Tel-Aviv University
Faculty of Exact Sciences
Exercises in Advanced Topics in ODE, 0372-4553-01

Lecturer: Prof. Arie Levant, January 9, 2022

1. Prove that the solution of the Cauchy problem $y'\cos x + y\sin x + \sin|y|^{\frac{1}{2}}\sin x - 2\sin x = 0$, $y(0) = -1$
satisfies the inequalities $1 - 2\cos x < y < 3 - 4\cos x$ for $0 < x < \frac{\pi}{2}$. Hint: compare with a more convenient equation. What does happen when $-\frac{\pi}{2} < x < 0$? Explain.

2. Find the solution of the Cauchy problem $\ddot{y} = 2\arctan\, y + \frac{3}{2}\dot{y}\cos\varepsilon - \varepsilon(e^{2t}y + \dot{y}^2) - \frac{1}{2}\pi$,
$y(0) = \cos\varepsilon, \dot{y}(0) = \varepsilon$, in the linear approximation with respect to $\varepsilon \approx 0$ for t varying in a fixed closed segment containing 0 as its internal point.

3. Find the general solution for the following system of differential equations (DEs) provided one partial solution is known:

$$\begin{cases} \dot{x} = (1 + e^{-t} - te^{-t})x + (te^{-t} - e^{-t})y \\ \dot{y} = (e^{-t} - te^{-t})x + (1 + te^{-t} - e^{-t})y \end{cases}, \text{ the particular solution} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} .$$

4. Find the general solution for the DE $(t^2 - 1)\ddot{x} + 4t\dot{x} + 2x = 6t$ provided two particular solutions are known $x_1 = t$, $x_2 = \dfrac{t^2 + t + 1}{t + 1}$.

5. Does the DE $\dddot{x} - 2\ddot{x} + 3\dot{x}\sin t - x\cos 5t = 0$ have stable solutions? Explain.

6. Check the asymptotic stability of the solutions for the following Cauchy problem:
$\dddot{x} - (t^2 + 1)^{-1}\ddot{x} + \dot{x} + tx = e^t$, $x(0) = 0, \dot{x}(0) = 1, \ddot{x}(0) = 0$.

7. Prove that any solution of the DE $\ddot{x} + (3t + 8)/(t + 1)\cdot x = 0$ has not more than two zeros in the segment $[20,23]$.

8. Prove that any solution of the DE $\ddot{x} + 2(t + 1/t)x = 0$ has at least one extremum in the segment $[3,7]$.

9. What is the area of the region to which transfers the square $|x| \le 0.5$, $|y| \le 0.5$ in 5 time units as the result of the motion described by the system of DEs

$$\begin{cases} \dot{x} = -3x + 3xy^2 \\ \dot{y} = \ln(x + 1) - y^3 \end{cases} .$$

10. What is the volume of the region to which transfers the cube $|x| \le 1$, $|y| \le 1$, $|z| \le 1$ in 3 time units as the result of the motion described by the system of DEs. What can be said on the stability of the constant zero solution? Explain.

$$\begin{cases} \dot{x} = -3x + 2xy/(1+y^2+z^2) + 5y \\ \dot{y} = 2y - z^3 - \ln(1+y^2+z^2) \\ \dot{z} = -x^7 + y + 3z \end{cases}$$

11.What are the values of $\alpha \in \mathbb{R}$ such that the critical point at the origin is asymptotically stable in the first approximation? $\begin{cases} \dot{x} = \sin(\alpha x - 3y) + x\cos y \\ \dot{y} = \alpha e^{\alpha x - 2y} + \ln(1+\alpha x) + 2x - \alpha \end{cases}$

12. What are the values of $\alpha \in \mathbb{R}$ such that there exists a constant solution stable in the first approximation. What is the solution?

$$\ddot{x} + (\alpha^2+1)^2 \cos(\alpha\ddot{x} + \dot{x}) + 3\alpha \sin\dot{x} - \ln(1 - 2\alpha\ddot{x}) - x^2 = 0$$

13. Draw the phase portrait for the DE $\ddot{x} = -x(x^2-1)$ and classify the critical points. What are the angles of the curves crossing the axis $x = 0$? Calculate the slopes of the curves entering or leaving the critical points.

14. Draw the phase portrait for the DE $\ddot{x} = (x^2-1)(x+2)$ and classify the critical points. What are the angles of the curves crossing the axis $x = 0$? Calculate the slopes of the curves entering or leaving the critical points.

15. Which of the following boundary-value problems are self adjoint? Explain.

$$\begin{cases} (u'\cos x)' + u\sin x = 0 \\ u(0) = 0, u(1) = 0 \end{cases}, \quad \begin{cases} u'' + u' + u = 0 \\ u(0) = 0, u'(1) + u(1) = 0 \end{cases}, \quad \begin{cases} (u'\sin x)' + u = 0 \\ u(0) = 0, u'(1) = 0 \end{cases}.$$

16. Find eigenvalues (graphic methods are allowed) and the eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} u'' + u + \lambda u = 0 \\ u(0) = 0, u(1) + 3u'(1) = 0 \end{cases}.$$

17. Find eigenvalues (graphic methods are allowed) and the eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} u'' + u + \lambda u = 0 \\ u(0) = 0,\ u(\pi/2) + u'(\pi/2) = 0 \end{cases}.$$

18. Find the Green function and express the solution for the boundary-value problem using definite integrals of concrete elementary functions (do not calculate the integrals),

$$\begin{cases} u'' + u = e^{-\sqrt[5]{x}} \\ u(0) = 0,\ u(\pi/2) + u'(\pi/2) = 0 \end{cases}.$$

19. **Not relevant since 2010.**
a. Find the value $\alpha \in \mathbb{R}$ for which there exists a solution in the form of Fourier series for a Sturm-Liouville problem compatible with the boundary-value problem $\begin{cases} u'' + u = 1 + \alpha \sin x \\ u(0) = 0, u(\pi) = 0 \end{cases}$, so that it satisfies the equation in $(0, \pi)$.
b. What is the number of such solutions? Find them.

20. **Not relevant since 2010.**
a. Find the value $\alpha \in \mathbb{R}$ for which there exists a solution in the form of Fourier series for a Sturm-Liouville problem compatible with the boundary-value problem $\begin{cases} u'' + u = \sin x + \alpha \cos x \\ u'(0) = 0, u'(\pi) = 0 \end{cases}$.
b. What is the number of such solutions? Find them.

21. Expand the function $f(x) = 1$ in the Fourier series with respect to the eigenfunctions of the Sturm-Liouville problem $\begin{cases} u'' + 4u + 4\lambda u = 0 \\ u(0) = 0, u'(\pi/2) = 0 \end{cases}$.

22. Find the critical points of the system $\begin{cases} \dot{x} = y - x - x^2 \\ \dot{y} = 3x - y - x^2 \end{cases}$ and check their stability.

23. Find the critical points of the system $\begin{cases} \dot{x} = 2x + y - 3x^2 + 4y^2 \\ \dot{y} = x + 2y + 3x^2 - 4y^2 \end{cases}$ and check their stability

24. Find constant solutions of the DE $\ddot{x} + \sin(x\dot{x}) + x^2 - 1 = 0$ and check their stability.

25. Draw the phase portraits of the systems. Classify the critical points.

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y + 2 \end{cases}, \begin{cases} \dot{x} = 4x - y + 1 \\ \dot{y} = 9x - 2y \end{cases}, \begin{cases} \dot{x} = -5x + 3y \\ \dot{y} = -9x + 7y \end{cases}, \begin{cases} \dot{x} = 3x - 7y \\ \dot{y} = 2x - 6y \end{cases},$$

$$\begin{cases} \dot{x} = 3x - 8y + 5 \\ \dot{y} = 2x - 5y + 3 \end{cases}, \begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = 4x - 5y \end{cases}, \begin{cases} \dot{x} = 3x - 2y \\ \dot{y} = 2x - 2y + 2 \end{cases}, \begin{cases} \dot{x} = -3x + y - 2 \\ \dot{y} = -5x + y \end{cases}.$$

26. Find a compact region such that all solutions enter it in finite time to stay in the region.

$$\begin{cases} \dot{x} = -4x - 2y + 0.01\cos(t^3 + xy) \\ \dot{y} = 3x + y \end{cases}$$

27. Find a compact region such that all solutions starting in it asymptotically converge to 0 without leaving the region.

$$\begin{cases} \dot{x} = x - 3y + |x|^{3/2} \sin(tx + 2019) \\ \dot{y} = 4x - 6y + y^2 \sin(tx) \end{cases}$$

28. Check the stability of solutions of the following DEs:

$$x^{(4)} + \dddot{x} - 2\ddot{x} + \dot{x} + x = \sin t; \qquad \dddot{x} + 2\ddot{x} + \dot{x} + 3x = 1; \qquad \ddot{x} + \pi\dot{x} + x = 0 \ ;$$

$$\dot{x} = -x^3, x(0) = 0 \ ; \qquad \dot{x} = -x^3 + x^2,\ x(0) = 0; \qquad \dot{x} = -|x|,\ x(0) = 0 \ ;$$

$$\dddot{x} + \ddot{x} + \pi\dot{x} + 3x = x^2,\ \ x(0) = \dot{x}(0) = \ddot{x}(0) = 0 .$$

29. What is the minimal order n of the DE $y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = 0$ with constant real coefficients, such that the functions $y = t^4$, $y = t^2 \cos t$ are its solutions (explain)? What is the DE (example)?

30. What is the minimal order k of the DE $y^{(k)} + a_1(t) y^{(k-1)} + ... + a_{k-1}(t)\dot{y} + a_k(t) y = 0$ with continuous real-valued coefficients, such that the function $y = t^6$ satisfies it for all t (explain)? What is the DE (example)?

31. It is known that the characteristic polynomial $p(\lambda)$ of the DE $y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = \sin t$ with constant real coefficients is divided by $(\lambda^2 + 1)^2$. What can be said on the stability of its solutions? Explain.

32. What can be said on the asymptotic stability and/or neutral (“just”) stability of solutions to the DE $y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = t$ with real constant coefficients, if it is known that $a_n = -1$? If it is known that $a_n = 0$? If it is known that $a_n = a_{n-1} = 0$? Explain

33. It is known that the DE $y^{(4)} + a_1 \dddot{y} + a_2 \ddot{y} + a_3 \dot{y} + a_4 y = \sin t$ with constant real coefficients has the particular solution $y_p = t\cos(2t) + \alpha \cos t + \beta \sin t$. Find the coefficients a_1, a_2, a_3, a_4 .

34.

a. Find the coefficients of the Taylor expansion $x = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + o(t^4)$ for the solution of the Cauchy problem $\ddot{x} - t\dot{x} + x\sin t = \ln(1+t), \quad x(0) = 1, \dot{x}(0) = 1$

b. Estimate the remainder $R(t) = x(t) - c_0 - c_1 t - c_2 t^2$ by finding two numbers $\varepsilon > 0$, $M > 0$, such that the inequality $|R(t)| \le M|t|^3$ holds for any $|t| \le \varepsilon$.