Vectors

Notation: vectors are **bold** Latin characters, while real numbers are Greece.

Main properties:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a},\ \ (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}),$$

$$0\,\boldsymbol{a} = \mathbf{0},\ \lambda\,(\boldsymbol{a} + \boldsymbol{b}) = \lambda\,\boldsymbol{a} + \lambda\,\boldsymbol{b},\ (\lambda + \mu)\,\boldsymbol{a} = \lambda\,\boldsymbol{a} + \mu\,\boldsymbol{a}$$

Linear independence (is similarly defined for any number of vectors):

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are called **linearly independent** if for any real numbers λ, μ, ν, η, not all of which are zero, $\lambda\,\boldsymbol{a} + \mu\,\boldsymbol{b} + \nu\,\boldsymbol{c} + \eta\,\boldsymbol{d} \neq \mathbf{0}$.

Or equivalently:

$$\lambda\,\boldsymbol{a} + \mu\,\boldsymbol{b} + \nu\,\boldsymbol{c} + \eta\,\boldsymbol{d} = \mathbf{0} \quad \Rightarrow \quad \lambda = \mu = \nu = \eta = \mathbf{0}.$$

Vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are called **linearly dependent** if there are real numbers λ, μ, ν, η, not all of which are zero, such that $\lambda\,\boldsymbol{a} + \mu\,\boldsymbol{b} + \nu\,\boldsymbol{c} + \eta\,\boldsymbol{d} = \mathbf{0}$.

If, for example, $\lambda \neq 0$, then $\boldsymbol{a} = -\frac{\mu}{\lambda}\boldsymbol{b} - \frac{\nu}{\lambda}\boldsymbol{c} - \frac{\eta}{\lambda}\boldsymbol{d}$.

In other words:

A group of vectors are linearly dependent iff one of them can be expressed as a linear combination of the others (with possibly zero coefficients).

In particular, any set of vectors containing zero are linearly dependent. Indeed if $\boldsymbol{a} = \mathbf{0}$ then

$$\boldsymbol{a} = \mathbf{0} = 0\boldsymbol{b} + 0\boldsymbol{c} + 0\boldsymbol{d}.$$

A group of vectors are *linearly independent* iff none of them is a linear combination of the others (with possibly zero coefficients).

The maximal number of linearly independent vectors is called the **dimension**. The dimension of the plane is 2, the dimension of the space is 3.

In particular, if $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are independent, then any other vector $\boldsymbol{r}$ can be expressed as

$$\boldsymbol{r} = \lambda\,\boldsymbol{a} + \mu\,\boldsymbol{b} + \nu\,\boldsymbol{c}\,.$$

Here λ, μ, ν are called the **coordinates** of $\boldsymbol{r}$ in the basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Similarly any 2 independent vectors constitute a basis in the plane.

Any 2 non-parallel vectors can be used as a basis in the plane. Any 3 vectors which cannot be put in one plane can be used as a basis in the space.

Projection of vector $\boldsymbol{v}$ on vector $\boldsymbol{a}$: $\mathrm{proj}_{\boldsymbol{a}}\boldsymbol{v} = |\boldsymbol{v}| \cos(\mathrm{angle}(\boldsymbol{a}, \boldsymbol{v}))$

Cartesian coordinates: ortogonal coordinates generated by some ortogonal unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ constituting a *right-hand* basis.

Notation: $r_x = \mathrm{proj}_{\boldsymbol{i}}\boldsymbol{r}$, $r_y = \mathrm{proj}_{\boldsymbol{j}}\boldsymbol{r}$, $r_z = \mathrm{proj}_{\boldsymbol{k}}\boldsymbol{r}$

$\boldsymbol{r} = r_x\, \boldsymbol{i} + r_y\, \boldsymbol{j} + r_z\, \boldsymbol{k}$, $|\boldsymbol{r}| = (r_x^2 + r_y^2 + r_z^2)^{1/2}$,

$\boldsymbol{r} = |\boldsymbol{r}|\, (\boldsymbol{i} \cos\alpha + \boldsymbol{j} \cos\beta + \boldsymbol{k} \cos\gamma)$, $\cos\alpha = r_x/|\boldsymbol{r}|$, $\cos\beta = r_y/|\boldsymbol{r}|$, $\cos\gamma = r_z/|\boldsymbol{r}|$.

Scalar product:

$\boldsymbol{a}\cdot\boldsymbol{b} = (\boldsymbol{a}, \boldsymbol{b}) = |\boldsymbol{a}|\, \mathrm{proj}_{\boldsymbol{a}}\boldsymbol{b} = |\boldsymbol{b}|\, \mathrm{proj}_{\boldsymbol{b}}\boldsymbol{a} = |\boldsymbol{a}|\, |\boldsymbol{b}| \cos(\mathrm{angle}(\boldsymbol{a}, \boldsymbol{b}))$

Main properties:

$(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{b}, \boldsymbol{a})$, $(\boldsymbol{a}, \boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a}, \boldsymbol{b}) + (\boldsymbol{a}, \boldsymbol{c})$, $(\lambda\boldsymbol{a}, \boldsymbol{b}) = \lambda(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{a}, \boldsymbol{a}) = |\boldsymbol{a}|^2 \geq 0$

$(\boldsymbol{a}, \boldsymbol{b}) = a_x\, b_x + a_y\, b_y + a_z\, b_z$,

$\cos(\mathrm{angle}(\boldsymbol{a}_1, \boldsymbol{a}_2)) = \dfrac{(\boldsymbol{a}_1, \boldsymbol{a}_2)}{|\boldsymbol{a}_1|\,|\boldsymbol{a}_2|} = \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$.

In particular $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Plane and straight line. Let $\boldsymbol{r} = (x, y, z)$ (or (x, y)) be a radius-vector in the space (plane).

A straight line in the plane: $a_1 x + a_2 y = c$ or $(\boldsymbol{a}, \boldsymbol{r}) = c$, $\boldsymbol{a}$ is ortogonal to the line

A plane in the space: $a_1 x + a_2 y + a_3 z = c$ or $(\boldsymbol{a}, \boldsymbol{r}) = c$, $\boldsymbol{a}$ is ortogonal to the plane

A straight line in the space is given as the intersection of 2 planes:

$$a_1 x + a_2 y + a_3 z = c$$
$$b_1 x + b_2 y + b_3 z = d$$

or $\quad \begin{aligned} (\boldsymbol{a}, \boldsymbol{r}) &= c ,\\ (\boldsymbol{b}, \boldsymbol{r}) &= d \end{aligned} \qquad$ $\boldsymbol{a}, \boldsymbol{b}$ are ortogonal to the line

Vector (cross) product.

$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$, $|\,\boldsymbol{a}\times\boldsymbol{b}\,| = |\,\boldsymbol{a}\,|\,|\,\boldsymbol{b}\,|\,|\sin \mathit{angle}(\boldsymbol{a},\boldsymbol{b})|$, $\boldsymbol{a}\times\boldsymbol{b}$ is ortogonal to both $\boldsymbol{a}$, $\boldsymbol{b}$.

$\boldsymbol{a}\times\boldsymbol{b} = -\,\boldsymbol{b}\times\boldsymbol{a}$

The area of the parallelogram generated by $\boldsymbol{a}$, $\boldsymbol{b}$ equals $|\,\boldsymbol{a}\times\boldsymbol{b}\,|$.

Two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ are linearly independent iff $\boldsymbol{a}\times\boldsymbol{b} \neq \mathbf{0}$.

Triple product.

$$\boldsymbol{abc} = (\boldsymbol{a}\times\boldsymbol{b},\boldsymbol{c}) = (\boldsymbol{a},\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$\boldsymbol{abc} = \boldsymbol{bca} = \boldsymbol{cab} = -\,\boldsymbol{bac} = -\,\boldsymbol{acb} = -\,\boldsymbol{cba}$.

The volume of the parallelepiped generated by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ equals $|\,\boldsymbol{abc}\,|$.

Three vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are linearly independent iff $\boldsymbol{abc} \neq \mathbf{0}$.