Lecture 8 28/11-2021

We continue the study of critical points in the plane

3, complex eigenvalues $\lambda_1 = \bar{\lambda}_2 = \alpha + \beta i$ (conjugate)

$\alpha, \beta \in \mathbb{R}, \quad \beta \neq 0$

There are two linearly independent complex eigenvectors, $A = \bar{A} \in \mathbb{R}^{2\times 2}$

$$(A - \lambda_1 I)e_1 = 0 \quad (A - \bar{\lambda}_1 I)\bar{e}_1 = 0$$

$\Rightarrow e_2 = \bar{e}_1$ can be taken

$$e_1 = u + vi, \quad e_2 = u - vi, \quad u, v \in \mathbb{R}^2$$

$$u = \frac{e_1 + e_2}{2}, \quad v = \frac{e_1 - e_2}{2i} \Rightarrow u, v \text{ linearly independent}$$

Choose them as the new basis

$$A(u + vi) = (\alpha + \beta i)(u + vi)$$

$$Au + Avi = \alpha u - \beta v + (\alpha v + \beta u)i$$

Thus
$$Au = \alpha u - \beta v$$
$$Av = \alpha v + \beta u, \quad \underbrace{x = \underset{\text{columns}}{[u\, v]} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{\text{coordinate change}}$$

From $\dot{x} = Ax$

$$[u, v]\dot{z} = A[u, v]z \qquad [u, v]^{-1}: \; u \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; v \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\dot{z} = [u, v]^{-1} A [u, v] z = [u, v]^{-1} [\alpha u - \beta v, \alpha v + \beta u] z$$

$$\dot{z} = \left[\begin{pmatrix} \alpha \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \begin{pmatrix} \beta \\ 0 \end{pmatrix} \right] z = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} z$$

$$\Rightarrow \quad z(t) = \exp\left(\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} t\right) z(0) \qquad \begin{pmatrix} \text{p. 61} \\ \text{Lecture 5} \end{pmatrix}$$

$$z(t) = e^{\alpha t} \begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

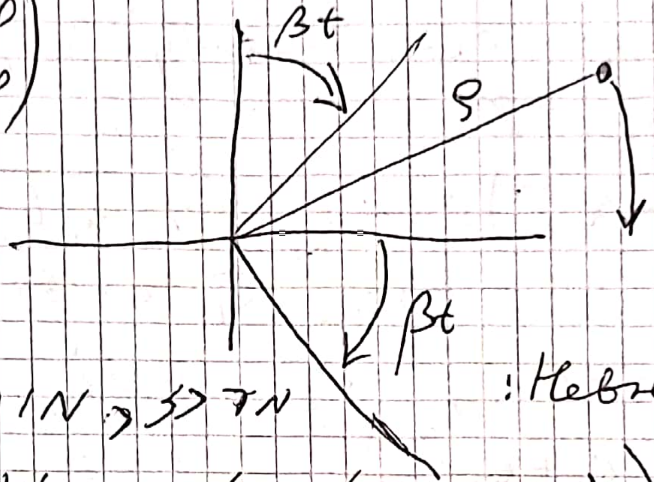
rotation by the angle $(-\beta t)$

Let:

$$z = \rho \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\rho(t) = \rho(0) e^{\alpha t}$$

$$\varphi(t) = \varphi(0) - \beta t$$



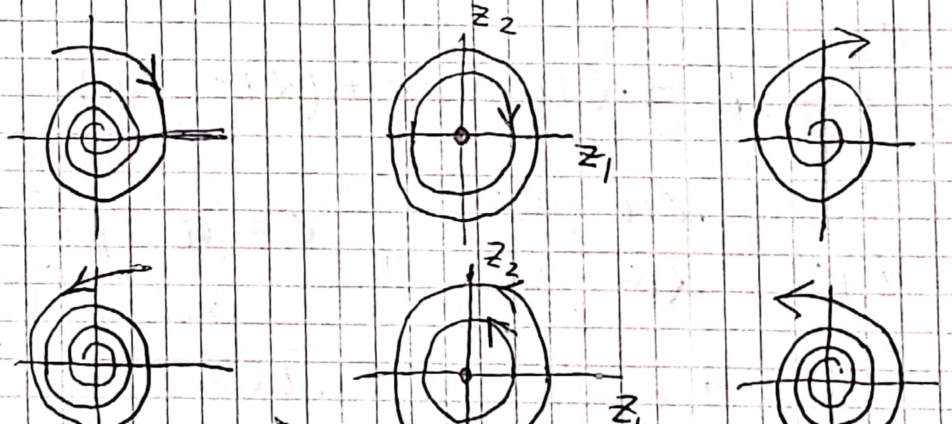
Hebrew: מרכז, מוקד יציב / לא יציב

$$z(t) = e^{\alpha t} \rho(0) \begin{pmatrix} \cos(\varphi(0) - \beta t) \\ \sin(\varphi(0) - \beta t) \end{pmatrix}$$

$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

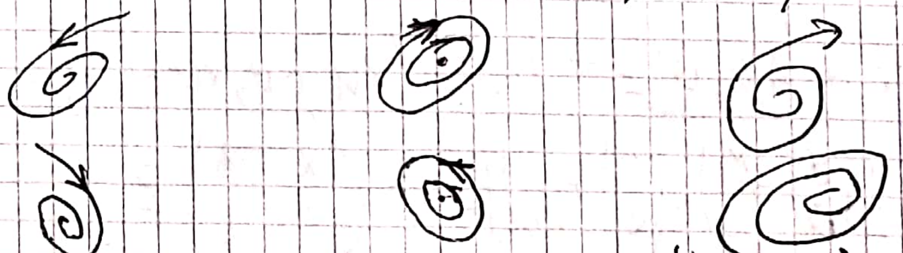
$\beta > 0$

$\beta < 0$



stable מוקד יציב
sink, focus, focal point   center מרכז   unstable מוקד לא יציב

x, y
general case



To find the rotation direction it is enough to find <u>one</u> velocity vector

Once more the names:

English: spiral point
focal point   stable / unstable
focus (foci plural)
focuses

Also: sink (i.e. stable), source (i.e. unstable)
center

<u>Hebrew</u>: מוקד יציב / לא יציב   מוקד (=לא יציב)
מרכז

<u>Examples</u>   Recall: $\dot{x} = Ax + b$, $Ax_0 + b = 0$

$\Rightarrow y = x - x_0$, x_0 critical point

$\dot{y} = Ay$

$\boxed{\text{1. It can be that there is no critical point}}$

~~2. There is~~

<u>Example</u>
$$\begin{cases} \dot{x} = 2x - 2 \\ \dot{y} = x + y \end{cases} \qquad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ critical point}$$

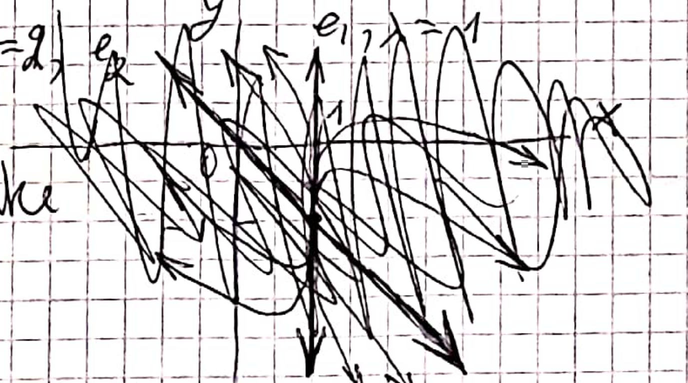
$$p(\lambda) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 1), \quad \lambda_{1,2} = 1, 2$$

<u>unstable knote</u>, <u>קשר לא יציב</u>

$$\lambda = 1 \qquad \begin{matrix} (2-1)x = 0 \\ x + y \cdot 0 = 0 \end{matrix} \qquad x = 0 \qquad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda = 2 \qquad \begin{cases} 0 = 0 \\ x + (1-2)y = 0 \end{cases} \qquad x = y \qquad e_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

escape along e_2 is faster

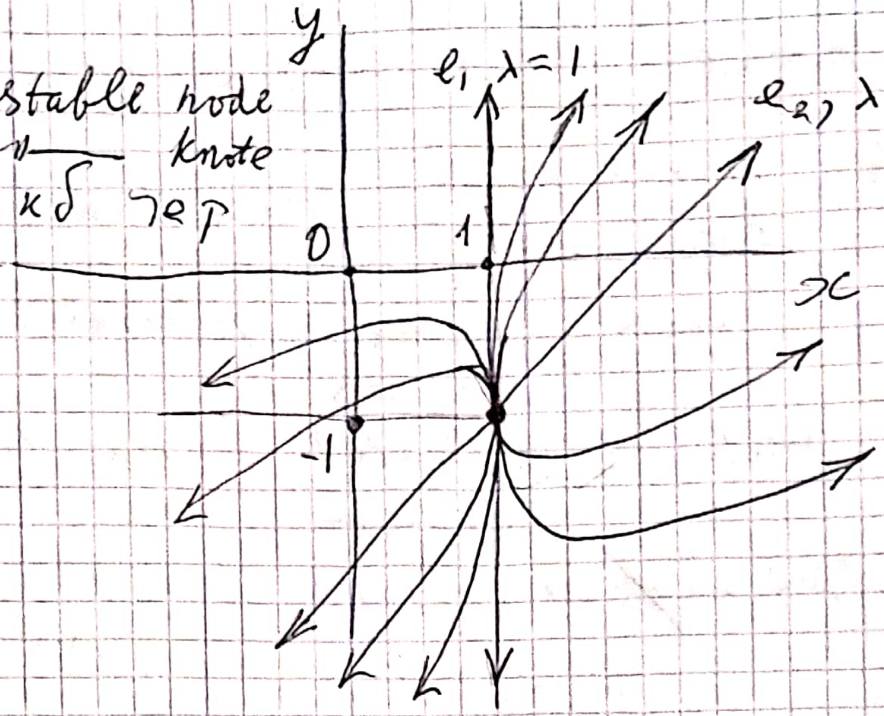


mistake

Unstable node

— " — knote

קשר לא יציב



2. $\begin{cases} \dot{x} = x - 2y \\ \dot{y} = 4x - 3y \end{cases}$ $\quad \begin{vmatrix} 1-\lambda & -2 \\ 4 & -3-\lambda \end{vmatrix} =$

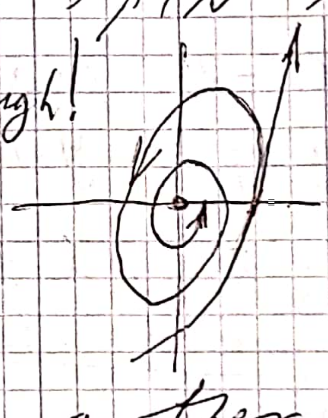
$= \lambda^2 + 2\lambda + 5$

~~Un~~stable focus

מוקד יציב

$\lambda = -1 \pm 2i$

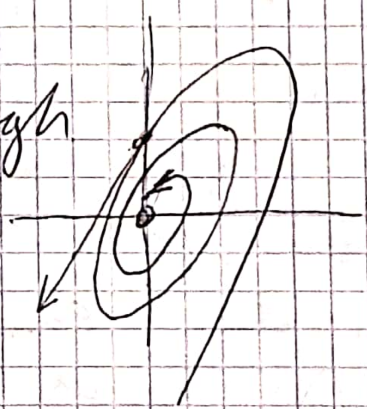
That's enough!



$\left.\begin{pmatrix} x-2y \\ 4x-3y \end{pmatrix}\right|_{(1,0)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

or taking another point: $\left.\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}\right|_{(0,1)} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$

also enough



stable spirale point

sink

→ right answers

3. $\begin{cases} \dot{x} = -3x+2y+1 \\ \dot{y} = -2x+y+1 \end{cases}$ $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

critical point (equilibrium)

$$\begin{vmatrix} -3-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2+2\lambda+1 = (\lambda+1)^2$$

There are 2 possibilities: star, $\lambda_{1,2}=-1$
or degenerate node (stable)

Star would be (always!) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ $\Rightarrow$ $\boxed{\text{stable degenerate node}}$ קשר מנוון

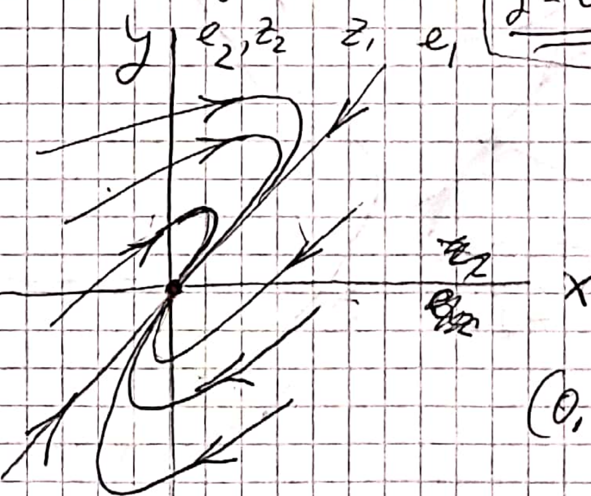
Recall: $\begin{cases} (A-\lambda I)e_1 = 0 \\ (A-\lambda I)e_2 = e_1 \end{cases}$
generalized eigenvector

$\lambda=-1$ $\begin{cases} -2x+2y=0 \\ -2x+2y=0 \end{cases}$ $x=y$ $e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{cases} -2x+2y=1 \\ -2x+2y=1 \end{cases}$ $\xrightarrow{x=0}$ $e_2 = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$

$\xrightarrow{y=0}$ $e_2' = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$

$z_1 z_2 \cdot (-1) > 0$



$z_1 z_2' \cdot (-1) > 0$

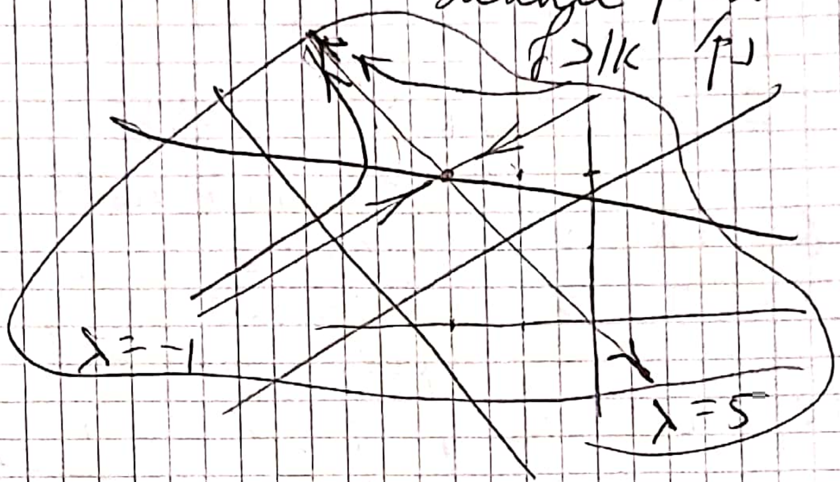
(0,0) – tangent point
השקה

4. $\begin{cases} \dot{x} = 3x+4y-2 \\ \dot{y} = 2x+y+2 \end{cases}$ $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

critical point

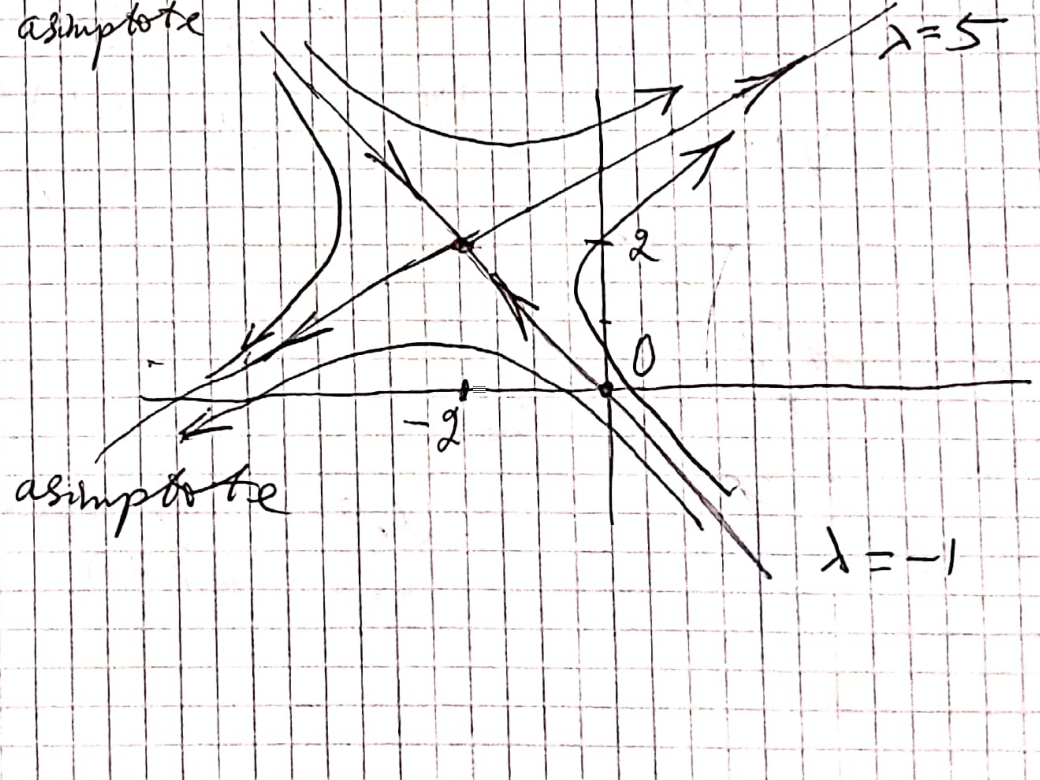
$$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda-5)(\lambda+1)$$

saddle point never stable



$\lambda = 5$: $\begin{cases} -2x+4y=0 \\ 2x-4y=0 \end{cases}$ $2y = x$ $e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda = -1$ $\begin{cases} 4x+4y=0 \\ 2x+2y=0 \end{cases}$ $y=-x$ $e_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



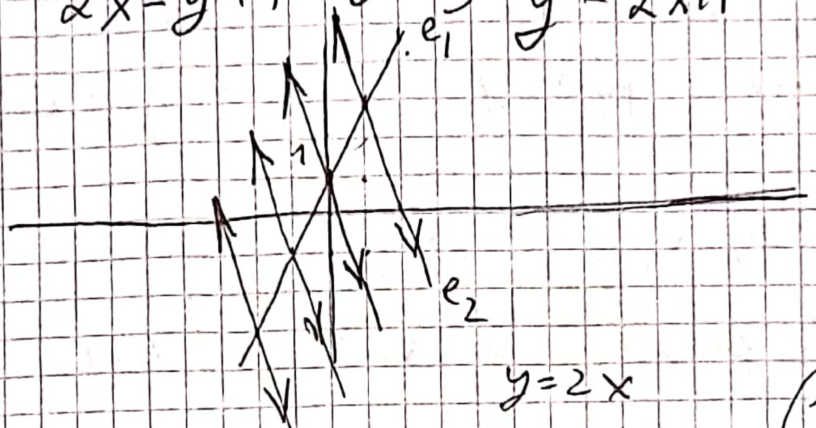
proportional!

111

5. $\begin{cases} \dot{x} = 2x - y + 1 \\ \dot{y} = -6x + 3y - 3 \end{cases}$ $\begin{vmatrix} 2-\lambda & -1 \\ -6 & 3-\lambda \end{vmatrix} = \lambda^2 - 5\lambda$

$\lambda_{1,2} = 0, 5$

$2x - y + 1 = 0 \Rightarrow y = 2x+1$ critical points



$y = 2x$

$\lambda = 0 \quad 2x - y = 0 \quad e_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\lambda = 5 \quad \begin{cases} -3x - y = 0 \\ -6x - 2y = 0 \end{cases} \quad -3\cancel{x} = y \quad e_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

6. $\begin{cases} \dot{x} = x - y - 2 \\ \dot{y} = x - y - 2 \end{cases}$ $\begin{vmatrix} 1-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2$ no name

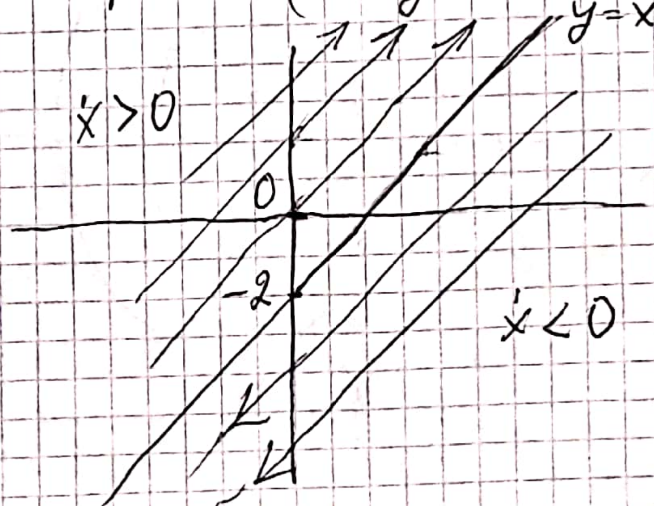
$\lambda = 0$

$\lambda = 0 \quad \begin{cases} x - y = 0 \\ x - y = 0 \end{cases} \quad x = y \quad e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x - y - 2 = 0$

$y = x - 2$

critical points $\begin{cases} x - y = 1 \\ x - y = 1 \end{cases} \quad y = 0 \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

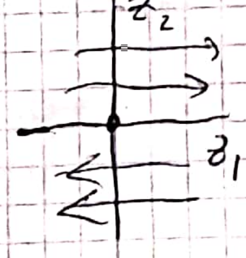


Jordan block

$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{matrix} z_1 \\ z_2 \end{matrix}$

$\dot{z}_1 = z_2$

$\dot{z}_2 = 0$



Nonlinear Case

$\dot{x} = f(x)$, $x \in \mathbb{R}^2$, $f(x_0) = 0$

$f \in C^2$

$\dot{x} = \underbrace{\frac{\partial f}{\partial x}}_{\text{matrix}}(x_0)(x - x_0) + o(\|x - x_0\|)$

$p(\lambda) = \det(f'(x_0) - \lambda I) = 0$

roots: $\lambda_{1,2}$

assume: $\boxed{\operatorname{Re}\lambda_1 \neq 0,\ \operatorname{Re}\lambda_2 \neq 0}$

Topological classification

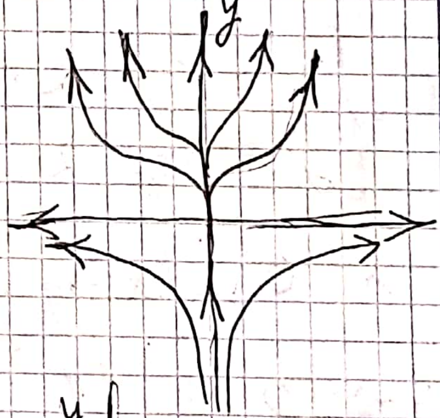
$\lambda_1\lambda_2 > 0$	$\lambda_1 = \bar{\lambda}_2$	$\lambda_1\lambda_2 < 0$
knote קשר	focus מוקד	saddle point אוכף

(star, degenerate node are here)

Example $\quad \dot{x} = x \quad \lambda_1 = 1$

$\dot{y} = y^2 \quad \lambda_2 = 0$

saddle-knote

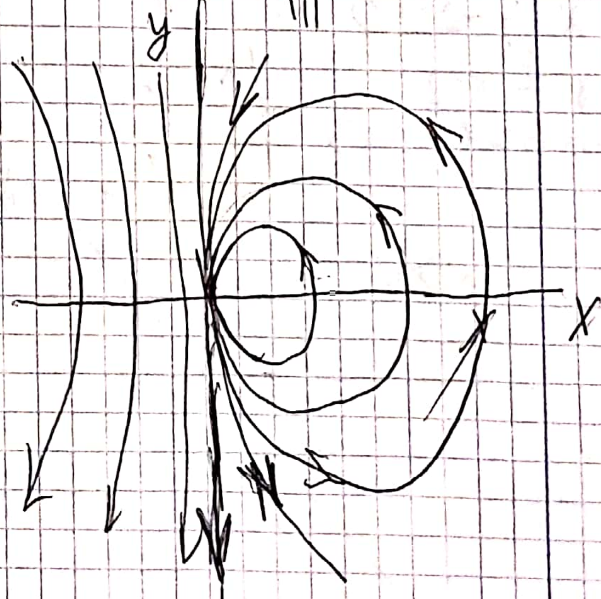


Example

$$\begin{cases} \dot{x} = -xy \\ \dot{y} = \frac{1}{2}x - y^2 \end{cases}$$

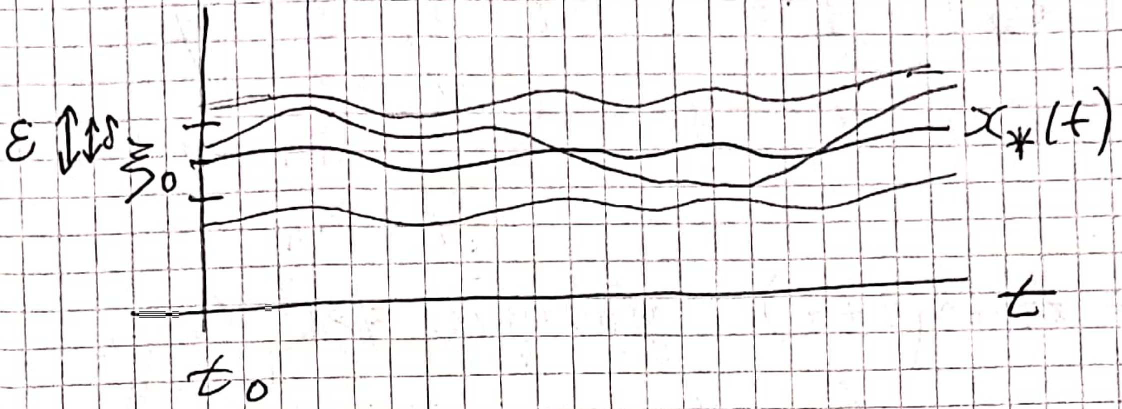
$f'(0) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$, $\lambda = 0$

Jordan block



Introduction to Stability Theory

$$\begin{cases} \dot{x} = f(t, x) \\ x(t_0) = \xi, \quad x \in \mathbb{R}^n, \quad f \in C \end{cases}$$ קושי Cauchy Problem



Does the solution continuously depend on the initial conditions over the infinite time interval $t_0 \le t < \infty$?

<u>Definition</u> Let solution $x_*(\cdot)$, $x_*: [t_0, \infty) \to \mathbb{R}^n$, $x_*(t_0) = \xi_0$, of the equation $\dot{x} = f(t, x)$ is called Lyapunov stable at the initial time t_0 ליפ' יציב

if 1. $\exists \delta_* > 0$: solution $x(t)$ exists for each $\|\xi - \xi_0\| < \delta_*$, $t \ge t_0$

2. $\forall \varepsilon > 0 \; \exists \delta > 0 : \|x(t_0) - \xi_0\| < \delta \Rightarrow$

$$\forall t \ge t_0 \quad \|x(t) - x_*(t)\| < \varepsilon$$

The case of the scalar equation

$$y^{(n)} = f(t, y, \dot{y}, \dots, y^{(n-1)}), \quad (y(t_0), \dots, y^{(n-1)}(t_0))^T = \xi$$

Solution y_*, $(y_*(t_0), \dots, y^{(n-1)}(t_0))^T = \xi_0$ is called stable if it is true for

$$\dot{x}_0 = x_1, \dots, \dot{x}_{n-2} = x_{n-1}, \quad \dot{x}_{n-1} = f(t, x_0, \dots, x_{n-1}), \quad x(t_0) = \xi,$$

and the solution x_*, $x_*(t_0) = \xi_0$.

Example $\ddot{x} = -x + t$, $x \in R$, $t_0 = 0$

$x(0) = \dot{x}(0) = 0$

Solution: $x_* = -\sin t + t$. Is it stable?

Denote $y = x - x_*$ $(= x(t) - x_*(t))$

$\Rightarrow (x - x_*)^{\cdot\cdot} = \ddot{y} = -x(t) + t + x_* - t = -y$

$\ddot{y} = -y$ $\left(X = X_h + x_* \right)$

$y \in X_h$

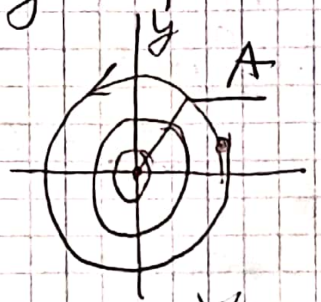
$\Rightarrow y = A\cos(t + \varphi)$ $\quad y^2(0) + \dot{y}^2(0) = \|\xi\|^2 = A \quad \|\xi - \xi_0\|^2$

$A\cos\varphi = y(0)$, $-A\sin\varphi = \dot{y}(0)$

Obviously $y^2 + \dot{y}^2 = A^2 = \|\xi - \xi_0\|^2$

$\Rightarrow \|(x - x_*), (\dot{x} - \dot{x}_*)\| =$

$= \|\xi - \xi_0\|$



$\Rightarrow$ Stability of x_* with $\delta = \varepsilon$

Asymptotic Stability, יציבות אסימפטוטית

x_* is called asymptotically stable (יציב אסימפטוטית)

if 1) it is Lyapunov stable (above)

2) $\exists \delta_1 \le \delta_*$:

$\underbrace{\|x(t_0)}_{\xi} - \underbrace{x_*(t_0)}_{\xi_0}\| < \delta_1 \Rightarrow \lim\limits_{t \to \infty} \|x(t) - x_*(t)\| = 0$

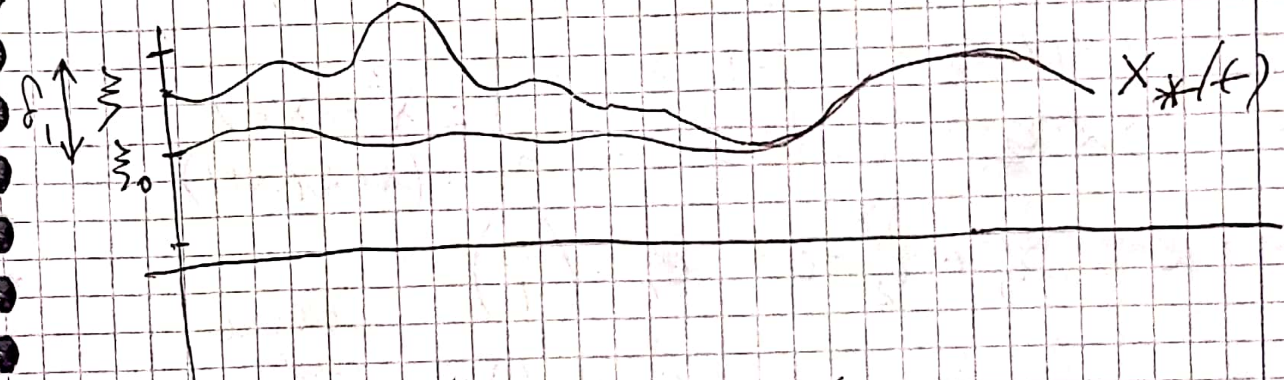
We have

Global Asymptotic Stability

If $\delta_1 = \delta_* = \infty$, i.e. if

1. Solution $x(t)$ exists for any initial condition $X(t_0) = \xi \in \mathbb{R}^n$
   $t \in [t_0, \infty)$

2. $x_*(t)$ is Lyapunov stable at t_0
   $x_*(t_0) = \xi_*$
   and $\forall x(t)$: $\lim\limits_{t\to\infty} \underbrace{\left(x(t) - x_*(t)\right)}_{\text{norm can be omitted here}} = 0$
   for any solution



<u>Example</u> $\ddot{x} + 2\dot{x} + x = t$, $x_* = t - 2$
$t_0 = 0$, $x(t_0) = -2$, $\dot{x}(t_0) = 1$

As previously $y = x - x_*$, $\ddot{y} + 2\dot{y} + y = 0$, $y_* = 0$

Degenerate node (Improper) $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

קשר מנוון

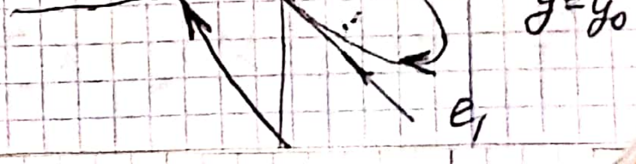
$$\begin{cases} y = c_1 e^{-t} + c_2 t e^{-t} \longrightarrow 0 \\ \dot{y} = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t} \longrightarrow 0 \end{cases}$$

$\Rightarrow$ We have asymptotic stability

$y_0 = y$, $y_1 = \dot{y}$; $\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$

$\lambda = -1$ $y_0 + y_1 = 0$ $e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{cases} y_0 + y_1 = 1 \\ -y_0 - y_1 = -1 \end{cases} \overset{y_1 = 0}{\Longrightarrow} e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



<u>Definition</u> Critical point x_0 of the autonomous equation $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $f \in C$, is called stable/asymptotically stable/globally asymptotically stable if the constant solution

$$x(t) \equiv x_0, \quad t \geq 0,$$

or any number t_0

is of the same stability.

<u>Definition</u> <u>linear system</u> $\dot{x} = Ax$, $x \in \mathbb{R}^n$, is called stable/as. st. if $x = 0$ is stable, as. st.

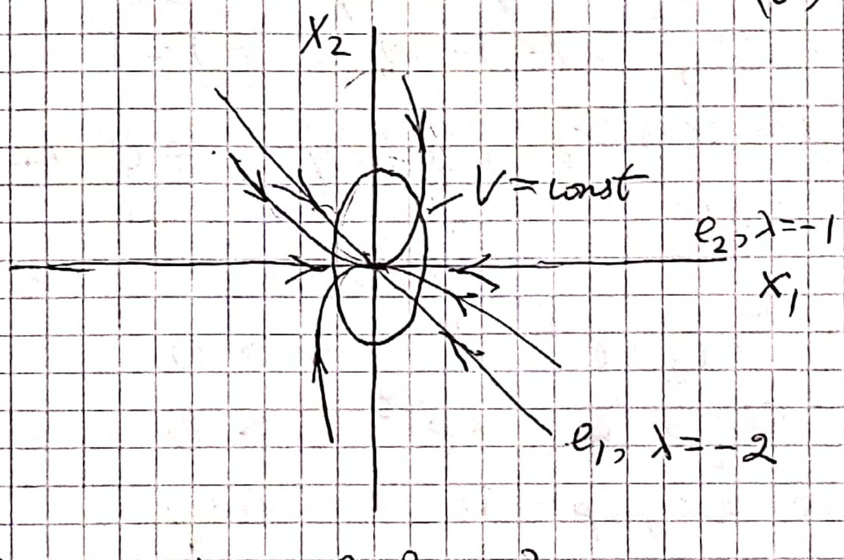
<u>Example</u> Is the origin stable?

$$\begin{cases} \dot{x}_1 = -x_1 + x_2 \\ \dot{x}_2 = -2x_2 \end{cases} \qquad \begin{vmatrix} -1-\lambda & 1 \\ 0 & -2-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

$\lambda = -2 \quad x_1 + x_2 = 0 \quad e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = -1 \quad x_2 = 0 \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

stable node



All solutions enter the set $V \leq \text{const}$. Velocities look inside. $\dot{x}$ - velocity

Define $V = 2x_1^2 + x_2^2$

$$\dot{V} = \frac{d}{dt}\left(2x_1^2(t) + x_2(t)^2\right) = 4x_1(-x_1 + x_2) + 2x_2(-2x_2)$$

$$= -4(x_1^2 - x_1x_2 + x_2^2) = -4\left[\left(x_1 - \tfrac{1}{2}x_2\right)^2 + \tfrac{3}{4}x_2^2\right] \leq 0$$

We see that $V \to 0 \Rightarrow x_1, x_2 \to 0$

The origin is an as. stable equilibrium (critical point)

The function V is called the Lyapunov function

Example $x^{(5)} - x^{(4)} + \cos t\, \dddot{x} - (t^2+1)x = \cos^2 t$

Whether there are stable solutions? (any t_0)

Solution Consider a solution x_*.
(all solutions are extendable to $t \in \mathbb{R}$ for any initial conditions)

Denote $y = x - x_*$, where x is any solution

Then $y(t)$ satisfies the homogeneous equation

$x = x_* \Leftrightarrow y = 0$, $y^{(5)} - y^{(4)} + \cos t\, \dddot{y} - (t^2+1) y = 0$

consider any 5 solutions $y_1, y_2, \dots, y_5 : \mathbb{R} \to \mathbb{R}$

let $w(t) = W[y_1, \dots, y_5](t)$ Wronskian

Abel-Liouville: $\dot{w} = w$, $w = w(0)e^t$

Thus $W(0) \neq 0 \Rightarrow |w| \to \infty$ $t \to \infty$

$$w(t) = \det \begin{pmatrix} y_1 & \dots & y_5 \\ \dot{y}_1 & \ddots & \dot{y}_5 \\ \vdots & & \vdots \\ y_1^{(4)} & \dots & y_5^{(4)} \end{pmatrix}$$

At least one of $y_1, \dots, y_5$ is unbounded (with derivatives)

$\Downarrow$

For example choose $\left.\begin{pmatrix} y_1 & & y_5 \\ \vdots & \ddots & \\ y_1^{(4)} & & y_5^{(4)} \end{pmatrix}\right|_{t=0} = \begin{pmatrix} \delta & & 0 \\ & \ddots & \\ 0 & & \delta \end{pmatrix}$, $w_0 = \delta^5 \neq 0$ $\Rightarrow$ There is no stability

No solution of this equation is stable.