No solution of this equation is stable.

Lecture 9. 5/12-2021

Aleksanr Mikhailovich Lyapunov

1857, Yaroslavl $\to$ 1918, Odessa
October 31

Stability 1892

Advisor: Chebyshev

Student: Steklov

His stability theory was forgotten until 1930s (Chetaev)

~~The~~ Lyapunov function (local)

$$\dot{x} = f(t,x), \quad f(t, \cancel{\xi_0}) \equiv 0, \; f \in C$$

$t \geq t_0$, $x \in R^n$

ξ_0 equilibrium,

Consider the vicinity $\|x - \xi_0\| \leq \gamma$ compact

Let the function $V: R^n \to R$ be defined for x, $\|x - \xi_0\| \leq \gamma$,

V is continuously differentiable, $V \in C^1$ .1

V is positive definite .2

$\forall x \; V(x) \geq 0, \; V(x) = 0 \Leftrightarrow x = \xi_0$

$\forall t \geq t_0 \quad \dot{V}(t,x) = \frac{d}{dt} V(x(t)) =$ .3

$$= \nabla V(x(t)) f(t, x(t)) \leq 0$$

Here $\underset{\text{gradient}}{\nabla} V = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}\right), \quad f = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix}$

$$\dot{V}(t,x) = \frac{\partial V}{\partial x_1}(x) f_1(t,x) + \dots + \frac{\partial V}{\partial x_n}(x) f_n(t,x)$$

Remark $\dot{V}$ is a function of the state x and t. It does not depend on solution

Theorem (Lyapunov, 1892 ~~1918~~)

If there exists a Lyapunov function then $x(t) \equiv \xi$, $t \geq t_0$, is stable.

Proof: Let $\xi = 0$ (otherwise $x := x - \xi$)

The idea: $V(x) < \varkappa$ defines a system of vicinities equivalent topologically to $\|x\| < \varepsilon$

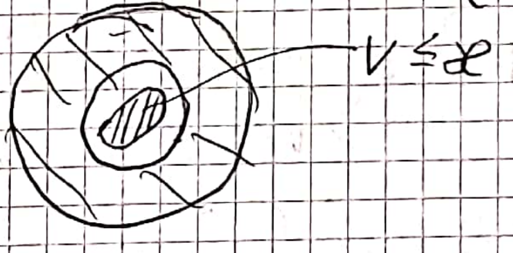
Proposition

1. $\forall \varepsilon > 0,\ \varepsilon < \gamma : \exists \varkappa > 0 : \left\{ x \,\middle|\, \begin{matrix} V(x) < \varkappa \\ \|x\| \leq \gamma \end{matrix} \right\} \subset \left\{ x \,\middle|\, \|x\| < \varepsilon \right\}$

2. $\forall \varkappa > 0,\ \exists \begin{matrix} \varepsilon > 0, \\ \varepsilon < \gamma \end{matrix} : \left\{ x \,\middle|\, \|x\| < \varepsilon \right\} \subset \left\{ x \,\middle|\, \begin{matrix} V(x) < \varkappa, \\ \|x\| \leq \gamma \end{matrix} \right\}$

Proof of the proposition:

1: $\varkappa = \frac{1}{2} \min \left\{ V(x) \,\middle|\, \varepsilon \leq \|x\| \leq \gamma \right\}$



2. Follows from the continuity of $V(x)$ at $x = 0$ (directly!)

QED

$\delta.\varepsilon.\partial.$ N

Proof of the Theorem

1, Solutions are ~~defined~~ extendable till $t=\infty$ for $t \geq t_0$ in some vicinity of $x=0$

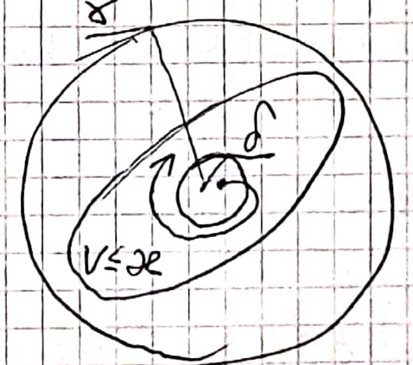
$\exists\, \varkappa, \delta > 0: \{\|x\| < \delta\} \subset \{V < \varkappa\} \subset \{\|x\| < \gamma\}$

$\Rightarrow x(t_0) \in \{\|x\| < \delta\} \Rightarrow V(x(t_0)) < \varkappa$ holds

$\dot V \leq 0 \Rightarrow \forall t \geq t_0 \quad V(x(t)) \leq \varkappa$

$\Rightarrow x(t) \in \{\|x\| < \gamma\}$

Each solution is extended till the boundary of $\{V(x) \leq \varkappa\}$ (impossible) or till $t = \infty$ $\Rightarrow$ till $t = \infty$



2. $\forall \varepsilon > 0, \varepsilon < \gamma \;\exists\, \delta > 0: \|x(t_0)\| < \delta$
$\Rightarrow \forall t \geq t_0 \;\|x(t)\| < \varepsilon$

Take $\varkappa$: $\{\|x\| < \delta\} \subset \{V < \varkappa\} \subset \{\|x\| < \varepsilon\}$

Choice order $\quad \leftarrow \quad \leftarrow$

$V(x(t_0)) < \varkappa \Rightarrow$ forever $\Rightarrow x(t) \in \{\|x\| < \varepsilon\}$

QED.

Local asymptotic stability

Theorem (Lyapunov 1892) (2)

Under the previous conditions 1, 2 for the Lyapunov function V

+ 3) $\dot{V} \leq -w(x) \leq 0$,

where $w \in C$, positive definite

$w : \mathbb{R}^n \to \mathbb{R}$, $w(x) \geq 0$

$w(x) = 0 \Leftrightarrow x = \xi$

Remark $\dot{V} = \dot{V}(t, x) = \nabla V(x) f(t, x)$ depends on t, the inequality establishes uniform separation of $\dot{V}$ from 0.

Proof Once more let $\xi = 0$.

According to the previous theorem the critical point $\xi = 0$ is stable. We "only" need to prove that all solutions $x(t)$ starting in some vicinity of $x = 0$ at $t = t_0$ converge to 0 as $t \to \infty$.

1. Idea It is enough to prove $V(x(t)) \xrightarrow[t \to \infty]{} 0$

Indeed $\lim_{t \to \infty} V(x(t)) = 0 \Leftrightarrow$ (means that)

$\forall \varkappa > 0 \ \exists t_1 \geq t_0 : \ \forall t \geq t_1 \ V(x(t)) < \varkappa$

$\lim_{t \to \infty} x(t) = 0$ means that

$\forall \varepsilon > 0 \ \exists t_1 \geq t_0 : \forall t \geq t_1 \ \|x(t)\| < \varepsilon$

Now from the topological equivalence of vicinities get

$$\forall \varepsilon > 0 \ \exists \varkappa > 0 : \quad \{V < \varkappa\} \subset \{\|x\| < \varepsilon\}$$
$$\varepsilon < \gamma$$

$$\Rightarrow \exists t_1 \geq t_0 : \forall t \geq t_0 \ V(x(t)) < \varkappa \Rightarrow \|x(t)\| < \varepsilon$$

2. Now prove that $V(x(t)) \to 0$
(We already know that solutions starting in $\|x\| < \delta$ are extended till $t = \infty$ in time) for some $\delta > 0$

We want: (definition)

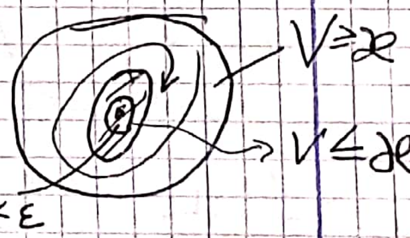
$$\forall \varkappa > 0 \ \exists t_1 \geq t_0 : \forall t \geq t_1 \ V(x(t)) < \varkappa$$

Suppose that it is not true for some solution $x_*(t)$, $x_*(\cdot) : [t_0, \infty) \to \mathbb{R}^n$
then

$$\exists \varkappa_0 > 0 \ \forall t_1 \geq t_0 \ \exists t_* \geq t_1 : V(x_*(t_*)) \geq \varkappa_0$$

Since $\dot{V} \leq 0$ we get $V(x_*(t_*)) \geq \varkappa_0$ $\forall t \geq t_0$

Thus $\forall t \geq t_0 : \ V(x_*(t)) \geq \varkappa_0$
(i.e. $t = t_1$)
from above



But $\dot{V}(x_*(t)) \leq -w(x_*(t))$

Let $\alpha = \min\limits_{\substack{V(x) \geq \varkappa_0 \\ \|x\| \leq \gamma}} w(x) \geq \min\limits_{\substack{\|x\| \geq \varepsilon \\ \|x\| \leq \gamma}} w > 0$ since w pos. def.

$$\Rightarrow \forall t \geq t_0 \quad \dot{V} \leq -\alpha < 0$$

$$\Rightarrow V(x_*(t)) = V(x_*(t_0)) + \int_{t_0}^{t} \dot{V}(s, x_*(s))\, ds \leq V(x_*(t_0)) -$$

$$V(x_*(t)) \leq V(x_*(t_0)) - \alpha(t-t_0)$$

$V(x_*(t)) < 0$ for $t - t_0 > \frac{1}{\alpha} V(x_*(t_0))$

contradiction QED

Global Asymptotic Stability

It is the case when $\gamma = \infty$

$\dot{x} = f(t, x)$, $f(t, \xi)_{t \geq t_0} = 0$, $x \in \mathbb{R}^n$

f is continuous $\forall t$

Lyapunov function: $V: \mathbb{R}^n \to \mathbb{R}$

1. $V \in C^1$ continuously differentiable ξ
2. Positive definite
   $\forall x \in \mathbb{R}^n \quad V(x) \geq 0$, $V(x) = 0 \Leftrightarrow x = \xi$
3. There exists $w: \mathbb{R}^n \to \mathbb{R}$
   w is continuous, $w \in C$
   and positive definite: $\forall x \quad w(x) \geq 0$
   $w(x) = 0 \Leftrightarrow x = \xi$

   and $\forall t \geq t_0$, $\forall x \in \mathbb{R}^n$
   $$\dot{V}(t, x) = \nabla V(x) f(t, x) \leq -w(x) \leq 0$$

4. New condition
   V is ~~rough~~ radially unbounded
   For any sequence $\{x_k\}$, $\lim_{k \to \infty} \|x_k\| = \infty$,
   also $V(x_k) \to \infty$

If exists such Lyapunov function $\Rightarrow$ global as. stability

It is easy to check $V \in C$ pos. def.

V is radially unbounded $\Leftrightarrow$

$\forall c \geq 0$ the set $\underbrace{\{x \in \mathbb{R}^n \mid V(x) \leq c\}}_{\text{level set}}$ is bounded (and compact)

<u>Proof</u>: "$\Rightarrow$" If $\Omega_c = \{V \leq c\}$ is unbounded

"$\Leftarrow$" then $\exists \{x_k\} \subset \Omega_c$, $\|x_k\| \to \infty$

i.e. V is not radially unbounded

<u>Example</u> $\underbrace{\|x\|^2}_{= x_1^2 + \dots + x_n^2}$ is radially unbounded

$V = \arctan \|x\|^2$ is not radially unbounded

but all its level sets are bounded for $c < \frac{\pi}{2}$

4') All level sets $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\}$ are bounded

<u>Theorem</u> Conditions 1–3 and 4' $\Rightarrow$ Global asymptotic stability of the equilibrium ξ

<u>Proof</u> Local as. stability is already proved. Consider some $x(t_0)$

Let $\gamma = \cancel{2V(x(t_0))} \max\limits_{\Omega_{2V(x(t_0))}} \|x - \xi\|$

Further all the same. QED

Example! $\begin{cases} \dot{x} = -x^3 - 2xy^3 \\ \dot{y} = x^2 - y \end{cases}$ 125

Is $\xi = (0,0)$ stable?

Solution! $v = x^2 + y^4$

$\dot{v} = 2x(-x^3 - 2xy^3) + 4y^3(x^2 - y)$

$= -2x^4 - \cancel{4x^2y^3} + \cancel{4y^3x^2} - 4y^4$

$\dot{v} = -2x^4 - 4y^4 = -w \le 0$

$w = 2x^4 + 4y^4$ is positive definite

Obviously v has bounded level sets $\left(x^2 + y^4 \le c \Rightarrow |x| \le c^{\frac{1}{2}}, |y| \le c^{\frac{1}{4}}\right)$

Global as. stability

It also is radially unbounded.

Obviously $\tilde{v} = (x^2 + y^4)^{\frac{1}{2}}$ is also a Lyapunov function, radially unbounded

$\tilde{\tilde{v}} = \arctan(x^2 + y^4)$ is a Lyapunov function with bounded level sets for any its value

$$\dot{\tilde{\tilde{v}}} = \frac{\dot{v}}{1 + (x^2 + y^4)^2} = \frac{\dot{v}}{1 + v^2}$$

Remark Obviously smooth coordinate transformation preserves the stability properties

~~II~~ Instability criterion

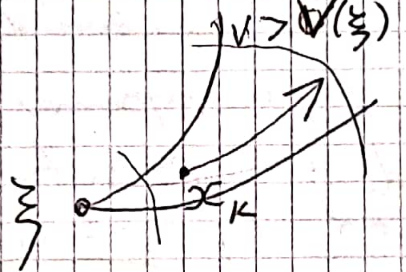
Chetaev Nikolay Gurievich (1902–1959)
Kazan University 1930–1940
theory of stability

The instability theorem

$\dot{x} = f(t,x)$, $\underset{t \geq t_0}{f(t,\xi) \equiv 0}$, $f \in C$
$\|x-\xi\| \leq \gamma$

1. There exists $v: \mathbb{R}^n \to \mathbb{R}$, $v \in C^1$, .1
2. $\exists \{x_k\}_{k=1}^{\infty}$, $\lim_{k\to\infty} x_k = \xi$, $v(x_k) > v(\xi)$ .2
(The way of escape)
3. $\exists w \in C$, $w(x) \geq 0$ nonnegative .3
$x \neq \xi \Rightarrow w(x) > 0$

$\Rightarrow$ The constant solution $x(t) \equiv \xi$
is not stable



Stability ~1,2,3~
$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x(t)$-solution
$\|x(t_0) - \xi\| < \delta \Rightarrow \forall t \geq t_0$
$\|x(t) - \xi\| < \varepsilon$

<u>Proof</u> We need to prove
$\exists \varepsilon > 0\ \forall \delta > 0\ \exists \underset{\text{solution}}{x_*(t)} : \|x_*(t_0) - \xi\| < \delta$
$\exists t_1 \geq t_0 : \|x_*(t_1) - \xi\| \geq \varepsilon$

Suppose in a vicinity solutions are extendable till $t = \infty$ (otherwise there is already no stability)

Let $\underbrace{\varepsilon}_{\|x-\xi\|\le 2\varepsilon}$ be some vicinity of ξ, $\cancel{x(t)}$ where solutions are extendable.

$\forall \delta > 0$ take x_k: $\|x_k - \xi\| < \delta$, $< \varepsilon$,

Let $x_*(t)$ be the solution satisfying $x_*(t_0) = x_k$, $t \ge t_0$,

Then $\dot V = \frac{\partial V}{\partial x}(x_*(t)) f(t, x_*(t)) \ge w(x_*) \ge 0$

$$x_k \in \left\{ x \in \mathbb{R}^n \,\middle|\, \frac{\|x_k - \xi\|}{2} \le \|x - \xi\| \le 2\varepsilon \right\} = \Omega$$

Ω is compact

$$\dot V \ge \min_{\Omega} w = \alpha > 0$$

$$\Rightarrow V(x_*(t)) \ge V(x_k) + \alpha(t - t_0) \xrightarrow[t\to\infty]{} \infty$$

in finite time get $V(x(t)) \ge \max_{\|x-\xi\|\le\varepsilon} V(x)$

QED

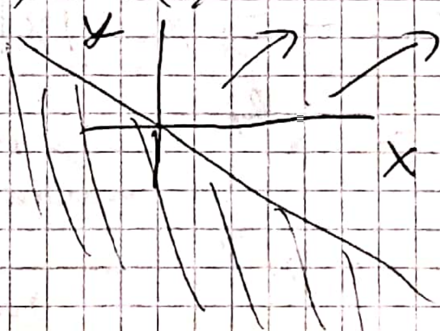
Example
$$\begin{cases} \dot x = y - x^3 y^3 + y^6 \\ \dot y = y - 2x^3 y^3 - x^6 \end{cases}, \quad \xi = (0,0)$$

Let $V = x - y$, $(x-y)^\cdot = x^6 + x^3 y^3 + y^6 = w$

$V(0) = 0$ $\quad w = \left(x^3 + \frac{1}{2} y^3\right)^2 + \frac{3}{4} y^6 \ge 0$ pos. def.

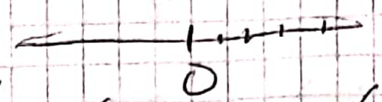
$V > 0$: $x > y$, $(x_k, y_k) = \left(\frac{1}{k}, 0\right)$

Unstable



$(x-y)' > 0$

Example $x \in \mathbb{R}$, $\dot{x} = x^2 = w$

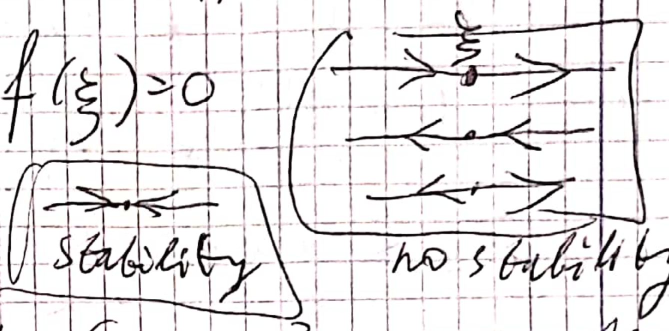


$V = x$

$\dot{V} = 2x$ Chetaev $\Rightarrow$ no stability

Simple case $x \in \mathbb{R}$.

$\dot{x} = f(x)$, $f(\xi) = 0$



formally: $V = (x-\xi)^2$

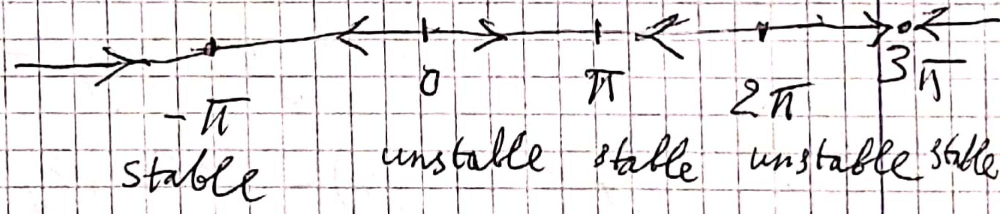
$\dot{V} = 2(x-\xi)f(x)$ neg. definite on a vicinity $\Rightarrow$ as. st.

sign $\dot{V}$ = sign $f(x) \cdot$ sign $(x-\xi)$.

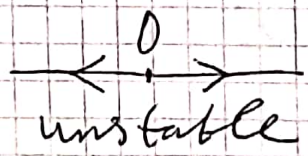
In scalar case one always takes

$V = (x-\xi)^2$

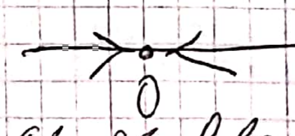
Examples $\dot{x} = \sin x$



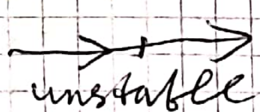
$\dot{x} = tg\, x^3$, $\xi = 0$



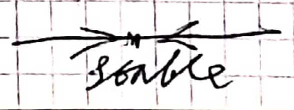
$\dot{x} = -tg\, x^3 \cdot |x|$, $\xi = 0$



$\dot{x} = x^4$



$\dot{x} = -x^5$



Kurzweil (1956)

<u>Theorem</u> $f: \mathbb{R}^n \to \mathbb{R}^n$, $f \in C$, $f(0)=0$

0 is the only critical point.

Local / global as. stability

$\Longleftrightarrow$ There exist (radially unbounded) Lyapunov function V and w, both pos. definite

$V \in C^\infty$, $w \in C^\infty$, $\underline{\dot V = -w}$

But there is no way to find V in the general case.

<u>Asymptotic stability in the first approximation</u>

$$\dot x = f(t,x) = A(x-\xi) + R(t,x)$$

$f \in C$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ constant

$R(t,x) = o(x-\xi)$, i.e.

$\forall \varepsilon > 0\ \exists \delta > 0 : \|x-\xi\| < \delta \Rightarrow \|R(t,x)\| < \varepsilon \|x-\xi\|$
$t \geq t_0$

Particular case: $\dot x = f(x) = A(x-\xi) + o(x-\xi)$

$A = \frac{\partial f}{\partial x}(\xi)$

$\dot z = A z$
linearization

4 Let $\mathbb{C}_+ = \{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0 \}$

$\mathbb{C}_- = \{ \lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0 \}$

Spec A — spectrum, spectre of A

<u>Theorem 1</u> Spec $A \subset \mathbb{C}_-$
$\Rightarrow x_*(t) \equiv \xi,\ t \geq t_0$, asymptotically stable

<u>Theorem 2</u> Spec $A \cap \mathbb{C}_+ \neq \emptyset$
$\Rightarrow x_*(t)$ is unstable

We will only prove Theorem 1.

<u>Examples</u>

1. All critical points in $\mathbb{R}^2$ which we learned:
stable/unstable node (knote)
stable/unstable ~~to~~ improper node
stable/unstable focal point
<u>stable/unstable star point</u>
"stable" $\Leftrightarrow$ asymptotically stable
"unstable" is indeed unstable

2. $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f(x,y) \quad \begin{cases} \dot{x} = \sin(x-2y-1) \\ \dot{y} = 3x-5y+xy-3 \end{cases} \qquad \xi = (1,0)$$

$$\left.\frac{\partial f(x,y)}{\partial (x,y)}\right|_{(1,0)} = \left.\begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}\right|_{(1,0)} = \left.\begin{pmatrix} \cos(x-2y-1), & -2\cos(x-2y-1) \\ 3+y & , -5+x \end{pmatrix}\right|_{(1,0)} =$$

$$= A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad P(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

<u>asymptotically stable</u>

3) $\dot{x} = \sin(x - 2y - 1)$ similar $\xi = (1, 0)$ |3|

$\dot{y} = 3x - 5y + 7xy - 3$

$$A = \frac{\partial f(x,y)}{\partial (x,y)}\Bigg|_{(x,y)=\xi} = \begin{pmatrix} \cos(x-2y-1), & -2\cos(x-2y-1) \\ 3+7y & , -5+7 \end{pmatrix}\Bigg|_{\substack{x=1 \\ y=0}}$$

$$A = \begin{pmatrix} 1 & , -2 \\ 3 & , 2 \end{pmatrix}, \quad P(\lambda) = \lambda^2 - 3\lambda + 8 =$$

$$\lambda_{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{29}\, i \in \mathbb{C}_+$$

$9 - 32 = -29$

unstable