Lecture 5 07/11-2021

Return to linear systems

$$\dot{x} = A(t)x + B(t), \quad x \in \mathbb{R}^n$$

$$X = X_h + x_p$$

particular case:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$ constant matrix

$$x(t_0) = \xi$$

In the scalar case $x \in \mathbb{R}$, $A = a \in \mathbb{R}$ we would get $x(t) = e^{a(t-t_0)}\xi$

Indeed: $\dot{x} = a e^{a(t-t_0)}\xi = ax$, $x(t_0) = \xi$

The same true in the vector case: $x(t) = e^{A(t-t_0)}\xi$

$$e^A \stackrel{def}{=} I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots$$

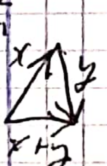
$$e^{At} \stackrel{def}{=} I + \frac{A}{1!}t + \frac{A^2}{2!}t^2 + \dots$$ How to understand this?

Recalling linear algebra:

$\langle x, y \rangle = x^t y = x_1y_1 + \dots + x_ny_n$ scalar (inner) product

Standard Euclidean norm: $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

$\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$, $\|\lambda x\| = |\lambda|\,\|x\|$, $\|x+y\| \leq \|x\| + \|y\|$ triangle inequality



$$\|A\| = \sup \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\| = \sqrt{\lambda_{max}(A^*A)}$$

the max. eigenvalue (nonnegative) A^TA over $\mathbb{R}$

Explanation: $\|Ax\| = (\underbrace{x^TA^TAx}_{})^{\frac{1}{2}}$ → quadratic form

$\|Ax\| = \sqrt{\lambda_1\tilde{x}_1^2 + \lambda_2\tilde{x}_2^2 + \dots + \lambda_n\tilde{x}_n^2}$ coordinate transformation in the orthogonal basis of the eigenvectors for A^TA

Theorem The series $e^{A} = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \ldots$ uniformly converges over any closed ball $\|A\| \le R$ (for any $R \geq 0$)

Cauchy criterion: $\left\| \frac{1}{m!}A^m + \frac{1}{(m+1)!}A^{m+1} + \ldots + \frac{1}{K!}A^K \right\|$

$$\le \frac{1}{m!}R^m + \frac{1}{(m+1)!}R^{m+1} + \ldots + \frac{1}{K!}R^K \xrightarrow[m,K\to\infty]{} 0$$

(scalar case)

Theorem $(e^{At})^{\cdot} = Ae^{At}$

Proof: $Ae^{At} = A(I + \frac{At}{1!} + \ldots)$ uniformly converges $\forall T > 0$ over $t \in [0,T]$

$$I + \int_0^t Ae^{At}dt = \underbrace{I + A(t + \frac{A}{2!}t^2 + \ldots)}_{\text{converges at one point } (t=0)} = e^{At}$$

$\Rightarrow$ converges uniformly

All proofs of the standard calculus results are transfered literally here

~~Ex~~ Corollary $\dot{x} = Ax$, $x \in \mathbb{R}^n$
$x(t_0) = \xi$

$\Rightarrow x(t) = e^{At}\xi$ (for $\dot{x} = Ae^{At}\xi = Ax$)

Example $\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{pmatrix}^n = \begin{pmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n & \\ 0 & & \ddots \ \lambda_n^n \end{pmatrix}$ obviously

$\Rightarrow \exp\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$

$\dot{x} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} x \Rightarrow x(t) = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} x(0)$

Example $\exp\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = ?$ $\quad \alpha, \beta \in \mathbb{R}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\exp(\alpha I + \beta J) = ?$

We need the theorem:

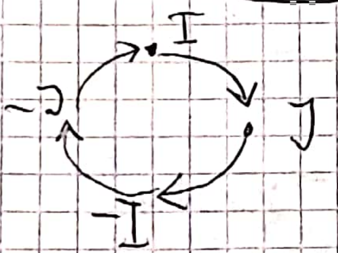
$$\boxed{AB = BA \Rightarrow e^{A+B} = e^A e^B}$$

Proof: Exactly as in the scalar case $e^{\alpha+\beta} = e^\alpha e^\beta$ by the multiplication of series

$$e^{\alpha I + \beta J} \underset{IJ = JI = J}{=} e^{\alpha I} \cdot e^{\beta J} = \begin{pmatrix} e^\alpha & 0 \\ 0 & e^\alpha \end{pmatrix} e^{\beta J}$$

$$e^{\alpha I + \beta J} = e^\alpha I e^{\beta J} = e^\alpha e^{\beta J}$$

Proposition: $J^0 = I$, $J^1 = J$, $J^2 = -I$, $J^3 = -J$, ...



multiplication diagram

$$J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$\Rightarrow e^{\beta J} = I + \frac{\beta}{1!} J + \frac{\beta^2}{2!} J^2 + \frac{\beta^3}{3!} J^3 + \frac{\beta^4}{4!} J^4 + \dots =$$

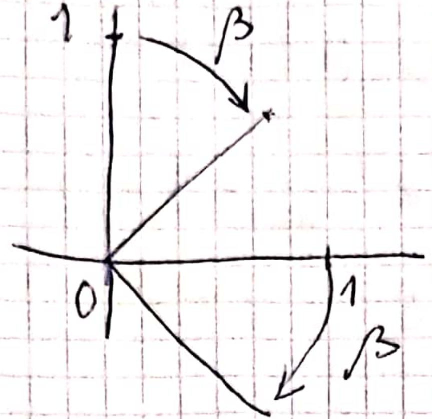
$$= I\left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \dots\right) + J\left(\frac{\beta}{1!} - \frac{\beta^3}{3!} + \dots\right)$$

$$e^{\beta J} = \cos\beta\, I + \sin\beta\, J$$

(compare: $e^{\beta i} = \cos\beta + i \sin\beta$)

$$e^{\alpha I + \beta J} = e^\alpha(\cos\beta\, I + \sin\beta\, J) = e^\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$$

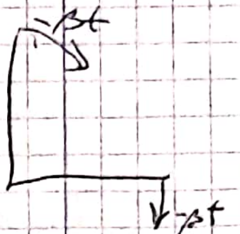
$$e^{\alpha I+\beta J} = e^{\alpha}\begin{pmatrix}\cos(-\beta) & \cos(\frac{\pi}{2}-\beta)\\ \sin(-\beta) & \sin(\frac{\pi}{2}-\beta)\end{pmatrix}$$



rotation by the angle β ~~against~~ in the clock direction

rotation by the angle $(-\beta)$ in the standard direction (against the clock)

<u>Corollary</u> $\quad \dot{\begin{pmatrix}x_1\\ x_2\end{pmatrix}} = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$



$$\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \underbrace{e^{\alpha t}}_{\substack{\text{extencion}\\ \alpha \text{ times}}}\underbrace{\begin{pmatrix}\cos\beta t & \sin\beta t\\ -\sin\beta t & \cos\beta t\end{pmatrix}}_{\text{rotation by the angle } (-\beta t)}\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix}$$

<u>Particular case:</u> $\qquad$ <u>Important</u>

$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -x_1\end{cases} \qquad \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

$$\begin{pmatrix}x_1\\ x_2\end{pmatrix} = x_1(0)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix} + x_2(0)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix}$$

$$\boxed{x\in\mathbb{R}, \quad \ddot{x} + x = 0 \Rightarrow x(t) = c_1\cos t + c_2\sin t, \quad c_1 = x_1(0),\ c_2 = \dot{x}_1(0)}$$

Harmonic equation

Theorem $\exp(KAK^{-1}) = K\exp(A)K^{-1}$

Proof: $I + \frac{1}{1!}(KAK^{-1}) + \frac{1}{2!}(KAK^{-1})^2 + \dots =$

$= I + \frac{1}{1!}KAK^{-1} + \frac{1}{2!}KA^2K^{-1} + \frac{1}{3!}KA^3K^{-1} + \dots$

$= K\left(I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots\right)K^{-1} = Ke^{A}K^{-1}$

$(KAK^{-1})^s = KA\cancel{K^{-1}}\cancel{K}A\cancel{K^{-1}}\dots\cancel{K}AK^{-1} = KA^sK^{-1}$

QED $\quad s \in \mathbb{N}$

Two methods of calculating e^{At}:

1) Coordinate transformation

$At = K\begin{pmatrix} \boxed{A_1} & 0 \\ 0 & \boxed{A_m} \end{pmatrix} t K^{-1}, \quad e^{At} = K\begin{pmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_m t} \end{pmatrix}K^{-1}$

2) $\dot{x} = Ax$ the solution $x(t) = e^{At}\xi$
$x(0) = \xi$

In other words one can also solve the DE $\Rightarrow$ obtain the exponent

Remark: $e^{A} = e^{A\cdot 1}, \; t=1$

Example $\exp\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} =$

$= \begin{pmatrix} e^2 & 0 & 0 & 0 \\ 0 & e\cos 2 & -e\sin 2 & 0 \\ 0 & +e\sin 2 & e\cos 2 & 0 \\ 0 & 0 & 0 & e^{\pi} \end{pmatrix}$

Example $\dot{x}_1 = 3x_1 - x_2$ $\quad \underset{\text{eigenvector}}{\overset{\text{eigenvalue}}{(A-\lambda I)e = 0}}$

$\dot{x}_2 = 4x_1 - 2x_2$

characteristic polynomial

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1)$$

$$\lambda_1 = 2 \quad \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad x_1 = x_2,\ e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \quad \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad 4x_1 = x_2,\ e_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$E = \begin{array}{cc} e_1 & e_2 \end{array} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad E^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \underline{\text{useful}}}$$

I $\quad AE = \underbrace{\big((\lambda_1 e_1)(\lambda_2 e_2)\big)}_{\text{columns}} = E\underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\Lambda} = E\Lambda$

$$A = E\Lambda E^{-1}$$

$$x(t) = e^{At}x(0) = E e^{\Lambda t} E^{-1} x(0)$$

$$x(t) = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}}_{e^{At}} x(0)$$

11

Another way: $\dot{x} = Ax$ 65

searching for solutions $x = e^{\lambda t} v$, $v \in \mathbb{R}^n$, $v \neq 0$

$\cancel{(e^{\lambda t} v)'} = \lambda e^{\lambda t} v = \dot{x} = A e^{\lambda t} v$

$\Rightarrow \quad \lambda v = A v, \quad (A - \lambda I) v = 0, \quad \lambda_1 \sim \lambda_n, \; v_1, \dots, v_n$

$p(\lambda) = \det(A - \lambda I) = 0$ if linearly independent

$$x = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + \dots + c_n e^{\lambda_n t} v_n$$

$$c_1, c_2, \dots, c_n \in \mathbb{R}, \quad C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$$

$$x(t) = \underbrace{\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}}_{E} \begin{pmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{pmatrix} C$$

$v_i = e_i$
another notation

$$x(0) = E \, \cancel{e^{\Lambda \cdot 0}}^{\,I} \, C$$

$$C = E^{-1} x(0)$$

$$x(t) = \underbrace{E \, e^{\Lambda t} E^{-1}}_{e^{At}} x(0) \qquad \underline{\text{once more}}$$

In the example:

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1}}_{e^{At}} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

In the general case

$A \to$ Jordan form over $\mathbb{C}$

or real numbers form Jordan form

$$\left.\begin{aligned}&(A-\lambda I)e_1 = 0 \quad \text{eigenvector}\\ &\left.\begin{aligned}&(A-\lambda I)e_2 = e_1\\ &\ldots\\ &(A-\lambda I)e_k = e_{k-1}\end{aligned}\right\} \text{generalized eigenvectors}\end{aligned}\right\} \text{Jordan chain}$$

There are a number of chains for each eigenvalue.

$$A = E\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k\end{pmatrix}E^{-1}$$

$$J_k = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda & 1\\ & & & & \lambda \end{pmatrix}_{k\times k} = \lambda I_k + \Delta_k, \quad \Delta_k = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1\\ & & & 0\end{pmatrix}$$

It is easy to check that $\Delta_k^s = \begin{array}{c} \begin{matrix} 1 & 2 & & s+1 & & \end{matrix} \\ \begin{pmatrix} 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ & & & & \ddots & & \\ & & & & & & 1 \\ & & & & & & \\ 0 & & & & & & 0 \end{pmatrix} \end{array}$

$$\Delta_k^k = 0$$

Thus $\exp J_k = \exp(\lambda I_k + \Delta_k) = e^{\lambda I_k} \cdot e^{\Delta_k}$

$$e^{J_k} = e^{\lambda} e^{\Delta_k} = \begin{pmatrix} e^{\lambda} & e^{\lambda} & \frac{1}{2}e^{\lambda} & \ldots & \frac{1}{(k-1)!}e^{\lambda} \\ & e^{\lambda} & \ddots & & \vdots \\ & & \ddots & & \frac{1}{2}e^{\lambda} \\ & & & & e^{\lambda} \\ & & & & e^{\lambda} \end{pmatrix} \qquad \frac{1}{(k-1)!}$$

$$e^{\Delta_k} = I_k + \Delta_k + \frac{\Delta_k^2}{2!} + \ldots + \frac{\Delta_k^{k-1}}{(k-1)!}$$

$$\boxed{\Delta_k^k = 0}$$

$$e^{J_k t} = e^{\lambda t}\begin{pmatrix} 1 & t & \ldots & \frac{t^{k-1}}{(k-1)!} \\ & 1 & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$$

example $\dot{x} = Ax$

$$\exp\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_{A} = e^2 \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp(At) = \begin{pmatrix} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{pmatrix}$$

$$x(t) = e^{At}x(0), \quad x(0) \in \mathbb{R}^3$$

It is the general solution (set of all solutions)

Nonhomogeneous linear systems
Method of variable parameters

$$\dot{x} = A(t)x + B(t), \quad x \in \mathbb{R}^n$$

A, B continuous

Suppose that we can solve the homogeneous DE $\dot{x} = A(t)x$ as $x(t) = \Phi(t)C$, $C \in \mathbb{R}^n$

C – free parameters. They can be the initial values, but it is not necessary $x(0) = \Phi(0)C$

$\Phi(t)$ is called the fundamental matrix.

$\Phi(t) = \underbrace{(\varphi_1(t), \dots, \varphi_n(t))}_{\text{columns}}$, $\varphi_1, \dots, \varphi_n$ – fundamental solutions

$\forall t \quad \det \Phi(t) = W[\varphi_1, \varphi_2, \dots, \varphi_n](t) \neq 0$

Wronskian , $\boxed{\dot{\Phi} = A(t)\Phi}$

What is the general solution of the nonhomogeneous DE (system)

$\dot{x} = A(t)x + B(t)$?

$x(t_0) = \xi$

Search a particular solution in the form $x_p(t) = \Phi(t)\underbrace{C_p(t)}$ Then $x(t) = \Phi C + x_p(t)$ $\to X_h$

It is the variation of the parameters C

Substitute

$$\dot{x}_p = \underbrace{A(t)\Phi(t)C_p(t)} + B(t)$$

$$\overset{\shortparallel}{\dot{\Phi}(t)C_p(t) + \Phi(t)\dot{C}_p(t)}$$

$$\overset{\shortparallel}{\underbrace{A(t)\Phi(t)C_p(t)}} \longrightarrow \text{canceled}$$

$$\Phi(t)\dot{C}_p(t) = B(t)$$

$$\dot{C}_p(t) = \Phi^{-1}(t)B(t) \quad \text{continuous}$$

$$C_p(t) = \int_{t_0}^{t} \Phi^{-1}(t)B(t)\,dt + \cancel{C_{p0}} \to 0$$

We only need one particular solution

$x_p(t) = \Phi(t)C_p(t)$, note! $C_p(t_0) = 0$

$$\underbrace{x(t) = \Phi(t)C + \Phi(t)C_p(t)}$$

General solution

Satisfy the initial conditions $x(t_0) = \xi$

$$\xi = x(t_0) = \Phi(t_0)C + \Phi(t)\cdot 0 = \Phi(t_0)C$$

Here we have used the special form of $C_p(t) = \int_{t_0}^{t} \Phi^{-1}(s) B(s)\, ds$

$\Rightarrow C_p(t_0) = 0$

$$\Rightarrow \boxed{C = \Phi(t_0)^{-1} \xi}$$

Important particular case: scalar differential equation

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y = b(t)$$

$$a_1, \dots, a_n, b \in C, \ \mathbb{R} \to \mathbb{R}$$

Suppose that we know fundamental solutions for $y^{(n)} + \dots + a_n(t) y = 0$: $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$

$f_1, f_2, \dots, f_n \in C^n$

Find the general solution

Rewrite the DE as a system

$$x_0 = y, \ x_1 = \dot{y}, \dots, x_{n-1} = y^{(n-1)}$$

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}^{\cdot} = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & \ddots \\ & & & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix}}_{A(t)} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ b(t) \end{pmatrix}}_{B(t)}$$

$$\dot{x} = A(t) \cdot x + B(t)$$

$$\Phi(t) = \begin{pmatrix} f_1 & \dots & f_n \\ \dot{f}_1 & & \dot{f}_n \\ \vdots & & \vdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix}$$

$$x_p = \Phi(t) C_p(t), \quad \left(y_p(t) = C_{p1}(t) f_1(t) + \dots + C_{pn}(t) f_n(t) \right)$$

Once more

$$\Phi \dot{C}_p + \cancel{\dot{\Phi} C_p} = \cancel{A\Phi C_p} + B$$

$$\overset{\shortparallel}{\cancel{A\Phi C_p}}$$

$$\Phi \dot{C}_p = B$$ (Beautiful explanation by Arnold)

$$\begin{cases} \dot{C}_{p1} f_1 + \dot{C}_{p2} f_2 + \dots + \dot{C}_{pn} f_n = 0 \\ \dot{C}_{p1} \dot{f}_1 + \dot{C}_{p2} \dot{f}_2 + \dots + \dot{C}_{pn} \dot{f}_n = 0 \\ \vdots \\ \dot{C}_{p1} f_1^{(n-1)} + \dot{C}_{p2} f_2^{(n-1)} + \dots + \dot{C}_{pn} f_n^{(n-1)} = b(t) \end{cases}$$

One can use here indefinite integrals, but the calculations are more complicated.

Example $\ddot{y} - 2\dot{y} + y = \frac{e^t}{t}$

$y(1) = 1, \quad \dot{y}(1) = 0, \quad t > 0$

$x_0 = y, \quad x_1 = \dot{y}, \quad y(t) = C_1 e^t + C_2 t e^t$ General Solution

We will learn it today

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{e^t}{t} \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \lambda^2 - 2\lambda + 1$$

$$x_p = C_{p1}(t) \begin{pmatrix} e^t \\ e^t \end{pmatrix} + C_{p2}(t) \begin{pmatrix} te^t \\ e^t + te^t \end{pmatrix}$$

$$y_p = C_{p1}(t) e^t + C_{p2}(t) te^t$$

$$\Phi(t) = \begin{pmatrix} e^t, & te^t \\ e^t, & e^t + te^t \end{pmatrix}, \det \Phi(t) = W(t) = e^{2t}$$

$$\dot{x}_p = A(t)\Phi(t)C_p(t) + B(t)$$

$$\underbrace{A(t)\Phi(t)}_{\dot{\Phi}(t)}C_p(t) + \Phi(t)\dot{C}_p(t)$$

$$\Phi(t)\dot{C}_p(t) = B(t)$$

$$\begin{cases} \dot{C}_{p1}e^t + \dot{C}_{p2}te^t = 0 \\ \dot{C}_{p1}e^t + \dot{C}_{p2}(e^t + te^t) = \frac{e^t}{t} \end{cases}$$

The Kramer rule:

$$\dot{C}_{p1} = \frac{\begin{vmatrix} 0 & te^t \\ \frac{e^t}{t} & e^t + te^t \end{vmatrix}}{e^{2t}} = -1, \quad C_{p1} = -\int_1^t dt = 1 - t$$

$$\dot{C}_{p2} = \frac{\begin{vmatrix} e^t & 0 \\ e^t & \frac{e^t}{t} \end{vmatrix}}{e^{2t}} = \frac{1}{t}, \quad C_{p2} = \int_1^t \frac{1}{s}ds = \ln|s|\Big|_1^t = \ln t$$

$\Rightarrow$ General solution

$$y(t) = c_1 e^t + c_2 te^t + (1-t)e^t + \ln t \cdot te^t, \quad c_1, c_2 \in \mathbb{R}$$

Initial conditions

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y(1) \\ \dot{y}(1) \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}\Bigg|_{t=1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \underbrace{0}_{x_p(1)}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e & e \\ e & 2e \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{e} \\ \frac{2}{e} \end{pmatrix}$$

72

Solutions of scalar linear differential equations with constant ~~parameters~~ ~~matrix~~ coefficients

Method of quasi-polynomials (Arnold)

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$$

$a_1, a_2, \dots, a_n \in \mathbb{R}$ (or $\mathbb{C}$) constant

$$x_0 = y,\ x_1 = \dot{y},\ \dots,\ x_{n-1} = y^{(n-1)}$$

$$\dot{x} = Ax, \quad A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix}$$

Proposition $\det(A - \lambda I) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots + a_n)$

$$\Delta_n = \det \begin{pmatrix} -\lambda & 1 & & & \\ & -\lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & -\lambda & 1 \\ -a_n & -a_{n-1} & \dots & -a_2 & -a_1-\lambda \end{pmatrix} = -\lambda \Delta_{n-1} + (-1)^{n-1}(-a_n) =$$

(Δ_{n-1})

$$= (-1)^n(\underbrace{\lambda \Delta_{n-1}}_{\text{Induction}} + a_n)$$

$$= (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots + a_{n-1}\lambda + a_n)$$

QED

Very convenient!

ODE: $P\left(\frac{d}{dt}\right) y = 0$, $P(\lambda)$ - characteristic polynomial

$$\cancel{\Phi(t) = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}}$$

$$\cancel{\det \Phi = W = e^{2t} + te^{2t} - te^{2t} = e^{2t}}$$

Quasi-polynomial

$\alpha, \beta \in \mathbb{R}$

$\lambda = \alpha$ eigen value

$$P_{\alpha,k}(t) = e^{\alpha t}\left[d_0 + d_1 t + \dots + d_k t^k\right] \in L_{\alpha,k}$$

$\dim L_{\alpha,k} = k+1$ linear space.

exactly the same! $-\ \alpha, d_0 \dots d_k, \lambda \in \mathbb{C}$

Proposition Basis $e^{\alpha t}, \frac{t}{1!}e^{\alpha t}, \dots, \frac{t^k}{k!}e^{\alpha t}$

$$\frac{d}{dt} - \alpha I_0 : \frac{t^k}{k!}e^{\alpha t} \mapsto \frac{t^{k-1}}{(k-1)!}e^{\alpha t} \mapsto \dots \mapsto e^{\alpha t} \mapsto 0$$

unit operator: $I_0 : f \mapsto f,\ f \in C^{\infty}$

~~Thus~~ **Proof** $\frac{d}{dt}\left(\frac{t^s}{s!}e^{\alpha t}\right) - \alpha \frac{t^s}{s!}e^{\alpha t} =$

$$= \frac{s}{s}\frac{t^{s-1}}{(s-1)!}e^{\alpha t} + \cancel{\alpha\frac{t^s}{s!}e^{\alpha t} - \alpha\frac{t^s}{s!}e^{\alpha t}}$$

QED

$\Rightarrow L_{\alpha,k}$ – solutions of $\left(\frac{d}{dt} - \alpha I_0\right)^{k+1} y = 0$ and only them

Proof: $\Rightarrow$ Trivial

$\Leftarrow$ dimensions coincide

QED

Theorem ~~$p\left(\frac{d}{dt}\right)$~~ $p(\lambda)=(\lambda-\alpha_1)^{k_1}\dots(\lambda-\alpha_m)^{k_m}$

~~$p\left(\frac{d}{dt}\right) = d$~~ $\alpha_1, \alpha_2, \dots, \alpha_m$ - different

74

Then the solution is

$$y(t) = L_{\alpha_1, k_1-1} \oplus L_{\alpha_2, k_2-1} \oplus \dots \oplus L_{\alpha_m, k_m-1}$$

Proof: $\left(\frac{d}{dt}-\alpha I_0\right)\left(\frac{d}{dt}-\beta I_0\right)$

commutate $\quad p(\lambda)=q(\lambda)(\lambda-\alpha_1)^{k_1}$

$$\Rightarrow p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)\left(\frac{d}{dt}-\alpha_1 I_0\right)^{k_1} y = 0$$

They do not intersect $\quad y \in L_{\alpha_1, k_1}$ - solutions

$$e_0, e_1, e_2, \dots, e_k = e^{\alpha t}, \frac{t}{1!}e^{\alpha t}, \dots, \frac{t^k}{k!}e^{\alpha t}$$

$$\frac{d}{dt} e_k = \alpha e_k + e_{k-1}$$

$k = 1, \dots, k$

$$\frac{d}{dt} e_0 = \alpha e_0$$

$$\begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & 1 & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix}$$

$$\left(\frac{d}{dt} - \beta I_0\right) e_k = (\alpha-\beta)e_k + e_{k-1}$$

$$\det \begin{pmatrix} \alpha-\beta & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \alpha-\beta \end{pmatrix} \neq 0$$

Example $(\lambda^2-1)^2 = \lambda^4 - 2\lambda^2 + 1$
$= (\lambda-1)^2(\lambda+1)^2$

$$y^{IV} - 2y'' + y = 0$$

Solution $y = e^{t}(c_1 + c_2 t) + e^{-t}(c_3 + c_4 t)$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$ (or $\mathbb{C}$)

The same logic is true over $\mathbb{C}$

$\cancel{(\lambda-2)(\lambda)}$ $\quad p(\lambda) = (\lambda+1)^3$

$$y''' + 3y'' + 3y' + y = 0$$

$y = e^{-t}(c_1 + c_2 t + c_3 t^2)$ solution

$$\lambda = \alpha \pm \beta i, \quad \alpha, \beta \in \mathbb{R}$$

$$P_{\alpha+\beta i, k} = e^{\alpha t}\Big[\cos(\beta t)(d_0 + d_1 t + \dots + d_k t^k) + \sin(\beta t)(f_0 + f_1 t + \dots + f_k t^k)\Big]$$

These are the solutions of the DE $p(\frac{d}{dt})y = 0$

$$p(\lambda) = (\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)^{k+1}$$

Example $\quad y'' + y = 0 \quad \alpha = 0, \beta = 1$

$$y = e^{0t}[\cos t \cdot c_1 + \sin t \cdot c_2]$$