Lecture 7 21/11-2021 9/

Comparison Theorem by Sturm

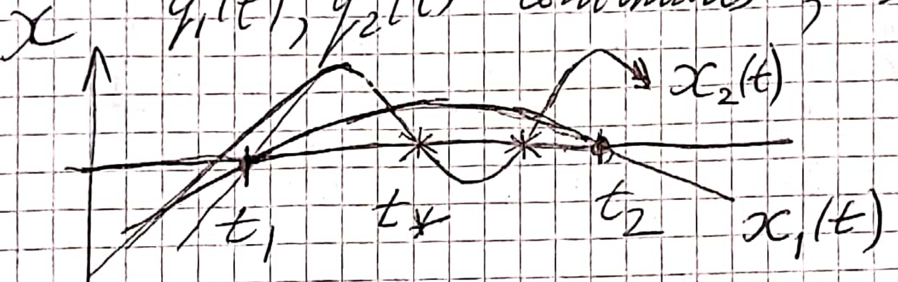
simpler version

~~$\ddot{y}_1 + q_1(t) y_1 = 0$~~ $\ddot{y}_2 + q_2(t) y_2 = 0$

change notation: x instead of y in order to keep the notation from previous years (exercises, tests)

$$\ddot{x}_1 + q_1(t) x_1 = 0 \qquad \ddot{x}_2 + q_2(t) x_2 = 0$$

$q_1(t), q_2(t)$ - continuous, $x_1, x_2 \in \mathbb{R}$



$x_1(t_1) = x_1(t_2) = 0$
$t_1 < t < t_2 \Rightarrow x_1(t) \neq 0$

t_1, t_2 are consecutive zeros of $x_1(t)$

$$q_1(t) \leq q_2(t) \text{ for any } t \in [t_1, t_2]$$

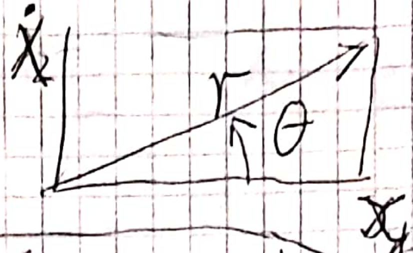
Theorem There are only 2 options:

1) There exists $t_* \in (t_1, t_2)$: $x_2(t_*) = 0$
(in particular $x_2(t) \equiv 0$ is possible)

2) $\exists \lambda \neq 0, \lambda \in \mathbb{R}$: $x_2(t) = \lambda x_1(t)$ over $[t_1, t_2]$
and $q_1(t) \equiv q_2(t)$ for any $t \in [t_1, t_2]$

"solutions of faster equations have more zeros"

Proof



$x_1 = r\cos\theta$

$x_2 = r\sin\theta$

$\ddot{x} + q(t)x = 0 \Rightarrow \boxed{\dot\theta = -(\sin^2\theta + q(t)\cos^2\theta)}$

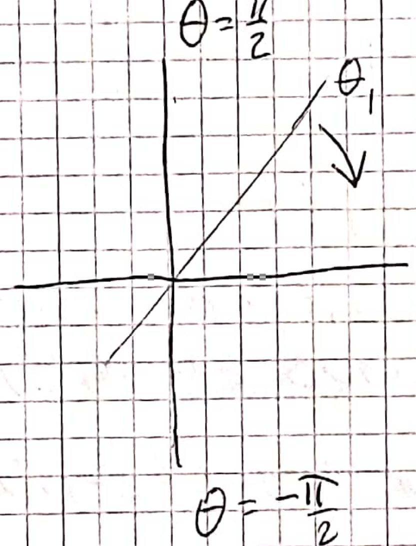
Thus let $\dot\theta_1 = -(\sin^2\theta_1 + q_1(t)\cos^2\theta_1)$

If: $x_2(t) \equiv 0$ $\quad \dot\theta_2 = -(\sin^2\theta_2 + q_2(t)\cos^2\theta_2)$

$x_1(t_1) = 0,\ \dot x_1(t_1) \neq 0$ since $x_1(t) \not\equiv 0$

By multiplication $\pm 1 \Rightarrow \theta_1(t_1) > 0,\ \theta_1(t_1) = \frac{\pi}{2}$

If $x_2(t) \not\equiv 0$ then $\Rightarrow \dot x_2(t_1) \geq 0,$ ~~$\theta_2(t_1) \leq \theta_1 \frac{\pi}{2}$~~



$\theta_2(t_1) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\boxed{\theta_2(t_1) \leq \theta_1(t_1) = \frac{\pi}{2}}$

$q_2(t) \geq q_1(t)$

$\boxed{\theta_1(t_2) = -\frac{\pi}{2}}$

$$\dot\theta_2(t) = -(\sin^2\theta_2(t) + q_2(t)\cos^2\theta_2(t))$$

$$\dot\theta_2(t) \leq -(\sin^2\theta_2(t) + q_1(t)\cos^2\theta_2(t)) = F(t, \theta_2(t))$$

$$\dot\theta_1(t) = -(\sin^2\theta_1(t) + q_1(t)\cos^2\theta_1(t)) = F(t, \theta_1(t))$$

According to the theorem we proved a week ago

1) $\theta_2(t_1) < \frac{\pi}{2} \Rightarrow \theta_2(t) < \theta_1(t)$ for ~~every~~ $t \in [t_1, t_2]$

$\Rightarrow \theta_2(t_2) < -\frac{\pi}{2} \Rightarrow$ on the way $\theta_2 = -\frac{\pi}{2}$ for some $t_* \in (t_1, t_2)$

$\Rightarrow x_2(t_*) = 0$

further let $\boxed{\theta_2(t_1) = \theta_1(t_1) = \frac{\pi}{2}}$

$\Rightarrow \quad \theta_2(t_2) \le \theta_1(t_2) = -\frac{\pi}{2}$

2) $\theta_2(t_2) < -\frac{\pi}{2} \Rightarrow \exists t_* \in (t_1, t_2): \theta_2 = -\frac{\pi}{2}$

$\Rightarrow x_2(t_*) = 0$

3) $\theta_2(t_2) = \theta_1(t_2) \Rightarrow \theta_1(t) \equiv \theta_2(t)$

$\Rightarrow \exists \lambda: x_2(t) = \lambda x_1(t)$ over $[t_1, t_2]$

$\Rightarrow \dot\theta_1 \equiv \dot\theta_2 \Rightarrow \sin^2\theta_1 + q_1(t)\cos^2\theta_1 = \sin^2\theta_1 + q_2(t)\cos^2\theta_1$

$= \theta_1(t)$ $\Rightarrow q_1 = q_2$ whenever $\cos^2\theta_1 \ne 0$

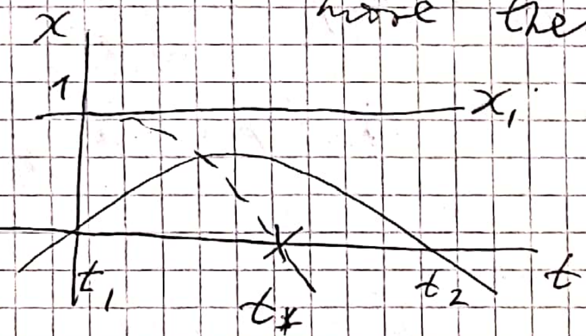
$\cos\theta_1 = 0 \Rightarrow x_1 = 0 \Rightarrow$ separated points in time

$\Rightarrow$ By continuity of $q_1(t), q_2(t)$ $q_1 = q_2$ also there

QED.

<u>Example</u> If $\ddot{x} + q(t)x = 0$, $q(t) \le 0$

$\Rightarrow$ solution $x(t)$ cannot have more then <u>one</u> zero if $x(t) \not\equiv 0$



Comparison: $\ddot{x}_1 = 0$

$q_1 \equiv 0$

solution $x_1(t) \equiv 1$

If $x_1(t_1) = x_1(t_2) = 0$

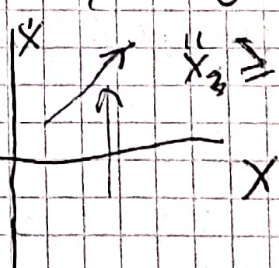
$\Rightarrow \exists t_*: x_2(t_*) = 0$

$1 = 0$

Another way: If $x(t_1) > 0$

$\dot{x}(t_1) > 0$

$\Rightarrow \dot{x}(t) > 0 \ \forall t > t_1$



$\ddot{x}_2 \ge 0$

$x(t_1) < 0$

$\dot{x}(t_1) < 0$

$\Rightarrow \dot{x}(t) < 0 \ \forall t > t_1$

rotation is impossible

"Large" (original) comparison theorem

Sturm (1836)

$$(p(t)\dot{x})^{\cdot} + q(t)x = 0, \quad p(t) > 0$$

$p \in C^1,\ q \in C$

obtained in the Sturm-Liouville theory

$p_1, p_2 > 0$

$$(p_1(t)\dot{x}_1)^{\cdot} + q_1(t)x_1 = 0 \qquad (p_2(t)\dot{x}_2)^{\cdot} + q_2(t)x_2 = 0$$

t_1, t_2 – consecutive zeros of x_1,

$x_1(t_1) = x_1(t_2) = 0$, $t_1 < t_2$, $\forall t \in (t_1, t_2): x_1(t) \neq 0$

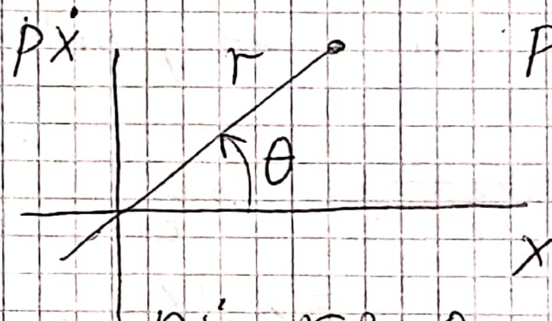
$p_1(t) \geq p_2(t)$, $q_1(t) \leq q_2(t)$ over $[t_1, t_2]$

Theorem: ~~Except of~~ There are two options only

1) $\exists\, t_* \in (t_1, t_2):\ x_2(t_*) = 0$

2) $\exists\, \lambda \in \mathbb{R}:\ x_2(t) \equiv \lambda x_1(t)$ (Picone 1910)
$\lambda \neq 0$

Proof



Prüfer coordinates

$X = r\cos\theta$, $\quad p\dot{X} = r\sin\theta$.

calculation

$$tg\,\theta = \frac{p\dot{X}}{X} \qquad\qquad ctg\,\theta = \frac{X}{p\dot{X}}$$

$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{(p\dot{X})^{\cdot}X - p\dot{X}^2}{X^2} = \qquad\qquad -\frac{\dot{\theta}}{\sin^2\theta} = \frac{\dot{X}^2 p - (p\dot{X})^{\cdot}X}{(p\dot{X})^2}$$

$$= \frac{-q r^2\cos^2\theta - \frac{1}{p}r^2\sin^2\theta}{r^2\cos^2\theta}$$

Will be used at the end of course

In both cases get

$$\boxed{\dot{\theta} = -\frac{1}{p(t)}\sin^2\theta - q(t)\cos^2\theta}$$

Like previously get $\theta_1(t_1) = \frac{\pi}{2}$, $\theta_1(t_2) = \frac{\pi}{2}$

$-\frac{\pi}{2} < \theta_2(t_1) \leq \frac{\pi}{2}$

$\dot{\theta}_2 \leq -\frac{1}{p_1(t)}\sin^2\theta_2 - q_1(t)\cos^2\theta_2 = F(t,\theta_2)$

$\dot{\theta}_1 = F(t,\theta_1)$ continuous, Lipschitzian with respect to θ_1 ~~and~~

$\Rightarrow$ 1) $\theta_2(t_1) < \frac{\pi}{2} \Rightarrow$ $\theta_2(t) < \theta_1(t)$ $\forall t \in [t_1, t_2]$ $\Rightarrow \exists t_*: \theta_2(t_*) = -\frac{\pi}{2}$, $t_* \in (t_1, t_2)$

$\cancel{x_2} = 0$

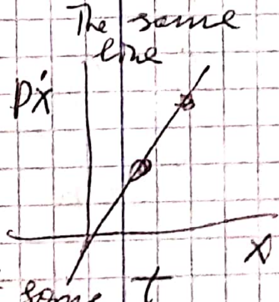
2) $\theta_2(t_1) = \frac{\pi}{2} \Rightarrow$ $\theta_2(t) \leq \theta_1(t)$ $\forall t \in [t_1, t_2]$

2a) $\theta_2(t_2) < -\frac{\pi}{2}$

$\Rightarrow \exists t_*$ $\theta_2(t_*) = -\frac{\pi}{2}$

$x_2(t_*) = 0$

2b) $\underset{\theta_1(t_1)}{\theta_2(t_1)} = \frac{\pi}{2}$, $\underset{\theta_1(t_2)}{\theta_2(t_2)} = -\frac{\pi}{2}$



$\Rightarrow \theta_1(t) \equiv \theta_2(t) \Rightarrow \exists \lambda \neq 0$

$\Rightarrow$ $\boxed{p\dot{x}_1 = \lambda p\dot{x}_2}$, $\boxed{x_1 = \lambda x_2}$ at some t $\Rightarrow$ is kept forever!

QED

Sturm Separation Theorem (1836): $q_1 \equiv q_2$, $p_1 \equiv p_2$

$x_1 \not\equiv x_2$, $x_1, x_2 \not\equiv 0$ $\Rightarrow$ exactly one zero t_* ~~between~~ in (t_1, t_2)

Proof: using twice $\Rightarrow$ not consecutive zeros

Example $\ddot{x} + \dfrac{1}{4(t^2+1)}\, x = 0$

There are not more than one positive zero to $x(t) \not\equiv 0$

Proof! $\dfrac{1}{4(t^2+1)} \le \dfrac{1}{4t^2}$ compare

$$\ddot{X} + \frac{1}{4t^2} X = 0 \qquad t^2\ddot{X} + \tfrac{1}{4}X = 0,\ t>0$$

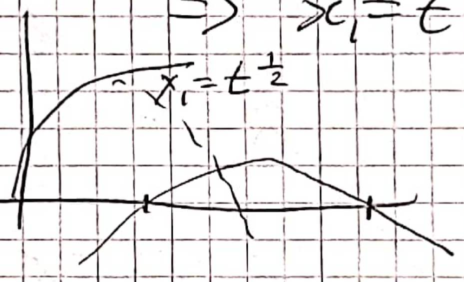
Euler equation

search the solution in the form $X = t^r$

$$t^2 r(r-1)t^{r-2} + \tfrac{1}{4} t^r = 0$$

$$r(r-1) + \tfrac{1}{4} = 0 \qquad r^2 - r + \tfrac{1}{4} = \left(r - \tfrac{1}{2}\right)^2 = 0$$

$$\Rightarrow x_1 = t^{\frac{1}{2}} \text{ is a solution}$$



Theorem $\ddot{x} + q(t)x = 0$

$$0 < m \le q(t) \le M$$

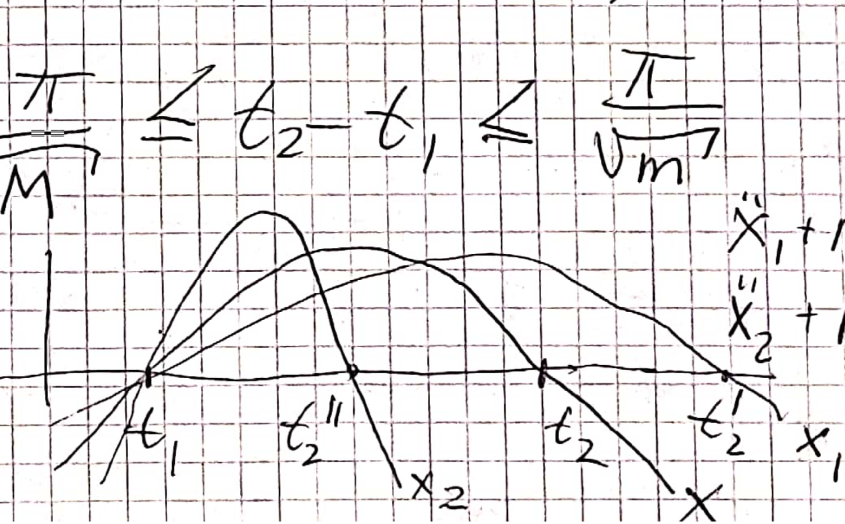
Let $x(t) \not\equiv 0$, t_1, t_2 be consecutive zeros $t_1 < t_2$, $X(t_1) = X(t_2) = 0$, $\forall t \in (t_1, t_2): x(t) \neq 0$

Then $\dfrac{\pi}{\sqrt{M}} \le t_2 - t_1 \le \dfrac{\pi}{\sqrt{m}}$

Proof

$t_2' - t_1 = \dfrac{\pi}{\sqrt{m}}$

$t_2'' - t_1 = \dfrac{\pi}{\sqrt{M}}$



$\ddot{X}_1 + m X_1 = 0$

$\ddot{X}_2 + M X_2 = 0$

Theorem $\ddot{x} + q(t)x = 0$ $\quad 0 < m \le q \le M$, $\quad q \in C$

The number of zeros of any nontrivial (nonzero) solution $x(t)$ in the segment $[a, b]$ satisfies the inequality $\left\lfloor \frac{b-a}{\delta_M} \right\rfloor \le N \le \left\lfloor \frac{b-a}{\delta_m} \right\rfloor + 1$,

$\delta_m = \frac{\pi}{\sqrt{M}}$, $\quad \delta_M = \frac{\pi}{\sqrt{m}}$.

small $\qquad$ larger value

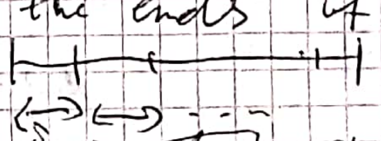
$\lfloor z \rfloor$ is the maximal integer number not exceeding z, $\lfloor 1 \rfloor = 1$, $\lfloor 0.99 \rfloor = 0$,

$\lfloor -0,5 \rfloor = -1$, $\quad \lfloor -2.99 \rfloor = -3$

Proof The distance between two consecutive zeros $\delta_m \le \Delta \le \delta_M$

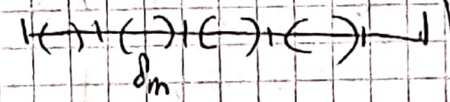
Moreover, there is a zero between any two numbers ξ, $\xi + \delta_M$. zeros can be on the ends if $q \equiv m$

$\Rightarrow$



$\left\lfloor \frac{b-a}{\delta_M} \right\rfloor$ segments

δ_M $\boxed{\text{ruler}}$ shifting the ruler a bit



obtain that none zero is on the markers $N \ge \left\lfloor \frac{b-a}{\delta_M} \right\rfloor$

On the other hand between each two consecutive zeros one can insert a segment of the length δ_m.

The number of such segments is $\left\lfloor \frac{b-a}{\delta_m} \right\rfloor$

The number of their ends $\left\lfloor \frac{b-a}{\delta_M} \right\rfloor + 1 \ge N$ $\quad$ QED

Example How many zeros have ~~a~~ nontrivial solutions of

$$\ddot{x} + 2\dot{x} + \frac{4t+3}{t+2}x = 0$$

on the segment $[100, 105]$?

Solution Transform the equation into the form $\ddot{z} + q(t)z = 0$

$\mu(t)$ – the integration factor

$z = \mu x, \quad \mu > 0, \quad \mu \in C^1$

Search for μ

$$\ddot{z} = \mu\ddot{x} + 2\dot{\mu}\dot{x} + \ddot{\mu}x$$

should be $\overset{?}{=}$

$$\underbrace{\mu\ddot{x} + 2\mu\dot{x} + \ddot{\mu}x}_{\ddot{z}} \underbrace{- \ddot{\mu}x + \frac{4t+3}{t+2}x}_{\left(\frac{4t+3}{t+2} - \frac{\ddot{\mu}}{\mu}\right)z} = 0$$

$\dot{\mu}\dot{x} = \mu\dot{x} \Rightarrow \dot{\mu} = \mu$

$\mu = e^t$ can be taken, $\ddot{\mu} = \dot{\mu} = \mu$

$$\ddot{z} + \left(\frac{4t+3}{t+2} - 1\right)z = 0$$

2.95 actually

$$\ddot{z} + \frac{3t+2}{t+2}z = 0, \quad 2 \le \frac{3t+2}{t+2} = q \le \cancel{4}\, 3$$

$t \in [100, 105]$

Remark $\ddot{x} + 2\dot{x} + \frac{4t+3\varphi(t)}{t+2}x = 0$

$\varphi \in [0, 1]$

$$\Rightarrow \ddot{z} + \underbrace{\left(\frac{4t+3\varphi(t)}{t+2} - 1\right)}_{\tilde{q}} z = 0$$

$2 \le \tilde{q} \le 3$ still

The distance between the consecutive zeros satisfies

$$\sqrt{3} = 1.7\ldots \quad \boxed{\frac{\pi}{\sqrt{3}} = \delta_m \le \Delta \le \delta_M = \frac{\pi}{\sqrt{2}}}, \quad \sqrt{2} = 1.41\ldots$$

$\pi = 3.14\ldots$

$$b - a = 105 - 100 = 5$$

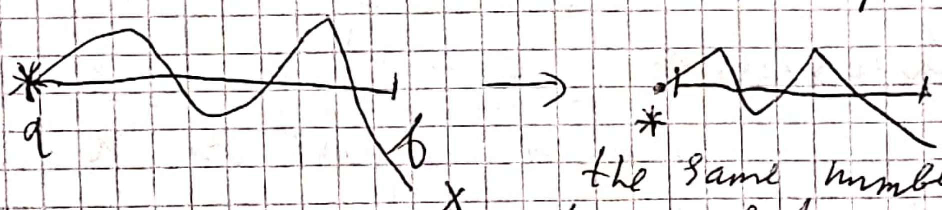
$$\left\lfloor \frac{b-a}{\delta_M} \right\rfloor \le N \le \left\lfloor \frac{b-a}{\delta_m} \right\rfloor + 1 = \left\lfloor \frac{\sqrt{3}\cdot 5}{\pi} \right\rfloor + 1$$

$$= \left\lfloor \frac{8.5}{3.1} \right\rfloor + 1 = 3$$

$$\left\lfloor \frac{b-a}{\delta_M} \right\rfloor = \left\lfloor \frac{5\cdot\sqrt{2}}{3.14} \right\rfloor = \left\lfloor \frac{7}{3.14} \right\rfloor = 2$$

strict!

$2 \le N \le 3$ The difference cannot be less!



the same number of zeros, but one is out

Question: For (*)

$$\ddot{x} + 2\dot{x} + \frac{4t + 3\varphi(t)}{t+2}x = 0, \quad \varphi(t) \in [0,1]$$

the inequality will be the same $2 \le N \le 3$, $\tilde{q} \in [2,3]$

Question

$$\ddot{x} + 2\dot{x} + \frac{4t + 3\cos^2(x^2 + 1000t)}{t+2}x = 0$$

Answer: consider any solution $x_*(t)$. Then ~~it~~ satisfies ~~the~~ equation (*) for $\varphi(t) = \cos^2(x_*(t)^2 + 1000t)$

$$\Rightarrow 2 \le N \le 3$$

Critical points of linear (autonomous) ODEs in the plane $\mathbb{R}^2$

Critical points, singular points, equilibrium (equilibria (many))

מצב מתמיד, נק' איזון

$$\dot{x} = Ax + b \quad , \quad x \in \mathbb{R}^2$$

Let $Ax_0 + b = 0 \Rightarrow x_0$ is a critical point

Then $y = x - x_0 \Rightarrow$ ~~$\dot{y} = Ax - Ax_0 + b =$~~ ~~$\dot{x} = Ax$~~ $x = y + x_0$

$$\dot{y} = \dot{x} = A(y + x_0) + b = Ay$$

$$\boxed{\dot{y} = Ay}$$

$\Rightarrow$ Consider $\boxed{\dot{x} = Ax \quad , \quad x \in \mathbb{R}^2}$

The critical point is $x = 0$

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 + \alpha_1 \lambda + \alpha_2$$

characteristic polynomial

λ_1, λ_2 - roots

1. $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$

$Ae_i = \lambda_i e_i$, $i = 1,2$, e_1, e_2 are eigenvectors

Coordinate transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x = [e_1\ e_2] \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Ez \quad , \quad E = [e_1\ e_2]$$

columns

$$[e_1\ e_2]\, \dot{z} = A[e_1, e_2]\, z = [\lambda_1 e_1, \lambda_2 e_2]\, z$$

$$E\dot{z} = E\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} z$$

$$\dot{z} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} z \qquad \begin{cases}\dot{z}_1 = \lambda_1 z_1\\ \dot{z}_2 = \lambda_2 z_2\end{cases}$$

$$z_1 = e^{\lambda_1 t} z_1(0), \quad z_2 = e^{\lambda_2 t} z_2(0)$$

$$z = z_1(0)\begin{pmatrix}1\\0\end{pmatrix} e^{\lambda_1 t} + z_2(0)\begin{pmatrix}0\\1\end{pmatrix} e^{\lambda_2 t}$$

$$x = [e_1\ e_2]\, z = z_1(0) e^{\lambda_1 t} e_1 + z_2(0) e^{\lambda_2 t} e_2$$

a. $\lambda_1 > \lambda_2 > 0$

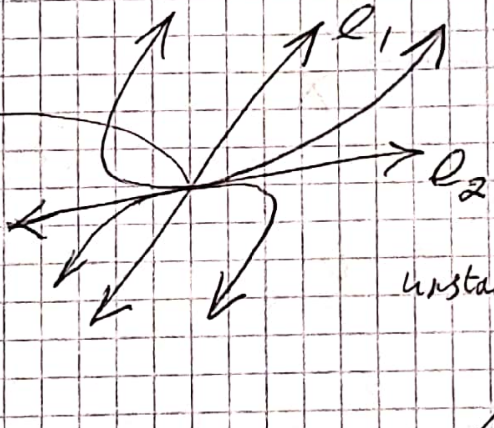
$t \to \infty \qquad \dfrac{z_2}{z_1} = \dfrac{z_2(0)}{z_1(0)} e^{(\lambda_2 - \lambda_1)t} \to 0$

$$\frac{\dot{z}_2}{\dot{z}_1} \to 0, \quad \frac{z_1}{z_2} \to \pm\infty$$

$t \to -\infty \qquad \dfrac{z_2}{z_1} \to \pm\infty, \quad \dfrac{z_1}{z_2} \to 0 \quad \dfrac{\dot{z}_1}{\dot{z}_2} \to 0$

tangent point

נק' השקה



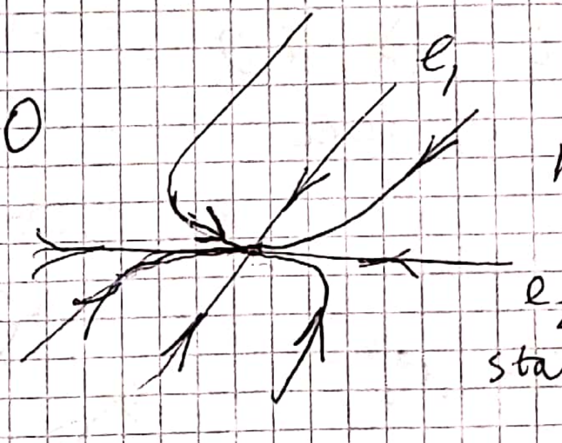
unstable knote

קשר לא יציב

unstable nodal point

nodal source

מקור,



b. $\lambda_1 < \lambda_2 < 0$

reversed time

stable knote point

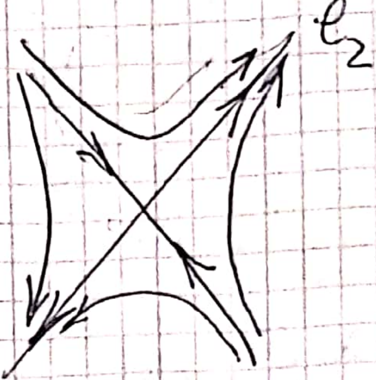
קשר יציב

stable nodal point

C_1 $\lambda_1 < 0 < \lambda_2$

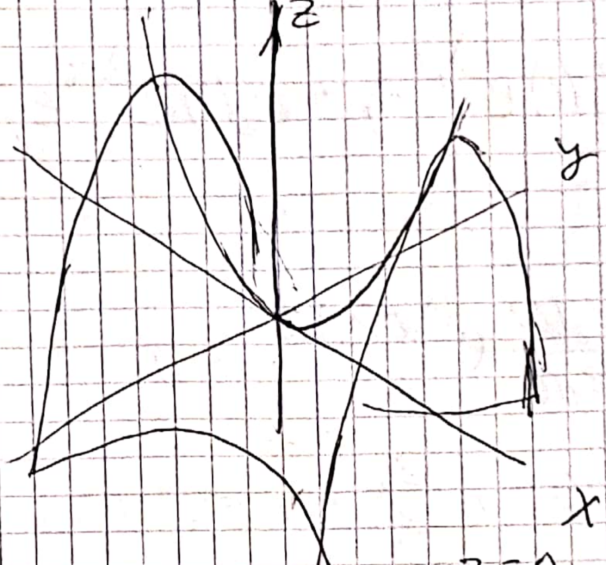
saddle point

f >1k /p)

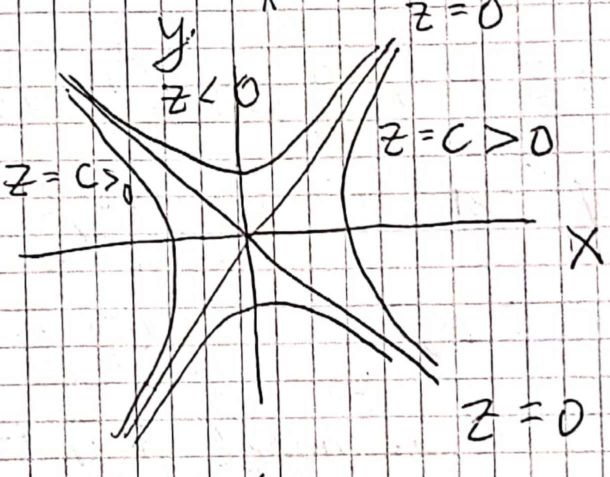


asymptotes

Mathematical saddle $z = x^2 - y^2$



parabole zy slides along the parabole zx

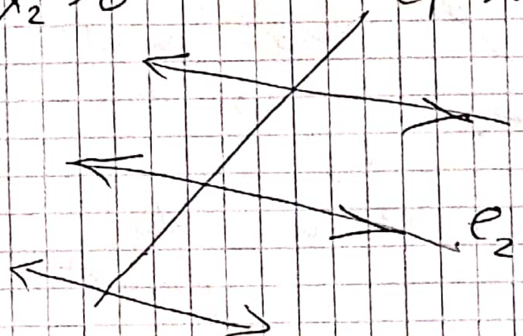


just similarity

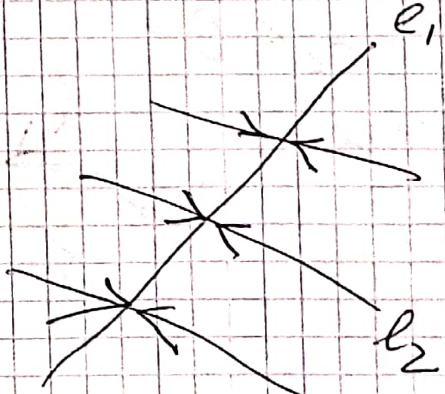
d_1 $\lambda_1 = 0$, $\lambda_2 \neq 0$

$\lambda_2 > 0$

e_1 static



$\lambda_2 < 0$

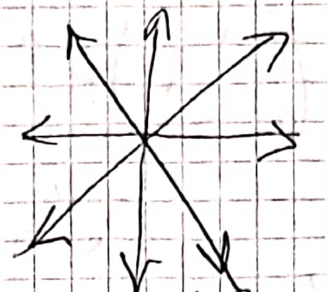


e_1 $\lambda_1 = \lambda_2 = \lambda$ $\quad x = z_1 e_1 + z_2 e_2$

$$Ax = z_1 \lambda e_1 + z_2 \lambda e_2 = \lambda x$$

Each vector $\neq 0$ is an eigenvector

$\lambda > 0$



unstable

star point

[illegible]

stable

2. $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, there is a generalized eigenvector

$Ae_1 = \lambda e_1$, $(A - \lambda I)e_1 = 0$ eigenvector

$Ae_2 = \lambda e_2 + e_1$, $(A - \lambda I)e_2 = e_1$ generalized e.v. corresponding to e_1

e_2 is not unique $\tilde{e}_2 = e_2 + \mu e_1$ also is a generalized e.v. for $\forall \mu \in \mathbb{R}$

$x = [e_1\ e_2]\, z = z_1 e_1 + z_2 e_2$ coordinate transformation

$$[e_1\ e_2]\,\dot{z} = Ax = [\lambda e_1,\ \lambda e_2 + e_1]\, z$$

$$[e_1\ e_2]^{-1}:\ e_1 \mapsto \begin{pmatrix}1\\0\end{pmatrix},\quad e_2 \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$\dot{z} = [e_1\ e_2]^{-1}[\lambda e_1,\ \lambda e_2 + e_1]\, z = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} z$$

Jordan block

$$\Lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda I + \Delta, \quad \Delta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$z(t) = e^{\Lambda t} z(0), \quad e^{\Lambda t} = e^{\lambda I t} \cdot e^{\Delta t}$$

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} e^{\Delta t} = e^{\lambda t} e^{\Delta t}$$

$$e^{\Delta t} = I + \frac{\Delta}{1!} t + \underbrace{\frac{\Delta^2}{2!} t^2 + \dots}_{0} \qquad \Delta^2 = 0$$

$$z(t) = e^{\lambda t}(I + \Delta t)\cancel{z(t)} = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} z(0)$$

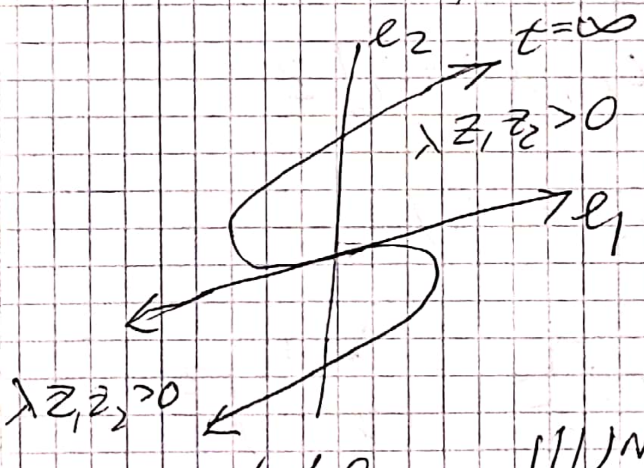
$$z_1(t) = z_1(0) e^{\lambda t} + z_2(0)\,t\,e^{\lambda t}$$

$$z_2(t) = z_2(0) e^{\lambda t}$$

$$x(t) = [e_1\ e_2] z = z_1(t) e_1 + z_2(t) e_2$$

$$x(t) = e^{\lambda t}\left[z_1(0) e_1 + z_2(0)(t e_1 + e_2)\right]$$

$\lambda > 0$, $z_2(0) > 0$ Improper node degenerate node $\lambda < 0$



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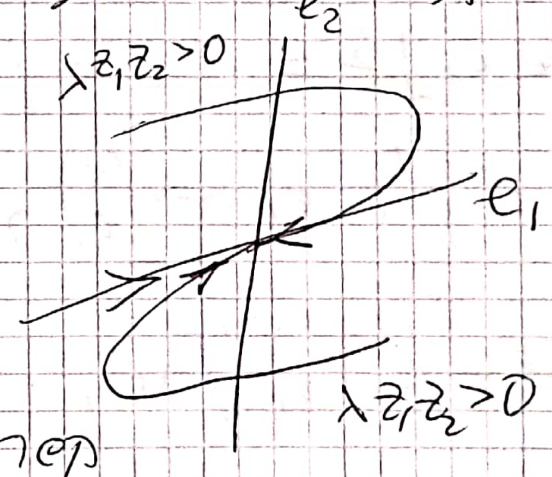

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