\tau, x), z_0(\tau, x))$ for any $x \in X$. System (3.16) generates a point mapping $\Psi(x, \eta, z)$ of the switching surface $s = 0$ into itself which has a fixed point $(\eta^*(x), z^*(x))$, $\Psi(x, \eta^*(x), z^*(x)) = (\eta^*(x), z^*(x))$. Moreover, the Frechet derivative of $\Psi(x, \eta, z)$ with respect to variables (η, z) calculated at $(\eta^*(x), z^*(x))$ is a contractive matrix for any $x \in \bar{X}$.

2. The averaged system

$$\dot{x} = \bar{h}(x) = \frac{1}{T(x)} \int_0^{T(x)} h(x, 0, \eta_0(\tau, x), z_0(\tau, x), u(s_0(\tau, x)))d\tau \tag{3.17}$$

has an unique equilibrium point $x = x_0$. This equilibrium point is exponentially stable.

Theorem 3 *[29]. Under conditions 1,2 system (3.15) has an isolated orbitally asymptotically stable periodic solution with the period* $\mu(T(x_0) + O(\mu))$ *near the closed curve*

$$(x_0, \mu s_0(t/\mu, x_0), \eta_0(t/\mu, x_0), z_0(t/\mu, x_0)).$$

Example

Consider a mathematical model of a control system with actuator and the overall relative degree 3

$$\dot{x} = -x - u,\ \dot{s} = z_1, \tag{3.18}$$

$$\mu \dot{z}_1 = z_2,\ \mu \dot{z}_2 = -2z_1 - 3z_2 - u. \tag{3.19}$$

Here $z_1, z_2, s, x \in \mathbf{R}$, $u(s) = \mathrm{sign} s$, μ is the actuator time constant. The fast motions taking place in (3.18),(3.19) are described by the system

$$\frac{d\xi}{d\tau} = z_1,\ \frac{dz_1}{d\tau} = z_2,$$

$$\frac{dz_2}{d\tau} = -2z_1 - 3z_2 - u,\ u = \mathrm{sign}\,\xi. \tag{3.20}$$

Then the solution of system (3.20) for $\xi > 0$ with initial condition $\xi(0) = 0,\ z_1(0) = z_{10},\ z_2(0) = z_{20}$ is as follows

$$\xi(\tau) = \frac{3}{2}z_{10} - 2z_{10}e^{-\tau} + \frac{1}{2}z_{10}e^{-2\tau} + \frac{1}{2}z_{20} - z_{20}e^{-\tau} + \frac{1}{2}z_{20}e^{-2\tau}$$
$$-\frac{1}{2}\tau + \frac{3}{4} - e^{-\tau} + \frac{1}{4}e^{-2\tau};$$
$$z_1(\tau) = 2z_{10}e^{-\tau} - z_{10}e^{-2\tau} + z_{20}e^{-\tau} - z_{20}e^{-2\tau} - \frac{1}{2} + e^{-\tau} - \frac{1}{2}e^{-2\tau};$$
$$z_2(\tau) = 2z_{10}e^{-2\tau} - 2z_{10}e^{-\tau} - z_{20}e^{-\tau} + 2z_{20}e^{-2\tau} - e^{-\tau} + e^{-2\tau}.$$

Consider the point mapping $\Xi(z_1, z_2)$ of the domain $z_1 > 0,\ z_2 > 0$ on the switching surface $\xi = 0$ into the domain $z_1 < 0,\ z_2 < 0$ with $\mathrm{sign}\,\xi > 0$ made by system (3.20). Then

$$\Xi(z_1, z_2) = (\Xi_1(z_1, z_2), \Xi_2(z_1, z_2));$$

$$\Xi_1(z_1, z_2) = 2z_1e^{-T} - z_1e^{-2T} + z_2e^{-T} - z_2e^{-2T} - \frac{1}{2} + e^{-T} - \frac{1}{2}e^{-2T};$$
$$\Xi_2(z_1, z_2) = 2z_1e^{-2T} - 2z_1e^{-T} - z_2e^{-T} + 2z_2e^{-2T} - e^{-T} + e^{-2T},$$

where $T(z_1, z_2)$ is the smallest root of equation

$$\xi(T(z_1, z_2)) = \frac{3}{2}z_1 - 2z_1e^{-T} + \frac{1}{2}z_1e^{-2T} + \frac{1}{2}z_2 - z_2e^{-T}$$

$$+\frac{1}{2}z_2e^{-2T} - \frac{1}{2}T + \frac{3}{4} - e^{-T} + \frac{1}{4}e^{-2T} = 0.$$

System (3.20) is symmetric with respect to the point $\xi = z_1 = z_2 = 0$. Thus, the initial condition $(0, z_1^*, z_2^*)$ and the semi-period $T^* = T(z_1^*, z_2^*)$ for the periodic solution of (3.20) are determined by the equations $\Xi(z_1^*, z_2^*) = -(z_1^*, z_2^*)$, $\xi(T(z_1^*, z_2^*)) = 0$ and consequently

$$\frac{3}{2}z_1^* - 2z_1^*e^{-T^*} + \frac{1}{2}z_1^*e^{-2T^*} + \frac{1}{2}z_2^* - z_2^*e^{-T^*} + \frac{1}{2}z_2^*e^{-2T^*}$$

$$-\frac{1}{2}T^* + \frac{3}{4} - e^{-T^*} + \frac{1}{4}e^{-2T^*} = 0.$$

$$2z_1^*e^{-T^*} - z_1^*e^{-2T^*} + z_2^*e^{-T^*} - z_2^*e^{-2T^*} - \frac{1}{2} + e^{-T} - \frac{1}{2}e^{-2T} = -z_1^*;$$

$$2z_1^*e^{-2T^*} - 2z_1^*e^{-T^*} - z_2^*e^{-T^*} + 2z_2^*e^{-2T^*} - e^{-T^*} + e^{-2T^*} = -z_2^*, \qquad (3.21)$$

Expressing z_1^*, z_2^* from the latter two equations of (3.21), achieve

$$T^*(e^{T^*} + e^{3T^*} + 1 + e^{2T^*}) - 5e^{T^*} - 3e^{3T^*} + 3 + 5e^{2T^*} = 0. \qquad (3.22)$$

Equations (3.22) and (3.21) have positive solution

$$T^* \approx 2.2755,\ z_1^* \approx 0.3241,\ z_2^* \approx 0.1654,$$

corresponding to the existence of a $2T^*$-periodic solution in system (3.20). Thus

$$\left(\frac{\partial T}{\partial z_1}, \frac{\partial T}{\partial z_2}\right) = \left(-\frac{\frac{3}{2} - 2e^{-T} + \frac{1}{2}e^{-2T}}{(2z_1 + z_2 + 1)e^{-T} - (z_1 + z_2 + \frac{1}{2})e^{-2T} - \frac{1}{2}}, -\frac{\frac{1}{2} - e^{-T} + \frac{1}{2}e^{-2T}}{(2z_1 + z_2 + 1)e^{-T} - (z_1 + z_2 + \frac{1}{2})e^{-2T} - \frac{1}{2}}\right)$$

and

$$\frac{\partial \Xi_1}{\partial z_1} = 2e^{-T} - e^{-2T} + [e^{-2T}(2z_1 + 2z_2 + 1) - e^{-T}(2z_1 + z_2 + 1)]\frac{\partial T}{\partial z_1};$$

$$\frac{\partial \Xi_1}{\partial z_2} = e^{-T} - e^{-2T} + [e^{-2T}(2z_1 + 2z_2 + 1) - e^{-T}(2z_1 + z_2 + 1)]\frac{\partial T}{\partial z_2};$$

$$\frac{\partial \Xi_2}{\partial z_1} = 2e^{-2T} - 2e^{-T} - [(e^{-T}(2z_1 + z_2 + 1) - 2e^{-2T}(2z_1 + 2z_2 + 1)] \frac{\partial T}{\partial z_1};$$
$$\frac{\partial \Xi_2}{\partial z_2} = 2e^{-2T} - e^{-T} - [(e^{-T}(2z_1 + z_2 + 1) - 2e^{-2T}(2z_1 + 2z_2 + 1)] \frac{\partial T}{\partial z_2}.$$

Calculating the value of Frechet derivative $\frac{\partial \Xi}{\partial z}$ at (z_1^*, z_2^*), using the found value of T^*, achieve

$$\frac{\partial \Xi}{\partial z}(z_1^*, z_2^*) = J = \begin{bmatrix} -0.4686 & -0.1133 \\ 0.3954 & 0.0979 \end{bmatrix}.$$

The eigenvalues of matrix J are -0.3736 and 0.0029. That implies existence and asymptotic stability of the periodic solution of (3.20). The averaged equation for system (3.18),(3.19) is

$$\dot{x} = -x,$$

and it has the asymptotically stable equilibrium point $x = 0$. Hence, system (3.18),(3.19) has an orbitally asymptotically stable periodic solution which lies in the $O(\mu)$-neighbourhood of the switching surface.

3.6 2-sliding controllers

We follow here [36, 35, 6].

3.6.1 2-sliding dynamics

Return to the system

$$\dot{x} = f(t, x, u), \quad s = s(t, x) \in \mathbb{R}, \; u = U(t, x) \in \mathbb{R}, \tag{3.23}$$

where $x \in \mathbf{R}^n$, t is time, u is control, and f, s are smooth functions. The control task is to keep output $s \equiv 0$.

Differentiating successively the output variable s achieve functions $\dot{s}, \ddot{s}, ...$ Depending on the relative degree [31] of the system different cases should be considered

a) relative degree $r = 1$, i.e., $\frac{\partial}{\partial u}\dot{s} \neq 0$;

b) relative degree $r \geq 2$, i.e., $\frac{\partial}{\partial u}s^{(i)} = 0$ $(i = 1, 2, \ldots, r-1)$, $\frac{\partial}{\partial u}s^{(r)} \neq 0$.

In case a) the classical VSS approach solves the control problem by means of 1–sliding mode control, nevertheless 2–sliding mode control can also be used in order to avoid chattering. For that purpose u is to become an output of some first order dynamic system [35]. For example, the time derivative of the plant control $\dot{u}(t)$ may be considered as the actual control variable. A discontinuous control $\dot{u}$ steers the sliding variable s to zero, keeping $s = 0$ in a 2–sliding mode, so that the plant control u is continuous and the chattering is avoided [35, 5]. In case b) the p–sliding mode approach, with $p \geq r$, is the control technique of choice.

Chattering avoidance: the generalized constraint fulfillment problem

When considering classical VSS the control variable $u(t)$ is a feedback-designed relay output. The most direct application of 2–sliding mode control is that of attaining sliding motion on the sliding manifold by means of a continuous bounded input $u(t)$ being a continuous output of a suitable first–order dynamical system driven by a proper discontinuous signal. Such first–order dynamics can be either inherent to the control device or specially introduced for chattering elimination purposes.

Assume that f and s are respectively $\mathcal{C}^1$ and $\mathcal{C}^2$ functions, and that the only available current information consists of the current values of t, $u(t)$, $s(t,x)$ and, possibly, of the sign of the time derivative of s. Differentiating the sliding variable s twice, the following relations are derived:

$$\dot{s} = \frac{\partial}{\partial t}s(t,x) + \frac{\partial}{\partial x}s(t,x)f(t,x,u), \tag{3.24}$$

$$\ddot{s}(t) = \frac{\partial}{\partial t}\dot{s}(t,x,u) + \frac{\partial}{\partial x}\dot{s}(t,x,u)f(t,x,u) + \frac{\partial}{\partial u}\dot{s}(t,x,u)\dot{u}(t). \tag{3.25}$$

The control goal for a 2–sliding mode controller is that of steering s to zero in finite time by means of control $u(t)$ continuously dependent on time. In order to state a rigorous control problem the following conditions are assumed:

1) Control values belong to the set $\mathcal{U} = \{u : |u| \leq U_M\}$, where $U_M > 1$ is a real constant; furthermore the solution of the system is well defined for all t, provided $u(t)$ is continuous and $\forall t \; u(t) \in \mathcal{U}$.

2) There exists $u_1 \in (0,1)$ such that for any continuous function $u(t)$ with $|u(t)| > u_1$, there is t_1, such that $s(t)u(t) > 0$ for each $t > t_1$. Hence, the control $u(t) = -\text{sign}(s(t_0))$, where t_0 is the initial value of time, provides hitting the manifold $s = 0$ in finite time.

3) Let $\dot{s}(t,x,u)$ be the total time derivative of the sliding variable $s(t,x)$. There are positive constants s_0, $u_0 < 1$, Γ_m, Γ_M such that if $|s(t,x)| < s_0$ then

$$0 < \Gamma_m \leq \frac{\partial}{\partial u}\dot{s}(t,x,u) \leq \Gamma_M \quad , \forall u \in \mathcal{U}, x \in \mathcal{X} \tag{3.26}$$

and the inequality $|u| > u_0$ entails $\dot{s}u > 0$.

4) There is a positive constant Φ such that within the region $|s| < s_0$ the following inequality holds $\forall t, x \in \mathcal{X}, u \in \mathcal{U}$

$$\left| \frac{\partial}{\partial t}\dot{s}(t,x,u) + \frac{\partial}{\partial x}\dot{s}(t,x,u) f(t,x,u) \right| \leq \Phi \tag{3.27}$$

The above condition 2 means that starting from any point of the state space it is possible to define a proper control $u(t)$ steering the sliding variable within a set such that the boundedness conditions on the sliding dynamics defined by conditions 3 and 4 are satisfied. In particular they state that the second time derivative of the sliding variable s, evaluated with fixed values of the control u, is uniformly bounded in a bounded domain.

It follows from the theorem on implicit function that there is a function $u_{eq}(t,x)$ which can be considered as equivalent control [55] satisfying the equation $\dot{s} = 0$. Once $s = 0$ is attained, the control $u = u_{eq}(t,x)$ would provide for the exact constraint fulfillment. Conditions 3 and 4 mean that $|s| < s_0$ implies $|u_{eq}| < u_0 < 1$, and that the velocity of the u_{eq} changing is bounded. This provides for a possibility to approximate u_{eq} by a Lipschitzian control.

The unit upper bound for u_0 and u_1 is actually a scaling factor. Note also that linear dependence on control u is not required here. The usual form of the uncertain systems dealt with by the VSS theory, i.e., systems affine in u and possibly in x, are a special case of the considered system and the corresponding constraint fulfillment problem may be reduced to the considered one [35, 20].

Relative degree two. In case of relative degree two the control problem statement could be derived from the above by considering the variable u as a state variable and $\dot{u}$ as the actual control. Indeed, let the controlled system be

$$f(t,x,u) = a(t,x) + b(t,x)u(t), \tag{3.28}$$

where $a : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $b : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are sufficiently smooth uncertain vector functions, $[\frac{\partial}{\partial x}s(t,x)]b(t,x) \equiv 0$. Calculating achieve that

$$\ddot{s} = \varphi(t,x) + \gamma(t,x)u. \tag{3.29}$$

It is assumed that $|\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M, \Phi > 0$.

Thus in a small vicinity of the manifold $s = 0$ the system is described by (3.28), (3.29) if the relative degree is 2 or by (3.23) and

$$\ddot{s} = \varphi(t,x) + \gamma(t,x)\dot{u}, \tag{3.30}$$

if the relative degree is 1.

3.6.2 Twisting algorithm

Let relative degree be 1. Consider local coordinates $y_1 = s$ and $y_2 = \dot{s}$, then after a proper initialization phase, the second order sliding mode control problem is equivalent to the finite time stabilization problem for the uncertain second order system with $|\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M, \Phi > 0$.

$$\begin{cases} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \varphi(t,x) + \gamma(t,x)\dot{u} \end{cases} \tag{3.31}$$

with $y_2(t)$ immeasurable but with a possibly known sign, and $\varphi(t,x)$ and $\gamma(t,x)$ uncertain functions with

$$\Phi > 0, |\varphi| \leq \Phi, 0 < \Gamma_m \leq \gamma \leq \Gamma_M. \tag{3.32}$$

Being historically the first known 2-sliding controller [34], that algorithm features twisting around the origin of the 2–sliding plane y_1Oy_2 (Fig.3.5). The trajectories perform an infinite number of rotations while converging in finite time to the origin. The vibration magnitudes along the axes as well as the rotation times decrease in geometric progression. The control derivative value commutes at each axis crossing, which requires availability of the sign of the sliding-variable time-derivative y_2.

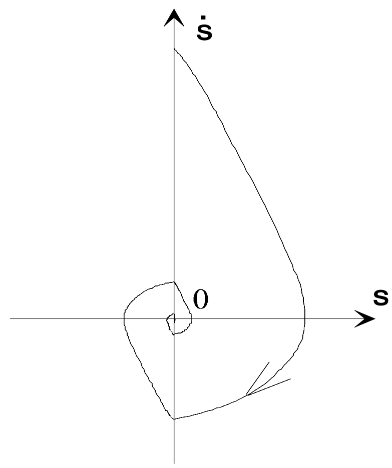

Figure 3.5: Twisting algorithm phase trajectory

The control algorithm is defined by the following control law [34, 35, 17, 20], in which the condition on $|u|$ provides for $|u| \leq 1$:

$$\dot{u}(t) = \begin{cases} -u & if \;\; |u| > 1, \\ -V_m \text{sign}(y_1) & if \;\; y_1 y_2 \leq 0; \;\; |u| \leq 1, \\ -V_M \text{sign}(y_1) & if \;\; y_1 y_2 > 0; \;\; |u| \leq 1. \end{cases} \tag{3.33}$$

The corresponding sufficient conditions for the finite time convergence to the sliding manifold are [35]

$$\begin{array}{l} V_M > V_m \\ V_m > \frac{4\Gamma_M}{s_0} \\ V_m > \frac{\Phi}{\Gamma_m} \\ \Gamma_m V_M - \Phi > \Gamma_M V_m + \Phi. \end{array} \tag{3.34}$$

The similar controller

$$u(t) = \begin{cases} -V_m \text{sign}(y_1) & if \;\; y_1 y_2 \leq 0 \\ -V_M \text{sign}(y_1) & if \;\; y_1 y_2 > 0 \end{cases}$$

is to be used in order to control system (3.28) when the relative degree is 2.

By taking into account the different limit trajectories arising from the uncertain dynamics of (3.29) and evaluating time intervals between successive crossings of the abscissa axis, it is possible to define the following upper bound for the convergence time [6]

$$t_{tw\infty} \leq t_{M_1} + \Theta_{tw} \frac{1}{1-\theta_{tw}} \sqrt{|y_{1_{M_1}}|}. \tag{3.35}$$

Here $y_{1_{M_1}}$ is the value of the y_1 variable at the first abscissa crossing in the $y_1 O y_2$ plane, t_{M_1} is the corresponding time instant and

$$\begin{array}{l} \Theta_{tw} = \sqrt{2} \frac{\Gamma_m V_M + \Gamma_M V_m}{(\Gamma_m V_M - \Phi)\sqrt{\Gamma_M V_m + \Phi}} \\ \theta_{tw} = \sqrt{\frac{\Gamma_M V_m + \Phi}{\Gamma_m V_M - \Phi}}. \end{array}$$

In practice when y_2 is immeasurable, its sign can be estimated by the sign of the first difference of the available sliding variable y_1 in a time interval τ, i.e., $\text{sign}(y_2(t))$ is estimated by $\text{sign}(y_1(t) - y_1(t-\tau))$. In that case the 2–sliding precision with respect to the measurement time interval is provided, and the size of the boundary layer of the sliding manifold is $\Delta \sim \mathcal{O}(\tau^2)$ [35]. Recall that it is the best possible accuracy asymptotics with discontinuous $\dot{y}_2 = \ddot{s}$.

3.6.3 Sub–optimal algorithm

That 2–sliding controller was developed as a sub–optimal feedback implementation of a classical time-optimal control for a double integrator. Let the relative degree be 2. The auxiliary system is

$$\left\{ \begin{array}{ll} \dot{y}_1 & = y_2 \\ \dot{y}_2 & = \varphi(t,x) + \gamma(t,x)u. \end{array} \right. \tag{3.36}$$

The trajectories on the $y_1 O y_2$ plane are confined within limit parabolic arcs which include the origin, so that both twisting and leaping (when y_1 and y_2 do not change sign) behaviors are possible (Fig.3.6). Also here the coordinates of the trajectory intersections with axis y_1 decrease in geometric progression. After an initialization phase the algorithm is defined by the following control law [4, 5, 6]:

$$\begin{array}{l} v(t) = -\alpha(t) V_M \text{sign}(y_1(t) - \frac{1}{2} y_{1_M}), \\ \alpha(t) = \left\{ \begin{array}{lll} \alpha^* & if & [y_1(t) - \frac{1}{2} y_{1_M}][y_{1_M} - y_1(t)] > 0 \\ 1 & if & [y_1(t) - \frac{1}{2} y_{1_M}][y_{1_M} - y_1(t)] \leq 0 \end{array} \right. , \end{array} \tag{3.37}$$

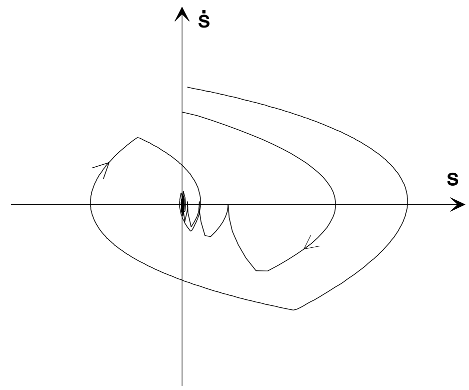

Figure 3.6: Sub–Optimal algorithm phase trajectories

where y_{1_M} is the latter singular value of the function $y_1(t)$, i.e. the latter value corresponding to the zero value of $y_2 = \dot{y}_1$. The corresponding sufficient conditions for the finite-time convergence to the sliding manifold are as follows [4]:

$$\begin{array}{l} \alpha^* \in (0,1] \cap (0, \frac{3\Gamma_m}{\Gamma_M}), \\ V_M > \max\left(\frac{\Phi}{\alpha^*\Gamma_m}, \frac{4\Phi}{3\Gamma_m - \alpha^*\Gamma_M}\right). \end{array} \tag{3.38}$$

Also in that case an upper bound for the convergence time can be determined [4]

$$t_{opt_\infty} \leq t_{M_1} + \Theta_{opt}\frac{1}{1-\theta_{opt}}\sqrt{|y_{1_{M_1}}|}. \tag{3.39}$$

Here $y_{1_{M_1}}$ and t_{M_1} are defined as for the twisting algorithm, and

$$\begin{array}{l} \Theta_{opt} = \frac{(\Gamma_m + \alpha^*\Gamma_M)V_M}{(\Gamma_m V_M - \Phi)\sqrt{\alpha^*\Gamma_M V_M + \Phi}}, \\ \theta_{opt} = \sqrt{\frac{(\alpha^*\Gamma_M - \Gamma_m)V_M + 2\Phi}{2(\Gamma_m V_M - \Phi)}}. \end{array}$$

The effectiveness of the above algorithm was extended to larger classes of uncertain systems [6]. It was proved [5] that in case of unit gain function the control law (3.37) can be simplified by setting $\alpha = 1$ and choosing $V_M > 2\Phi$.

The sub–optimal algorithm requires some device in order to detect the singular values of the available sliding variable $y_1 = s$. In the most practical case y_{1_M} can be estimated by checking the sign of the quantity $D(t) = [y_1(t-\tau) - y_1(t)]\, y_1(t)$ in which $\frac{\tau}{2}$ is the estimation delay. In that case the control amplitude V_M has to belong to an interval instead of a half-line:

$$V_M \in \left(\max \left(\frac{\Phi}{\alpha^* \Gamma_m}, V_{M_1}(\tau, y_{1_M}) \right), V_{M_2}(\tau; y_{1_M}) \right) \tag{3.40}$$

Here $V_{M_1} < V_{M_2}$ are the solutions of the second order algebraic equation

$$\left[(3\Gamma_m - \alpha^* \Gamma_M) \frac{V_{M_i}}{\Phi} - 4 \right] \frac{y_{1_M}}{\Phi \delta^2} - \frac{V_{M_i}}{8\Phi} \left[\Gamma_m + \Gamma_M (2 - \alpha^*) \right] \left(\Gamma_M \frac{V_{M_i}}{\Phi} + 1 \right) = 0$$

In the case of approximated evaluation of y_{1_M} the second order real sliding mode is achieved, and the size of the boundary layer of the sliding manifold is $\Delta \sim \mathcal{O}(\tau^2)$. It can be minimized by choosing V_M as follows [6]:

$$V_M = \frac{4\Phi}{3\Gamma_m - \alpha^* \Gamma_M} \left[1 + \sqrt{1 + \frac{3\Gamma_m - \alpha^* \Gamma_M}{4\Gamma_M}} \right]$$

An extension of the sub–optimal 2–sliding controller to a class of sampled data systems such that the gain function in (3.29) is constant, i.e., $\gamma(\cdot) = 1$, was recently presented [6].

3.6.4 Super–twisting algorithm

That algorithm has been developed to control systems with relative degree one in order to avoid chattering in VSC. Also in that case the trajectories on the 2–sliding plane are characterized by twisting around the origin (Fig.3.7), but the continuous control law $u(t)$ is constituted by two terms. The first is defined by means of its discontinuous time derivative, while the other is a continuous function of the available sliding variable.

The control algorithm is defined by the following control law [35]:

$$\begin{aligned} u(t) &= u_1(t) + u_2(t) \\ \dot{u}_1(t) &= \begin{cases} -u & if \ \ |u| > 1 \\ -W \text{sign}(y_1) & if \ \ |u| \le 1 \end{cases} \\ u_2(t) &= \begin{cases} -\lambda |s_0|^\rho \text{sign}(y_1) & if \ \ |y_1| > s_0 \\ -\lambda |y_1|^\rho \text{sign}(y_1) & if \ \ |y_1| \le s_0 \end{cases} \end{aligned} \tag{3.41}$$

and the corresponding sufficient conditions for the finite time convergence to the sliding manifold are [35]

$$\begin{aligned} & W > \tfrac{\Phi}{\Gamma_m} \\ & \lambda^2 \ge \tfrac{4\Phi}{\Gamma_m^2} \tfrac{\Gamma_M (W + \Phi)}{\Gamma_m (W - \Phi)} \\ & 0 < \rho \le 0.5 \end{aligned} \tag{3.42}$$

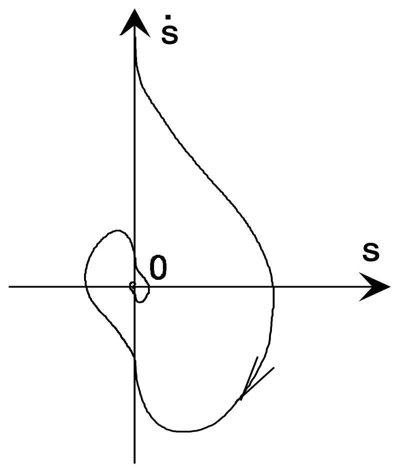

Figure 3.7: Super–Twisting algorithm phase trajectory

That controller may be simplified when controlled systems (3.28) are linearly dependent on control, u does not need to be bounded and $s_0 = \infty$:

$$u = -\lambda |s|^\rho \mathrm{sign}(y_1) + u_1,$$
$$\dot{u}_1 = -W \mathrm{sign}(y_1).$$

The super-twisting algorithm does not need any information on the time derivative of the sliding variable. An exponentially stable 2–sliding mode arrives if the control law (3.41) with $\rho = 1$ is used. The choice $\rho = 0.5$ assures that the maximal possible for 2–sliding realization real-sliding order 2 is achieved. Being extremely robust, that controller is successfully used for real-time robust exact differentiation [37] (see further).

3.6.5 Drift algorithm

The idea of the controller is to steer the trajectory to the 2-sliding mode $s = 0$ while keeping $\dot{s}$ relatively small, i.e. to cause "drift" towards the origin along axis y_1. When using the drift algorithm, the phase trajectories on the 2-sliding plane are characterized by loops with constant sign of the sliding variable y_1 (Fig.3.8). That controller intentionally yields real 2-sliding and uses sampled values of the available signal y_1 with sampling period τ. The control algorithm is defined by the following control law [35, 16, 18] (relative

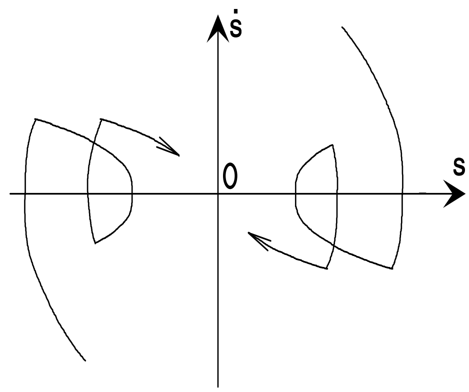

Figure 3.8: Drift algorithm phase trajectories

degree is 1):

$$\dot{u} = \begin{cases} -u & if \ \ |u| > 1 \\ -V_m \mathrm{sign}(\Delta y_{1_i}) & if \ \ y_1 \Delta y_{1_i} \leq 0; \ |u| \leq 1 \\ -V_M \mathrm{sign}(\Delta y_{1_i}) & if \ \ y_1 \Delta y_{1_i} > 0; \ |u| \leq 1 \end{cases} \tag{3.43}$$

where V_m and V_M are proper positive constants such that $V_m < V_M$ and $\frac{V_M}{V_m}$ is sufficiently large, and $\Delta y_{1_i} = y_1(t_i) - y_1(t_i - \tau)$, $t \in [t_i, t_{i+1})$. The corresponding sufficient conditions for the convergence to the sliding manifold are rather cumbersome [18] and are omitted here for the sake of simplicity. Also here a similar controller corresponds to relative degree 2:

$$\dot{u} = \begin{cases} -V_m \mathrm{sign}(\Delta y_{1_i}) & if \ \ y_1 \Delta y_{1_i} \leq 0 \\ -V_M \mathrm{sign}(\Delta y_{1_i}) & if \ \ y_1 \Delta y_{1_i} > 0 \end{cases}$$

After substituting y_2 for Δy_{1_i} a first order sliding mode on $y_2 = 0$ would be achieved. That implies $y_1 = \mathrm{const}$, but since an artificial switching time delay appears, we ensure a real sliding on y_2 with most of time spent in the region $y_1 y_2 < 0$. Therefore, $y_1 \to 0$. The accuracy of the real sliding on $y_2 = 0$ is proportional to the sampling time interval τ; hence, the duration of the transient process is proportional to τ^{-1}.

Such an algorithm does not satisfy the definition of a real sliding algorithm (Section 3.3) requiring the convergence time to be uniformly bounded with respect to τ. Consider a variable sampling time $\tau_{i+1}[y_1(t_i)] = t_{i+1} - t_i$, $i = 0, 1, 2, \ldots$ with $\tau = \max(\tau_M, \min(\tau_m, \eta |y_1(t_i)|^\rho))$, where $0.5 \leq \rho \leq 1$, $\tau_M > \tau_m > 0$, $\eta > 0$. Then with η, $\frac{V_m}{V_M}$ sufficiently small and V_m sufficiently large the drift algorithm constitutes a second order real sliding algorithm with respect to $\tau \to 0$. That algorithm has no overshoot if the parameters are chosen properly [18].

3.6.6 Algorithm with a prescribed convergence law

That class of sliding controllers features the possibility of choosing a transient process trajectory: the switching of $\dot{u}$ depends on a suitable function of s.

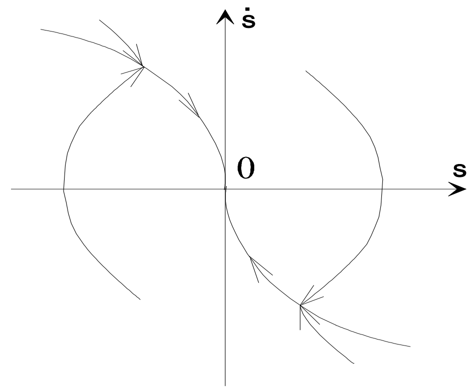

Figure 3.9: Phase trajectories for the algorithm with prescribed law of variation of s

The general formulation of such a class of 2–sliding control algorithms is as follows :

$$\dot{u} = \begin{cases} -u & if \;\; |u| > 1 \\ -V_M \mathrm{sign}(y_2 - g(y_1)) & if \;\; |u| \leq 1 \end{cases} \tag{3.44}$$

Here V_M is a positive constant and the continuous function $g(y_1)$ is smooth everywhere but in $y_1 = 0$. A controller for the relative degree 2 is formed in an obvious way:

$$\dot{u} = -V_M \mathrm{sign}(y_2 - g(y_1)).$$

Function g must be chosen in such a way that all solutions of the equation $\dot{y}_1 = g(y_1)$ vanish in finite time and the function $g' \cdot g$ be bounded. For example, the following function can be used

$$g(y_1) = -\lambda_1 |y_1|^{\rho} \mathrm{sign}(y_1), \lambda > 0 \;,\; 0.5 \leq \rho < 1.$$

The sufficient condition for the finite time convergence to the sliding manifold is defined by the following inequality

$$V_M > \frac{\Phi + \sup(g'(y_1)g(y_1))}{\Gamma_m}, \qquad (3.45)$$

and the convergence time depends on the function g [16, 35, 56].

That algorithm needs y_2 to be available, which is not always the case. The substitution of the first difference of $y_1$for y_2 i.e. $\mathrm{sign}[\Delta y_{1_i} - \tau_i g(y_1)]$ instead of $\mathrm{sign}[y_2 - g(y_1)]$ $(t \in [t_i, t_{i+1}),\ \ \tau_i = t_i - t_{i-1})$, turns the algorithm into a real sliding algorithm. The real sliding order equals two if $g(\cdot)$ is chosen as in the above example with $\rho = 0.5$ [35].

Important remark. All the above-listed discretized 2-sliding controllers, except for the super-twisting one, are sensitive to the choice of the measurement interval τ. Indeed, given any measurement error magnitude, any information significance of the first difference Δy_{1_i} is eliminated with sufficiently small τ, and the algorithm convergence is disturbed. That problem was shown to be solved [40] by a special feedback determinating τ as a function of the real-time measured value of y_1. In particular, it was shown that the feedback $\tau = \max(\tau_M, \min(\tau_m, \eta|y_1(t_i)|^\rho))$, $0.5 \le \rho \le 1$, $\tau_M > \tau_m > 0$, $\eta > 0$, makes the twisting controller robust with respect to measurements errors. Moreover, the choice $\rho = 1/2$ is proved to be the best one. It provides for keeping the second order real-sliding accuracy $s = O(\tau^2)$ in the absence of measurement errors and for sliding accuracy proportional to the maximal error magnitude otherwise. Note that the super-twisting controller is robust due to its own nature and does not need such auxiliary constructions.

3.6.7 Examples

Practical implementation of 2-sliding controllers is described in [42]. Continue the example series 3.5.1, 3.5.2. The process is given by

$$\dot{x} = u, \quad x, u \in \mathbb{R}, \quad s = x - f(t), \quad f : \mathbb{R} \to \mathbb{R},$$

so that the problem is to track a signal $f(t)$ given in real time, where $|f|, |\dot{f}|, |\ddot{f}| < 0.5$. Only values of x, f, u are available. Following is the appropriate discretized twisting controller:

$$\dot{u} = \begin{cases} -u(t_i), & |u(t_i)| > 1, \\ -5\,\mathrm{sign}\, s(t_i), \quad s(t_i)\Delta s_i > 0, & |u(t_i)| \le 1, \\ -\mathrm{sign}\, s(t_i), \quad s(t_i)\Delta s_i \le 0, & |u(t_i)| \le 1. \end{cases}$$

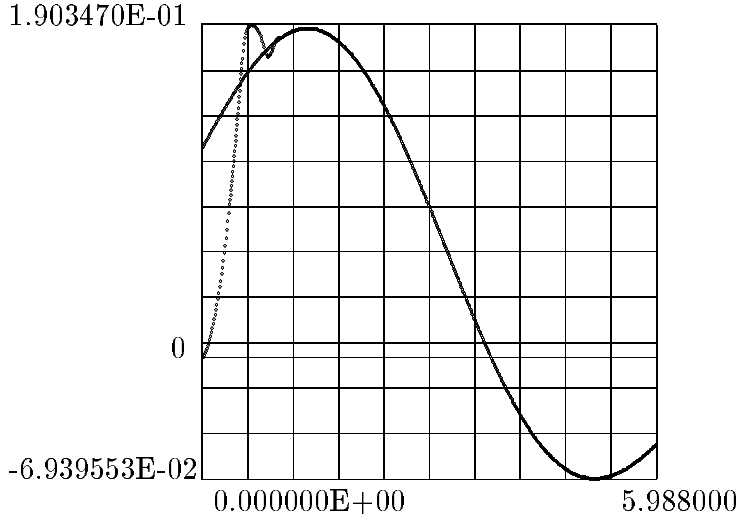

Figure 3.10: Twisting 2-sliding algorithm. Tracking: $x(t)$ and $f(t)$.

Here $t_i \leq t < t_{i+1}$. Let function f be chosen like in examples 3.5.1, 3.5.2:

$$f(t) = 0.08 \sin t + 0.12 \cos 0.3t \ , \quad x(0) = 0, \ v(0) = 0 \ .$$

The corresponding simulation results are shown in Fig. 3.10, 3.11.

The discretized super-twisting controller [19, 35, 37] serving the same goal is the algorithm

$$u = -2\sqrt{|s(t_i)|}\operatorname{sign} s(t_i) + u_1, \quad \dot{u}_1 = \begin{cases} -u(t_i), & |u(t_i)| > 1, \\ -\operatorname{sign} s(t_i), & |u(t_i)| \leq 1, \end{cases}$$

Its simulation results are shown in Fig.3.12, 3.13.

3.7 Arbitrary-order sliding controllers

We follow here [38, 39, 41].

3.7.1 The problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t, x) + b(t, x)u, s = s(t, x), \tag{3.46}$$

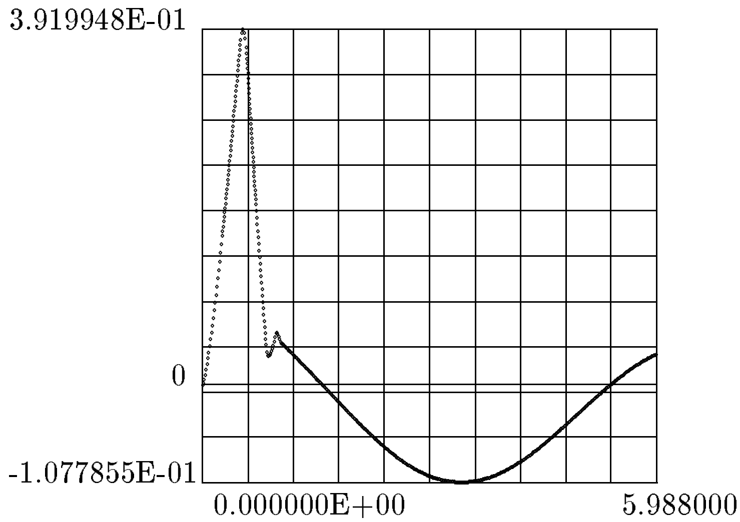

Figure 3.11: Twisting 2-sliding algorithm. Control $u(t)$.

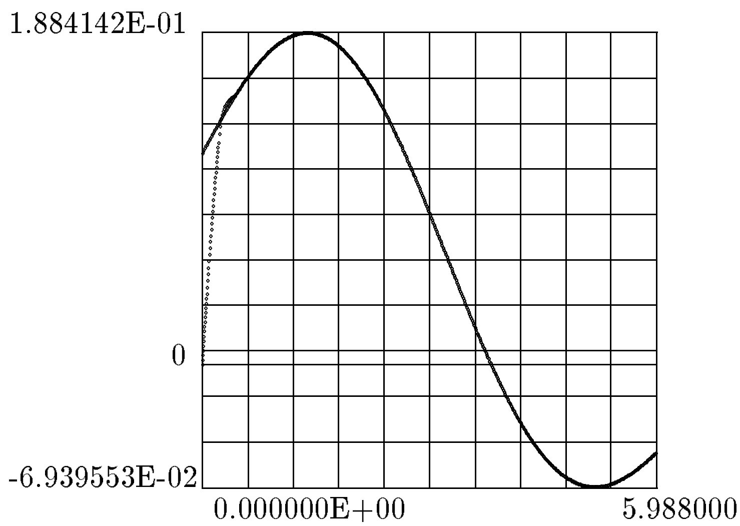

Figure 3.12: Super-twisting 2-sliding controller. Tracking: $x(t)$ and $f(t)$.

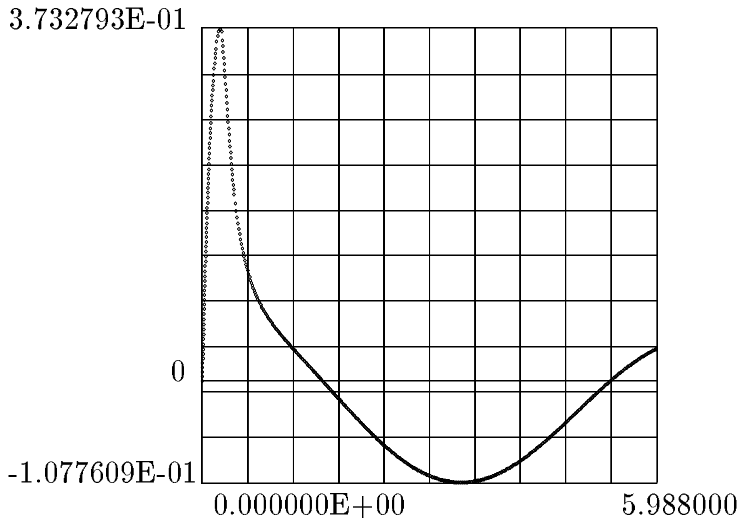

Figure 3.13: Super-twisting 2-sliding controller. Control $u(t)$.

where $x \in \mathbb{R}^n$, a, b, s are smooth functions, $u \in \mathbb{R}$. The relative degree r of the system is assumed to be constant and known. That means, in a simplified way, that u first appears explicitly only in the r-th total derivative of s and $\frac{d}{du}s^{(r)} \neq 0$ at the given point. The task is to fulfill the constraint $s(t,x) = 0$ in finite time and to keep it exactly by discontinuous feedback control. Since $s, \dot{s}, ..., s^{(r-1)}$ are continuous functions of t and x, the corresponding motion will correspond to an r-sliding mode.

Introduce new local coordinates $y = (y_1, ..., y_n)$, where $y_1 = s, y_2 = \dot{s}, ..., y_r = s^{(r-1)}$. Then

$$s^{(r)} = h(t,y) + g(t,y)u, \quad g(t,y) \neq 0,$$

$$\xi = \eta(t, s, \dot{s}, ..., s^{(r-1)}, \xi) + \gamma(t, s, \dot{s}, ..., s^{(r-1)}, \xi)u, \quad \xi = (y_{r+1}, ..., y_n). \quad (3.47)$$

Let a trivial controller $u = -K \text{sign} s$ be chosen with $K > \sup|u_{eq}|$, $u_{eq} = -h(t,y)/g(t,y)$ [55]. Then the substitution $u = u_{eq}$ defines a differential equation on the r-sliding manifold of (3.46). Its solution provides for the r-sliding motion. Usually, however, such a mode is not stable.

It is easy to check that $g = L_b L_a^{r-1} s = \frac{d}{du}s^{(r)}$. Obviously, $h = L_a^r s$ is the rth total time derivative of s calculated with $u = 0$. In other words, functions h and g may be defined in terms of input-output relations. Therefore, dynamic system (3.46) may be considered as a "black box".

The problem is to find a discontinuous feedback $u = U(t,x)$ causing finite-time convergence to an r- sliding mode. That controller has to generalize the

1-sliding relay controller $u = -K\text{sign}s$. Hence, $g(t, y)$ and $h(t, y)$ in (3.47) are to be bounded, $h > 0$. Thus, we require that for some $K_m, K_M, C > 0$

$$0 < K_m \leq \frac{\partial}{\partial u} s^{(r)} \leq K_M, \quad |L_a^r s| \leq C. \tag{3.48}$$

3.7.2 Controller construction

Let p be a positive number. Denote

$$N_{1,r} = |s|^{(r-1)/r},$$

$$N_{i,r} = \left(|s|^{p/r} + |\dot{s}|^{p/(r-1)} + ... + |s^{(i-1)}|^{p/(r-i+1)}\right)^{(r-i)/p},\ i = 1, ..., r-1,$$

$$N_{r-1,r} = \left(|s|^p/r + |\dot{s}|^{p/(r-1)} + ... + |s^{(r-2)}|^{p/2}\right)^{1/p}.$$

$$\psi_{0,r} = s,$$

$$\psi_{1,r} = \dot{s} + \beta_1 N_{1,r}\text{sign}(s),$$

$$\psi_{i,r} = s^{(i)} + \beta_i N_{i,r}\text{sign}(\psi_{i-1,r}),\ i = 1, ..., r-1,$$

where $\beta_1, ..., \beta_{r-1}$ are positive numbers.

Theorem 4 *Let system (3.46) have relative degree r with respect to the output function s and (3.48) be fulfilled. Then with properly chosen positive parameters $\beta_1, ..., \beta_{r-1}$ controller*

$$u = -\alpha\, \text{sign}(\psi_{r-1,r}(s, \dot{s}, ..., s^{(r-1)})). \tag{3.49}$$

provides for the appearance of r-sliding mode $s \equiv 0$ attracting trajectories in finite time.

The positive parameters $\beta_1, ..., \beta_{r-1}$ are to be chosen sufficiently large in the index order. Each choice determines a controller family applicable to all systems (3.46) of relative degree r. Parameter $\alpha > 0$ is to be chosen specifically for any fixed C, K_m, K_M. The proposed controller is easily generalized: coefficients of $N_{i,r}$ may be any positive numbers etc. Obviously, α is to be negative with $\frac{\partial}{\partial u} s^{(r)} < 0$.

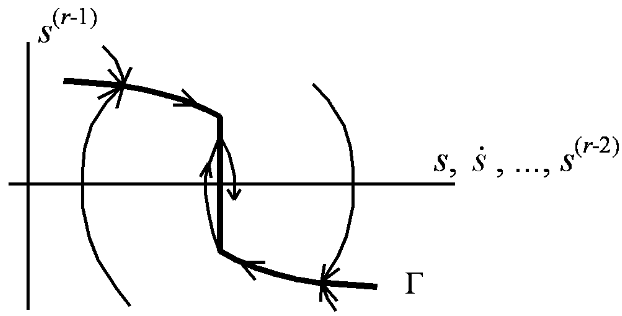

Figure 3.14: The idea of r-sliding controller

Certainly, the number of choices of β_i is infinite. Here are a few examples with β_i tested for $r \leq 4, p$ being the least common multiple of $1, 2, ..., r$. The first is the relay controller, the second is listed in section 3.6.

$$\begin{aligned}
&1. u = -\alpha \,\mathrm{sign}\, s \\
&2. u = -\alpha \,\mathrm{sign}(s + |s|^{1/2} \mathrm{sign} s), \\
&3. u = -\alpha \,\mathrm{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6} \mathrm{sign}(\dot{s} + |s|^{2/3} \mathrm{sign} s), \\
&4. u = -\alpha \,\mathrm{sign}\{s^{(3)} + 3(\ddot{s}^6 + \dot{s}^4 + |s|^3)^{1/12} \mathrm{sign}[\ddot{s} + \\
&\quad (\dot{s}^4 + |s|^3)^{1/6} \mathrm{sign}(\dot{s} + 0.5|s|^{3/4} \mathrm{sign} s)]\}, \\
&5. u = -\alpha \,\mathrm{sign}(s^{(4)} + \beta_4(|s|^{12} + |\ddot{s}|^{15} + |s|^{20} + \\
&\quad |s^{(3)}|^{30})^{1/60} \mathrm{sign}(s^{(3)} + \beta_3(|s|^{12} + |\dot{s}|^{15} + |\ddot{s}|^{20})^{1/30} \mathrm{sign}(\ddot{s} \\
&\quad + \beta_2(|s|^{12} + |\dot{s}|^{15})^{1/20} \mathrm{sign}(\dot{s} + \beta_1 |s|^{4/5} \mathrm{sign} s))))
\end{aligned}$$

The idea of the controller is that a 1-sliding mode is established on the smooth parts of the discontinuity set Γ of (3.49) (Fig.3.14). That sliding mode is described by the differential equation $\psi_{r-1,r} = 0$ providing in its turn for the existence of a 1-sliding mode $\psi_{r-1,r} = 0$. But the primary sliding mode disappears at the moment when the secondary one is to appear. The resulting movement takes place in some vicinity of the subset of Γ satisfying $\psi_{r-2,r} = 0$, transfers in finite time into some vicinity of the subset satisfying $\psi_{r-3,r} = 0$ and so on. While the trajectory approaches the r-sliding set, set Γ retracts to the origin in the coordinates $s, \dot{s}, ..., s^{(r-1)}$. Set Γ with $r = 3$ is shown in Fig. 3.15.

An interesting controller, so-called "terminal sliding mode controller", was proposed by [56]. In the 2-dimensional case it coincides with a particular case of the 2-sliding controller with given convergence law (Section

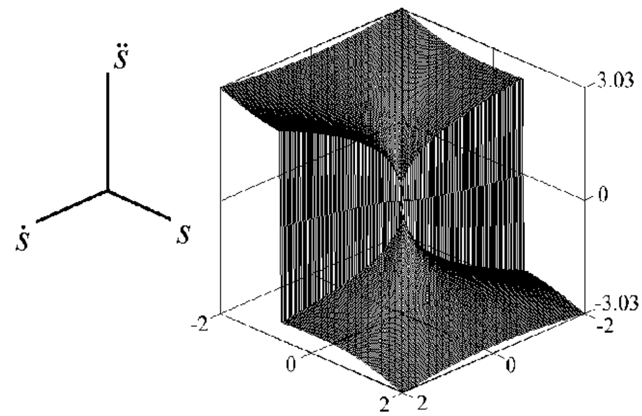

Figure 3.15: The discontinuity set of the 3-sliding controller

3.6). In the r-dimensional case a mode is produced at the origin similar to the r-sliding mode. The problem is that a closed-loop system with terminal sliding mode does not satisfy the Filippov conditions [22] for the solution existence with $r > 2$. Indeed, the control influence is unbounded in vicinities of a number of hyper-surfaces intersecting at the origin. The corresponding Filippov velocity sets are unbounded as well. Thus, some special solution definition is to be elaborated, the stability of the corresponding quasi-sliding mode at the origin and the very existence of solutions are to be shown.

Controller (3.49) requires the availability of $s, \dot{s}, ..., s^{(r-1)}$. The needed information may be reduced if the measurements are carried out at times t_i with constant step $\tau > 0$. Consider the controller

$$u(t) = -\alpha \,\mathrm{sign}(\Delta s_i^{(r-2)} + \beta_{r-1} \tau N_{r-1,r}(s_i, \dot{s}_i, ..., s_i^{(r-2)}) \\ \mathrm{sign}(\psi_{r-2,r}(s_i, \dot{s}_i, ..., s_i^{(r-2)})) \tag{3.50}$$

Theorem 5 *Under conditions of Theorem 4 with discrete measurements both algorithms (3.49) and (3.50) provide in finite time for some positive constants* $a_0, a_1, ..., a_{r-1}$ *for fulfillment of inequalities*

$$|s| < a_0 \tau^r, |\dot{s}| < a_1 \tau^{r-1}, ..., |s^{(r-1)}| < a_{r-1} \tau.$$

That is the best possible accuracy attainable with discontinuous $s^{(r)}$. Convergence time may be reduced by changing coefficients β_j. Another way is to substitute $\lambda^{-j}s^{(j)}$ for $s^{(j)}$, $\lambda^r\alpha$ for α and $\alpha\tau$ for τ in (3.49) and (3.50), $\lambda > 0$, causing convergence time to be diminished approximately by λ times.

Implementation of r-sliding controller when the relative degree is less than r. Introducing successive time derivatives $u, \dot{u}, ..., u^{(r-k-1)}$ as new auxiliary variables and $u^{(r-k)}$ as a new control, achieve different modifications of each r-sliding controller intended to control systems with relative degrees $k = 1, 2, ..., r$. The resulting control is $(r-k-1)$-smooth function of time with $k < r$, a Lipschitz function with $k = r-1$ and a bounded "infinite-frequency switching" function with $k = r$.

Chattering removal. The same trick removes the chattering effect. For example, substituting $u^{(r-1)}$ for u in (3.50), receive a local r - sliding controller to be used instead of the relay controller $u = -\mathrm{sign}\, s$ and attain rth order sliding precision with respect to τ by means of $(r-2)$-smooth control with Lipschitz $(r-2)$th time derivative. It has to be modified for global usage.

Controlling systems nonlinear on control. Consider a system $\dot{x} = f(t, x, u)$ nonlinear on control. Let $\frac{\partial}{\partial u}s^{(i)}(t, x, u) = 0$ for $i = 1, ..., r-1$, $\frac{\partial}{\partial u}s^{(r)}(t, x, u) > 0$. It is easy to check that

$$s^{(r+1)} = \Lambda_u^{r+1}s + \frac{\partial}{\partial u}s^{(r)}\dot{u}, \quad \Lambda_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot)f(t, x, u).$$

The problem is now reduced to that considered above with relative degree $r+1$ by introducing a new auxiliary variable u and a new control $v = \dot{u}$.

Discontinuity regularization. The complicated discontinuity structure of the above-listed controllers may be smoothed by replacing the discontinuities under the sign-function with their finite-slope approximations. As a result, the transient process becomes smoother. Consider, for example, the above-listed 3-sliding controller. The function $\mathrm{sign}(\dot{s} + |s|^{2/3}\mathrm{sign}\, s)$ may be replaced by the function $\max[-1, \min(1, |s|^{-2/3}(\dot{s} + |s|^{2/3}\mathrm{sign}\, s)/\varepsilon)]$ for some sufficiently small $\varepsilon > 0$. For $\varepsilon = 0.1$ the resulting tested controller is

$$u = -\alpha\, \mathrm{sign}(\ddot{s} + 2(|\dot{s}|^3 + |s|^2)^{1/6} \max[-1, \min(1, 10|s|^{-2/3}(\dot{s} + |s|^{2/3}\mathrm{sign}\, s))]). \tag{3.51}$$

Controller (3.51) provides for the existence of a standard 1-sliding mode on the corresponding continuous piece-wise smooth surface.

Theorem 6 *Theorems 4, 5 remain valid for controller (3.51).*

Real-time control of output variables.

The implementation of the above-listed r-sliding controllers requires real-time observation of the successive derivatives $s, \dot{s}, ..., s^{(r-1)}$. Thus, theoretically no model of the controlled process needs to be known. Only the relative degree and 3 constants are needed in order to adjust the controller. Unfortunately, the problem of successive real-time exact differentiation is usually considered to be practically unsolvable. Nevertheless, under some assumptions the real-time exact robust differentiation is possible. Indeed, let input signal $\eta(t)$ be a Lebesgue-measurable locally bounded function defined on $[0, \infty)$ and let it consist of a base signal $\eta_0(t)$ having a derivative with Lipschitz's constant $C > 0$ and a bounded measurable noise $N(t)$. Then the following system realizes *a real-time differentiator* [37]:

$$\dot{\nu} = v,\ v = \nu_1 - \lambda|\nu - \eta(t)|^{1/2}\mathrm{sign}(\nu - \eta(t)),\ \dot{\nu}_1 = -\mu\,\mathrm{sign}(\nu - \eta(t))$$

where $\mu, \lambda > 0$. Here $v(t)$ is the output of the differentiator. Solutions of the system are understood in the Filippov sense. Parameters may be chosen in the form $\mu = 1.1C$, $\lambda = 1.5C^{1/2}$, for example (it is only one of possible choices). That differentiator provides for finite-time convergence to the exact derivative of $\eta_0(t)$ if $N(t) = 0$. Otherwise, if sup $N(t) = \varepsilon$ it provides for accuracy proportional to $C^{1/2}\varepsilon^{1/2}$. Therefore, having been implemented k times successively, that differentiator will provide for kth order differentiation accuracy of the order of $\varepsilon^{(2^{-k})}$. Thus, full local real-time robust control of output variables is possible, using only output variable measurements and knowledge of the relative degree [41].

When the base signal $\eta_0(t)$ has (r-1)th derivative with Lipschitz's constant $C > 0$, the best possible kth order differentiation accuracy is $d_k\ C^{k/r}\ \varepsilon^{(r-k)/r}$, where $d_k > 1$ may be estimated (this asymptotics may be improved with additional restrictions on $\eta_0(t)$). Moreover, it is proved that such a robust exact differentiator really exists [37]. The corresponding differentiator has been submitted by A. Levant for possible presentation at the European Control Conference in Portugal (2001).

Theorem 7 *An optimal k-th order differentiator having been applied, r - sliding controller (3.49) provides locally for the sliding accuracy* $|s^{(i)}| < c_i\varepsilon^{(r-i)/r}$, $i = 0, 1, ..., r-1$, *where* ε *is the maximal possible error of real-time measurements of* s *and* c_i *are some positive constants.*

Theorem 7 probably determines the best sliding asymptotics attainable when only s is available.

3.7.3 Examples

Car control.

Consider a simple kinematic model of car control [45]

$$\dot{x} = v \cos\varphi, \ \dot{y} = v \sin\varphi,$$

$$\dot{\varphi} = \frac{v}{l} \tan\delta,$$

$$\dot{\delta} = u,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and δ is the steering angle. The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while x, y and φ are assumed to be measured in real time. Define

$$s = y - g(x).$$

Let $v = \text{const} = 10m/s$, $l = 5m$, $g(x) = 10 \sin 0.05x + 5$, $x = y = \varphi = \delta = 0$ at $t = 0$. The relative degree of the system is 3 and both 3-sliding controller *No.*3 and its regularized form (3.51) may be applied here. It was taken $\alpha = 20$. The corresponding trajectories are the same, but the performance is different. The trajectory and function $y = g(x)$ with measurement step $\tau = 2 \cdot 10^{-4}$ are shown in Fig.3.16. Graphs of $s, \dot{s}, \ddot{s}$ are shown in Fig. 3.16, 3.17 for regularized and not regularized controllers respectively.

4-sliding control.

Consider a model example of a tracking system. Let input $z(t)$ and the control system satisfy equations

$$z^{(4)} + 3\ddot{z} + 2z = 0,$$

$$x^{(4)} = u.$$

The task is to track z by x, $s = x - z$, the 4th controller with $\alpha = 40$ is used. Initial conditions for z and x at time $t = 0$ are

$$z(0) = 0, \ \dot{z}(0) = 0, \ \ddot{z}(0) = 2, \ z^{(3)}(0) = 0;$$

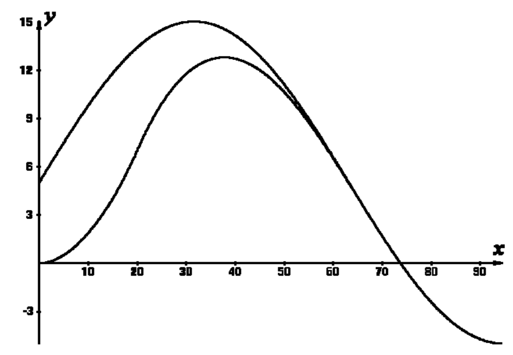

Figure 3.16: Car trajectory tracking

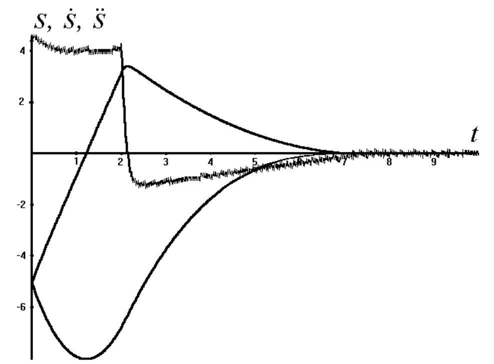

Figure 3.17: Regularized 3-sliding controller

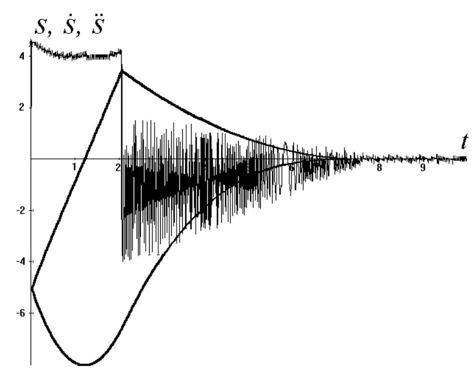

Figure 3.18: Standard 3-sliding controller

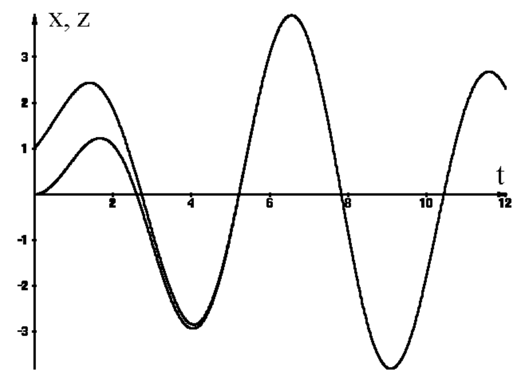

Figure 3.19: 4 - sliding tracking

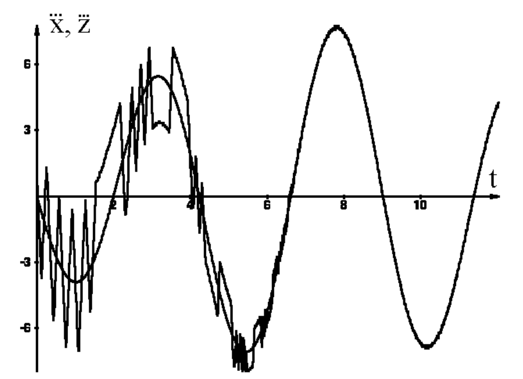

Figure 3.20: Third derivative tracking

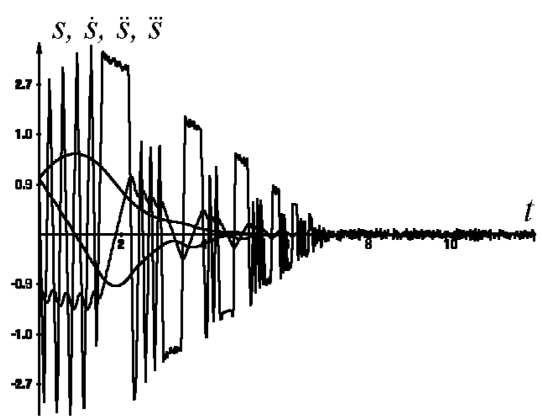

Figure 3.21: Tracking deviation and its three derivatives

$$x(0) = 1, \dot{x}(0) = 1, \ddot{x}(0) = 1, \; x^{(3)}(0) = 1.$$

A mutual graph of x and z with $\tau = 0.01$ is shown in Fig. 3.19. A mutual graph of $x^{(3)}$ and $z^{(3)}$ with $\tau = 0.001$ is shown in Fig. 3.20. Mutual graphs of $s, \dot{s}, \ddot{s}, s^{(3)}$ with $\tau = 0.001$ are demonstrated in Fig. 3.21. The attained accuracies are $|s| \leq 1.33 \cdot 10^{-4}$ with $\tau = 0.01$ and $|s| \leq 1.49 \cdot 10^{-12}$ with $\tau = 0.0001$.

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3.8 Conclusions

- A general review of the current state of the higher order sliding theory, its main notions and results were presented.
- It was demonstrated that higher order sliding modes are natural phenomena for relay control systems if the relative degree of the system is more than 1 or a dynamic actuator is present.
- Stability was studied of second order sliding modes in relay systems with fast stable dynamic actuators of relative degree 1.
- Instability of higher order sliding modes was shown in relay systems with dynamic actuators of relative degree 2 and more.
- A number of the most popular 2-sliding controllers were listed and compared.
- A family of arbitrary order sliding controllers with finite time convergence was presented.
- The discrete switching modification of presented sliding controllers provide for the sliding precision of their order with respect to the measurement time interval.
- A robust exact differentiator was presented allowing for full control of output variables using only measurements of their current values.
- A number of simulation examples were presented.

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