Chapter 1

INTRODUCTION TO HIGH-ORDER SLIDING MODES

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1.1 Introduction

One of the most important control problems is control under heavy uncertainty conditions. While there are a number of sophisticated methods like adaptation based on identification and observation, or absolute stability methods, the most obvious way to withstand the uncertainty is to keep some constraints by "brutal force". Indeed any strictly kept equality removes one "uncertainty dimension". The most simple way to keep a constraint is to react immediately to any deviation of the system stirring it back to the constraint by a sufficiently energetic effort. Implemented directly, the approach leads to so-called sliding modes, which became main operation modes in the variable structure systems (VSS) [52]. Having proved their high accuracy and robustness with respect to various internal and external disturbances, they also reveal their main drawback: the so-called chattering effect, i.e. dangerous high-frequency vibrations of the controlled system. Such an effect was considered as an obvious intrinsic feature of the very idea of immediate powerful reaction to a minutest deviation from the chosen constraint. Another important feature is proportionality of the maximal deviation from the constraint to the sampling time interval (or to the switching delay).

To avoid chattering some approaches were proposed . The main idea was to change the dynamics in a small vicinity of the discontinuity surface in order to avoid real discontinuity and at the same time to preserve the main properties of the whole system. In particular, high-gain control with saturation approximates the sign-function and diminishes the chattering, while on-line estimation of the so-called equivalent control [52] is used to reduce the discontinuous-control component [49]; the sliding-sector method [25] is suitable to control disturbed linear time-invariant systems; the changing speed of the control value is artificially bounded, removing the chattering effect [13]. However, the ultimate accuracy and robustness of the sliding mode were partially lost. On the contrary, higher order sliding modes (HOSM) generalize the basic sliding mode idea acting on the higher order time derivatives of the system deviation from the constraint instead of influencing the first deviation derivative like it happens in standard sliding modes. Keeping the main advantages of the original approach, at the same time they totally remove the chattering effect and provide for even higher accuracy in realization. A number of such controllers were described in the literature [29, 14, 30, 33, 3, 5, 37, 24] . As we will see soon, these modes may not only remove the chattering effect, but also completely solve a number of "black-box" control problems when actually only the relative degree of the system is known [40]. One of the sudden applications of these modes is construction of an exact robust differentiator with finite-time convergence [32, 38].

The purpose of this Chapter is not to present an accurate review of the current achievements in the field, but to provide a simple introduction to the theory and to demonstrate the main features and abilities of higher order sliding modes.

1.2 Elementary Introduction to Higher-Order Sliding Modes

HOSM is actually a movement on a discontinuity set of a dynamic system understood in Filippov's sense [20]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the mode. If the task is to provide for keeping a constraint given by equality of a smooth function σ to zero, the sliding order is a number of continuous total derivatives of σ (including the zero one) in the vicinity of the sliding mode. Hence, the rth order sliding mode is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = ... = \sigma^{(r-1)} = 0. \tag{1.1}$$

forming an r-dimensional condition on the state of the dynamic system. The words "rth order sliding" are often abridged to "r-sliding". Standard sliding mode is called in this notation 1-sliding mode, for $\dot{\sigma}$ is discontinuous. A simple introduction to 2-sliding modes is presented in this Section.

1.2.1 Twisting controller

Consider a simple example. Let an uncertain dynamic system be given by the following differential equation

$$\ddot{x} = a(t) + b(t)u, \tag{1.2}$$

where

$$|a| \leq 1, \qquad 1 \leq b \leq 2 \tag{1.3}$$

are unknown, $u \in \mathbf{R}$ is the control. The goal is to stabilize the system at the origin. Mark that any approach except for the VSS can hardly do the job. The standard VSS way is to define a constraint like $\sigma = x + \dot{x} = 0$ and to keep it by the controller

$$\sigma = x + \dot{x}, \quad u = -2\,\mathrm{sign}\,\sigma. \tag{1.4}$$

That controller solves the problem locally. Indeed, the calculation

$$\dot{\sigma} = \dot{x} + \ddot{x} = \dot{x} + a(t) - 2b(t)\,\mathrm{sign}\,\sigma$$

shows that with $|\dot{x}| < 1$ the control dominates, $\sigma\dot{\sigma} < 0$ and σ vanishes in finite time. The global controller is also easily constructed:

$$u = -2(|x| + |\dot{x}| + 1)\,\mathrm{sign}\,\sigma,$$

but it is unbounded. The transient time is infinite and in the presence of a small switching delay the accuracy is proportional to the delay [49]. The phase portrait is given in Fig. 1.1.

Consider now a simple 2-sliding controller, namely the "twisting" controller [29, 14, 30]

$$u = -5 \operatorname{sign} x - 3 \operatorname{sign} \dot{x}. \tag{1.5}$$

It will be shown a little bit later that it provides for the global finite time convergence to the origin. The control is obviously bounded here.

The 2-sliding mode exists in the both systems (1.2), (1.4) and (1.2), (1.5) at the origin $x = \dot{x} = 0$ only. Indeed, the functions x, $\dot{x}$ are obviously continuous functions of the state space coordinates t, x, $\dot{x}$, and the origin is described by the equations $x = \dot{x} = 0$. Thus, there is an asymptotically stable 2-sliding mode with respect to the constraint $x = 0$ in system (1.2), (1.4) (Fig. 1.1) and a 2-sliding mode attracting in finite time in (1.2), (1.5) (Fig. 1.2). At the same time there is a 1-sliding mode with respect to the constraint $x + \dot{x} = 0$ in (1.2), (1.4). It exists on the line $x + \dot{x} = 0$.

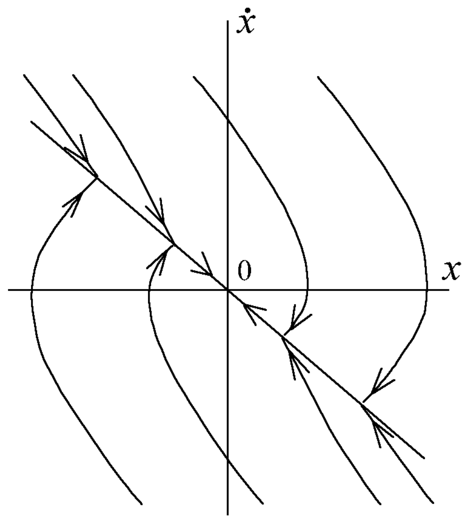

Figure 1.1: Classical 2-dimensional VSS

In order to demonstrate the chattering elimination, consider even a simpler dynamic system

$$\dot{x} = a(t) + b(t)u, \tag{1.6}$$

where the condition $|\dot{a}| \leq 1$, $|\dot{b}| \leq 1$ is added to (1.3). The traditional VSS controller is $u = -2 \operatorname{sign} x$. As a result a global finite-time attracting 1-sliding mode appears at the origin. The 2-sliding controller removing the chattering

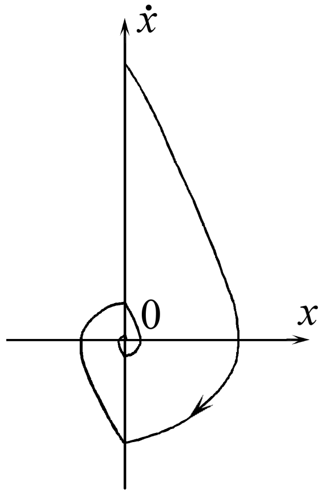

Figure 1.2: Twisting controller: phase portrait

is given by

$$\dot{u} = \begin{cases} -u & \text{with } |u| > 2, \\ -10 \operatorname{sign} x - 6 \operatorname{sign} \dot{x} & \text{with } |u| \leq 2. \end{cases} \qquad (1.7)$$

The state space is extended here to include the coordinate u. The functions x and $\dot{x} = a(t) + b(t)u$ are obviously continuous functions of the state space coordinates t, x, u, and the 2-sliding mode is described by the equations $x = 0$, $a(t) + b(t)u = 0$. Thus, in the 2-sliding mode $u = a(t)/b(t)$, which means that the control is continuous and coincides with the equivalent control [52]. The problem in realization of this approach is that $\dot{x}$ is often not available. In that case $\operatorname{sign} \dot{x}$ is replaced by $\operatorname{sign} \Delta x_i$ where $\Delta x_i = x(t_i) - x(t_{i-1})$ and the current time $t \in [t_i, t_{i+1})$.

Consider now a general case when

$$|a(t, x)| \leq C, \quad 0 \leq K_m \leq b(t, x) \leq K_M, \qquad (1.8)$$

$$u = -r_1 \operatorname{sign} x - r_2 \operatorname{sign} \dot{x}, \quad r_1 > r_2 > 0. \qquad (1.9)$$

Lemma 1 *Let r_1 and r_2 satisfy the conditions*

$$K_m(r_1 + r_2) - C > K_M(r_1 - r_2) + C, \; K_m(r_1 - r_2) > C. \qquad (1.10)$$

Then controller (1.9) provides for the appearance of a 2-sliding mode $x = \dot{x} = 0$ attracting the trajectories in finite time.

Proof. It is easy to see that every trajectory of the system crosses the axis $x = 0$ in finite time. Indeed, due to (1.8), (1.10) $\operatorname{sign} x \operatorname{sign} \ddot{x} < 0$ and with $\operatorname{sign} x$ being constant for a long time, $\operatorname{sign} x \operatorname{sign} \dot{x} < 0$ is established, while the absolute value of $\dot{x}$ tends to infinity. It follows from (1.10) that with $x \neq 0$

$$\begin{array}{ll} -[K_M(r_1+r_2)+C] \leq \ddot{x} \operatorname{sign} x \leq -[K_m(r_1+r_2)-C] < 0 & \text{with } \dot{x}x > 0, \\ -[K_M(r_1-r_2)+C] \leq \ddot{x} \operatorname{sign} x \leq -[K_m(r_1-r_2)-C] < 0 & \text{with } \dot{x}x < 0. \end{array} \tag{1.11}$$

As always with the Filippov definitions the values taken on a set of the measure 0 do not matter. Let $\dot{x}_0 x_M \dot{x}_M$ (Fig. 1.3) be the trajectory of the differential equation

$$\ddot{x} = \begin{cases} -[K_m(r_1+r_2)-C] \operatorname{sign} x & \text{with } \dot{x}x > 0, \\ -[K_M(r_1-r_2)+C] \operatorname{sign} x & \text{with } \dot{x}x \leq 0. \end{cases} \tag{1.12}$$

with the same initial conditions. Assume now for simplicity that the initial values are $x = 0$, $\dot{x} = \dot{x}_0 > 0$ at $t = 0$. Thus, the trajectory enters the half-plane $x > 0$. Simple calculation shows that with $x > 0$ the solution of (1.12) is determined by the equalities

$$\begin{array}{ll} x = x_M - \frac{\dot{x}^2}{2[K_m(r_1+r_2)-C]} & \text{with } \dot{x} > 0, \\ x = x_M - \frac{\dot{x}^2}{2[K_M(r_1-r_2)+C]} & \text{with } \dot{x} \leq 0, \end{array}$$

where $2[K_m(r_1+r_2)-C]x_M = \dot{x}_0^2$. Consider any point $P(x_P, \dot{x}_P)$of this curve (Fig. 1.3). The velocity of (1.2), (1.9) at this point has coordinates $(\dot{x}_P, \ddot{x}_P)$. Hence, the horizontal component of the velocity depends only on the point itself. Since the vertical component satisfies the inequalities (1.11), the velocity of (1.2), (1.9) always "looks" into the region bounded by the axis $x = 0$ and curve (1.12). That curve is called the majorant [30]. Let the trajectory of (1.2), (1.9) intersect the next time with the axis $x = 0$ at the point $\dot{x}_1$. Then, obviously, $|\dot{x}_1| \leq |\dot{x}_M|$ and

$$|\dot{x}_1|/|\dot{x}_0| \leq [K_M(r_1-r_2)+C]/[K_m(r_1+r_2)-C] = q < 1.$$

Extending the trajectory into the half=plane $x < 0$ after a similar reasoning achieve that the successive crossings of the axis $x = 0$ satisfy the inequality $|\dot{x}_{i+1}|/|\dot{x}_i| \leq q < 1$ (Fig. 1.2). Therefore, the algorithm obviously converges. The convergence time is to be estimated now. The real trajectory

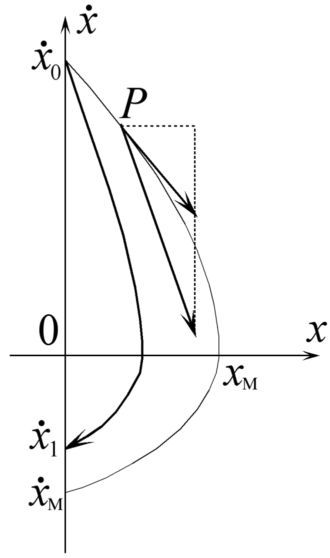

Figure 1.3: Majorant curve of the twisting controller

consists of infinite number of segments belonging to the half-planes $x \geq 0$ and $x \geq 0$. On each of these segments $\dot{x}$ changes monotonously according to (1.11). The total variance of the function $\dot{x}(t)$ is

$$Var(\dot{x}(\cdot)) = \sum |\dot{x}_i| \leq |\dot{x}_0|(1 + q + q^2 + ...) = \frac{|\dot{x}_0|}{1-q},$$

and the total convergence time is estimated as

$$T \leq \sum \frac{|\dot{x}_i|}{[K_m(r_1 - r_2) - C]} \leq \frac{|\dot{x}_0|}{(1-q)[K_m(r_1 - r_2) - C]}$$

■

Important remark: in practice the parameters are *never* assigned according to inequalities (1.10). Usually the real system is not exactly known, the model itself is not really adequate, and the estimations of parameters K_M,K_m,C are much larger than the actual values (often 100 times larger!). The larger the controller parameters, the more sensitive is the controller to any switching imperfections and measurement noises. Thus, the right way is to adjust the controller parameters during computer simulation. That remark is true with respect to all controllers described in this Chapter.

1.2.2 Super-twisting controller

As it was shown, the twisting controller (1.7) requires real-time measurements of $\dot{x}$ or just of $\operatorname{sign} \dot{x}$, when used to remove chattering. In other words, in order to provide for $x = \dot{x} = 0$ both x and $\dot{x}$ measurements are needed. That is reasonable, but, nevertheless, not inevitable. Consider once more the dynamic system

$$\dot{x} = a(t) + b(t)u, \tag{1.13}$$

and suppose that for some positive constants C, K_M, K_m, U_M, q

$$|\dot{a}| + U_M|\dot{b}| \leq C, \quad 0 \leq K_m \leq b(t,x) \leq K_M, \quad |a/b| < qU_M, \quad 0 < q < 1. \tag{1.14}$$

The following controller does not need measurements of $\dot{x}$. Let

$$u = -\lambda |x|^{1/2} \operatorname{sign} x + u_1, \qquad \dot{u}_1 = \begin{cases} -u, & |u| > U_M, \\ -\alpha \operatorname{sign} x, & |u| \leq U_M. \end{cases}$$

Lemma 2 *With $K_m \alpha > C$ and λ sufficiently large the controller provides for the appearance of a 2-sliding mode $x = \dot{x} = 0$ attracting the trajectories in finite time. The control u enters in finite time the segment $[-U_M, U_M]$ and stays there. It never leaves the segment if the initial value is inside at the beginning.*

The latter controller is called super-twisting controller. The corresponding phase portrait is shown in Fig. 1.4 . A sufficient (*very crude!*) condition for validity of the Lemma is

$$\lambda > \sqrt{\frac{2}{(K_m\alpha - C)}} \frac{(K_m\alpha + C)K_M(1+q)}{K_m^2(1-q)}. \tag{1.15}$$

Proof. Calculate $\dot{u}$ with $|u| > U_M$ and obtain $\dot{u} = -\frac{1}{2}\lambda \dot{x}|x|^{-1/2} - u$. It follows from (1.13), (1.14) that $\dot{x}u > 0$ with $|u| > U_M$. Thus, $\dot{u}u < 0$, $|\dot{u}| > U_M$ when $|u| > U_M$, and $|u| \leq U_M$ is established in finite time. Nevertheless, a 1-sliding mode $u = -\operatorname{sign} x$ is still possible during time intervals with constant $\operatorname{sign} x$. The following equation is satisfied with $|u| < U_M$, $x \neq 0$:

$$\ddot{x} = \dot{a} + \dot{b}u - b\frac{1}{2}\lambda \frac{\dot{x}}{|x|^{1/2}} - b\alpha \operatorname{sign} x.$$

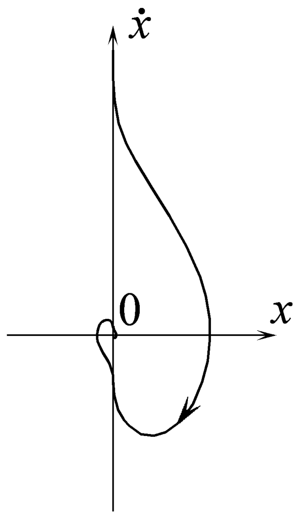

Figure 1.4: Super-twisting controller: phase portrait

The trivial identity $\frac{d}{dt}|x| = \dot{x}\,\mathrm{sign}\,x$ is used here. Note once more that the values taken on sets of measure 0 are not accounted for, thus the differentiation is performed when $\mathrm{sign}\,x = \mathrm{const}$. The latter equation may be rewritten as

$$\ddot{x} \in [-C, C] - [K_m, K_M](\frac{1}{2}\lambda\frac{\dot{x}}{|x|^{1/2}} + \alpha\,\mathrm{sign}\,x). \tag{1.16}$$

This inclusion does not "remember" anything on the original system. Similarly to the proof of Lemma 1 with $x > 0$, $\dot{x} > 0$ the real trajectory is confined by the axes $x = 0$, $\dot{x} = 0$ and the trajectory of the equation $\ddot{x} = -(K_m\alpha - C)$. Let x_M be the intersection of this curve with axis $\dot{x} = 0$. Obviously, $2(K_m\alpha - C)x_M = \dot{x}_0^2$ (Fig. 1.5). It is easy to see from (1.5) that

$$x > 0, \quad \dot{x} > 0, \ \frac{1}{2}\lambda\frac{|\dot{x}|}{|x|^{1/2}} > \frac{C}{K_m} + \alpha \quad \Longrightarrow \ \ddot{x} \ > 0.$$

Thus, the majorant curve with $x > 0$ may be taken as follows (Fig. 1.5):

$$\begin{aligned} \dot{x}^2 &= 2(K_m\alpha - C)(x_M - x) \text{ with } \dot{x} > 0, \\ x &= x_M \ \text{ with } 0 \geq \dot{x} \geq -\frac{2}{\lambda}(\frac{C}{K_m} + \alpha)x^{1/2}, \\ \dot{x} &= \dot{x}_M = -\frac{2}{\lambda}(\frac{C}{K_m} + \alpha)x_M^{1/2} \text{ with } \dot{x} > -\frac{2}{\lambda}(\frac{C}{K_m} + \alpha)x^{1/2}. \end{aligned}$$

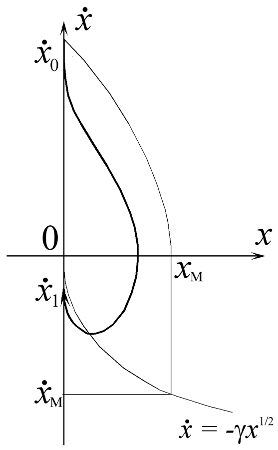

Figure 1.5: Super-twisting controller: the proof

The condition $|\dot{x}_M/\dot{x}_0| < 1$ is sufficient for the algorithm convergence while $|u| < U_M$. That condition is

$$\frac{2(K_m\alpha + C)^2}{\lambda^2 K_m^2(K_m\alpha - C)} < 1.$$

Unfortunately, the latter inequality is still not sufficient, for this consideration does not include possible 1-sliding mode keeping $u = \pm U_M$. It is easy to see that such a mode is not possible with $x > 0$, $\dot{x} > 0$. Indeed, in that case $\dot{u}$ stays negative and does not allow any sign switching. On the other hand, as follows from (1.13), (1.14) and $|u| \leq U_M$

$$|\dot{x}| = b|(a/b + u)| \leq K_M(1+q)U_M, \ |\dot{x}| = b|(a/b + u)| \geq K_m(1-q)U_M.$$

Thus, $\dot{x}_0 \leq K_M(1+q)U_M$, and the condition

$$|\frac{\dot{x}_M}{\dot{x}_0}| < \frac{K_m(1-q)U_M}{K_M(1+q)U_M} = \frac{K_m(1-q)}{K_M(1+q)}$$

is sufficient to avoid keeping $u = \pm U_M$ in sliding mode. The resulting final condition coincides with (1.15).

It is required now to prove the finite-time convergence. It is sufficient to consider only sufficiently small vicinity of the origin where $|u| < U_M$ is guaranteed. Consider an auxiliary variable $\xi = a(t) + b(t)u_1$. Obviously,

$\xi = \dot{x}$ at the moments when $x = 0$, and $u_1 \to -a/b$ as $t \to \infty$. Thus, $\xi = b(a/b + u_1)$ tends to zero. Starting from the moment when $|u_1| < U_M$ holds, its derivative $\dot{\xi} = \dot{a} + \dot{b}u_1 - b\alpha \,\mathrm{sign}\, x$ satisfies the inequalities

$$0 < K_m\alpha - C \leq -\dot{\xi} \,\mathrm{sign}\, x \leq K_M\alpha + C.$$

Like in the proof of Lemma 1, the total variation of ξ equals $\sum |\dot{x}_i|$, is majored by a geometric series and therefore converges. The total convergence time $T \leq \sum |\dot{x}_i|/(K_m\alpha - C)$. ■

1.2.3 First-order differentiator

The super-twisting controller is used to control systems of the relative degree 1. In other words it can be used instead of a standard 1-sliding-mode controller in order to avoid chattering. Nevertheless, with the relative degree 2 the 2-sliding controller, like a twisting one, is needed to stabilize system (1.2) in finite time. In order to avoid usage of $\dot{x}$ measurements a differentiator (observer) is needed. Popular linear high-gain observers [1] cannot do the job here, for they provide only for asymptotic stabilization at an equilibrium state. The latter requirement is the most restrictive, for it actually means here that $a(t) \equiv 0$ or $a = \mathrm{const.}$ $b = \mathrm{const.}$ The differentiator needed here has to feature robust exact differentiation with finite-time convergence.

Let input signal $f(t)$ be a function defined on $[0, \infty)$ consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown base signal $f_0(t)$ with the first derivative having a known Lipschitz constant $L > 0$. The problem is to find real-time robust estimations of $f_0(t)$, $\dot{f}_0(t)$ being exact in the absence of measurement noises.

Consider the auxiliary system $\dot{z} = v$, where v is control. Let $x = z - f_0$ and let the task be to keep $x = 0$ in 2-sliding mode. In that case $\dot{x} = x = 0$, which means that $z = f_0$ and $\dot{f}_0 = v$. The system can be rewritten as

$$\dot{x} = -\dot{f}_0(t) + v, \quad |\ddot{f}_0| \leq L.$$

The function $\dot{f}_0$ does not need to be smooth here, its derivative $\ddot{f}_0$ exists almost everywhere due to the Lipschitz property of $\dot{f}_0$. A modification of the super-twisting controller can be applied here:

$$\begin{aligned} v &= -\lambda_1 |x|^{1/2} \,\mathrm{sign}\, x + z_1, \\ \dot{z}_1 &= -\lambda_2 \,\mathrm{sign}\, x. \end{aligned}$$

The modification is needed here, for neither $\dot{f}_0(t)$ nor v is bounded. The differentiator takes on the resulting form

$$\begin{aligned} \dot{z} &= v = -\lambda_1 |z - f(t)|^{1/2} \operatorname{sign}(z - f(t)) + z_1, \\ \dot{z}_1 &= -\lambda_2 \operatorname{sign}(z - f(t)), \end{aligned} \tag{1.17}$$

where both v and z_1 can be taken as the differentiator outputs.

Theorem 3 *[32] For any $\lambda_2 > L$ with every sufficiently large λ_1 both v and z_1 converge in finite time to $\dot{f}_0(t)$.*

The proof of the Theorem is actually contained in the proof of Lemma 2. The sufficient convergence conditions are

$$\lambda_2 > L, \quad \frac{2(\lambda_2 + L)^2}{\lambda_1^2(\lambda_2 - L)} < 1. \tag{1.18}$$

Theorem 4 *[32] Let the input noise satisfy the inequality $|f(t) - f_0(t)| \leq \varepsilon$. Then the following inequalities are established in finite time for some positive constants μ, ν, η depending exclusively on the parameters of the differentiator:*

$$|z - f_0(t)| \leq \mu\varepsilon, \quad |z_1 - \dot{f}_0(t)| \leq \nu\varepsilon^{1/2}, \quad |v - \dot{f}_0(t)| \leq \eta\varepsilon^{1/2}.$$

Sketch of the proof. Let $\sigma = z - f_0(t)$, $\xi = z_1 - \dot{f}_0(t)$, then

$$\dot{\xi} = -\ddot{f}_0(t) - \lambda_2 \operatorname{sign} \sigma \in [-L, L] - \lambda_2 \operatorname{sign} \sigma,$$

and the differentiator equations in the absence of the input noise may be replaced by the inclusion

$$\begin{aligned} \dot{\sigma} &= -\lambda_1 |\sigma|^{1/2} \operatorname{sign} \sigma + \xi, \\ \dot{\xi} &\in -[\lambda_2 - L, \lambda_2 + L] \operatorname{sign} \sigma. \end{aligned} \tag{1.19}$$

Its solutions converge to the origin $\sigma = 0$, $\xi = 0$ in finite time. With $\varepsilon \neq 0$ inclusion (1.19) turns into

$$\begin{aligned} \dot{\sigma} &= -\lambda_1 |\sigma + [-\varepsilon, \varepsilon]|^{1/2} \operatorname{sign}(\sigma + [-\varepsilon, \varepsilon]) + \xi, \\ \dot{\xi} &\in -[\lambda_2 - L, \lambda_2 + L] \operatorname{sign}(\sigma + [-\varepsilon, \varepsilon]). \end{aligned} \tag{1.20}$$

With small $\varepsilon = \varepsilon_0$ the trajectories are concentrated in a small set $\sigma \leq \varkappa_1$, $\xi \leq \varkappa_2$ and stay there forever. Apply combined transformation of coordinates, time and parameters

$$G_\nu : (\sigma, \xi, t, \varepsilon_0) \mapsto (\nu^2\sigma, \nu\xi, \nu t, \nu^2\varepsilon_0).$$

It is easy to see that the trajectories of inclusion (1.20) are transferred into the trajectories of the same inclusion but with the different noise magnitude $\varepsilon = \nu^2\varepsilon_0$. Thus, $\nu = \sqrt{\varepsilon/\varepsilon_0}$ and the new attracting invariant set is given by the inequalities $\sigma \leq \nu^2\varkappa_1 = (\varkappa_1/\varepsilon_0)\varepsilon$, $\xi \leq \nu\varkappa_2 = (\varkappa_2/\sqrt{\varepsilon_0})\varepsilon$. ■

Lemma 5 *[32] Let parameters $\lambda_1 = \Lambda_1$, $\lambda_2 = \Lambda_2$ of differentiator (1.17) provide for exact differentiation with $L = 1$. Then the parameters $\lambda_1 = \Lambda_1 L^{1/2}$, $\lambda_2 = \Lambda_2 L$ are valid for any $L > 0$ and provide for the accuracy*

$$|z - f_0(t)| \leq \mu\varepsilon, \quad |z_1 - \dot{f}_0(t)| \leq \nu L^{1/2}\varepsilon^{1/2}, \quad |v - \dot{f}_0(t)| \leq \eta L^{1/2}\varepsilon^{1/2}.$$

for some positive constants μ, ν, η.

Proof. Denote $\tilde{f} = f/L$, then the following differentiator provides for the exact differentiation of $\tilde{f}(t)$:

$$\begin{aligned} \dot{\tilde{z}} &= -\Lambda_1|\tilde{z} - \tilde{f}(t)|^{1/2}\operatorname{sign}(\tilde{z} - \tilde{f}(t)) + \tilde{z}_1, \\ \dot{\tilde{z}}_1 &= -\Lambda_2 \operatorname{sign}(\tilde{z} - \tilde{f}(t)). \end{aligned} \quad (1.21)$$

Multiplying by L and defining $z = L\tilde{z}$, $z_1 = L\tilde{z}_1$, achieve the statement of the Lemma.■

Convergence condition (1.18) is very conservative. A special integral convergence criterion is developed in [32]. In particular, the parameter choices $\lambda_1 = 1.5L^{1/2}$, $\lambda_2 = 1.1L$ and $\lambda_1 = L^{1/2}$, $\lambda_2 = 2L$ are valid, though they do not satisfy (1.18). Note that while v is noisy in the presence of the input noises, z_1 is a Lipschitzian signal, but input noises lead to small phase delay.

Computer simulation

It was taken that $t_0 = 0$, initial values of the internal variable $x(0)$ and the measured input signal $f(0)$ coincide, initial value of the output signal $u(0)$ is zero. The simulation was carried out by the Euler method with measurement and integration steps equaling 10^{-4}.

Compare the proposed differentiator (1.17) with a simple linear differentiator described by the transfer function $p/(0.1p+1)^2$. Such a differentiator is actually a combination of the ideal differentiator and a low-pass filter. Let $\lambda_1 = 6$, $\lambda_2 = 8$. The output signals for inputs $f(t) = \sin t + 5t$, $f(t) = \sin t + 5t + 0.01 \cos 10t$, and $f(t) = \sin t + 5t + 0.001 \cos 30t$ and ideal derivatives $\dot{f}_0(t)$ are shown in Fig. 1.6. The linear differentiator is seen not to differentiate exactly. At the same time it is highly insensitive to any signals with frequency above 30. The proposed differentiator handles properly any input signal f with $\ddot{f} \le 7$ regardless the signal spectrum.

1.2.4 Output-feedback control

We are able now to construct a 2-sliding controller for system (1.2)

$$\begin{aligned} \ddot{x} &= a(t) + b(t)u, \\ |a(t,x)| &\le C, \quad 0 \le K_m \le b(t,x) \le K_M, \end{aligned} \tag{1.22}$$

which solves the stabilization problem in finite time using only measurements of x. It is possible due to the boundedness of $\ddot{x}$. Combining the twisting controller and the differentiator achieve

$$\begin{aligned} &u = -r_1 \operatorname{sign} z - r_2 \operatorname{sign} z_1, \quad r_1 > r_2 > 0. \\ &\dot{z} = -\lambda_1 |z - x|^{1/2} \operatorname{sign}(z - x) + z_1, \\ &\dot{z}_1 = -\lambda_2 \operatorname{sign}(z - x), \\ &\lambda_1 = 1.5(K_m(r_1 + r_2) + C)^{1/2}, \lambda_2 = 1.1(K_m(r_1 + r_2) + C) \end{aligned} \tag{1.23}$$

where the convergence condition is

$$K_m(r_1 + r_2) - C > K_M(r_1 - r_2) + C, \; K_m(r_1 - r_2) > C. \tag{1.24}$$

As a consequence of Lemma 1 and Theorem 3 achieve exact stabilization and finite-time convergence. It is proved [39] that in the presence of a bounded Lebesgue-measurable noise with the maximal magnitude ε the steady state accuracies $\sup |x|$ and $\sup |\dot{x}|$ are proportional to ε and $\sqrt{\varepsilon}$ respectively.

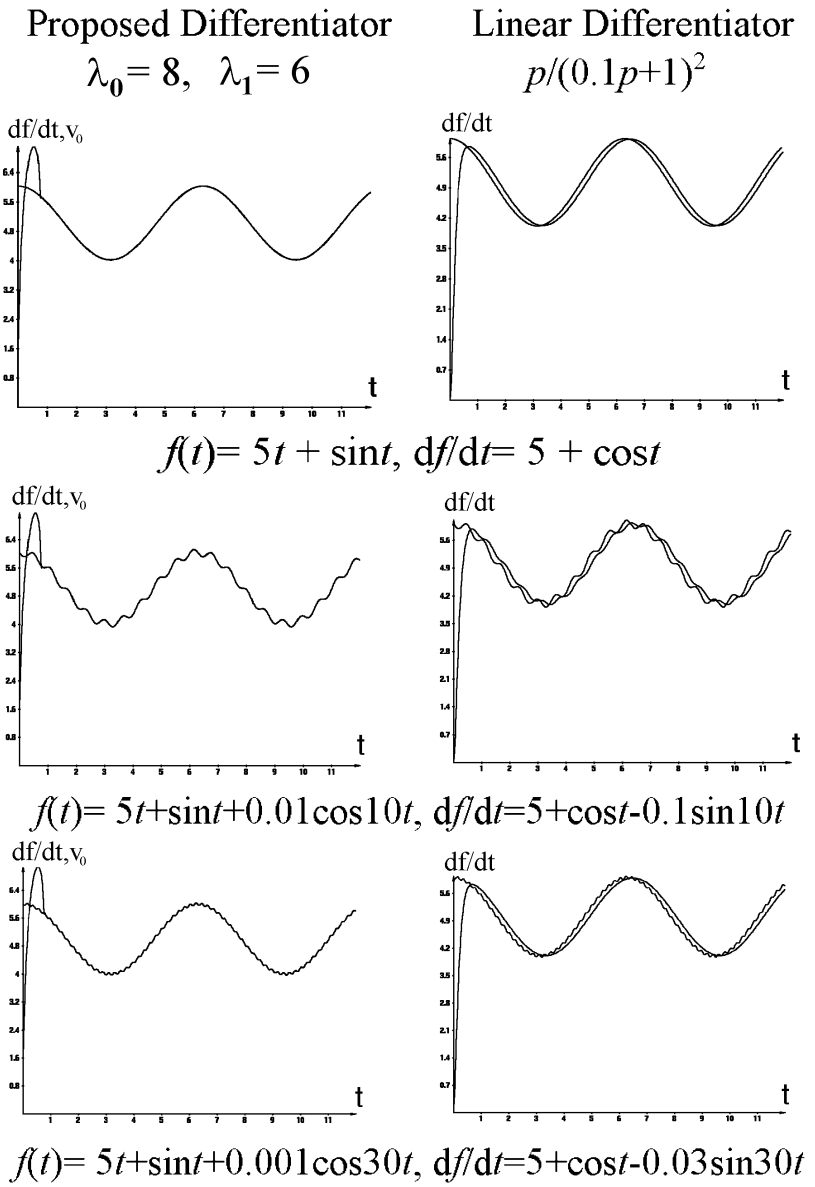

Figure 1.6: Comparison of the proposed and linear differentiators

Example

Consider system (1.2) with restrictions (1.3)

$$\ddot{x} = a(t) + b(t)u, \quad |a| \leq 1, \quad 1 \leq b \leq 2$$

and compose an output-feedback control of form (1.23) from (1.5):

$$\begin{aligned} u &= -5\,\mathrm{sign}\,z - 3\,\mathrm{sign}\,z_1, \\ \dot{z} &= -7|z - x|^{1/2}\,\mathrm{sign}(z - x) + z_1, \\ \dot{z}_1 &= -18\,\mathrm{sign}(z - x). \end{aligned} \tag{1.25}$$

At the moment $t = 0$ the initial values $z(0) = x(0)$, $z_1 = 0$ were taken. The concrete dynamic system

$$\ddot{x} = \sin 14.12t + (1.5 + 0.5\cos 21t)u,$$

was taken for simulation. The trajectory in the plane $x\dot{x}$ and the mutual graph of x, $\dot{x}$ and z_1 are shown in Figs. 1.7a,b respectively. The graph of z cannot be shown, since one cannot distinguish it from x. Convergence in the presence of an input high-frequency noise with magnitude 0.01 and graphs of x, $\dot{x}$ and z_1 are shown in Figs. 1.7c,d respectively. The resulting steady-state accuracies are $|x| \leq 0.041$, and $|\dot{x}| \leq 0.79$.

1.3 Definitions of higher order sliding modes

Standard sliding mode features few special properties. It is reached in finite time, which means that a number of trajectories meet at any sliding point. In other words, the shift operator along the phase trajectory exists, but is not invertible in time at any sliding point. Other important features are that the manifold of sliding motions has a nonzero codimension and that any sliding motion is performed on a system discontinuity surface and may be understood only as a limit of motions when switching imperfections vanish and switching frequency tends to infinity. Any generalization of the sliding mode notion has to inherit some of these properties [11].

Let us recall first what Filippov's solutions [19, 20] are of a discontinuous differential equation

$$\dot{x} = v(x),$$

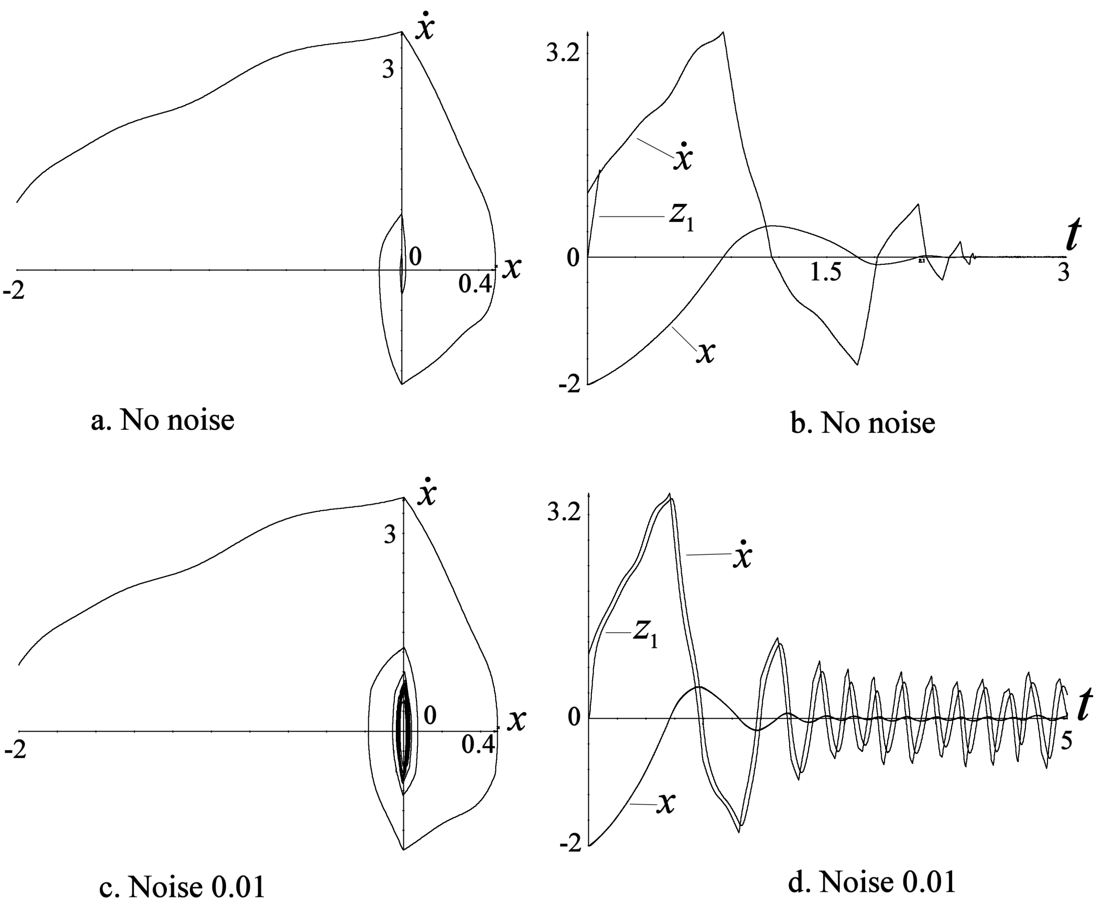

Figure 1.7: Output-feedback twisting controller

where $x \in \mathbb{R}^n$, v is a locally bounded measurable (Lebesgue) vector function. In that case, the equation is replaced by an equivalent differential inclusion

$$\dot{x} \in \mathcal{V}(x).$$

In the particular case when the vector-field v is continuous almost everywhere, the set-valued function $\mathcal{V}(x)$ is the convex closure of the set of all possible limits of $v(y)$ as $y \to x$, while $\{y\}$ are continuity points of v. Any solution of the equation is defined as an absolutely continuous function $x(t)$, satisfying the differential inclusion almost everywhere.

The following Definitions are based on [29, 14, 15, 17, 30, 23]. Recall that the word combinations "rth order sliding" and "r-sliding" are equivalent.

1.3.1 Sliding modes on manifolds

Let $\mathcal{S}$ be a smooth manifold. Set $\mathcal{S}$ itself is called the 1-sliding set with respect to $\mathcal{S}$. The 2-sliding set is defined as the set of points $x \in \mathcal{S}$, where $\mathcal{V}(x)$ lies entirely in tangential space $\mathbf{T}_x$ to manifold $\mathcal{S}$ at point x (Fig.1.8).

Definition 1 *It is said that there exists a first (or second) order sliding mode on manifold $\mathcal{S}$ in a vicinity of a first (or second) order sliding point x, if in this vicinity of point x the first (or second) order sliding set is an integral set, i.e. it consists of Filippov's sense trajectories.*

Let $\mathcal{S}_1 = \mathcal{S}$. Denote by $\mathcal{S}_2$ the set of 2-sliding points with respect to manifold $\mathcal{S}$. Assume that $\mathcal{S}_2$ may itself be considered as a sufficiently smooth manifold. Then the same construction may be considered with respect to $\mathcal{S}_2$. Denote by $\mathcal{S}_3$ the corresponding 2-sliding set with respect to $\mathcal{S}_2$. $\mathcal{S}_3$ is called *the 3-sliding set with respect to manifold $\mathcal{S}$* . Continuing the process, achieve sliding sets of any order.

Definition 2 *It is said that there exists an r-sliding mode on manifold $\mathcal{S}$ in a vicinity of an r-sliding point $x \in \mathcal{S}_r$, if in this vicinity of point x the r-sliding set $\mathcal{S}_r$ is an integral set, i.e. it consists of Filippov's sense trajectories.*

1.3.2 Sliding modes with respect to constraint functions

Let a constraint be given by an equation $\sigma(x) = 0$, where $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a sufficiently smooth constraint function. It is also supposed that total time derivatives along the trajectories $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-1)}$ exist and are single-valued functions of x, which is not trivial for discontinuous dynamic systems. In other words, this means that discontinuity does not appear in the first $r-1$ total time derivatives of the constraint function σ. Then the rth order sliding set is determined by the equalities

$$\sigma = \dot{\sigma} = \ddot{\sigma} = \ldots = \sigma^{(r-1)} = 0. \tag{1.26}$$

Here (1.26) is an r-dimensional condition on the state of the dynamic system.

Definition 3 *Let the r-sliding set* (1.26) *be non-empty and assume that it is locally an integral set in Filippov's sense (i.e. it consists of Filippov's trajectories of the discontinuous dynamic system). Then the corresponding motion satisfying* (1.26) *is called an r-sliding mode with respect to the constraint function σ (Fig.1.8).*

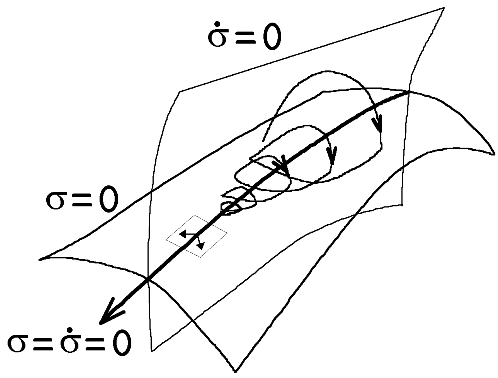

Figure 1.8: Second order sliding mode trajectory

To exhibit the relation with the previous Definitions, consider a manifold $\mathcal{S}$ given by the equation $\sigma(x) = 0$. Suppose that $\sigma, \dot{\sigma}, \ddot{\sigma}, \ldots, \sigma^{(r-2)}$ are differentiable functions of x and that

$$\text{rank}\{\nabla\sigma, \nabla\dot{\sigma}, \ldots, \nabla\sigma^{(r-2)}\} = r - 1 \tag{1.27}$$

holds locally (here rank $\mathcal{V}$ is a notation for the rank of the vector set $\mathcal{V}$). Then $\mathcal{S}_r$ is determined by (1.26) and all $\mathcal{S}_i, i = 1, \ldots, r-1$ are smooth manifolds. If in its turn $\mathcal{S}_r$ is required to be a differentiable manifold, then the latter condition is extended to

$$\mathrm{rank}\{\nabla\sigma, \nabla\dot{\sigma}, \dots, \nabla\sigma^{(r-1)}\} = r \tag{1.28}$$

Equality (1.28) together with the requirement for the corresponding derivatives of σ to be differentiable functions of x will be referred to as *the sliding regularity condition*, whereas condition (1.27) will be called *the weak sliding regularity condition*.

With the weak regularity condition satisfied and $\mathcal{S}$ given by equation $\sigma = 0$ Definition 3 is equivalent to Definition 2. If regularity condition (1.28) holds, then new local coordinates may be taken. In these coordinates the system will take the form

$$y_1 = \sigma, \;\; \dot{y}_1 = y_2; \;\; \dots; \;\; \dot{y}_{r-1} = y_r;$$
$$\sigma^{(r)} = \dot{y}_r = \Phi(y, \xi);$$
$$\dot{\xi} = \Psi(y, \xi), \;\; \xi \in \mathbf{R}^{n-r}.$$

Proposition 6 *Let regularity condition* (1.28) *be fulfilled and r-sliding manifold* (1.26) *be non-empty. Then an r-sliding mode with respect to the constraint function σ exists if and only if the intersection of the Filippov vector-set field with the tangential space to manifold* (1.26) *is not empty for any r-sliding point.*

Proof. The intersection of the Filippov set of admissible velocities with the tangential space to the sliding manifold (1.26), mentioned in the Proposition, induces a differential inclusion on this manifold. This inclusion satisfies all the conditions by Filippov [19, 20] for solution existence. Therefore, manifold (1.26) is an integral one.■

Let now σ be a smooth vector function, $\sigma : \mathbb{R}^n \to \mathbb{R}^m$, $\sigma = (\sigma_1, \dots, \sigma_m)$, and also $r = (r_1, \dots, r_m)$, where r_i are natural numbers.

Definition 4 *Assume that the first r_i successive full time derivatives of σ_i are smooth functions, and a set given by the equalities*

$$\sigma_i = \dot{\sigma}_i = \ddot{\sigma}_i = \dots = \sigma_i^{(r_i-1)} = 0, \;\; i = 1, \dots, m,$$

is locally an integral set in Filippov's sense. Then the motion mode existing on this set is called a sliding mode with vector sliding order r with respect to the vector constraint function σ.

The corresponding sliding regularity condition has the form

$$\text{rank}\{\nabla\sigma_i, \ldots, \nabla\sigma_i^{(r_i-1)} | i = 1, \ldots, m\} = r_1 + \ldots + r_m.$$

Definition 4 corresponds to Definition 2 in the case when $r_1 = \ldots = r_m$ and the appropriate weak regularity condition holds.

A sliding mode is called *stable* if the corresponding integral sliding set is stable.

Remarks

1. These definitions also include trivial cases of an integral manifold in a smooth system. To exclude them we may, for example, call a sliding mode "not trivial" if the corresponding Filippov set of admissible velocities $V(x)$ consists of more than one vector.
2. The above definitions are easily extended to include non-autonomous differential equations by introduction of the fictitious equation $\dot{t} = 1$. Note that this differs slightly from the Filippov definition considering time and space coordinates separately.

1.3.3 Higher-order sliding in control systems

Single out two cases: *ideal sliding* occurring when the constraint is ideally kept and *real sliding* taking place when switching imperfections are taken into account and the constraint is kept only approximately.

Ideal sliding

All the previous considerations are translated literally to the case of a process controlled

$$\dot{x} = f(t, x, u), \quad \sigma = \sigma(t, x) \in \mathbb{R}, \; u = U(t, x) \in \mathbb{R},$$

where $x \in \mathbb{R}^n$, t is time, u is control, and f, σ are smooth functions. Control u is determined here by a feedback $u = U(t, x)$, where U is a discontinuous function. For simplicity we restrict ourselves to the case when σ and u are scalars. Nevertheless, all statements below may also be formulated for the case of vector sliding order.

Let the system to be controlled have the form

$$\begin{aligned} \dot{x} &= a(t,x) + b(t,x)u, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}, \\ \sigma &: (t,x) \longmapsto \sigma(t,x) \in \mathbf{R}, \end{aligned} \tag{1.29}$$

where σ is the output of the system. We confine ourseves to the Single-Input-Single-Output (SISO) case. The task is to make σ vanish by means of possibly discontinuous feedback.

Extend the system by means of a fictituous equation

$$\dot{t} = 1.$$

Let $\tilde{x} = (x,t)^t$, $\tilde{a}(\tilde{x}) = (a(t,x),1)^t$, $\tilde{b}(\tilde{x}) = (b(t,x),0)^t$. Then the system takes on the form

$$\dot{\tilde{x}} = \tilde{a}(\tilde{x}) + \tilde{b}(\tilde{x})u, \ \sigma = \sigma(\tilde{x}). \tag{1.30}$$

According to [26] the equality of the relative degree of system (1.30) to r means that the Lie derivatives $L_{\tilde{b}}\sigma, L_{\tilde{b}}L_{\tilde{a}}\sigma, \ldots, L_{\tilde{b}}L_{\tilde{a}}^{r-2}\sigma$ equal zero identically in a vicinity of a given point and $L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma$ is not zero at the point. The equality of the relative degree to r means, in a simplified way, that u first appears explicitly only in the rth total time derivative of σ. It is known that in that case $\sigma^{(i)} = L_{\tilde{a}}^i\sigma$ for $i = 1, \ldots, r-1$, regularity condition (1.28) is satisfied automatically and also $\frac{\partial}{\partial u}\sigma^{(r)} = L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma \neq 0$. Each time when it is mentioned that system (1.29) has relative degree r it is meant that system (1.30) has the same relative degree.

As follows from [26] in the case of the relative degree r the equation

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad g(t,x) \neq 0,$$

holds locally, where $h(t,x) = L_{\tilde{a}}^r\sigma = \sigma^{(r)}|_{u=0}$, $g(t,x) = L_{\tilde{b}}L_{\tilde{a}}^{r-1}\sigma = \frac{\partial}{\partial u}\sigma^{(r)}$. Thus, h, g may be defined on the basis of the input-output relations. There is a direct analogy between the relative degree notion and the sliding regularity condition. Loosely speaking, it may be said that the sliding regularity condition (1.28) means that the "relative degree with respect to discontinuity" is not less than r. Similarly, the rth order sliding mode notion is analogous to the zero-dynamics notion [26].

Theorem 7 *Let the system have relative degree r with respect to the output function σ at some r-sliding point (t_0, x_0). Let, also, the discontinuous* *function U take on values from sets $[K, \infty)$ and $(-\infty, -K]$ on some sets of non-zero measure in any vicinity of any r-sliding point near point (t_0, x_0). Then it provides, with sufficiently large K, for the existence of r-sliding mode in some vicinity of point (t_0, x_0). r-sliding motion satisfies the zero-dynamics equations.*

Proof. This Theorem is an immediate consequence of Proposition 6, nevertheless, we will detail the proof. Consider some new local coordinates $y = (y_1, \dots, y_n)$, where $y_1 = \sigma, y_2 = \dot{\sigma}, \dots, y_r = \sigma^{(r-1)}$. In these coordinates manifold L_r is given by the equalities $y_1 = y_2 = \dots = y_r = 0$ and the dynamics of the system is as follows:

$$\begin{array}{l} \dot{y}_1 = y_2, \ \dots, \ \dot{y}_{r-1} = y_r, \\ \dot{y}_r = h(t, y) + g(t, y)u, \quad g(t, y) \neq 0, \\ \dot{\xi} = \Psi_1(t, y) + \Psi_2(t, y)u, \ \ \xi = (y_{r+1}, \dots, y_n). \end{array} \tag{1.31}$$

Denote $U_{eq} = -h(t, y)/g(t, y)$. It is obvious that with initial conditions being on the r-th order sliding manifold $\mathcal{S}_r$ equivalent control $u = U_{eq}(t, y)$ provides for keeping the system within manifold $\mathcal{S}_r$. It is also easy to see that the substitution of all possible values from $[-K, K]$ for u gives us a subset of values from Filippov's set of the possible velocities. Let $|U_{eq}|$ be less than K_0, then with $K > K_0$ the substitution $u = U_{eq}$ determines a Filippov's solution of the discontinuous system which proves the Theorem.■

The trivial control algorithm $u = -K \operatorname{sign} \sigma$ satisfies Theorem 7. Usually, however, such a mode will not be stable. In particular, such HOSMs arise in VSSs with actuators. Such r-sliding modes are always unstable with $r > 2$ [21, 22, 24]. It follows from the proof above that the equivalent control method [51] is applicable to r-sliding mode and produces equations coinciding with the zero-dynamics equations for the corresponding system:

$$\sigma = \dot{\sigma} = \dots = \sigma^{(r-1)} = 0, \quad y = (0, \xi), \quad \dot{\xi} = \Psi_1(t, 0, \xi) - \Psi_2(t, 0, \xi)\frac{h(t, 0, \xi)}{g(t, 0, \xi)}.$$

The sliding mode order notion [9, 12] seems to be understood in a very close sense (the authors had no possibility to acquaint themselves with the work by Chang). A number of papers approach the higher order sliding mode technique in a very general way from the differential-algebraic point of view [45, 46, 47, 41]. In these papers so-called "dynamic sliding modes" are not distinguished from the algorithms generating them. Consider that approach.

Let the following equality be fulfilled identically as a consequence of the dynamic system equations [47]:

$$P(\sigma^{(r)}, \ldots, \dot{\sigma}, \sigma, x, u^{(k)}, \ldots, \dot{u}, u) = 0. \tag{1.32}$$

Equation (1.32) is supposed to be solvable with respect to $\sigma^{(r)}$ and $u^{(k)}$. Function σ may itself depend on u. The rth order sliding mode is considered as a steady state $\sigma \equiv 0$ to be achieved by a controller satisfying (1.32). In order to achieve for σ some stable dynamics

$$\Sigma = \sigma^{(r-1)} + a_1\sigma^{(r-2)} + \ldots + a_{r-1}\sigma = 0$$

the discontinuous dynamics

$$\dot{\Sigma} = -\mathrm{sign}\,\Sigma \tag{1.33}$$

is provided. For this purpose the corresponding value of $\sigma^{(r)}$ is evaluated from (1.33) and substituted into (1.32). The obtained equation is solved for $u^{(k)}$.

Thus, a dynamic controller is constituted by the obtained differential equation for u which has a discontinuous right hand side. With this controller successive derivatives $\sigma, \ldots, \sigma^{(r-1)}$ are smooth functions of closed system state space variables. The steady state of the resulting system satisfies at least (1.26) and under some relevant conditions also the regularity requirement (1.28), and therefore Definition 3 will hold. Note that there are two different sliding modes in system (1.32), (1.33): a standard sliding mode of the first order which is kept on the manifold $\Sigma = 0$, and an asymptotically stable r-sliding mode with respect to the constraint $\sigma = 0$ which is kept in the points of the r-sliding manifold $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$.

1.3.4 Real sliding and finite time convergence

Recall that the objective is synthesis of such a control u that the constraint $\sigma(t, x) = 0$ holds. The quality of the control design is closely related to the sliding accuracy. In reality, no approaches to this design problem may provide for ideal keeping of the prescribed constraint. Therefore, there is a need to introduce some means in order to provide a capability for comparison of different controllers.

Any ideal sliding mode should be understood as a limit of motions when switching imperfections vanish and the switching frequency tends to infinity

(Filippov [19, 20]). Let ε be some measure of these switching imperfections. Then sliding precision of any sliding mode technique may be featured by a sliding precision asymptotics with $\varepsilon \to 0$ [30]:

Definition 5 *Let $(t, x(t,\varepsilon))$ be a family of trajectories, indexed by $\varepsilon \in \mathbb{R}^{\mu}$, with common initial condition $(t_0, x(t_0))$, and let $t \geq t_0$ (or $t \in [t_0, T]$). Assume that there exists $t_1 \geq t_0$ (or $t_1 \in [t_0, T]$) such that on every segment $[t', t'']$, where $t' \geq t_1$, (or on $[t_1, T]$) the function $\sigma(t, x(t,\varepsilon))$ tends uniformly to zero with ε tending to zero. In that case we call such a family a real-sliding family on the constraint $\sigma = 0$. We call the motion on the interval $[t_0, t_1]$ a transient process, and the motion on the interval $[t_1, \infty)$ (or $[t_1, T]$) a steady state process.*

Definition 6 *A control algorithm, dependent on a parameter $\varepsilon \in \mathbb{R}^{\mu}$, is called a real-sliding algorithm on the constraint $\sigma = 0$ if, with $\varepsilon \to 0$, it forms a real-sliding family for any initial condition.*

Definition 7 *Let $\gamma(\varepsilon)$ be a real-valued function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. A real-sliding algorithm on the constraint $\sigma = 0$ is said to be of order r ($r > 0$) with respect to $\gamma(\varepsilon)$ if for any compact set of initial conditions and for any time interval $[T_1, T_2]$ there exists a constant C, such that the steady state process for $t \in [T_1, T_2]$ satisfies*

$$|\sigma(t, x(t,\varepsilon))| \leq C|\gamma(\varepsilon)|^r.$$

In the particular case when $\gamma(\varepsilon)$ is the smallest time interval of control smoothness, the words "with respect to γ" may be omitted. This is the case when real sliding appears as a result of switching discretization.

As follows from [30], with the r-sliding regularity condition satisfied, in order to get the rth order of real sliding with discrete switching it is necessary to get at least the rth order in ideal sliding (provided by infinite switching frequency). Thus, the real sliding order does not exceed the corresponding sliding mode order. The standard sliding modes provide, therefore, for the first order real sliding only. The second order of real sliding was really achieved by discrete switching modifications of the second order sliding algorithms [29, 14, 15, 16, 17, 30]. An arbitrary order of real sliding can be achieved by discretization of the same order sliding algorithms from [33, 34, 37](see Section 1.5).

Real sliding may also be achieved in a way different from the discrete switching realization of sliding mode. For example, high gain feedback systems [44] constitute real sliding algorithms of the first order with respect to k^{-1}, where k is a large gain. A special discrete-switching algorithm providing for the second order real sliding were constructed in [50], another example of a second order real sliding controller is the drift algorithm [16, 30]. A third order real-sliding controller exploiting only measurements of σ was recently presented [7].

It is true that in practice the final sliding accuracy is always achieved in finite time. Nevertheless, besides the pure theoretical interest there are also some practical reasons to search for sliding modes attracting in finite time. Consider a system with an r-sliding mode. Assume that with minimal switching interval τ the maximal r-th order of real sliding is provided. That means that the corresponding sliding precision $|\sigma| \sim \tau^r$ is kept, if the r-th order sliding condition holds at the initial moment. Suppose that the r-sliding mode in the continuous switching system is asymptotically stable and does not attract the trajectories in finite time. It is reasonable to conclude in that case that with $\tau \to 0$ the transient process time for fixed general case initial conditions will tend to infinity. If, for example, the sliding mode were exponentially stable, the transient process time would be proportional to $r \ln(\tau^{-1})$. Therefore, it is impossible to observe such an accuracy in practice, if the sliding mode is only asymptotically stable. At the same time, the time of the transient process will not change drastically if it was finite from the very beginning. It has to be mentioned, also, that the authors are not aware of a case when a higher real-sliding order is achieved with infinite-time convergence.

1.4 Second-order sliding controllers: general consideration

Problem of controlling general dynamic system of the second relative degree is considered in this Section. Like previously only the twisting controller will be considered. Other 2-sliding controllers may be found in [30, 23, 5, 6, 42].

Let the system to be controlled have the form (1.29)

$$\begin{aligned} \dot{x} &= a(t,x) + b(t,x)u, \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}, \\ \sigma &: (t,x) \longmapsto \sigma(t,x) \in \mathbf{R}. \end{aligned} \tag{1.34}$$

The functions a, b, σ as well as n are supposed unknown. It is assumed that the system does not explode in finite time when any Lebesque-measurable bounded feedback is applied. In practical applications it is enough that the zero-dynamics be stable (asymptotical stability is not needed), for the convergence of the proposed controllers can be made arbitrarily fast. It is assumed also that the relative degree is constant and equals 2. Hence the equation

$$\ddot{\sigma} = h(t,x) + g(t,x)u, \quad g(t,x) \neq 0,$$

holds. Recall that $h(t,x) = L_{\tilde{a}}^2 \sigma = \ddot{\sigma}|_{u=0}$, $g(t,x) = L_{\tilde{b}} L_{\tilde{a}} \sigma = \frac{\partial}{\partial u} \ddot{\sigma}$. The functions h, g are unknown, It is assumed, nevertheless, that they are bounded:

$$|\ddot{\sigma}|_{u=0}| \leq C, \quad 0 < K_m \leq \frac{\partial}{\partial u} \ddot{\sigma} \leq K_M \tag{1.35}$$

The following results are mostly simple consequences of Section 1.2.

1.4.1 Twisting controller with relative degree 2

Theorem 8 *Let r_1 and r_2 satisfy the conditions*

$$K_m(r_1 + r_2) - C > K_M(r_1 - r_2) + C, \quad K_m(r_1 - r_2) > C. \tag{1.36}$$

Then controller

$$u = -r_1 \operatorname{sign} \sigma - r_2 \operatorname{sign} \dot{\sigma} \tag{1.37}$$

provides for the appearance of a 2-sliding mode $\sigma = \dot{\sigma} = 0$ attracting the trajectories in finite time.

The proof actually coincides with the proof of Lemma 1. the following Theorem is proved in [30, 18]. Let the measurements be carried out at discrete times t_i, $t \in [t_i, t_{i+1})$ with sampling step $t_{i+1} - t_i = \tau > 0$, $i = 0, 1, ...$ Then $\dot{\sigma}$ can be replaced in (1.37) by the finite difference $\Delta \sigma_i = \sigma(t_i, x(t_i)) - \sigma(t_{i-1}, x(t_{i-1}))$.

Theorem 9 *Let $\tau > 0$ be the constant input sampling interval, the noises be absent and the controller*

$$u(t) = -r_1 \operatorname{sign} \sigma_i - r_2 \operatorname{sign} \Delta\sigma_i, \quad t \in [t_i, t_{i+1}), \quad t_{i+1} - t_i = \tau \tag{1.38}$$

be applied. Then the inequalities $|\sigma| \leq \mu_0 \tau^2$, $|\dot{\sigma}| \leq \mu_1 \tau$ are established in finite time for some positive constants μ_0, μ_1.

1.4.2 2-sliding controller with a given convergence law (terminal sliding mode)

The following controller [14, 30] exploits the standard VSS approach. The idea is to attain the 2-sliding mode keeping a suitable constraint in the standard 1-sliding mode. the following controller is close to the independently developed "terminal sliding mode" controller [42]. Let

$$u = -\alpha \operatorname{sign}(\dot{\sigma} - \beta|\sigma|^{1/2} \operatorname{sign} \sigma), \quad \alpha, \beta > 0. \tag{1.39}$$

Theorem 8 is valid also for this controller for any sufficiently large α. The trajectory of the controller in the plane $\sigma\dot{\sigma}$ is shown in Fig. 1.9. Theorem 9 holds for the discrete-sampling version of controller (1.39)

$$u = -\alpha \operatorname{sign}(\Delta\sigma_i - \beta\tau|\sigma_i|^{1/2} \operatorname{sign} \sigma).$$

1.4.3 2-sliding control with relative degree 1

In the case of the relative degree 1 when the problem can be solved by the traditional relay controller $u = -U_M \operatorname{sign} \sigma$, the chattering can be avoided by application of the super-twisting controller

$$u = -\lambda|x|^{1/2} \operatorname{sign} x + u_1, \qquad \dot{u}_1 = \begin{cases} -u, & |u| > U_M, \\ -\alpha \operatorname{sign} x, & |u| \leq U_M. \end{cases}$$

or by means of the twisting controller

$$\dot{u} = \begin{cases} -u & \text{with } |u| > U_M, \\ -r_1 \operatorname{sign} \sigma - r_2 \operatorname{sign} \dot{\sigma} & \text{with } |u| \leq U_M. \end{cases}$$

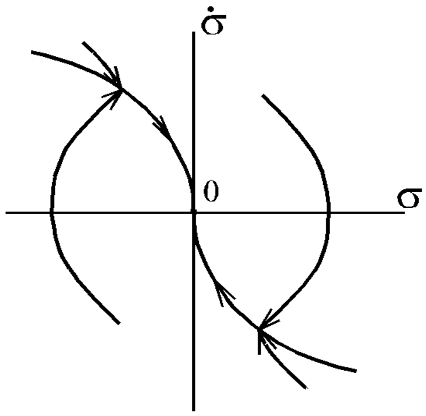

Figure 1.9: Phase portrait of the 2-sliding controller with a given convergence law (terminal sliding mode)

The formal problem statement is published, for example, in [30, 23, 6, 35]. All other known 2-sliding controllers including (1.39) may be used in such a way. Replacing $\dot{\sigma}$ with finite difference an output-feedback controller is achieved effective with small input noises, with significant noises or too small sampling step variable sampling step is used [35].

1.4.4 Output-feedback 2-sliding control

The following theorems are simple consequences of the Theorems 3, 8.

Theorem 10 *Under the conditions of Theorem 8 the output-feedback controller*

$$\begin{aligned} &u = -r_1 \operatorname{sign} z - r_2 \operatorname{sign} z_1, \quad r_1 > r_2 > 0. \qquad (1.40)\\ &\dot{z} = -\lambda_1 |z - \sigma|^{1/2} \operatorname{sign}(z - \sigma) + z_1,\\ &\dot{z}_1 = -\lambda_2 \operatorname{sign}(z - \sigma),\\ &\lambda_1 = 1.5(K_m(r_1 + r_2) + C)^{1/2}, \lambda_2 = 1.1(K_m(r_1 + r_2) + C) \end{aligned}$$

provides for the establishment of a finite-time attracting 2-sliding mode $\sigma = 0$.

1.4.5 Example

We follow here [39]. Consider a variable-length pendulum control problem. All motions are restricted to some vertical plane. A load of some known mass

m is moving along the pendulum rod (Fig. 1.10). Its distance from O equals $R(t)$ and is not measured. There is no friction. An engine transmits a torque w which is considered as control. The task is to track some function x_c given in real time by the angular coordinate x of the rod. The system is described by the equation

$$\ddot{x} = -2\frac{\dot{R}}{R}\dot{x} - g\frac{1}{R}sinx + \frac{1}{mR^2}w \tag{1.41}$$

where $g = 9.81$ is the gravitational constant, $m = 1$ was taken. Let $0 < Rm \leq R \leq RM$, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, $\sigma = x - x_c$ be available. The initial conditions are $x(0) = \dot{x}(0) = 0$. Following are the functions R and x_c considered in the simulation:

$$\begin{aligned} R &= 1 + 0.25\sin 4t + 0.5\cos t, \\ x_c &= 0.5\sin 0.5t + 0.5\cos t. \end{aligned}$$

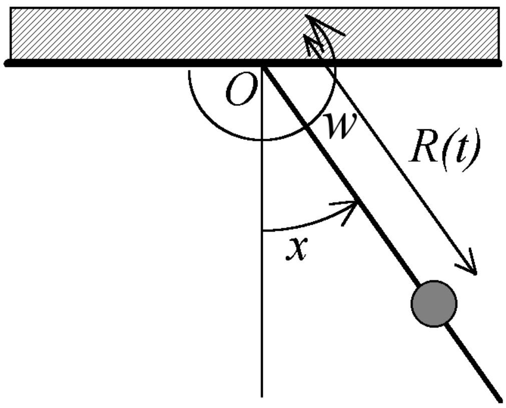

Figure 1.10: Variable-length pendulum

In case the torque chattering is unacceptable, $u = \dot{w}$ is considered as a new control. Define $\sigma = \dot{x} - \dot{x}_c + 2(x - x_c)$. The relative degree of the system is 2. Condition (1.35) holds here only locally: $\ddot{\sigma}|_{u=0}$ depends on $\dot{x}$ and is not uniformly bounded. Thus, the controllers are effective only in a bounded vicinity of the origin $x = \dot{x} = w = 0$. Their global application requires some standard tricks [30, 35], not implemented here.

The applied output-feedback controller is of form (1.40):

$$\begin{aligned} \dot{w} &= u = -15\,\mathrm{sign}\,z_0 + 10\,\mathrm{sign}\,z_1, & (1.42)\\ \dot{z}_0 &= -35|z_0-\sigma|^{1/2}\,\mathrm{sign}(z_0-\sigma)+z_1,\ \ \dot{z}_1 = -70\,\mathrm{sign}(z_0-\sigma), & (1.43)\\ \sigma &= \dot{x}-\dot{x}_c+2(x-xc). & (1.44)\end{aligned}$$

The angular velocity is considered here to be directly measured. Otherwise, a 3-sliding controller can be applied together with a second order differentiator (see the next Section) producing both - and . In the case when discontinuous torque is acceptable, another way is to directly implement a 2-sliding controller considering $x - x_c$ as the output to be nullified. Indeed, the corresponding relative degree is also 2, and the appropriate discontinuous controller of form (1.40) is

$$\begin{aligned} w &= -10\,\mathrm{sign}\,z_0 + 5\,\mathrm{sign}\,z_1, & (1.45)\\ \dot{z}_0 &= -6|z_0-\sigma|^{1/2}\,\mathrm{sign}(z_0-\sigma)+z_1,\ \ \dot{z}_1 = -35\,\mathrm{sign}(z_0-\sigma), & (1.46)\\ \sigma &= x-x_c. & (1.47)\end{aligned}$$

Initial values $x(0) = \dot{x}(0) = 0$ were taken, $w(0) = 0$ is taken for controller (1.42) - (1.44), the sampling step t = 0.0001. The trajectories in the coordinates x - xc and - in the absence of noises are shown for systems (1.41) - (1.44) and (1.41), (1.45) - (1.47) in Figs. 1.11a, b respectively, the corresponding accuracies being $|x - xc| \leq 1.6 \cdot 10^{-6}$, $|\dot{x} - \dot{x}_c| \leq 1.8 \cdot 10^{-5}$ and $|x-xc| \leq 6.7 \cdot 10^{-6}$, $|\dot{x}-\dot{x}_c| \leq 0.01$. The trajectory of (1.41) - (1.44) in the presence of a noise with magnitude 0.02 in σ-measurements is shown in Fig. 1.11c, the tracking results are shown in Fig. 1.11d, the tracking accuracy being $|x - xc| \leq 0.018$, $|\dot{x} - \dot{x}_c| \leq 0.16$. The performance does not differ when the frequency of the noise changes from 10 to 10000.

1.5 Higher-order sliding controllers

We follow here [33, 34, 37].

1.5.1 The problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t,x), \tag{1.48}$$

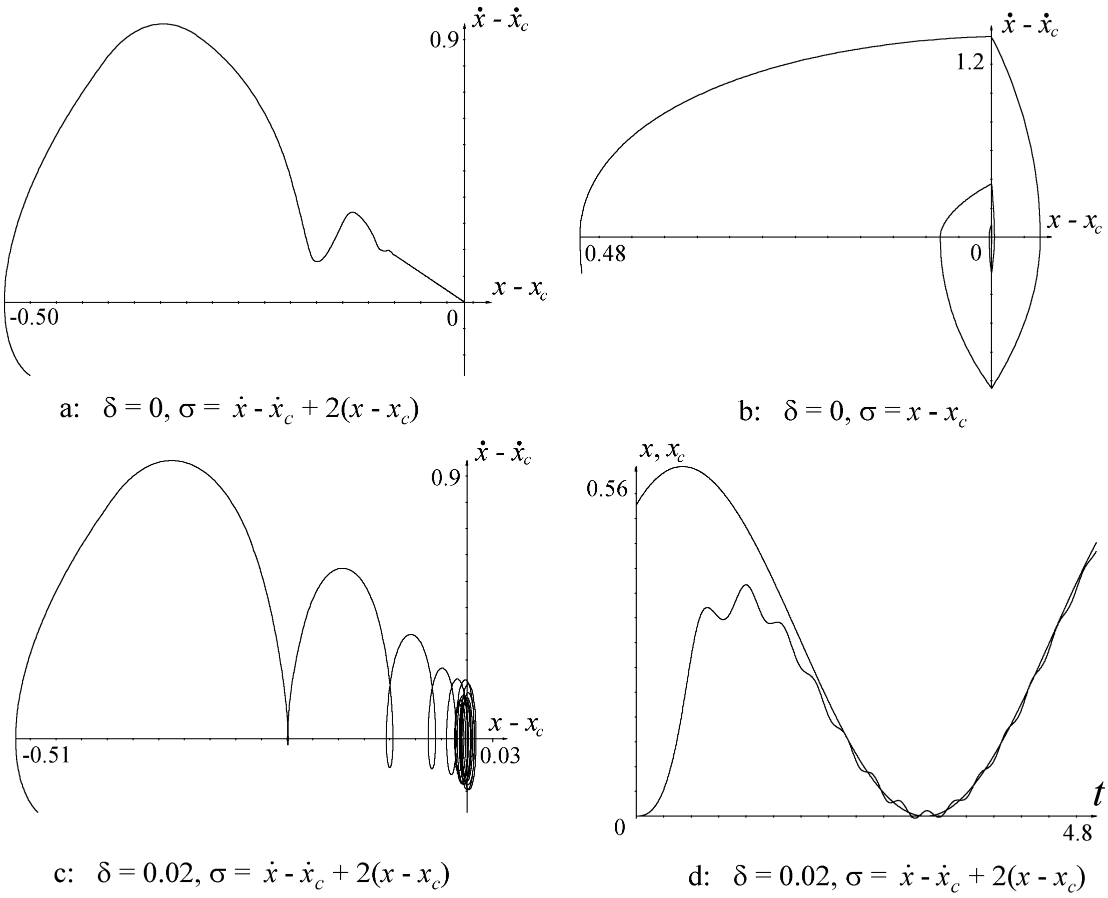

Figure 1.11: Output-feedback 2-sliding control of the pendulum

where $x \in \mathbb{R}^n, a, b, \sigma$ are smooth unknown functions, $u \in \mathbb{R}$. The relative degree r of the system is assumed to be constant and known. That means, in a simplified way, that u first appears explicitly only in the r-th total derivative of σ and $\frac{d}{du}\sigma^{(r)} \neq 0$ at the given point. The task is to fulfill the constraint $\sigma(t, x) = 0$ in finite time and to keep it exactly by discontinuous feedback control. Since $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are continuous functions of t and x, the corresponding motion will correspond to an r-sliding mode. The equation

$$\sigma^{(r)} = h(t, x) + g(t, x)u, \quad g(t, x) \neq 0, \tag{1.49}$$

holds for the relative degree is r. Recall that $h(t, x) = L^r_{\tilde{a}}\sigma = \sigma^{(r)}|_{u=0}$, $g(t, x) = L_{\tilde{b}} L^{r-1}_{\tilde{a}}\sigma = \frac{\partial}{\partial u}\sigma^{(r)}$. In other words, the unknown functions h and g may be defined in terms of input-output relations. Therefore, dynamic system (1.48) may be considered as a "black box". The resulting controller has to generalize the 1-sliding relay controller $u = -K\text{sign}\sigma$. Hence, $g(t, y)$ and $h(t, y)$ in (1.49) are to be bounded, $h > 0$. Thus, we require that for some $K_m, K_M, C > 0$

$$|\sigma^{(r)}|_{u=0}| \leq C, \quad 0 < K_m \leq \frac{\partial}{\partial u}\sigma^{(r)} \leq K_M. \tag{1.50}$$

Let a trivial controller $u = -K\text{sign}\sigma$ be chosen with $K > \sup|u_{eq}|$, $u_{eq} = -h(t, y)/g(t, y)$ [52]. Then the substitution $u = u_{eq}$ defines a differential equation on the r-sliding manifold of (1.48). Its solution corresponds to the r-sliding motion. Usually, however, such a mode is not stable.

1.5.2 Arbitrary order sliding controller

Let $p \geq r$. Denote

$$N_{1,r} = |\sigma|^{(r-1)/r},$$

$$N_{i,r} = (|\sigma|^{p/r} + |\dot{\sigma}|^{p/(r-1)} + ... + |\sigma^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p}, \ i = 1, ..., r-1,$$

$$N_{r-1,r} = (|\sigma|^p/r + |\dot{\sigma}|^{p/(r-1)} + ... + |\sigma^{(r-2)}|^{p/2})^{1/p}.$$

$$\psi_{0,r} = \sigma,$$

$$\psi_{1,r} = \dot{\sigma} + \beta_1 N_{1,r}\text{sign}\sigma,$$

$$\psi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r}\text{sign}(\psi_{i-1,r}), \ i = 1, ..., r-1,$$

where $\beta_1, ..., \beta_{r-1}$ are positive numbers.

Theorem 11 *Let system (1.48) have relative degree r with respect to the output function* σ *and (1.50) be fulfilled. Then with properly chosen positive parameters* $\beta_1, ..., \beta_{r-1}$ *the controller*

$$u = -\alpha \operatorname{sign}(\psi_{r-1,r}(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})). \tag{1.51}$$

provides for the appearance of r*-sliding mode* $\sigma \equiv 0$ *attracting trajectories in finite time.*

The positive parameters $\beta_1, ..., \beta_{r-1}$ are to be chosen sufficiently large in the index order. Each choice determines a controller family applicable to all systems (1.48) of relative degree r. Parameter $\alpha > 0$ is to be chosen specifically for any fixed C, K_m, K_M. The proposed controller is easily generalized: coefficients of $N_{i,r}$ may be any positive numbers etc. Obviously, α is to be negative with $\frac{\partial}{\partial u}\sigma^{(r)} < 0$.

Certainly, the number of choices of β_i is infinite. Here are a few examples with β_i tested for $r \leq 4, p$ being the least common multiple of $1, 2, ..., r$. The first is the relay controller, the second is 1.39.

1. $u = -\alpha \operatorname{sign} \sigma$
2. $u = -\alpha \operatorname{sign}(\sigma + |\sigma|^{1/2}\operatorname{sign}\sigma)$,
3. $u = -\alpha \operatorname{sign}(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{1/6}\operatorname{sign}(\dot{\sigma} + |\sigma|^{2/3}\operatorname{sign}\sigma)$,
4. $u = -\alpha \operatorname{sign}\{\sigma^{(3)} + 3(\ddot{\sigma}^6 + \dot{\sigma}^4 + |\sigma|^3)^{1/12}\operatorname{sign}[\ddot{\sigma} + (\dot{\sigma}^4 + |\sigma|^3)^{1/6}\operatorname{sign}(\dot{\sigma} + 0.5|\sigma|^{3/4}\operatorname{sign}\sigma)]\}$,
5. $u = -\alpha \operatorname{sign}(\sigma^{(4)} + \beta_4(|\sigma|^{12} + |\ddot{\sigma}|^{15} + |\sigma|^{20} + |\sigma^{(3)}|^{30})^{1/60}\operatorname{sign}(\sigma^{(3)} + \beta_3(|\sigma|^{12} + |\sigma|^{15} + |\ddot{\sigma}|^{20})^{1/30}\operatorname{sign}(\ddot{\sigma} + \beta_2(|\sigma|^{12} + |\dot{\sigma}|^{15})^{1/20}\operatorname{sign}(\dot{\sigma} + \beta_1|\sigma|^{4/5}\operatorname{sign}\sigma))))$

The idea of the controller is that a 1-sliding mode is established on the smooth parts of the discontinuity set Γ of (1.51) (Fig.1.12). That sliding mode is described by the differential equation $\psi_{r-1,r} = 0$ providing in its turn for the existence of a 1-sliding mode $\psi_{r-1,r} = 0$. But the primary sliding mode disappears at the moment when the secondary one is to appear. The resulting movement takes place in some vicinity of the subset of Γ satisfying $\psi_{r-2,r} = 0$, transfers in finite time into some vicinity of the subset satisfying $\psi_{r-3,r} = 0$ and so on. While the trajectory approaches the r-sliding set, set Γ retracts to the origin in the coordinates $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$. Set Γ with $r = 3$ is shown in Fig. 1.13.

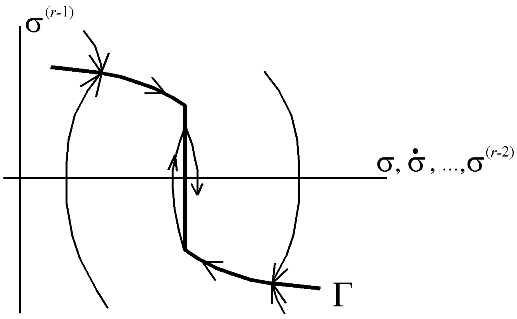

Figure 1.12: The idea of r-sliding controller

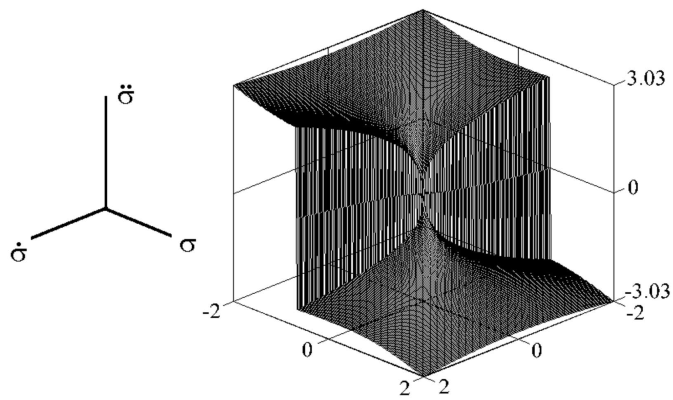

Figure 1.13: The discontinuity set of the 3-sliding controller

An interesting controller, so-called "terminal sliding mode controller", was proposed by [53]. In the 2-dimensional case it coincides with a particular case of the 2-sliding controller with given convergence law (1.39). In the r-dimensional case a mode is produced at the origin similar to the r-sliding mode. The problem is that a closed-loop system with terminal sliding mode does not satisfy the Filippov conditions [20] for the solution existence with $r > 2$. Indeed, the control influence is unbounded in vicinities of a number of hyper-surfaces intersecting at the origin. The corresponding Filippov velocity sets are unbounded as well. Thus, some special solution definition is to be elaborated, the stability of the corresponding quasi-sliding mode at the origin and the very existence of solutions are to be shown.

Controller (1.51) requires the availability of $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$. The needed information may be reduced if the measurements are carried out at times t_i with constant step $\tau > 0$. Consider the controller

$$\begin{aligned} u(t) = -\alpha \,\mathrm{sign}(\Delta\sigma_i^{(r-2)} + \beta_{r-1}\tau N_{r-1,r}(\sigma_i, \dot{\sigma}_i, ..., \sigma_i^{(r-2)}) \\ \mathrm{sign}(\psi_{r-2,r}(\sigma_i, \dot{\sigma}_i, ..., \sigma_i^{(r-2)})) \end{aligned} \tag{1.52}$$

Theorem 12 *Under conditions of Theorem 11 with discrete measurements both algorithms (1.51) and (1.52) provide in finite time for some positive constants $a_0, a_1, ..., a_{r-1}$ for fulfillment of inequalities*

$$|\sigma| < a_0\tau^r, |\dot{\sigma}| < a_1\tau^{r-1}, ..., |\sigma^{(r-1)}| < a_{r-1}\tau.$$

That is the best possible accuracy attainable with discontinuous $\sigma^{(r)}$. Convergence time may be reduced by changing coefficients β_j. Another way is to substitute $\lambda^{-j}\sigma^{(j)}$ for $\sigma^{(j)}$, $\lambda^r\alpha$ for α and $\alpha\tau$ for τ in (1.51) and (1.52), $\lambda > 0$, causing convergence time to be diminished approximately by λ times.

Implementation of r-sliding controller when the relative degree is less than r. Introducing successive time derivatives $u, \dot{u}, ..., u^{(r-k-1)}$ as new auxiliary variables and $u^{(r-k)}$ as a new control, achieve different modifications of each r-sliding controller intended to control systems with relative degrees $k = 1, 2, ..., r$. The resulting control is $(r-k-1)$-smooth function of time with $k < r$, a Lipschitz function with $k = r-1$ and a bounded "infinite-frequency switching" function with $k = r$.

Chattering removal. The same trick removes the chattering effect. For example, substituting $u^{(r-1)}$ for u in (1.52), receive a local r - sliding controller to be used instead of the relay controller $u = -\mathrm{sign}\sigma$ and attain rth order sliding precision with respect to τ by means of $(r-2)$-smooth control with Lipschitz $(r-2)$th time derivative. It has to be modified for global usage.

Controlling systems nonlinear on control. Consider a system $\dot{x} = f(t, x, u)$ nonlinear on control. Let $\frac{\partial}{\partial u}\sigma^{(i)}(t, x, u) = 0$ for $i = 1, ..., r-1$, $\frac{\partial}{\partial u}\sigma^{(r)}(t, x, u) > 0$. It is easy to check that

$$\sigma^{(r+1)} = \Lambda_u^{r+1}\sigma + \frac{\partial}{\partial u}\sigma^{(r)}\dot{u}, \quad \Lambda_u(\cdot) = \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot)f(t, x, u).$$

The problem is now reduced to that considered above with relative degree $r + 1$ by introducing a new auxiliary variable u and a new control $v = \dot{u}$.

Discontinuity regularization. The complicated discontinuity structure of the above-listed controllers may be smoothed by replacing the discontinuities under the sign-function with their finite-slope approximations. As a result, the transient process becomes smoother. Consider, for example, the above-listed 3-sliding controller. The function $\mathrm{sign}(\dot{\sigma} + |\sigma|^{2/3}\mathrm{sign}\sigma)$ may be replaced by the function $\max[-1, \min(1, |\sigma|^{-2/3}(\dot{\sigma} + |\sigma|^{2/3}\mathrm{sign}\sigma)/\varepsilon)]$ for some sufficiently small $\varepsilon > 0$. For $\varepsilon = 0.1$ the resulting tested controller is

$$u = -\alpha\,\mathrm{sign}(\ddot{\sigma} + 2(|\dot{\sigma}|^3 + |\sigma|^2)^{1/6}\max[-1, \min(1, 10|\sigma|^{-2/3}(\dot{\sigma} + |\sigma|^{2/3}\mathrm{sign}\sigma))]). \tag{1.53}$$

Controller (1.53) provides for the existence of a standard 1-sliding mode on the corresponding continuous piece-wise smooth surface.

Theorem 13 *Theorems 11, 12 remain valid for controller (1.53).*

1.5.3 Arbitrary-order exact robust differentiation

The implementation of the above-listed r-sliding controllers requires real-time observation of the successive derivatives $\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$. Thus, theoretically no model of the controlled process needs to be known. Only the relative degree and 3 constants are needed in order to adjust the controller. Unfortunately, the problem of successive real-time exact differentiation is usually considered to be practically unsolvable. Nevertheless, as we have seen, under some assumptions the real-time exact robust differentiation is possible. Differentiator (1.17)

$$\begin{aligned} \dot{z} &= v = -\lambda_1 |z - f(t)|^{1/2}\,\mathrm{sign}(z - f(t)) + z_1, \\ \dot{z}_1 &= -\lambda_2\,\mathrm{sign}(z - f(t)), \end{aligned} \tag{1.54}$$

provides for finite-time convergence to the exact derivative of $f_0(t)$ if the input noise $\eta = f(t) - f_0(t) = 0$. Otherwise, if sup $\eta(t) = \varepsilon$ it provides for accuracy proportional to $C^{1/2}\varepsilon^{1/2}$. Therefore, having been implemented k times successively, that differentiator will provide for kth order differentiation accuracy of the order of $\varepsilon^{(2^{-k})}$. Thus, full local real-time robust control of output variables is possible, using only output variable measurements and knowledge of the relative degree [37].

When the base signal $f_0(t)$ has (r-1)th derivative with Lipschitz's constant $L > 0$, the best possible kth order differentiation accuracy is $d_k\ L^{k/r}\ \varepsilon^{(r-k)/r}$, where $d_k > 1$ may be estimated (this asymptotics may be improved with additional restrictions on $f_0(t)$). Moreover, it is proved that such a robust exact differentiator really exists [32]. The corresponding differentiator has been recently presented by Levant ([38]).

The aim is to find real-time robust estimations of $f_0(t), f_0(t), ..., f_0^{(p)}(t)$, being exact in the absence of measurement noise and continuously depending on it. A recursive design scheme is proposed. Let a $(p-1)$th-order differentiator $D_{p-1}(f(t), C_{p-1})$ produce outputs D^i_{p-1} $(i = 0, 1, ..., p-1)$ which are estimates of $f_0, \dot{f}_0, , ..., f_0^{(p-1)}$ for any input signal f with $f_0^{(p-1)}$ having Lipschitz constant $L > 0$.

Then, the pth order differentiator has the outputs $z_i = D^i_p$, $i = 0, 1, ..., p$, defined as follows:

$$\begin{array}{l} \dot{z}_0 = \nu, \quad \nu = -\lambda|z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - f(t)) + z_1, \\ z_1 = D^0_{p-1}(\nu, L), \quad \ldots, \quad z_p = D^{p-1}_{p-1}(\nu, L) \end{array} \tag{1.55}$$

Here $D_0(h(t), L)$ is a simple nonlinear filter

$$D_0 : \qquad \dot{z} = -\lambda \operatorname{sign}(z - f(t)) \ , \qquad \lambda > L. \tag{1.56}$$

In other words it has the form

$$\begin{array}{l} \dot{z}_0 = \nu_0, \quad \nu_0 = -\lambda_0|z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - f(t)) + z_1, \\ \ldots \\ \dot{z}_i = \nu_i, \quad \nu_i = -\lambda_i|z_i - \nu_{i-1}|^{\frac{p-i}{p-i+1}} \operatorname{sign}(z_i - \nu_{i-1}) + z_{i+1}, \\ \ldots \\ \dot{z}_p = -\lambda_p \operatorname{sign}(z_p - \nu_{p-1}) \end{array} \tag{1.57}$$

It is easy to check that the above-presented pth order differentiator can be expressed in the non-recursive form

$$\begin{array}{l} \dot{z}_0 = z_1 - \kappa_0|z_0 - f(t)|^{\frac{p}{p+1}} \operatorname{sign}(z_0 - h(t)) \\ \dot{z}_1 = z_2 - \kappa_1|z_0 - f(t)|^{\frac{p-1}{p+1}} \operatorname{sign}(z_0 - h(t)) \\ \ldots \\ \dot{z}_i = z_i - \kappa_i|z_0 - f(t)|^{\frac{p-i}{p+1}} \operatorname{sign}(z_0 - h(t)) \\ \ldots \\ \dot{z}_p = -\kappa_p \operatorname{sign}(z_0 - f(t)) \end{array} \tag{1.58}$$

for suitable positive constant coefficients κ_i. The coefficients are easier to be found for form (1.57), for in that case the pth order differentiator requires only one parameter to be found, if the lower-order differentiators are already built. Having been found for $L = 1$, the parameters are easily recalculated for any L. In the following Theorems [38] the performance of the proposed differentiator in the presence of bounded measurement noises, and with discrete-time implementation, is studied.

Theorem 14 *Let the input noise satisfy the inequality* $|f(t) - f_0(t)| \leq \varepsilon$. *Then the following inequalities are established in finite time for some positive constants* μ_i, ν_i *depending only on the parameters of differentiator (1.57)*

$$\begin{aligned} |z_i - f_0^{(i)}(t)| &\leq \mu_i \, \varepsilon^{\frac{(p-i+1)}{(p+1)}}, \ i = 0, ..., p; \\ |v_i - f_0^{(i+1)}(t)| &\leq \nu_i \, \varepsilon^{\frac{(p-i)}{(p+1)}}, \ i = 0, ..., p-1. \end{aligned}$$

Exact differentiation is provided with $\varepsilon = 0$. Using recursive high-order differentiators the noise propagation is obviously counteracted as compared with the cascade implementation of first-order differentiators. Consider the discrete-sampling case, when $z_0(t_j) - f(t_j)$ is substituted for $z_0 - f(t)$, with $t_j \leq t < t_{j+1}$, $t_{j+1} - t_j = \tau > 0$.

Theorem 15 *Let* $\tau > 0$ *be the constant sampling interval in the absence of noises. Then the following inequalities are established in finite time for some positive constants* μ_i, ν_i *depending exclusively on the parameters of differentiator (1.57)*

$$\begin{aligned} |z_i - f_0^{(i)}(t)| &\leq \mu_i \, \tau^{p-i+1}, \ i = 0, ..., p; \\ |v_i - f_0^{(i+1)}(t)| &\leq \nu_i \, \tau^{p-i}, \ i = 0, ..., p-1. \end{aligned}$$

Theorem 16 *Let parameters* λ_{0i}, $i = 0, 1, ..., p$, *of differentiator (1.57) provide for exact* $p-th$ *order differentiation with* $L = 1$. *Then the parameters* $\lambda_i = \lambda_{0i} L^{1/(n-i+1)}$ *are valid for any* $L > 0$ *and provide for the accuracy* $|z_i - f_0(i)(t)| \leq \mu_i L^{i/(n+1)} \varepsilon^{(n-i+1)/(n+1)}$ *for some* $\mu_i \geq 1$.

Parameters λ_{0i} are easily found by computer simulation, successively rising the differentiator order according to (1.55). A set of such parameters is demonstrated in the following simulation example.

Simulation example

Differentiator (1.57) of order 5 with $L = 1$ and coefficients $\lambda_0 = 12$, $\lambda_1 = 8$, $\lambda_2 = 5$, $\lambda_3 = 3$, $\lambda_4 = 1.5$, $\lambda_5 = 1.1$ has been tested. As it was mentioned above (see (1.55)), it contains also all differentiators of the lower orders. These parameters can be easily changed, for the differentiator is not very sensitive to these values. The tradeoff is as follows: the larger the parameters, the faster the convergence and the higher sensitivity to the input noise and the sampling step. The estimation of the i-th derivative achieved by means of the k-th order differentiator is denoted as $D_k^i(t)$.

Initial values of the differentiator state were taken zero with exception for the initial estimation z0 of f, which is taken equal to the initial measured value of f. The base input signal

$$f_0(t) = 0.5 \sin 0.5t + 0.5 \cos t.$$

was taken for the differentiator testing. Derivatives of $f_0(t)$ do not exceed 1 in absolute value.

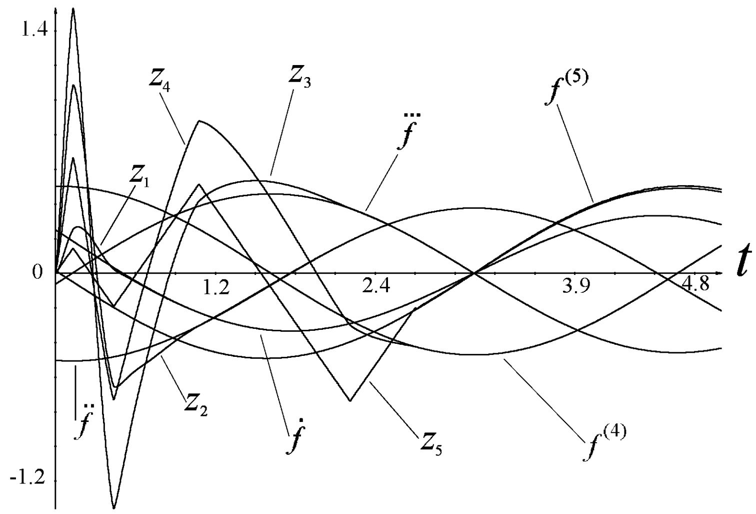

Figure 1.14: 5th order differentiation

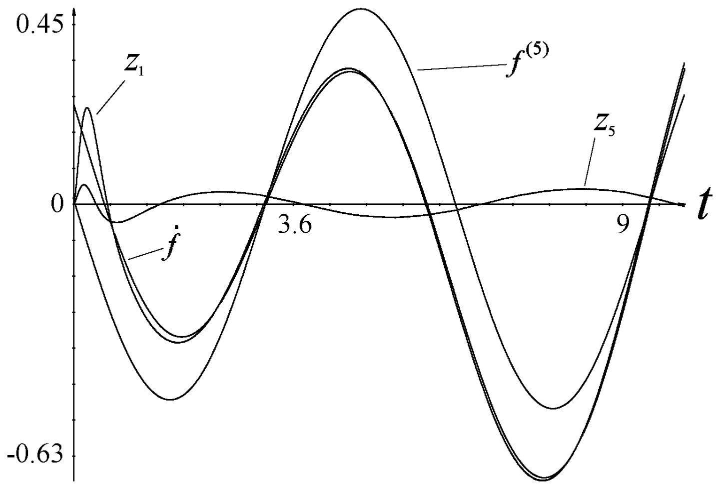

Figure 1.15: Noisy 5th order differentiation

The attained accuracies are $1.1 \cdot 10^{-16}$, $1.29 \cdot 10^{-12}$, $7.87 \cdot 10^{-10}$, $5.3 \cdot 10^{-7}$, $2.0 \cdot 10^{-4}$ and 0.014 for tracking the signal, the first, second, third, fourth and fifth derivatives respectively with $\tau = 10^{-4}$ (Fig. 1.14). Recall that z_i are estimations of $f_0^{(i)}$. There is no significant improvement with further reduction of τ. The author wanted to demonstrate the 10th order differentiation, but found that differentiation of the order exceeding 5 is unlikely to be performed with the standard software. Further calculations are to be carried out with precision higher than the standard long double precision (128 bits per number).

Sensitivity to noises. The main problem of the differentiation is certainly its well-known sensitivity to noises. As we have seen, even small computer calculation errors appear to be a considerable noise in the calculation of the fifth derivative. Recall that, when the pth derivative has the Lipschitz constant 1 and the noise magnitude is ε, the best possible accuracy of the ith order differentiation, $i \leq p$, is $k(i,p)\varepsilon^{(p-i+1)/(p+1)}$ (Levant 1998a), where $k(i,p) > 1$ is a constant independent of the differentiation realization. That is a minimax (worst case) evaluation. Since differentiator (1.57) assumes this Lipschitz input condition, it satisfies this accuracy restriction as well (see also Theorems 14, 16). In particular, with the noise magnitude $\varepsilon = 10^{-6}$

the maximal 5th derivative error exceeds $\varepsilon^{1/6} = 0.1$. For comparison, if the successive first-order differentiation were used, the respective maximal error would be at least $\varepsilon^{(2^{-5})} = 0.649$ and some additional conditions on the input signal would be required. Taking 10% as a border, achieve that the direct successive differentiation does not give reliable results starting with the order 3, while the proposed differentiator may be used up to the order 5.

With the noise magnitude 0.01 and the noise frequency about 1000 the 5th-order differentiator produces estimation errors 0.00042, 0.0088, 0.076, 0.20, 0.34 and 0.52 for the signal and its 5 derivatives respectively (Fig. 1.15). The differentiator performance does not significantly depend on the noise frequency.

1.5.4 Universal output-feedback SISO controller

These results have been just recently obtained ([40]), and the author supposes to describe them in a special paper. Thus, only a brief description is provided. Consider uncertain system (1.48), (1.50). Combining controller (1.51) and differentiator (1.57) achieve a combined SISO controller

$$\begin{aligned}
&u = -\alpha \operatorname{sign}(\psi_{r-1,r}(z_0, z_1, ..., z_{r-1})),\\
&\dot{z}_0 = \nu_0, \quad \nu_0 = -\lambda_0 |z_0 - \sigma|^{\frac{r-1}{r}} \operatorname{sign}(z_0 - \sigma) + z_1,\\
&\dots\\
&\dot{z}_i = \nu_i, \quad \nu_i = -\lambda_i |z_i - \nu_{i-1}|^{\frac{r-2}{r-1}} \operatorname{sign}(z_i - \nu_{i-1}) + z_{i+1},\\
&\dots\\
&\dot{z}_{r-1} = -\lambda_{r-1} \operatorname{sign}(z_{r-1} - \nu_{r-1})
\end{aligned}$$

where parameters λ_i of the differentiator are chosen according to the condition $|\sigma^{(r)}| \leq L$, $L \geq C + \alpha K_M$. As noted above, relations $\lambda_i = \lambda_{0i} L^{1/(r-i)}$ may be used, where λ_{0i} are chosen in advance for $L = 1$ (Theorem 16). Thus, parameters of controller (1.51) are chosen separately of the differentiator. In case when C and K_M are known, only one parameter α is really needed to be tuned, otherwise both L and a might be found in computer simulation. Theorems 11, 12 hold also for the combined output-feedback controller. In particular, under the conditions of Theorem 11 the combined controller provides for the global convergence to the r-sliding mode $\sigma \equiv 0$ with the transient time being a locally bounded function of the initial conditions.

On the other hand, let the initial conditions of the differentiator belong to some compact set. Than for any 2 embedded disks centered at the origin of the space σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ the parameters of the combined controller can be chosen in such a way that all trajectories starting in the smaller disk do not leave the larger disk during their finite-time convergence to the origin. The maximal convergence time can be made arbitrarily small. That allows for the local controller application.

With discrete measurements, in the absence of input noises, the controller provides for the rth-order real sliding $\sup|\sigma|\tilde{}\tau^r$, where τ is the sampling interval. Therefore, the differentiator does not spoil the r-sliding asymptotics if the input noises are absent. It is also proved that the resulting controller is robust and provides for the accuracy proportional to the maximal error of the input measurement (the input noise magnitude). Note once more that the proposed controller does not require detailed mathematical model of the process to be known.

1.5.5 Example

Car control.

Consider a simple kinematic model of car control [43]

$$\dot{x} = v\cos\varphi,\ \ \dot{y} = v\sin\varphi,$$

$$\dot{\varphi} = \frac{v}{l}\tan\theta,$$

$$\dot{\theta} = u,$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1.16). The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while x, y and φ are assumed to be measured in real time. Note that the actual control here is θ and $\dot{\theta} = u$ is used as a new control in order to avoid discontinuities of θ. Any practical implementation of the developed here controller would require some real-time coordinate transformation with φ approaching $\pm\pi/2$. Define

$$\sigma = y - g(x).$$

Let $v = \text{const} = 10m/s$, $l = 5m$, $g(x) = 10\sin 0.05x + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$. The relative degree of the system is 3 and both 3-sliding controller $No.3$ and its regularized form (1.53) may be applied here.

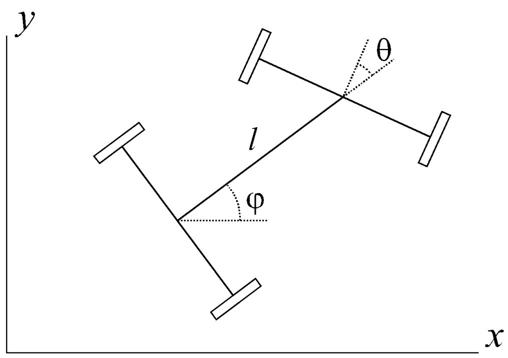

Figure 1.16: Kinematic car model

The resulting steering angle dependence on time is not sufficiently smooth ([37]), therefore the relative degree is artificially increased up to 4, $\dot{u}$ having been considered as a new control. The 4-sliding controller from the list in Subsection 1.5.2 is applied now, $\alpha = 20$ is taken. The following 3rd order differentiator was implemented:

$$\begin{aligned}
\dot{z}_0 &= v_0, v_0 = -25|z_0 - \sigma|^{3/4}\,\mathrm{sign}(z_0 - \sigma) + z_1,\\
\dot{z}_1 &= v_1, v_1 = -25|z_1 - v_0|^{2/3}\,\mathrm{sign}(z_1 - v_0) + z_2,\\
\dot{z}_2 &= v_2, v_2 = -33|z_2 - v_1|^{1/2}\,\mathrm{sign}(z_2 - v_1) + z_3,\\
\dot{z}_3 &= -500\,\mathrm{sign}(z_3 - v_2).
\end{aligned} \qquad (1.59)$$

The coefficient in (1.59) is large due to the large values of $\sigma^{(4)}$, other coefficients were taken according to Theorem 16 and the parameters defined for L=1 in the simulation example of Subsection 1.5.3. During the first half-second the control is not applied in order to allow the convergence of the differentiator. Substituting z_0, z_1, z_2 and z_3 for σ, $\dot{\sigma}$, $\ddot{\sigma}$ and $\dddot{\sigma}$ respectively, obtain the following 4-sliding controller:

$$\begin{aligned}
&u = 0, \qquad\qquad 0 \le t < 0.5,\\
&u = -20\,\mathrm{sign}\{z_3 + 3(z_2^6 + z_1^4 + |z_0|^3)^{1/12}\,\mathrm{sign}[z_2 + (z_1^4 + |z_0|^3)^{1/6}\,\mathrm{sign}(z_1 +\\
&\qquad\qquad 0.5|z_0|^{3/4}\,\mathrm{sign}\, z_0)]\}, \qquad t \ge 0.5.
\end{aligned}$$

The trajectory and function $y = g(x)$ with the sampling step $\tau = 10^{-4}$ are shown in Figure 1.17a. The integration was carried out according

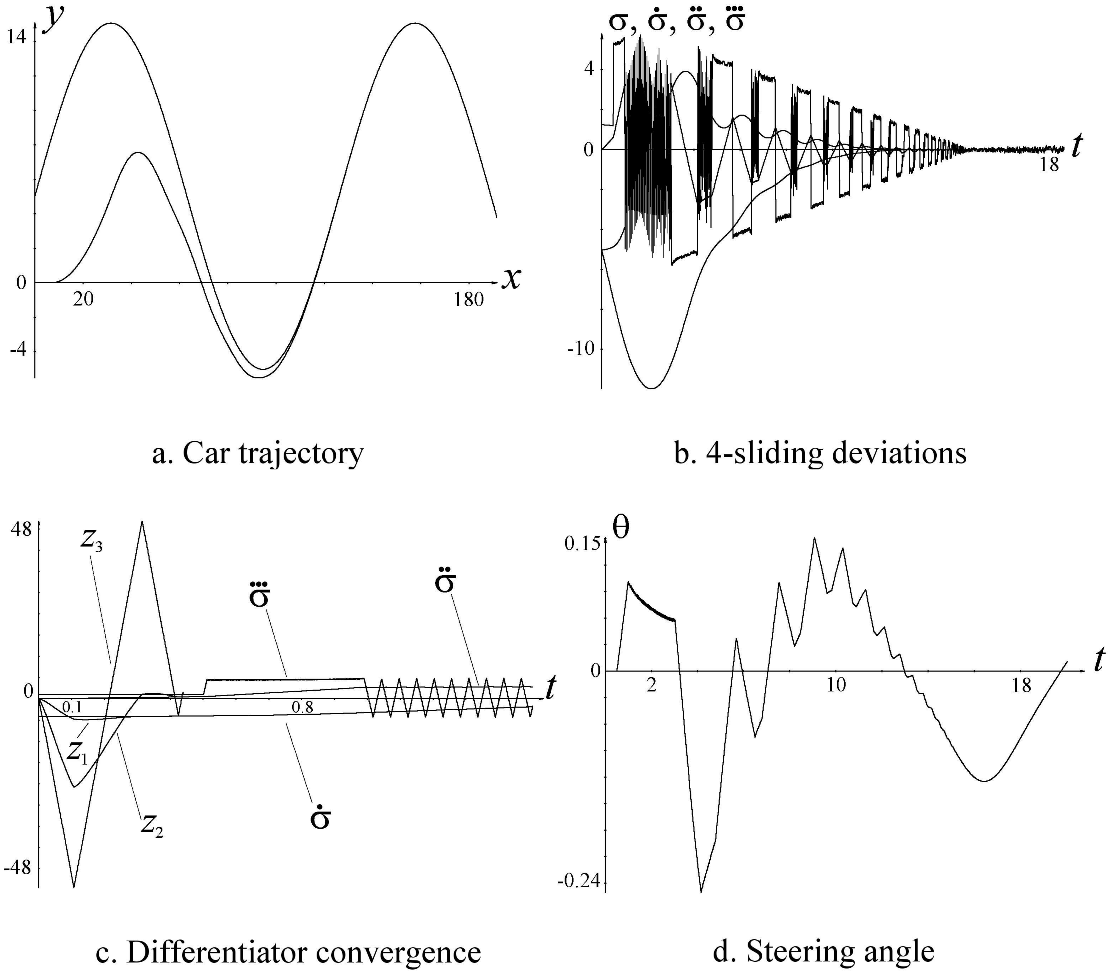

Figure 1.17: 4-sliding car control

to the Euler method, the only method effective for sliding-mode simulation. Graphs of σ, $\dot{\sigma}$, $\ddot{\sigma}$, $\sigma^{(3)}$ are shown in Figure 1.17b. The differentiators' performance within the first 1.5 seconds is demonstrated in Figure 1.17c. The steering angle graph (actual control) is presented in Figure 1.17d. The sliding accuracies $|\sigma| \leq 9.3 \cdot 10^{-8}$, $|\dot{\sigma}| \leq 7.8 \cdot 10^{-5}$, $|\ddot{\sigma}| \leq 6.6 \cdot 10^{-4}$, $|\sigma^{(3)}| \leq 0.43$ were attained with the sampling time step $\tau = 10^{-4}$.

1.6 Conclusions

- An elementary introduction to the higher order sliding theory, its main notions and results is presented.
- Some detailed proofs for the twisting and super-twisting controller are published for the first time.
- A robust first-order exact differentiator is presented. Otput-feedback 2-sliding control is demonstrated.
- A family of arbitrary-order sliding controllers with finite time convergence was presented.
- Arbitrary-order robust exact differentiator with finite-time convergence is presented.
- Output-feedback arbitrary-order sliding controllers are presented.
- The discrete switching modification of presented sliding controllers provide for the sliding precision of their order with respect to the measurement time interval.
- A number of simulation examples were presented.

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