Quasi Polynomials: complex numbers.

All the results are valid over $\mathbb{C}$:

$$y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=0,\quad a_1,a_2,\dots,a_n\in\mathbb{C}$$

$\lambda\in\mathbb{C}$ $\qquad y:\mathbb{R}\to\mathbb{C}$

$$L^{\mathbb{C}}_{\lambda,k}=\left\{e^{\lambda t}(d_0+d_1t+\dots+d_kt^k)\,\middle|\,d_0,\dots,d_k\in\mathbb{C}\right\}$$

$$L^{\mathbb{C}}_{\lambda,k}=\text{solutions of }\left(\frac{d}{dt}-\lambda I_G\right)^{k+1}y=0$$

$$\dim L^{\mathbb{C}}_{\lambda,k}=k+1$$

* Real numbers: complex eigenvalues (roots)

$$y^{(n)}+a_1y^{(n-1)}+\dots+a_ny=0,\quad a_1,\dots,a_n\in\mathbb{R}$$

$y:\mathbb{R}\to\mathbb{R}$

complex roots come in conjugate pairs

$\alpha\pm i\beta,\quad \alpha,\beta\in\mathbb{R},\quad \beta\neq 0$

The minimal polynomial with the roots $\alpha\pm i\beta$ is $s^2-2\alpha s+\alpha^2+\beta^2$

$$=(s-\alpha)^2-(\beta i)^2$$

The solutions of $\left(\frac{d^2}{dt^2}-2\alpha\frac{d}{dt}+(\alpha^2+\beta^2)I_G\right)^{k+1}y=0$

are $L^{\mathbb{C}}_{\alpha+\beta i,k}\oplus L^{\mathbb{C}}_{\alpha-\beta i,k}$ over the field $\mathbb{C}$

Its basis consists of

$$e^{(\alpha+\beta i)t}\cdot\left\{1,\frac{t}{1!},\dots,\frac{t^k}{k!}\right\}\cup e^{(\alpha-\beta i)t}\left\{1,\frac{t}{1!},\dots,\frac{t^k}{k!}\right\}$$

complex functions

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$$

$$e^{(\alpha-\beta i)t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$$

are linear combinations of

$$e^{\alpha t}\cos(\beta t) = \frac{1}{2}\left(e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}\right)$$

$$e^{\alpha t}\sin(\beta t) = \frac{1}{2i}\left(e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}\right)$$

Thus $L^{\mathbb{C}}_{\alpha+\beta i, k} + L^{\mathbb{C}}_{\alpha-\beta i, k} =$

$$\left\{ e^{\alpha t}\left[\cos(\beta t)(d_0 + d_1 t + \dots + d_k t^k) + \sin(\beta t)(f_0 + f_1 t + \dots + f_k t^k)\right] \,\middle|\, \begin{array}{l} d_0,\dots,d_k \in \mathbb{C} \\ f_0,\dots,f_k \in \mathbb{C} \end{array} \right\}$$

<u>Definition</u> Generalized (real) quasi-polynomial: $\alpha, \beta \in \mathbb{R}$

$$L_{\alpha\pm\beta i, k} = \left\{ e^{\alpha t}\left[\cos(\beta t)(d_0 + \dots + d_k t^k) + \sin(\beta t)[f_0 + \dots + f_k t^k]\right], \; d_0, d_1, \dots, d_k, f_0, f_1, \dots, f_k \in \mathbb{R} \right\}$$

Obviously they constitute the general solution of the equation

$$\left(\frac{d^2}{dt^2} - 2\alpha\frac{d}{dt} + (\alpha^2+\beta^2) I_G\right)^{k+1} y = 0, \quad \beta \neq 0$$

or

$$\left(\frac{d}{dt} - \alpha I_G\right)^{k+1} y = 0 \quad \text{for } \beta = 0$$

since $\sin(\beta t) \equiv 0$

over $\mathbb{R}$

Example! $\lambda = \pm 1, \; -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$(\lambda^2-1)(\lambda^2+\lambda+1)^2 = \lambda^6+2\lambda^5+2\lambda^4-2\lambda^2-1$$

$$y^{(6)}+2y^{(5)}+2y^{(4)}-2\ddot{y}-y=0$$

Solution:

$$y = c_1 e^{t} + c_2 e^{-t} + e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)(c_3+c_4 t)$$
$$+ e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)(c_5+c_6 t)$$

$c_1, c_2, \dots, c_6 \in \mathbb{R}$ over $\mathbb{R}$

$c_1, c_2, \dots, c_6 \in \mathbb{C}$ - solutions over $\mathbb{C}$

Resonance

Nonhomogeneous ODE

$$y^{(n)}+a_1 y^{(n-1)}+\dots+a_n y = b(t) = e^{\alpha t}\left[\cos(\beta t)(\hat{d}_0+\dots+\hat{d}_k t^k) + \sin(\beta t)(\hat{f}_0+\dots+\hat{f}_k t^k)\right]$$

$$p\left(\frac{d}{dt}\right)y = b(t) \in L_{\mu,k}, \quad \mu = \alpha \pm \beta i$$

$\hat{d}_0, \dots, \hat{d}_k, \hat{f}_0 + \dots + \hat{f}_k \in \mathbb{R}$ - concrete numbers

We know the general solution of $p\left(\frac{d}{dt}\right)y = 0$ ~~if~~ (provided the roots of $p(\lambda)$ are known)

Only one particular solution is needed.

Let the multiplicity of the root μ be m

$m=0 \Leftrightarrow p(\mu) \neq 0$, $m>0 \Rightarrow p(s) = (s-\mu)^m q(s)$
$q(\mu) \neq 0, \beta = 0$

or $p(s) = (s^2 - 2\alpha s + \alpha^2 + \beta^2)^m q(s)$, $\beta \neq 0$

Definition

$$\boxed{\begin{array}{l} m>0 \\ \Rightarrow \text{resonance} \end{array}}$$

Theorem There exists a particular solution of the form $y_p \in t^m L_{\mu,k}$, i.e.

$$y_p = t^m e^{\alpha t}\left[\cos(\beta t)(d_0+d_1 t+\dots+d_k t^k) + \sin(\beta t)(f_0+f_1 t+\dots+f_k t^k)\right]$$

$d_0, \dots, d_k, f_0, \dots, f_k \in \mathbb{R}$

Proof: $p\left(\frac{d}{dt}\right) y = b(t)$ $\boxed{\text{Let } \mu = \alpha, \ \beta = 0}$

$$q\left(\frac{d}{dt}\right) \left(\frac{d}{dt} - \mu I_G\right)^m y = b(t) \in L_{\mu,k} \quad (79)$$

$$q\left(\frac{d}{dt}\right) : L_{\mu,k} \to L_{\mu,k}$$ bijection one-to-one mapping

$$\Rightarrow \exists! \hat{b} \in L_{\mu,k} : q\left(\frac{d}{dt}\right) \hat{b} = b$$

$$\Rightarrow \left(\frac{d}{dt} - \mu I_G\right)^m y = \hat{b}$$

$$\frac{d}{dt} - \mu I_G : \frac{t^{m+k}}{(m+k)!} e^{\mu t} \mapsto \frac{t^{m+k-1}}{(m+k-1)!} e^{\mu t} \mapsto \ldots \mapsto e^{\mu t} \mapsto 0$$

Each application of $\frac{d}{dt} - \mu I_G$ lowers the polynomial degree, $\boxed{\text{surjection "onto"}}$ exactly by 1. Therefore there exists a non unique polynomical $y_p \in L_{\mu, k+m}$

$$\left(\frac{d}{dt} - \mu I_G\right) y_p = b$$

One can remove all terms from y_p of the degree less than m, since

$$\left(\frac{d}{dt} - \mu I_G\right)^m L_{\mu, m-1} = \{0\}$$

Q.E.D.

Example $\ddot{y} + 3\dot{y} + 2y = e^{-t}$ $\quad \mu = -1, \ m = 1$

$\lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$

search for $y_p = f t e^{-t}, \ f = ?$

$\dot{y}_p = f(e^{-t} - t e^{-t}), \ \ddot{y}_p = -2f e^{-t} + f t e^{-t}$

$\dot{y}_p = f e^{-t}(1-t), \ \ddot{y}_p = f e^{-t}(t-2)$

Substitute $y_p, \dot{y}_p, \ddot{y}_p$: 1

$$[f(t-2)+3f(1-t)+2ft]\cancel{e^{-t}} = \cancel{e^{-t}}$$

$$0 \quad (\cancel{f-3f+2f})t + f(-2+3) = 1$$

$t \;|\; 0=0$ according to the theory!

$1 \;|\; f=1 \qquad y_p = te^{-t}$

$$y = \underbrace{c_1 e^{-2t} + c_2 e^{-t}}_{y_h} + \underbrace{te^{-t}}_{y_p} \qquad c_1, c_2 \in \mathbb{R} \quad \left(\begin{matrix} c_1, c_2 \in \mathbb{C} \\ \text{over } \mathbb{C} \end{matrix}\right)$$

general solution of the homogeneous equation

Over $\mathbb{C}$ everything is simple:

$$p\left(\frac{d}{dt}\right) y = b, \quad b \in L^{\mathbb{C}}_{\mu,\kappa}, \qquad m\text{-multiplicity} \quad p(\lambda) = (\lambda-\mu)^m q(\lambda), \quad q(\mu) \neq 0$$

$$\Rightarrow y_p \in t^m L^{\mathbb{C}}_{\mu,\kappa}$$

That's the end of ODE-1

Comparison Theorems (Birkhoff, Rota)

Theorem 1 $y_1(t_0) = y_2(t_0)$, $\dot{y}_1 \le \dot{y}_2$

$\Rightarrow \forall t > t_0 \quad y_1 \le y_2$

$\forall t < t_0 \quad y_1 \ge y_2$

Integration:

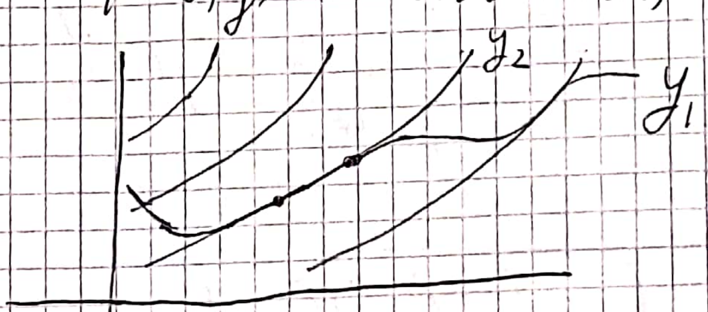
$t > t_0$ $\quad y_1(t) = y_1(t_0) + \int_{t_0}^{t} \dot{y}_1(s)\,ds \le y_2(t_0) + \int_{t_0}^{t} \dot{y}_2(s)\,ds$, $\quad y_2(t_0) = y_1(t_0)$

$t < t_0$ $\quad$ — " — $\; - \int_{t}^{t_0} \dot{y}_1(s)\,ds \ge y_2(t_0) - \int_{t}^{t_0} \dot{y}_2(s)\,ds$

Q.E.D. Trivial

$$\dot{y}_1 \le F(t, y_1), \quad \dot{y}_2 = F(t, y_2)$$

$F(t,y)$ is continuous, Lipschitzian with respect to y



$$\boxed{|F(t,y_1) - F(t,y_2)| \le L\,|y_1 - y_2|, \quad L > 0 \quad \forall y_1, y_2, t}$$

The claim: y_1 cannot cross the graphs of solutions for $\dot{y} = F(t,y)$ up from below, ~~for~~ but can glue to a solution of $\dot{y} = F(t,y)$ and then leave it down

Theorem 2. 1. $y_1(t_0) = y_2(t_0) \Rightarrow \forall t \ge t_0 \;\; y_1 \le y_2$

$\forall t < t_0 \;\; y_1 \ge y_2$

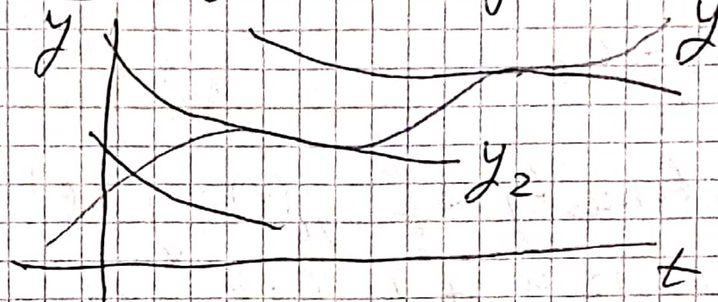
2, $y_1(t_0) < y_2(t_0) \Rightarrow \forall t > t_0 \;\; y_1(t) < y_2(t)$

$y_1(t_0) > y_2(t_0) \Rightarrow \forall t < t_0 \;\; y_1(t) > y_2(t)$

3, $y_1(t_0) = y_2(t_0)$, $y_1(t_1) = y_2(t_1)$, $t_1 > t_0$

$\Rightarrow \forall t \in [t_0, t_1] : \; y_1(t) = y_2(t)$

Theorem 2' $\quad \dot{y}_1 \geq F(t, y_1)$, $\dot{y}_2 = F(t, y_2)$



1, $y_1(t_0) = y_2(t_0) \Rightarrow$ $\quad t > t_0 \;\; y_1(t) \geq y_2(t)$

$\quad t < t_0 \;\; y_1(t) \leq y_2(t)$

2, $y_1(t_0) > y_2(t_0) \Rightarrow \forall t > t_0 \;\; y_1(t) > y_2(t)$

$y_1(t_0) < y_2(t_0) \Rightarrow \forall t < t_0 \;\; y_1(t) < y_2(t)$

3. $y_1(t_0) = y_2(t_0)$, $y_1(t_1) = y_2(t_1)$, $t_1 > t_2$

$\Rightarrow \forall t \in [t_0, t_1] : \; y_1(t) = y_2(t)$

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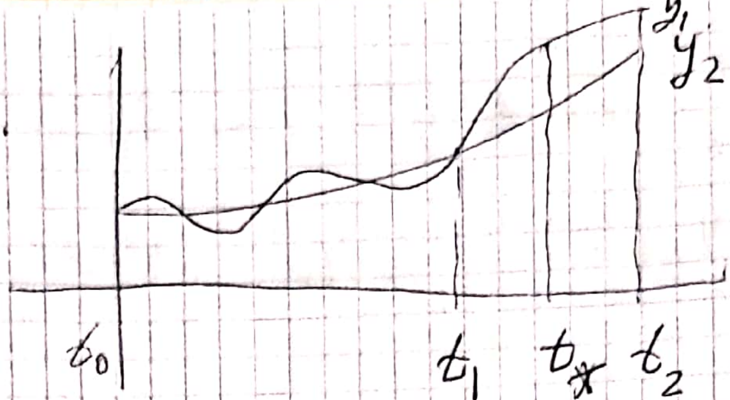
Proof 1.

Let $t > t_0$



The proof is similar to the proof of uniqueness. The planarity is used.

Assume by contradiction that

$\exists t_2 > t_0 : y_1(t_2) > y_2(t_2)$, $t_1 = \sup\{t \in [t_0, t_2],\ y_1(t) = y_2(t)\}$

$\Rightarrow t_0 \le t_1 < t_2$ $\qquad t_0 \in \Rightarrow$ nonempty

From continuity of $y_1(t) - y_2(t)$ obtain: $y_1(t_1) = y_2(t_1)$ (otherwise the inequality would hold in a vicinity)

$\forall t \in (t_1, t_2] : y_1(t) > y_2(t)$

$\exists t_* \overset{def}{=} \arg\max\limits_{[t_1, t_2]} (y_1(t) - y_2(t))$, $t_* \in (t_1, t_2]$

$M = \max\limits_{t \in [t_1, t_2]} y_1(t) - y_2(t)$, $M = y_1(t_*) - y_2(t_*)$

$$M = y_1(t_*) - y_2(t_*) = \int_{t_1}^{t_*} (\dot{y}_1(s) - \dot{y}_2(s))\, ds$$

$$\le \int_{t_1}^{t_*} [F(s, y_1(s)) - F(s, y_2(s))]\, ds \le \int_{t_1}^{t_*} L(y_1(s) - y_2(s))\, ds$$

$$\le LM(t_* - t_1) \le \boxed{LM(t_2 - t_1)}$$

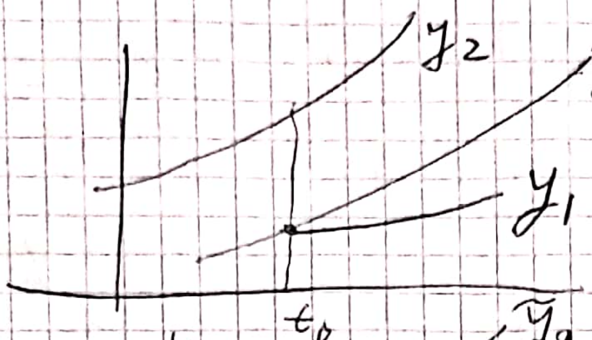
$$\Rightarrow 1 \le L(t_2 - t_1)$$

Now move t_2 closer to t_1

so that $t_2 - t_1 < \frac{1}{2L} \Rightarrow 1 < \frac{1}{2}$

Q.E.D. Contradiction

2.

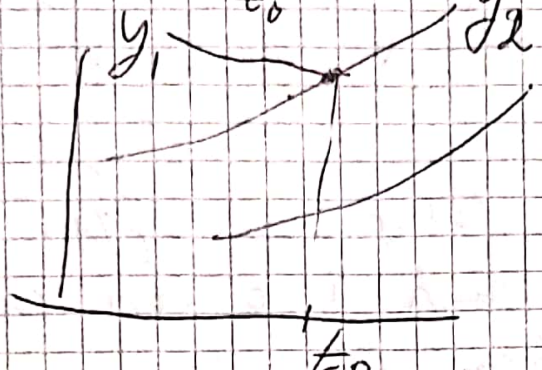


$\tilde{y}_2' = F(t, \tilde{y}_2)$

$\tilde{y}_2(t_0) = y_1(t_0)$

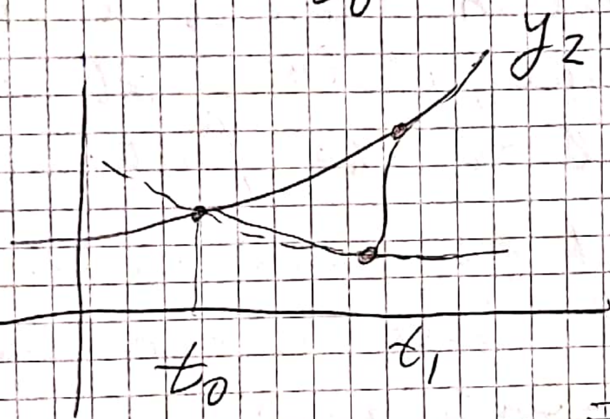
$\boxed{t > t_0}$

or



$\boxed{t < t_0}$

3.



$y_1(t_0) = y_2(t_0)$

$y_1(t_1) = y_2(t_1)$

Suppose $\exists t_* \in (t_0, t_1)$

$y_1(t_*) \neq y_2(t_*)$

$\Rightarrow$ contradiction according to 2.

Q.E.D.

The proof of Theorem 2' is similar or alternatively one can apply the time transformation:

$$\tau = t_0 - t \Rightarrow \tau_0 = 0$$

$$\tilde{y}_1 = y_1(t_0 - \tau), \quad \tilde{y}_2 = y_2(t_0 - \tau)$$

$$\tilde{y}_1' \geq \tilde{F}(\tau, \tilde{y}_1) = -F(t_0 - \tau, \tilde{y}_1), \quad \tilde{y}_2' = \tilde{F}(\tau, \tilde{y}_2)$$

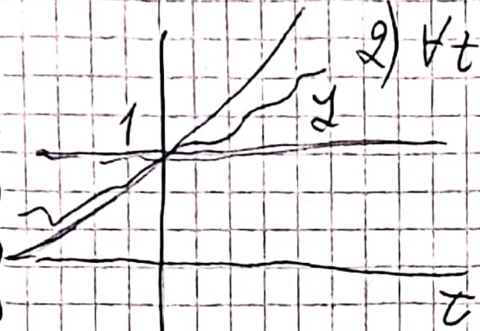
At the same time it proves the cases $t < t_0$

Example $\dot y = (\cos t)^2 y$, $|t| \le 1$

$y(0) = 1$

Prove: 1) $\forall t \in \cancel{[0,1]}$ >0 : $1 < y(t) < e^t$

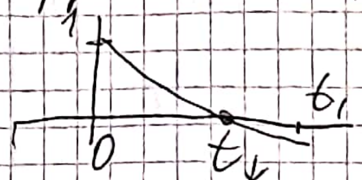
2) $\forall t \in \cancel{[-1,0)}$ <0 : $e^t < y(t) < 1$



Proof: $1 \ge \cos^2 t \ge 0$

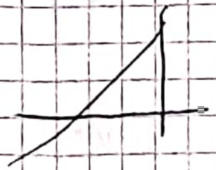
claim: $\forall t \quad y(t) > 0$

Suppose the opposite: $y(t_1) \le 0$, $t_1 > 0$ for example



$t_* = \inf\{t \in [0, t_1] \mid y(t) = 0\}$

not empty due to the continuity of $y(t)$



$\Rightarrow y(t_*) = 0$, $\forall t \in [0, t_*)$: $y(t) > 0$

Thus over $[0, t_*]$ get $\dot y = (\cos t)^2 y \ge 0$

$\left.\begin{array}{l}\Rightarrow \dot y \ge 0 \\ y(0) = 1\end{array}\right\} \Rightarrow y(t) \ge 1 > 0 \quad \forall t \in [0, t_*]$

contradiction

Similarly $t_1 < 0$

$t_* = \sup\{t \in [t_1, 0] \mid y(t) = 0\}$

$\Rightarrow \dot y \ge y$, $y(0) = 1$

$\Rightarrow y(t) \ge e^t$, $t \in [t_*, 0]$

contradiction

Or the simplest way:

$\dot y = (\cos t)^2 y$ satisfies the $\exists!$ Theorem

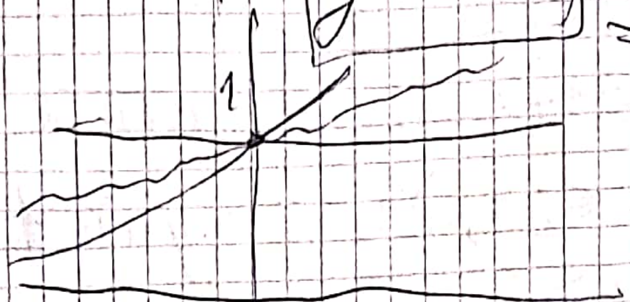
$\hat y(t) = 0$ — solution for $y(0) = 0$

$\Rightarrow y > 0$ holds

Thus $\boxed{\dot y \le y}$ $\forall t$

$\boxed{\dot y \ge 0}$ $\forall t$

$\Rightarrow$



$\Rightarrow$ $\boxed{\begin{array}{l} y \ge 1 \\ y_t \ge 0 \\ y \le 1 \\ y_t \le 0 \end{array}}$

$\Rightarrow$ $y \le e^t$ $t > 0$

$y \ge e^t$ $t < 0$

It still is needed to prove that the inequalities are strict for $t \neq 0$

Suppose for example that

$$y(t_1) = e^{t_1}, \quad t_1 > 0$$

then from $\dot y = (\cos t)^2 y \le y$, $y(0)=1$

obtain $y(t) \equiv e^t$ $t \in [0, t_1]$

$$\Rightarrow \quad e^t = \dot y = (\cos t)^2 y = (\cos t)^2 e^t$$

$$\Rightarrow \quad (\cos t)^2 \equiv 1 \quad \text{— wrong} \quad \left(\cos \tfrac{\pi}{3}\right)^2 = \tfrac{1}{4}$$

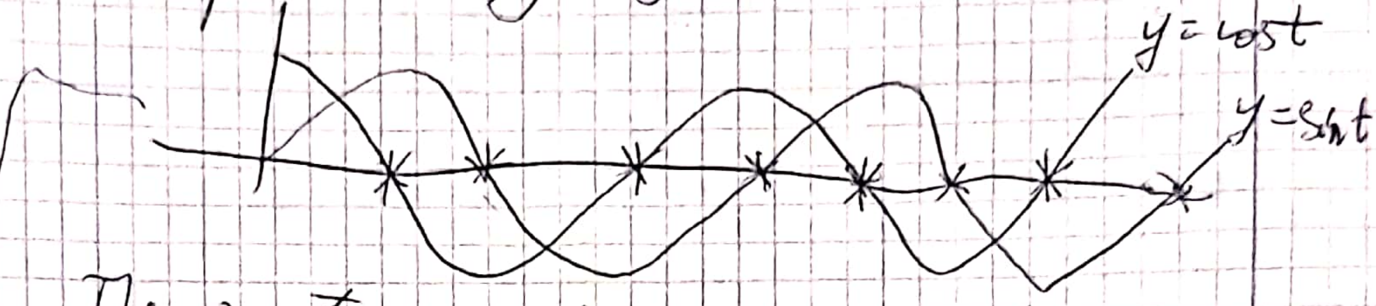
Similarly other cases.

Oscillation theorems by Sturm

We still use the plane properties

$$\ddot{y} + a(t)\dot{y} + b(t)y = 0, \quad a, b \in C$$

Example: $\ddot{y} + y = 0$



The ~~root~~ zeros come one after another. Between each two consecutive zeros of one solution there is a zero of each other one, if it is not zero or proportional to the first one.

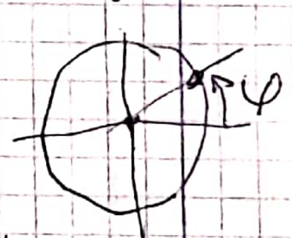
We will see that the same is true in the general case ~~under the additional condition~~ $a \in C^1$

Explanation:

$$y = c_1 \cos t + c_2 \sin t = A\cos(t+\varphi)$$

$$A = \sqrt{c_1^2 + c_2^2}, \quad \cos\varphi = \frac{c_1}{A}, \quad \sin\varphi = -\frac{c_2}{\sqrt{\ }}$$

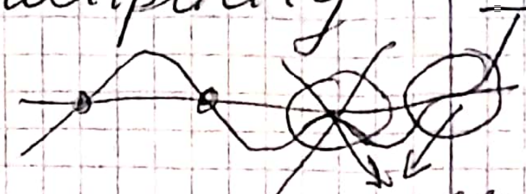
$$y = \sqrt{c_1^2+c_2^2}\left(\frac{c_1}{\sqrt{\ }}\cos t + \frac{c_2}{\sqrt{\ }}\sin t\right)$$



Theorem $a, b \in C$ continuous

1. $y(t) \not\equiv 0$, $y(t_0) = 0 \Rightarrow \dot{y}(t_0) \neq 0$
2. All zeros are isolated if $y(t) \not\equiv 0$

Interpretation: multiplicity of each zero is 1



Proof: 1) If $y(t_0)=\dot y(t_0)=0 \Rightarrow y\equiv 0$ (E! Theorem)

2) Suppose zeros are not isolated $\Rightarrow$ There are infinitely many different points t_i at a closed segment I where $y(t_i)=0$, $y\not\equiv 0$

$\Rightarrow$ Compactness: $\exists\, t_{i_k} \to t_* \in I$

$\Rightarrow y(t_*)=0$ from continuity

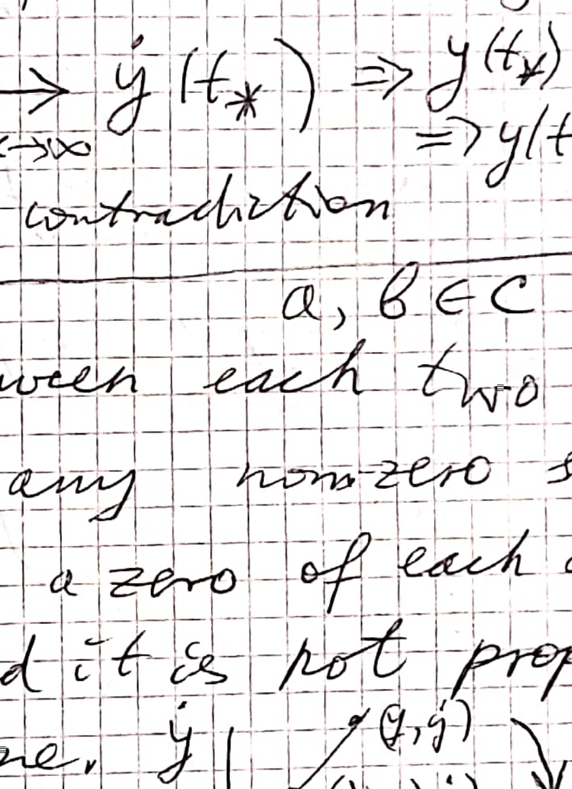
$$0=\frac{y(t_{i_k})-y(t_*)}{t_{i_k}-t_*}\xrightarrow[k\to\infty]{}\dot y(t_*) \Rightarrow y(t_*)=\dot y(t_*)=0$$
$$\Rightarrow y(t)\equiv 0$$

Q.E.D. contradiction

$a, b \in \mathbb{C}$

Theorem Between each two consecutive zeros of ~~each~~ any non-zero solution there is always a zero of each other solution provided it is not proportional to the first one.

Proof:



Each direct line moves as a single object rotating around zero in the clock direction

Two direct lines different at some time remain different for each other $\forall$ time ($\exists!$ Theorem)

Indeed: $\lambda(y_1, \dot{y}_1)\big|_{t=t_0} = (y_2, \dot{y}_2)\big|_{t=t_0} \neq (0,0)$

$\Rightarrow \quad y_2(t) \equiv \lambda y_1(t) \quad \forall t$ QED

That proof can be formalized. (not done)

Add additional condition: $a \in C$ continuously differentiable

Variable transformation:

transfer to $\ddot{y} + q(t)y = 0, \; q \in C$

Method of the integration factor

$\mu\ddot{y} + a\mu\dot{y} + b\mu y = 0, \; z = \mu y$ new variable

$\ddot{z} = (\mu y)'' = \mu\ddot{y} + 2\dot{\mu}\dot{y} + \ddot{\mu}y$

$\underbrace{\mu\ddot{y} + a\mu\dot{y} + \ddot{\mu}y}_{\ddot{z}} \underbrace{- \ddot{\mu}y + b\mu y}_{\left(b - \frac{\ddot{\mu}}{\mu}\right)z} = 0$

$\rightarrow \quad a\mu\dot{y} = 2\dot{\mu}\dot{y} \quad \Rightarrow \quad a\mu = 2\dot{\mu}$

(we need only one solution $\mu(t)$)

$\frac{\dot{\mu}}{\mu} = \frac{a}{2}, \quad$ let $\mu > 0 \quad \Rightarrow \ln\mu = \frac{1}{2}\int_{t_0}^{t} a(s)\,ds + \cancel{C} \nearrow 0$

$\boxed{\mu = e^{\frac{1}{2}\int_{t_0}^{t} a(s)\,ds}}$

$$\mu = \frac{1}{2} a \mu \Rightarrow \ddot{\mu} = \frac{1}{2}\dot{a}\mu + \frac{1}{2} a \dot{\mu}$$

$$\ddot{\mu} = \frac{1}{2}\dot{a}\mu + \frac{1}{4} a^2 \mu$$

$$\ddot{z} + \left(b - \frac{\ddot{\mu}}{\mu}\right) z = 0$$

$$\Rightarrow \quad \ddot{z} + \underbrace{\overbrace{\left(b(t) - \frac{1}{2}\dot{a}(t) - \frac{1}{4} a^2(t)\right)}^{q(t)}}_{\text{continuous}} z = 0$$

Now consider $\ddot{y} + q(t) y = 0$, $q \in C$

it has the same zeros.

<u>The comparison theorem</u>

$\ddot{y}_1 + q_1(t) y_1 = 0$, $\ddot{y}_2 + q_2(t) y_2 = 0$

Let $q_1, q_2 \in C$, $t_1 < t_2$ be consecutive zeros of $y_1(t)$,

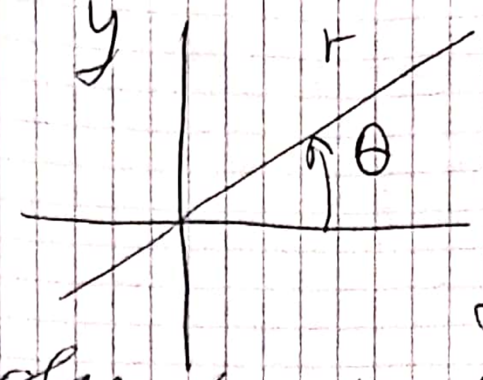
$\forall t \in \underset{\text{closed}}{[t_1, t_2]}$: $q_1(t) \le q_2(t)$, $y_1(t) \not\equiv 0$

<u>Theorem</u> There exists only 2 options:

1) $\exists t_* \in (t_1, t_2)$: $y_2(t_*) = 0$

2) $\exists \lambda \neq 0$: $y_2(t) \equiv \lambda y_1(t)$, $q_1(t) \equiv q_2(t)$ over $[t_1, t_2]$

The previous theorem is the particular case.



polar coordinates

Proof

$\ddot y + q(t)\, y = 0$

$y = r\cos\theta$

$\dot y = r\sin\theta$

$y \neq 0 \quad \tan\theta = \frac{\dot y}{y}$

$$\dot\theta\,\frac{1}{\cos^2\theta} = \frac{\ddot y y - \dot y^2}{y^2} = \frac{-q\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

$\dot y \neq 0 \quad \cot\theta = \frac{y}{\dot y}$

$$-\dot\theta\,\frac{1}{\sin^2\theta} = \frac{\dot y^2 - \ddot y y}{\dot y^2} = \frac{\sin^2\theta + q\cos^2\theta}{\sin^2\theta}$$

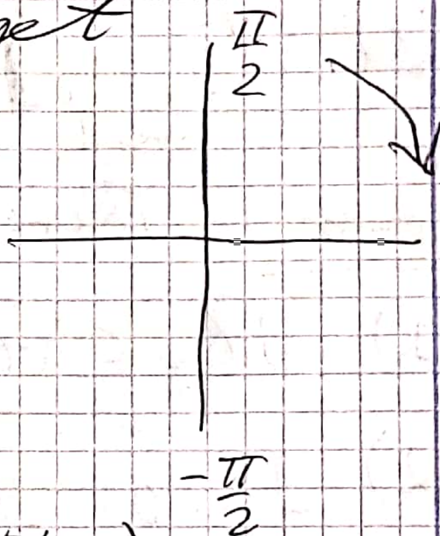
$$\dot\theta = -\left(\sin^2\theta + q(t)\cos^2\theta\right)$$

multiplicating by ± 1 get

$y_1(t_1) = y_1(t_2) = 0$

$\theta_1(t_1) = \frac{\pi}{2} \qquad \theta_1(t_2) = -\frac{\pi}{2}$

$\theta_2(t_1) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$$\dot\theta_2 = -\left(\sin^2\theta_2 + q_2(t)\cos^2\theta_2\right) \le -\left(\sin^2\theta_2 + q_1(t)\cos^2\theta_2\right) = F(t,\theta_2)$$

$$\dot\theta_1 = F(t,\theta_1) = -\left(\sin^2\theta_1 + q_1(t)\cos^2\theta_1\right)$$

$F \in C$, Lipschitzian with respect to θ

Now the theorem follows from the comparison theorems we proved