31/10-2021 Lecture 4 (43)

Definition $\varphi_1(t) = \begin{pmatrix} \varphi_{11}(t) \\ \vdots \\ \varphi_{1n}(t) \end{pmatrix}, \dots, \varphi_n(t) = \begin{pmatrix} \varphi_{n1}(t) \\ \vdots \\ \varphi_{nn}(t) \end{pmatrix}$

Wronskian

$$W[\varphi_1, \varphi_2, \dots, \varphi_n](t) \overset{def}{=} \begin{vmatrix} \varphi_{11}(t) & \dots & \varphi_{n1}(t) \\ \vdots & & \vdots \\ \varphi_{1n}(t) & \dots & \varphi_{nn}(t) \end{vmatrix}$$

Proposition For each n solutions $\varphi_1 \dots \varphi_n$ of a homogeneous equation $\dot{x} = A(t)x$ one has only 2 options

1. $W[\varphi_1 \dots \varphi_n](t) \equiv 0$ for any t

or

2. $\forall t$: $W[\varphi_1 \dots \varphi_n](t) \neq 0$

Conclusion If $W[\varphi_1 \dots \varphi_n](t_1) = 0$
$W[\varphi_1 \dots \varphi_n](t_2) \neq 0$
for some t_1, t_2, then $\varphi_1 \dots \varphi_n$ cannot be solutions of any ODE $\dot{x} = A(t)x$, $x \in \mathbb{R}^n$, $A \in C$

Example $\dot{x}_1 = 5x_1 - 2x_2$ $\quad p(\lambda) = \lambda^2 - 3\lambda + 2$
$\dot{x}_2 = 6x_1 - 2x_2$ $\quad \lambda_{1,2} = 2, 1$

$\lambda = 1$ $\begin{cases} 3x_1 - 2x_2 = 0 \\ 6x_1 - 4x_2 = 0 \end{cases}$ $\quad 3x_1 = 2x_2$ $\quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = 2$ $\begin{cases} 4x_1 - 2x_2 = 0 \\ 6x_1 - 3x_2 = 0 \end{cases}$ $\quad 2x_1 = x_2$ $\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{\begin{aligned} &\dot{x} = Ax \\ &x = \xi e^{\lambda t} \\ &\lambda \xi = A\xi \\ &(A - \lambda E)\xi = 0 \end{aligned}}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t}$$

$$W\left[\begin{pmatrix} 2e^{2t} \\ 3e^{2t} \end{pmatrix}, \begin{pmatrix} e^{t} \\ 2e^{t} \end{pmatrix}\right] = \begin{vmatrix} 2e^{2t} & e^{t} \\ 3e^{2t} & 2e^{t} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} e^{3t} = e^{3t} \neq 0$$

Example a) $\varphi_* = \begin{pmatrix} t-1 \\ \sin(t-1) \end{pmatrix}$ cannot be a solution of the ODE $\dot{x} = A(t)x$, $x \in \mathbb{R}^2$, $A \in C$

for $\varphi_*(1) = 0$ and $X(t) \equiv 0$ - another solution

b) $\varphi_1(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$ and $\varphi_2 = \begin{pmatrix} 1 \\ \sin(t-1)+1 \end{pmatrix}$

cannot be solutions of any system of the form $\dot{x} = A(t)x$, $A \in C$, $x \in \mathbb{R}^2$,

for (1) $\varphi_1 - \varphi_2 = \varphi_*$

or equivalently for (2)

$$W[\varphi_1, \varphi_2](t) = \begin{vmatrix} t & 1 \\ 1 & 1-\sin(t-1) \end{vmatrix} = t^2 - t\sin(t-1) - 1 \not\equiv 0$$

and $W[\varphi_1, \varphi_2](1) = 1 - 1\cdot 0 - 1 = 0$

The case of one equation

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y = 0, \quad y \in \mathbb{R}$$

$a_1, a_2, \dots, a_n : \mathbb{R} \to \mathbb{R}$, continuous

$\Rightarrow$ Equivalent to $\dot{x} = A(t)x$, $A = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & 1 & \\ & & \ddots & \\ 0 & 0 & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix}$

here $x_1 = y,\ x_2 = \dot{y}, \dots, x_n = y^{(n-1)}$

Wronskian of scalar functions is defined so that it will be the Wronskian of the corresponding solutions for the equivalent system of 1st-order equations

Definition $f_1, \dots, f_n : \mathbb{R} \to \mathbb{R},\ f_i \in C^{n-1}$

$$W[f_1, f_2, \dots, f_n](t) \overset{\text{def}}{=} \begin{vmatrix} f_1(t) & \dots & f_n(t) \\ \dot{f}_1(t) & \dots & \dot{f}_n(t) \\ \vdots & & \\ f_1^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{vmatrix}$$

Example $\ddot{x} + x = 0 \quad x = c_1 \cos t + c_2 \sin t$

$$W[\cos t, \sin t] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

Proposition $f_1, \dots, f_n$ solutions of

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y = 0 \quad (*)$$

$\Rightarrow \quad \forall t \ W[f_1 \dots f_n](t) = 0$

or $\forall t \ W[f_1 \dots f_n](t) \neq 0$

Example: 1) t, t^2 cannot be solutions of any equation $(*)$ of the order $n=2$

since $W[t, t^2]\big|_{t=0} = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}_{t=0} = 0$, but $\exists t : \underset{= t^2}{W(t)} \neq 0$

2) $(t-1)^2, \cos(t-1)$ — " — for $W(1) = \begin{vmatrix} \dots \\ 0 \ 0 \end{vmatrix} = 0$

Definition Functions $f_1,\dots,f_n:\mathbb{R}\to\mathbb{R}$ are called *linearly dependent* if

$$\exists\, \lambda_1,\dots,\lambda_n \in \mathbb{R}: \quad \forall t \quad \lambda_1 f_1(t)+\dots+\lambda_n f_n(t)=0$$

$\neq 0,0,\dots,0$ (not all zero)

Vector functions $\varphi_1,\dots,\varphi_n:\mathbb{R}\to\mathbb{R}^n$

— " — $\quad \forall t \quad \lambda_1\varphi_1(t)+\dots+\lambda_n\varphi_n(t)=0\in\mathbb{R}^n$

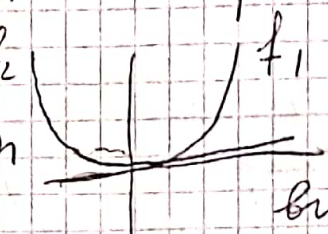
Obviously: $\exists t: W[f_1,\dots,f_n](t)\neq 0$

or $\exists t: W[\varphi_1,\dots,\varphi_n](t)\neq 0$

$\Rightarrow$ linear ~~inpendence~~ independence

The opposite is wrong:

Example $f_1(t)=\begin{cases}0 & t<0\\ t^2 & t\geq 0\end{cases}$, $\quad f_2(t)=\begin{cases}t^2, & t<0\\ 0, & t\geq 0\end{cases}$



Then $\quad W[f_1,f_2](t)\equiv 0$

but linearly independent

They cannot be solutions of any equation (*) with continuous a_1, a_2, $n=2$,

$$y^{(n)}+a_{n-1}(t)y^{(n-1)}+a_{n-2}(t)y^{(n-2)}+\dots+a_0(t)y=0 \quad (*)$$

Example $\{t, 1\}$ can be solutions,

~~for~~ since $W[t,1]=\begin{vmatrix}t, & 1\\ 1, & 0\end{vmatrix}=-1\neq 0$. Indeed:

The equation: $\ddot{y}=0$

~~Example~~

Question Given the functions $f_1, \dots, f_n : R \to \mathbb{R}$, $f_i \in C^n[a,b]$, can they be fundamental solutions (i.e., constituting a basis) for an ODE

$$y^{(n)} + a_{n-1}(t) y^{(n-1)} + \dots + a_0(t) y = 0, \quad (*)$$

$a_0, \dots, a_{n-1} \in C[a,b]$?

Answer: The necessary and sufficient condition: $\underbrace{W[f_1, f_2, \dots, f_n](t)}_{=W_*(t)} \neq 0 \quad \forall t \in [a,b]$

The necessity is obvious. Construct the equation.

$f_1, \dots, f_n$ – basis $\Rightarrow$ ~~$\forall$~~ Any solution $y(t)$ of $(*)$ should be their linear combination

$$\Rightarrow W[y, f_1, f_2, \dots, f_n](t) \equiv 0 \quad t \in [a,b]$$

It is the ODE!

$$W = \begin{vmatrix} y & f_1 & \cdots & f_n \\ \dot{y} & \dot{f}_1 & \cdots & \dot{f}_n \\ \vdots & \vdots & & \vdots \\ y^{(n)} & f_1^{(n)} & \cdots & f_n^{(n)} \end{vmatrix} = 0$$

Expand the determinant with respect to the first column:

$$(-1)^n \underbrace{W_*(t)}_{\neq 0} y^{(n)} + \dots = 0$$

Moreover: There is only one such ODE of the form $(*)$. Indeed,

$$\begin{array}{l} y^{(n)} + a_{n-1}(t) y^{(n-1)} + \dots + a_0(t) y = 0 \\ - \; y^{(n)} + \tilde{a}_{n-1}(t) y^{(n-1)} + \dots + \tilde{a}_0(t) y = 0 \\ \hline \Delta a_{n-1} y^{(n-1)} + \dots + \Delta a_0 y = 0 \end{array}$$

suppose $a_{n-1} \neq \tilde{a}_{n-1}$ at $t_* \in [a,b]$

$\Delta a_k = a_k - \tilde{a}_k$

If $\Delta a_{n-1}(t_*) \neq 0 \Rightarrow$ in a vicinity

$$y^{(n-1)} + \frac{\Delta a_{n-2}}{\Delta a_{n-1}} y^{(n-2)} + \dots + \frac{\Delta a_0}{\Delta a_{n-1}} y = 0$$

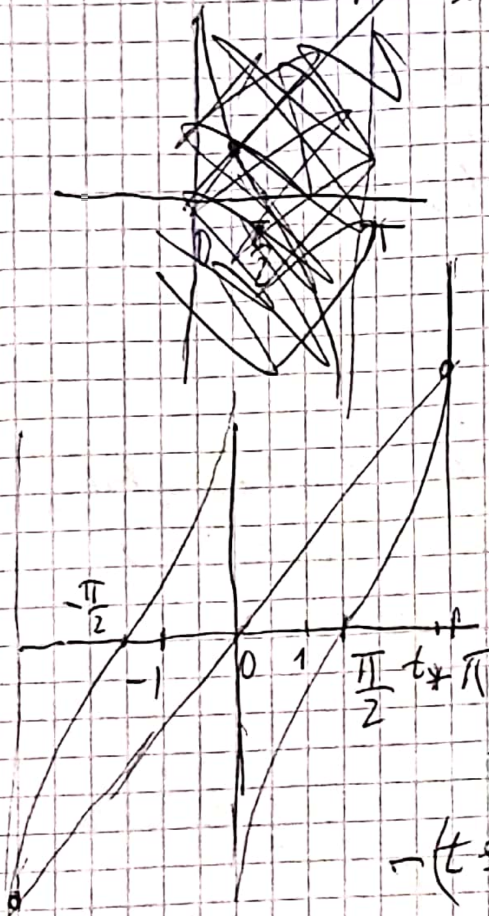
has n linearly independent solutions $f_1, f_2, \dots, f_n \Rightarrow$ contradiction the dimension is $n-1$!

The same if $\Delta a_{n-1}, \dots, \Delta a_{k+1} \equiv 0$, $\Delta a_k(t_*) \neq 0$ ...

<u>Example</u> Can t, ~~t^2+1~~ $\cos t$ be solutions of ODE (*) for $n=2$? over $\left[\frac{\pi}{2}, \pi\right]$, ~~$\left[0, \frac{\pi}{2}\right]$~~ $[0,1]$?

$$W(t) = \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix} = -(t\sin t + \cos t)$$

$t_* \sin t_* + \cos t_* = 0 \Rightarrow t_* = -\frac{\cos t_*}{\sin t_*} = -\cot t_*$



<u>Yes</u>: The equation over $[0,1]$

$$\begin{vmatrix} y & t & \cos t \\ \dot{y} & 1 & -\sin t \\ \ddot{y} & 0 & -\cos t \end{vmatrix} = 0$$

Also $[-1,1]$

<u>No</u> over ~~$\left[0, \frac{\pi}{2}\right]$~~ $\left[\frac{\pi}{2}, \pi\right]$

$t_* \in \left[\frac{\pi}{2}, \pi\right]$, $t_* \notin [0,1]$

The equation itself is

$$-(t\sin t + \cos t)\ddot{y} - t\cos t\, \dot{y} + \cos t\, y = 0$$

$$\ddot{y} + \frac{t\cos t}{t\sin t + \cos t} \dot{y} - \frac{\cos t}{t\sin t + \cos t} y = 0$$

<u>Question</u> Given the functions

$\varphi_1, \varphi_2, \ldots, \varphi_n : R \to \mathbb{R}^n$, $\varphi_i \in C^1[a,b]$

whether there is an ODE

$\dot{x} = A(t)x$, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$, $A \in C$,

for which they are fundamental solutions?

<u>Answer</u> Yes. If and only if (iff)

$\forall t \in [a,b]: W[\varphi_1, \varphi_2, \ldots, \varphi_n](t) \neq 0$

Indeed, $(\dot{\varphi}_1, \dot{\varphi}_2, \ldots, \dot{\varphi}_n) = \overbrace{(\varphi_1, \varphi_2, \ldots, \varphi_n)}^{\Phi}$ matrix $n\times n$

$\dot{\Phi} = \underbrace{\dot{\Phi}\Phi^{-1}}_{A}\Phi$ $\qquad A(t) = \dot{\Phi}(t)\Phi^{-1}(t)$

$\det \Phi(t) = W(t) \neq 0 \Rightarrow \exists\, \Phi^{-1}$

Obviously it is unique.

<u>Example</u> Which pair of functions can be fundamental sol. for $t \in (-\infty, \infty)$

$n = 2$

$\begin{pmatrix} t \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ t+1 \end{pmatrix}$ or $\begin{pmatrix} t \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ t \end{pmatrix}$

$\boxed{W = t^2 + t + 1 \neq 0}$ $\qquad \boxed{W = t - 1,\ W(1) = 0 \Rightarrow \underline{no}}$

<u>Yes</u> $\downarrow$

$A(t) = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\dot{\Phi}} \overbrace{\begin{pmatrix} t & 1 \\ -1 & t+1 \end{pmatrix}^{-1}}^{\Phi^{-1}} = \frac{1}{t^2+t+1}\begin{pmatrix} t+1 & -1 \\ 1 & t \end{pmatrix}$

$\dot{x} = A(t)x$, $x \in \mathbb{R}^2$

Theorem of Abel-Liouville (Ø eN)

$$\dot{x} = A(t)x, \; x \in \mathbb{R}^n, \; A \in C$$
continuous

Let $\varphi_1(t), \dots, \varphi_n(t)$ be solutions (not necessarily fundamental) any solutions

$$A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \dots & a_{nn}(t) \end{pmatrix}$$

Theorem

$$\frac{d}{dt} W[\varphi_1, \varphi_2, \dots, \varphi_n](t) = T_r A(t) \cdot W[\varphi_1 .. \varphi_n](t)$$

Here $T_r A(t) = a_{11}(t) + a_{22}(t) + \dots + a_{nn}(t)$

Proof Denote $W(t) = W[\varphi_1 .. \varphi_n](t) = \det \Phi(t)$

$$\Phi(t) = (\varphi_1(t) .. \varphi_n(t)) = \begin{pmatrix} \Phi_1(t) \\ \vdots \\ \Phi_n(t) \end{pmatrix}$$
columns rows

$\dot{\varphi}_i = A\varphi_i, \; i = 1, \dots, n$

$\Rightarrow \dot{\Phi} = A\Phi, \quad \dot{\Phi}_i = (a_{i1}, a_{i2}, \dots, a_{in})\Phi$

$$\dot{\Phi}_i = a_{i1}\Phi_1 + a_{i2}\Phi_2 + \dots + a_{in}\Phi_n, \; i = 1, \dots, n$$

It is known that

$$(u_1 u_2 \dots u_n)^{\cdot} = \dot{u}_1 u_2 \dots u_n + u_1 \dot{u}_2 u_3 \dots u_n + \dots + u_1 \dots u_{n-1} \dot{u}_n$$

$$\det \begin{pmatrix} d_{11} & \dots & d_{1n} \\ d_{n1} & \dots & d_{nn} \end{pmatrix} = \sum (d_{1j_1} \dots d_{nj_n}) (-1)^{\text{Permutation degree}}$$

$$\Rightarrow \dot{W} = \frac{d}{dt} \det \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} = \det \begin{pmatrix} \dot{\Phi}_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} + \det \begin{pmatrix} \Phi_1 \\ \dot{\Phi}_2 \\ \vdots \\ \Phi_n \end{pmatrix} + \dots + \det \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \dot{\Phi}_n \end{pmatrix}$$

$$= \sum_i \det \begin{pmatrix} \Phi_1 \\ \vdots \\ a_{i1}\Phi_1 + \dots + a_{in}\Phi_n \\ \vdots \\ \Phi_n \end{pmatrix} = a_1 W + a_2 W + \dots + a_n W = T_r A \cdot W$$

QED

Further $\dot{W} = Tr\,A(t)\,W$, $W(t_0) = W_0$

$W = 0$ - solution, let $W(t) \neq 0$ (then never vanishes)

$$\int \frac{\dot{W}\,dt}{W} = \int Tr\,A(t)\,dt$$

$$\ln|W| = \int_{t_0}^{t} Tr\,A(s)\,ds + C, \quad C \in \mathbb{R}$$

$$W(t) = \pm e^{C} e^{\int_{t_0}^{t} Tr\,A(s)} \neq 0$$

$$\Rightarrow \boxed{W(t) = W(t_0) \exp \int_{t_0}^{t} Tr\,(A(s))\,ds}$$

Order Reduction

Just examples

$$\begin{cases} \dot{x} = (1+e^{-t} - te^{-t})x + (te^{-t} - e^{-t})y \\ \dot{y} = (e^{-t} - te^{-t})x + (1+te^{-t} - e^{-t})y \end{cases}$$

One solution is known $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix}$

Find the general solution

$Tr\,A(t) = 2 \Rightarrow \det \begin{pmatrix} e^t & x \\ e^t & y \end{pmatrix} = c\,e^{2t}$ (Abel)

We only need to find one solution linearly independent with $\begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

Thus we can take $c = 1$.

$$e^t y - e^t x = e^{2t} \Rightarrow y = e^t + x$$

Order reduction: substitute

$y = e^t + x$ in the equation for $\dot{x}$

$\Rightarrow$ get a scalar equation

$$\dot{x} = (1 + e^{-t} - te^{-t})x + (te^{-t} - e^{-t})(e^t + x)$$
$$= x + t - 1$$

$$\dot{x} - x = t - 1.$$

char. polynomial: $\lambda - 1$; no resonance (we learn it further)

$\Rightarrow x(t) = X_h + x_p$, x_p – particular solution

X_h general solution of the homogeneous equation

$$\dot{x} - x = 0 \Rightarrow x = C_1 e^t$$

search for $x_p(t)$ in the form

$$x_p = \alpha t + \beta$$

$$\alpha - \underbrace{\alpha t}_{} - \beta = \underbrace{t}_{} - 1 \Rightarrow \alpha = -1$$
$$-1 - \beta = -1, \ \beta = 0$$

$$x_p = t$$

$$x = C_1 e^t - t$$

$$\Rightarrow y = e^t + x = e^t - t + C_1 e^t$$

Thus

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{C_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix}}_{\text{old solution}} + \underbrace{\begin{pmatrix} -t \\ e^t - t \end{pmatrix}}_{\text{new solution}}$$

Finally

$$\begin{pmatrix} x \\ y \end{pmatrix} = \tilde{C}_1 \begin{pmatrix} e^t \\ e^t \end{pmatrix} + \tilde{C}_2 \begin{pmatrix} -t \\ e^t - t \end{pmatrix}$$

For simplicity: check that $\begin{pmatrix} -t \\ e^t - t \end{pmatrix}$ is a solution by substitution.

In fact, one can show that it has to be.

Scalar equation, ~~n=2~~

~~example~~ $\cancel{\ddot{y} + a(t)\dot{y} + b(t)y = 0}$

$$y^{(n)} + a_1(t)\,y^{(n-1)} + a_2(t)\,y^{(n-2)} + \dots + a_n(t)\,y = 0$$

is equivalent to $\quad x_i = y^{(i)}, \; i = 0, \dots, n-1$

$$\begin{pmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & 1 \\ -a_n(t) & -a_{n-1}(t) & -a_{n-2}(t) & \cdots & -a_1(t) \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix}$$

$f_1(t), f_2(t), \dots, f_n(t)$ – scalar solutions

$$\varphi_i(t) = \begin{pmatrix} f_i(t) \\ \dot{f}_i(t) \\ \vdots \\ f_i^{(n-1)}(t) \end{pmatrix}$$ – the corresponding vector solutions

$$W(t) = W[f_1 \dots f_n](t) = W[\varphi_1 \dots \varphi_n](t) = \det(\varphi_1 \dots \varphi_n)$$

$$\dot{W} = -\operatorname{Tr} A(t) \cdot W = -a_1(t)\, W$$

$$\boxed{\begin{gathered} \dot{W} = -a_1(t)\, W \\ W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} a_1(s)\, ds\right) \end{gathered}}$$

Example $n=2$

$$\ddot{y}+a(t)\dot{y}+b(t)y=0$$

A known solution: $f(t)$

$$\begin{vmatrix} f & y \\ \dot{f} & \dot{y} \end{vmatrix} = c_1 e^{-\int_{t_0}^{t} a(t)\,dt}$$

One can take $c_1=1$, or continue as is

Order reduction

$$-y\dot{f}+\dot{y}f = c_1\alpha(t), \quad \alpha(t) \overset{\text{def}}{=} e^{-\int_{t_0}^{t} a(s)\,ds}$$

Let $f\neq 0$ (otherwise $f=0$ at separate points)

$$\frac{\dot{y}f-y\dot{f}}{f^2} = \left(\frac{y}{f}\right)^{\cdot} = c_1\frac{\alpha(t)}{f^2(t)}$$

$$\Rightarrow y = c_1 f(t)\int_{t_0}^{t}\frac{\alpha(s)}{f^2(s)}\,ds + c_2 f(t)$$

$$\left(c_1\int\frac{\alpha(s)}{f^2(s)}\,ds = c_1\int_{t_0}^{t}\frac{\alpha(s)}{f^2(s)}\,ds + \underbrace{c_1 C}_{c_2}\right)$$

One only needs to remember the way!

Example $(t^2+1)\ddot{y}-2t\dot{y}+2y=0$, $f(t)=t$ solution

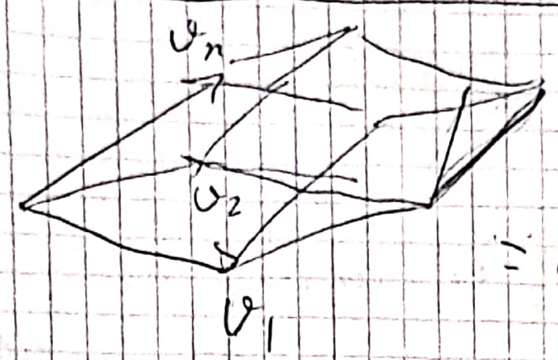
$$\begin{vmatrix} t & y \\ 1 & \dot{y} \end{vmatrix} = c_1 e^{\int_0^t \frac{2s}{s^2+1}ds} = c_1 e^{\ln(t^2+1)} = c_1(t^2+1)$$

Divide by t^2: $\left(\dfrac{y}{t}\right)^{\cdot} = c_1\left(1+\frac{1}{t^2}\right)$

$$y = t\left[c_1\left(t-\frac{1}{t}\right)+\overbrace{c_1\tilde{c}_2}^{c_2}\right] = c_1(t^2-1)+c_2 t$$

~~Ge~~ Abel, Niels Henrik
Norwey, 1802–1829, died 26 years old

General Theorem of Abel-Liouville



$$\text{Volume} = \left|\det \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}\right| \quad \text{rows}$$

$$= \text{abs} \det(v_1, v_2 \ldots v_n) \quad \text{columns}$$

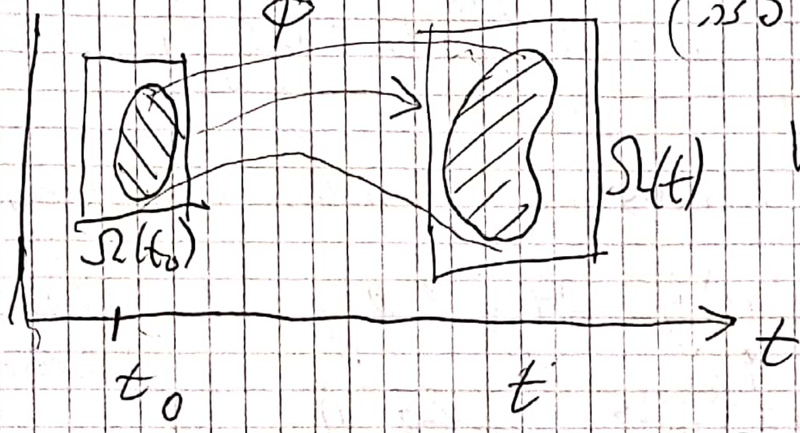
That is the geometrical sense.

$W[\varphi_1 \ldots \varphi_n](t) = \pm$ Volume (area) of the parallelepiped enclosing the vectors $\varphi_1(t) \ldots \varphi_n(t)$

$$\begin{cases} \dot{x} = f(t, x) = \begin{pmatrix} f_1(t,x) \\ \vdots \\ f_n(t,x) \end{pmatrix} & f \in C^k, \; k \geq 1 \\ x(t_0) = x_0 \end{cases}$$

$x = \Phi(t, t_0, x_0) \in C^k$

Phase stream (זרם הפאזות)



$$V(t) = \int_{\Omega(t)} dx_1 \ldots dx_n$$

The compact region $\Omega(t_0) \subset \mathbb{R}^n$ transfers ~~at~~ to $\Omega(t) \subset \mathbb{R}^n$ by the phase stream in the time $t - t_0$

$V(t)$ is its volume

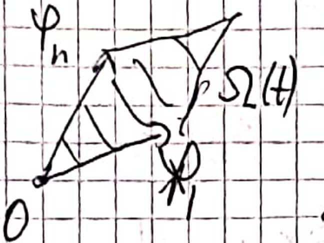
Theorem

$$\dot V(t) = \int\limits_{\Omega(t)} \sum \frac{\partial f_i}{\partial x_i}(t,x)\, dx_1 \dots dx_n = \int\limits_{\Omega(t)} \mathrm{Tr}\, \frac{\partial f}{\partial x}(x)\, dx_1 \dots dx_n$$

$t,$

$$= \int\limits_{\Omega(t)} \underbrace{\mathrm{div}_x}_{\text{divergence}} f(t,x)\, dx_1 \dots dx_n$$

In the classical case $\dot x = A(t)x$



$V(t) = \underbrace{|W(t)|}_{\text{Wronskian}}$, $\dot V = \mathrm{Tr}\, A(t)\, V$

do not change the sign

Proof Let $x_0 = \begin{pmatrix} x_{01} \\ \vdots \\ x_{0n} \end{pmatrix}$ - coordinates at $t = t_0$

$x = x(t,\xi) = \varphi(t,\xi)$, $x_0 = \xi$ init. value

denote $x_*(t) = \varphi(t,\xi)$

$x(t,x_0) = x_*(t) + \Delta x(t,x_0)$, $\dot x = f(t,x)$

differentiate with respect to x_0 at $x_0 = \xi$

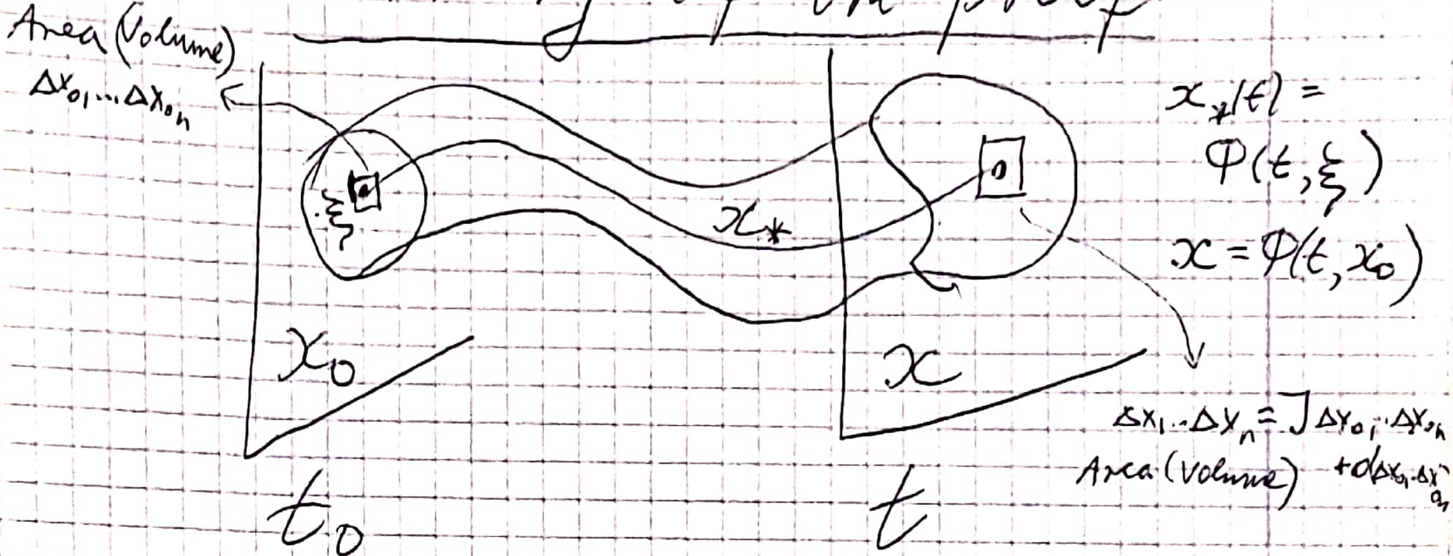
the point

$$\frac{\partial^2}{\partial t\, \partial x_0} x(t,\xi) = \left(\frac{\partial x}{\partial x_0}\right)^{\cdot} = \frac{\partial f}{\partial x}(t, \underbrace{x(t,\xi)}_{x_*(t)}) \frac{\partial x}{\partial x_0}(t,\xi)$$

Denote $Z(t) = \dfrac{\partial x}{\partial x_0}(t,\xi)$ $\qquad Z \in \mathbb{R}^{n\times n}$

$$\dot Z = \frac{\partial f}{\partial x}(t, x_*(t))\, Z, \quad Z = \underbrace{\left(\frac{\partial x}{\partial x_1}, \dots, \frac{\partial x}{\partial x_n}\right)}_{\text{columns}}(t,\xi)$$

Summary of the proof



$$V(t) = \int_{\Omega(t)} dx_1 dx_2 \dots dx_n = \int_{\Omega(t_0)} \underbrace{\left| \det \frac{\partial \Phi}{\partial x_0}(t, x_0) \right|}_{\text{Jacobian } J} dx_{01} \dots dx_{0n}$$

The Jacobian happens to be the Wronskian of the linear DE

$$\dot{\zeta} = \frac{\partial f}{\partial x}(t, x_*(t)) \zeta, \quad \zeta \in \mathbb{R}^n,$$

zeta

for the unique solution $\begin{cases} \dot{x}_* = f(t, x_*) \\ x_*(t_0) = \xi \in \Omega(t_0) \end{cases}$

where ξ is any fixed value of x_0

~~with~~ and the solutions $\zeta_1, \dots, \zeta_n$ correspond to $\zeta_i(t_0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$

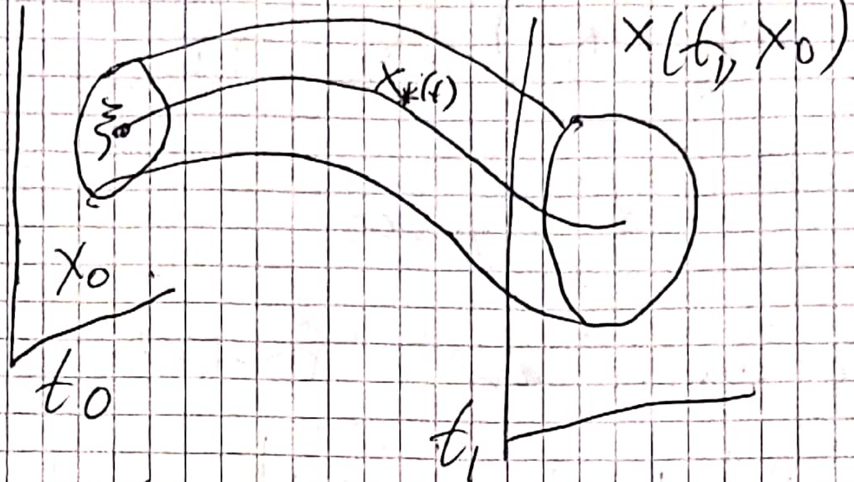
ζ_i are the columns of the matrix $Z \in \mathbb{R}^{n \times n}$

$$\dot{Z} = \frac{\partial f}{\partial x}(t, x_*(t)) Z, \quad Z(0) = I \in \mathbb{R}^{n \times n}$$

$$Z = \begin{pmatrix} \xi_1 & \dots & \xi_n \\ Z_1, & \dots, & Z_n \end{pmatrix}, \quad \overset{\xi_i}{Z_i} = \frac{\partial x}{\partial x_i}(t,\xi)$$

$$\dot{Z}_i = \frac{\partial f}{\partial x}(t, x_*(t))\, Z_i \quad \text{— linear DE}$$

$W = \det Z$ — Wronskian, $\dot{W} = \mathrm{Tr}\,\frac{\partial f}{\partial x}(t, x_*(t))\, W$



Change the coordinates to x_0

$$V(t) = \int_{\Omega(t)} dx_1 \dots dx_n = \int_{\Omega(t_0)} \left|\det \frac{\partial x}{\partial x_0}(t, x_0)\right| dx_{01} \dots dx_{0n}$$

Theorem Abel-Liouville: $\det \frac{\partial x}{\partial x_0}(t, x_0)$ does not vanish

$t = t_0$ $\quad \det \frac{\partial x}{\partial x_0}(t_0, x_0) = \det I = 1$

$x(t_0) = x_0 \}$

$$\Rightarrow V(t) = \int_{\Omega(t_0)} \overbrace{\det \frac{\partial x}{\partial x_0}(t, x_0)}^{W(t, x_0)}\, dx_{01} \dots dx_{0n}$$

(no abs. value)

$$\dot{V}(t) = \int_{\Omega(t_0)} \mathrm{Tr}\,\frac{\partial f}{\partial x}(t, x_0)\, \underbrace{W(t, x_0)}_{\text{unknown}}\, dx_{01} \dots dx_{0n}$$

, $W(t_0, x_0) = 1$

Substitute $t = t_0$

$$\dot{V}(t_0) = \int_{\Omega(t_0)} \mathrm{Tr}\,\frac{\partial f}{\partial x}(t_0, x_0)\, dx_{01} \dots dx_{0n}$$

But t_0 is any moment
QED.

Example:

$$\dot{x}_1 = t x_2 \sin x_3 - t x_1 - \cos x_2 \cos t$$

$$\dot{x}_2 = t x_1 \sin x_3 - t x_2 + t$$

$$\dot{x}_3 = t x_1 \cos(x_2^2) - t x_3$$

What is the volume of the region in x_1, x_2, x_3 to which the cube $x_i \in [0,1]$, $i=1,2,3$, is transfered from $t=0$ to $t=2$?

$$\dot{V} = \int_{\Omega(t)} (-3)t \, dx_1 dx_2 dx_3 = -3tV$$

$$V \neq 0 \Rightarrow \quad \frac{\dot{V}}{V} = -3t$$

$$\ln V(t) - \ln V(0) = -\frac{3}{2}t^2$$

$$V(t) = \underset{\overset{\parallel}{1}}{V(0)} e^{-\frac{3}{2}t^2}, \quad \boxed{V(2) = e^{-6}}$$