Linearization

$$\dot{x} = f(x), \quad f(\xi) = 0,$$

$x \in \mathbb{R}^n$

ξ is called a critical point, equilibrium, singular point

(נק' קריטית, שיווי משקל, נק' איזון)
סינגולרית

Suppose $f \in C^2$, $x \approx \xi$

Then $\dot{x} = f(\xi) + f'(\xi)(x - \xi) + O(\|x-\xi\|^2)$

$o(\|x - \xi\|)$ (if $f \in C^1$)

From the engineering point of view

$$\underbrace{\dot{z} = f'(\xi)\, z}_{\text{Linearization}}, \quad \text{where } z = x - \xi \approx 0$$

wrong!

We are interested in the solutions and not in the right-hand side estimation

Remarks Here $f(x) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \dots \\ f_n(x_1, \dots, x_n) \end{pmatrix}$

$$f'(\xi) = \frac{\partial f}{\partial x}(\xi) = \begin{pmatrix} f'_{1x_1}(\xi), & f'_{1x_2}(\xi), & \dots, & f'_{1x_n}(\xi) \\ \dots & & & \\ f'_{nx_1}(\xi), & f'_{nx_2}(\xi), & \dots, & f'_{nx_n}(\xi) \end{pmatrix}$$

Square matrix
מטריצה

Example $\ddot{x} = -\sin x$, $x \in \mathbb{R}$

$x = k\pi$ – equilibrium

In the standard form $x = x_1$, $\dot{x} = x_2$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin x_1 \end{cases} \qquad f'(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -\cos x_1 & 0 \end{pmatrix}$$

equilibria: $(k\pi, 0)$

for $\xi = (0,0)$ get $f'(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Linearization $\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = -z_1 \end{cases} \qquad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

more or less OK

for $\xi = (\pi, 0)$ get $f'(\xi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = +z_1 \end{cases} \qquad \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = c_1 \begin{pmatrix} \cosh t \\ \sinh t \end{pmatrix} + c_2 \begin{pmatrix} \sinh t \\ \cosh t \end{pmatrix}$$

is not bounded and escapes to ∞ exponentially!

$\boxed{\text{exponential rotation acceleration}}$

So what is the reliability of linearization?

Example $\dot{x} = x^2$, $x \in \mathbb{R}$

$x = 0$ equilibrium

$\dot{x} = f(x) = x^2$, $f'(0) = 0$

Linearization: $\boxed{\dot{z} = 0}$ $\quad z = \text{const}$

no fin. time escape to infinity

see p. 23 graph

Once more

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$$\begin{cases} \dot{x} = f(x), \quad f(\xi) = 0, \ \varepsilon \in \mathbb{R} \\ \qquad f \in C^2 \qquad \|v\| = 1 \\ x(0) = \xi + \varepsilon v \quad \text{Initial condition} \end{cases}$$

$$x(t,\varepsilon) = \varphi(t,\varepsilon) = \underset{\varphi(t,0)}{\xi} + \underbrace{\varphi'_{\varepsilon}(t,0)}_{z(t)}\varepsilon + O(\varepsilon^2)$$

$$\left.\begin{matrix} t \in [-T, T] \\ |\varepsilon| \le \varepsilon_M \end{matrix}\right\} \text{compact}$$

Once more $z = \varphi'_{\varepsilon}(t,0)$, $x = \varphi(t,\varepsilon)$

$$\begin{matrix} \varphi''_{t\varepsilon}(t,\varepsilon) = \\ \varphi'_{\varepsilon}(0,\varepsilon) = \end{matrix} \begin{cases} \dot{x}'_{\varepsilon} = f'(x)\cdot \varphi'_{\varepsilon}(t,\varepsilon), \quad x = \varphi(t,\varepsilon) \\ x'_{\varepsilon}(0) = v \end{cases}$$

Substitute $\varepsilon = 0$

$$\begin{cases} \dot{z} = f'(\xi)\, z \\ z(0) = v \end{cases}$$

$$x(t) = \xi + \underbrace{z(t)\,\varepsilon}_{\tilde{z}} + O(\varepsilon^2)$$

Replace: $\tilde{z} = \varepsilon z$

multiply by ε:

Obtain

$$\begin{cases} \boxed{\dot{\tilde{z}} = f'(\xi)\, \tilde{z}} \quad \text{Linearization!} \\ \tilde{z}(0) = \varepsilon v \end{cases}$$

$$x(t) = \xi + \tilde{z}(t) + O(\|\tilde{z}(0)\|^2)$$

So it is only valid over a bounded time interval for small enough $\tilde{z}(0) = x(0) - \xi$

Conservative system, Newton equation

$x \in \mathbb{R}^n, \quad \ddot{x} = F(x) = -\left(\frac{\partial U(x)}{\partial x}\right)^T$ (33)

$U : \mathbb{R}^n \to \mathbb{R}$ a potential field
(potential energy)

$\frac{\partial U}{\partial x}(x) = \nabla U = \left(\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}, \dots, \frac{\partial U}{\partial x_n}\right)$ gradient
a row vector

Kinetic energy: $\frac{1}{2}\|\dot{x}\|^2 = \frac{1}{2}\underbrace{(\dot{x}_1^2 + \dots + \dot{x}_n^2)}$

Complete energy: $E = U(x) + \frac{1}{2}\langle \dot{x}, \dot{x} \rangle$
scalar product

$\frac{d}{dt} E(x) = \frac{\partial U}{\partial x}(x)\cdot\dot{x} + \frac{1}{2}\langle \ddot{x}, \dot{x}\rangle + \frac{1}{2}\langle \dot{x}, \ddot{x}\rangle$

here $\langle x, y\rangle \overset{\text{def}}{=} x_1 y_1 + \dots + x_n y_n = x^T y$

$= \frac{\partial U}{\partial x}\dot{x} + \langle \underset{\text{column}}{\ddot{x}}, \dot{x}\rangle = \Big(\underbrace{\frac{\partial U}{\partial x} + \ddot{x}^T}_{\text{row}}\Big) x = 0$

The energy is preserved!

$\boxed{\begin{array}{l}\text{A function } I(t,x) \text{ which remains constant} \\ \text{along each trajectory } x(t) \text{ of} \\ \text{the ODE } \dot{x} = f(t,x), \text{ is called a first} \\ \text{integral. } \dot{I} = \frac{\partial I}{\partial t}(t,x) + \frac{\partial I}{\partial x}(x)\cdot f(t,x) = 0\end{array}}$

E is the first integral

Proposition Let V be bounded from below. Then each solution of $\ddot{x} = -\left(\frac{\partial V}{\partial x}\right)^T$ is extendable till $\pm\infty$ in time

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Proof Let $V \geq C$

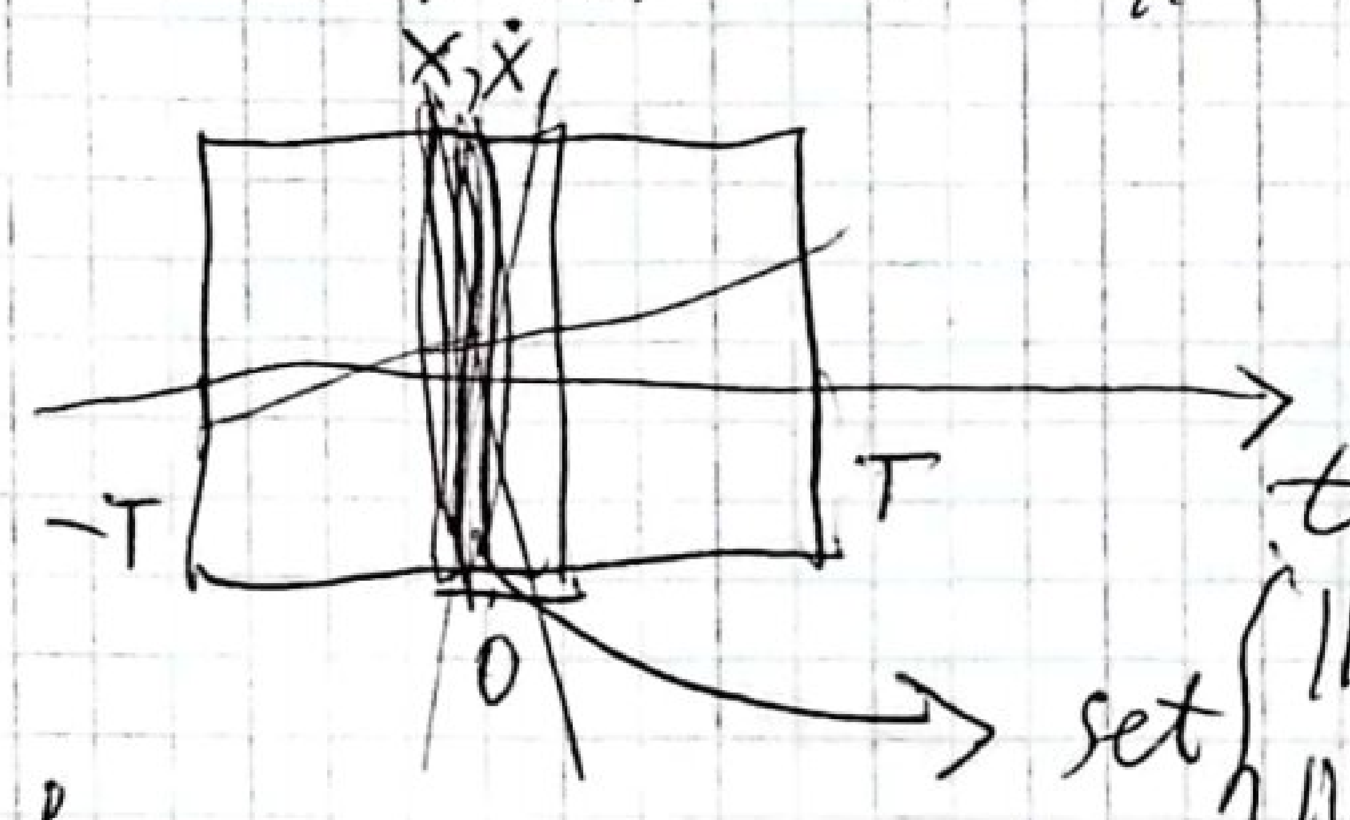
Redefine $V := V - C \geq 0$

(allowed in lectures, not in math. texts)

$$\text{const} = E = \frac{1}{2}\|\dot{x}\|^2 + \underbrace{V}_{\geq 0} \geq 0$$

$$\Rightarrow \|\dot{x}\| \leq \sqrt{2E} < \sqrt{2E} + 1$$

$$|t| \leq T \Rightarrow \|x\| \leq \|x(0)\| + \sqrt{2E}\cdot T < \|x(0)\| + \sqrt{2E}\,T + 1$$



set $\begin{cases} \|x\| \leq \|x(0)\| + \sqrt{2E}\,T + 1 \\ \|\dot{x}\| \leq \sqrt{2E} + 1 \end{cases}$

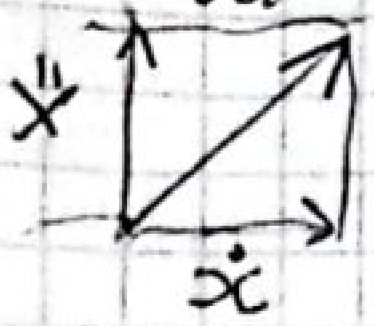
Remark

The state of the system is $(\underset{\mathbb{R}^n}{x}, \underset{\mathbb{R}^n}{\dot{x}}) \in \mathbb{R}^{2n}$

QED

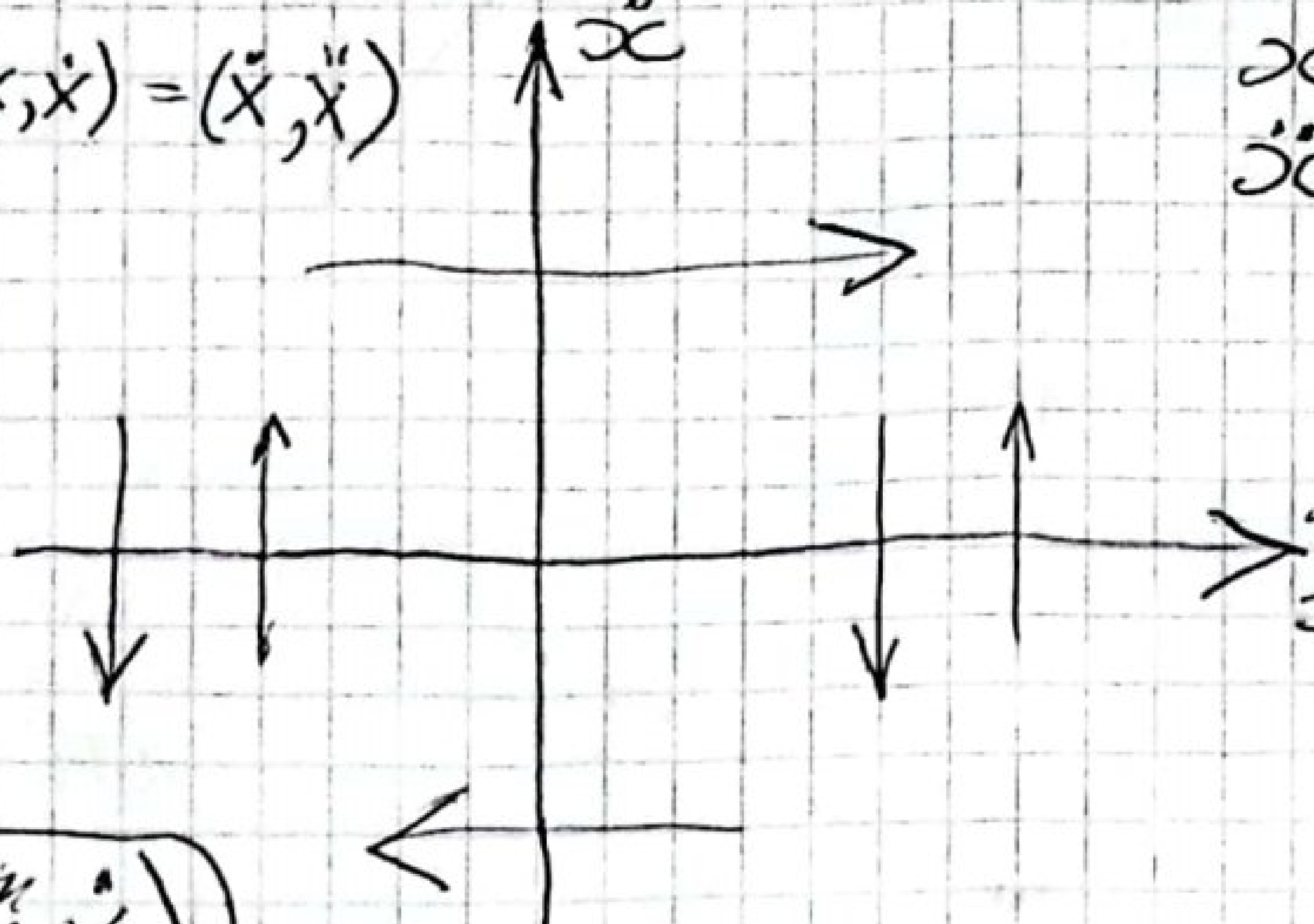
Phase Plane   שיעור כג'

$\frac{d}{dt}(x,\dot{x}) = (\dot{x},\ddot{x})$



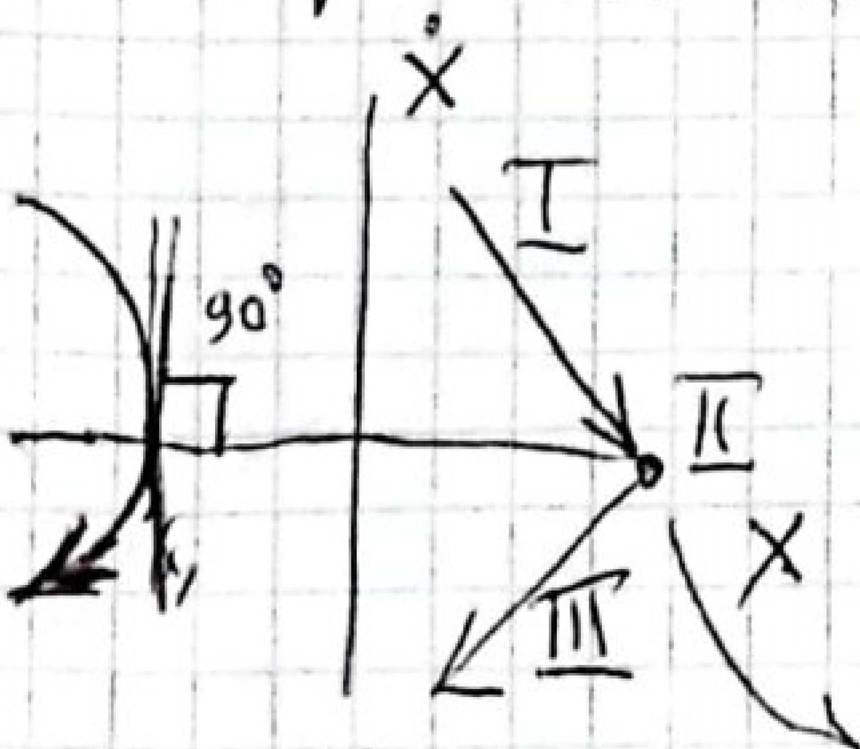
$x \in \mathbb{R}$

$\ddot{x} = f(t, x, \dot{x})$

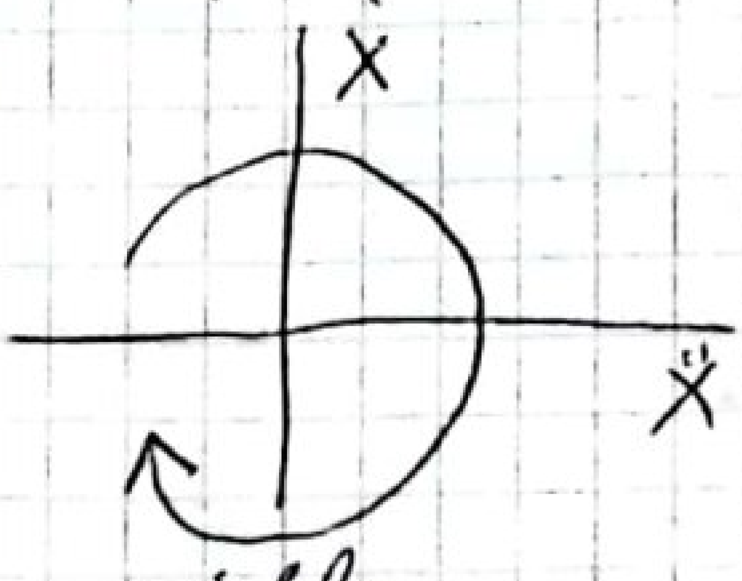


$\boxed{\ddot{x} = f(x,\dot{x})}$

In particular:



possible



possible

Impossible

3 solutions

Stop at the point with $\dot{x} = 0$

Newton Equation, $n = 1$

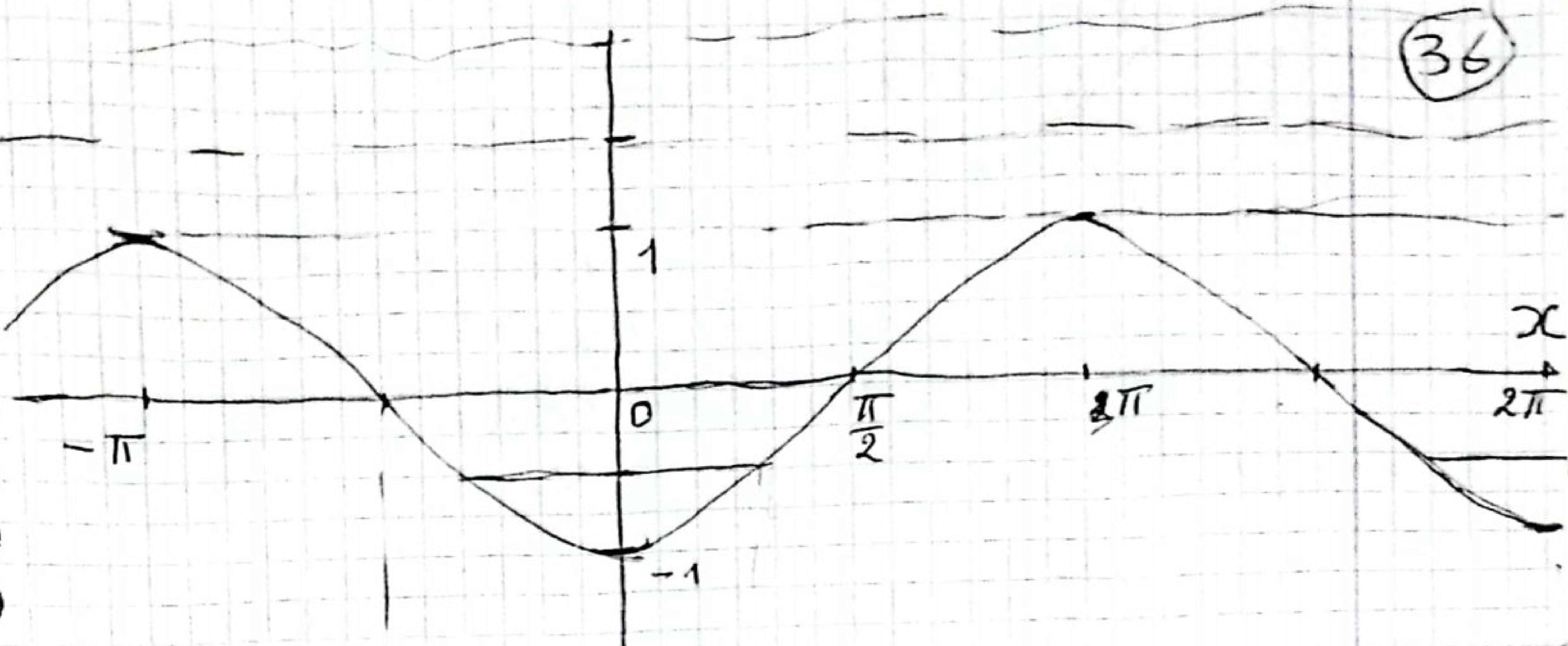
$$\ddot{x} = -\frac{\partial V}{\partial x}(x), \quad x \in \mathbb{R}, \ V: \mathbb{R} \to \mathbb{R}, \ V \in C^2$$

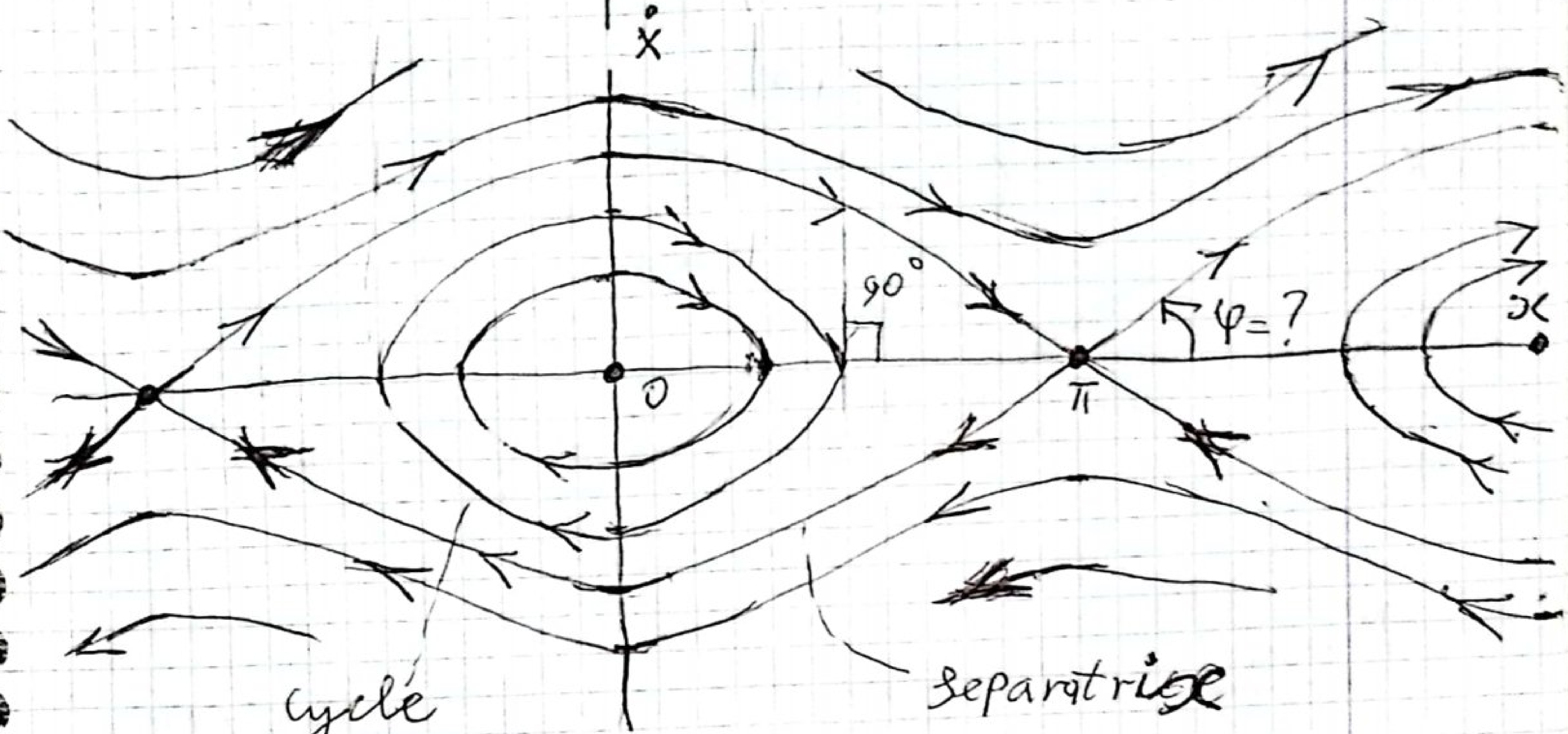
Example $\ddot{x} = -\sin x$

$$\ddot{x} = -\frac{\partial}{\partial x}(\cos x)$$

$$\Rightarrow E = \frac{\dot{x}^2}{2} - \cos x = \text{const}$$

The energy





$$\left.\begin{aligned}\frac{\partial U}{\partial x} = 0 \Rightarrow \ddot{x} = 0 \\ \dot{x} = 0\end{aligned}\right\} \Rightarrow \text{equilibrium}, \quad E = \frac{\dot{x}^2}{2} - \cos x = \text{const}$$

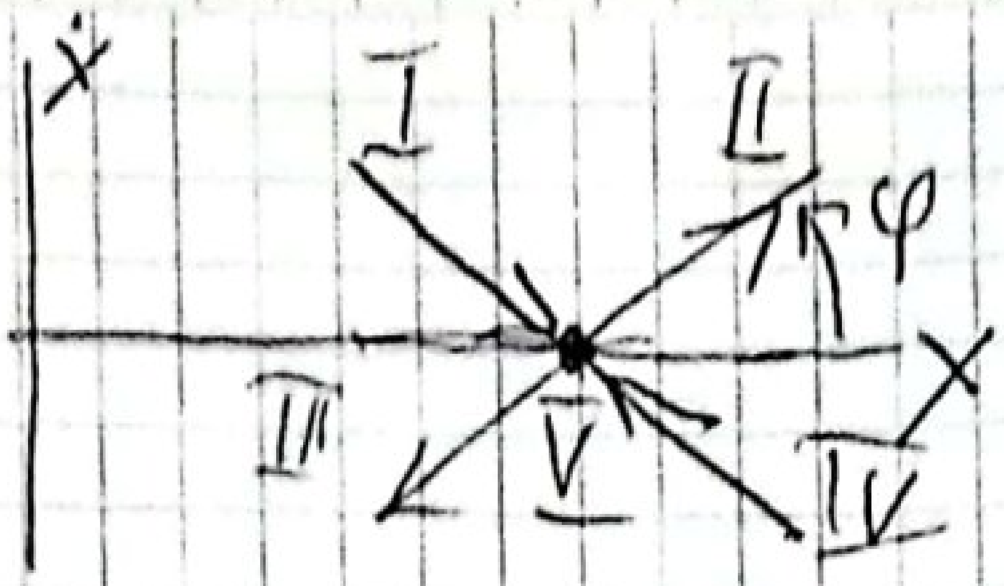
$$\Rightarrow \dot{x} = \pm\sqrt{2E + \cos x}$$

separatrix ([illegible]) connects two equilibria

cycle ([illegible]) → periodical solution

It is important to check that a trajectory passes through each point.

There is one path at each point.
and only one



There are 5 solutions (paths) here

V – $x(t) \equiv$ const
equilibrium

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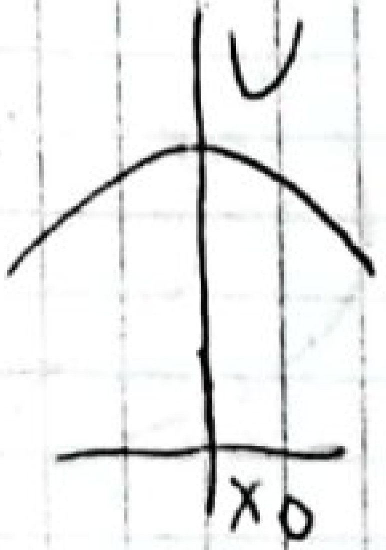
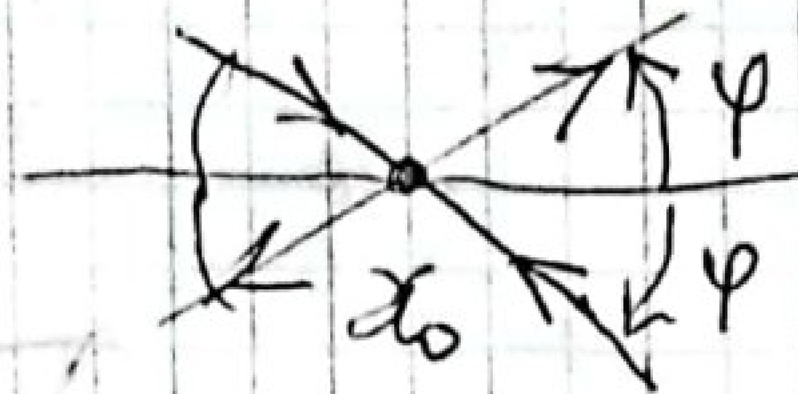
4 other enter or leave the equilibrium in infinite time

Find the angle φ (the slope)

Theorem $\ddot{x} = -\frac{\partial V}{\partial x}(x)$, $x \in \mathbb{R}$, $V \in C^2$

$V''(x_0) < 0$, $\frac{\partial V}{\partial x}(x_0) = 0 \Rightarrow \left.\begin{array}{l} x(t) \equiv x_0 \\ \dot{x}(t) = 0 \end{array}\right\}$ solution equilibrium





$\dot{x} = 0$

$$\boxed{\operatorname{tg} \varphi = \sqrt{|V''(x_0)|}}$$

Proof $V(x_0 + \Delta x) = \underset{V_0}{\underset{\|}{V(x_0)}} + \frac{1}{2} V''(x_0) \Delta x^2 + o(\Delta x^2)$

$\frac{\dot{x}^2}{2} + V(x) = E = V_0 = \text{const}$

$\cancel{\dot{x}^2} \; \dot{x} = \pm \sqrt{2V_0 - 2V(x)} = \pm \sqrt{|V''(x_0)| \Delta x^2 + o(\Delta x^2)}$

$= \pm \Delta x \sqrt{|V''(x_0)|} \left(1 + o(1)\right)^{\frac{1}{2}} = \pm \Delta x \sqrt{|V''(x_0)|} + o(\Delta x)$

$(1 + w)^{\frac{1}{2}} = 1 + \frac{1}{2} w + o(w)$, $w \to 0$

QED

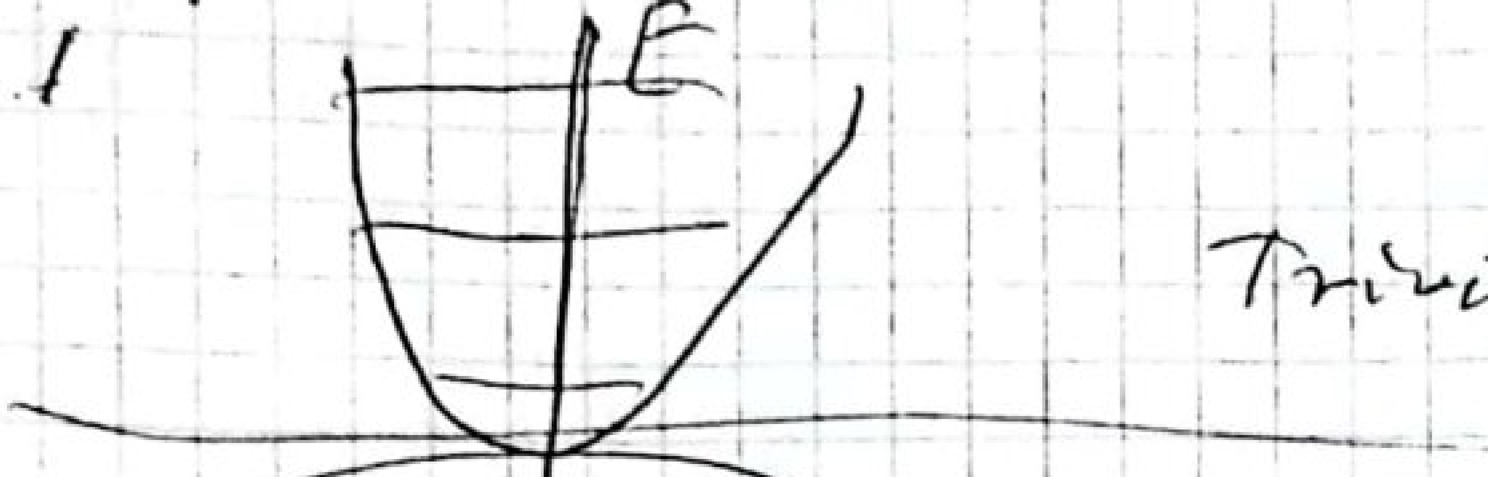
Similarly: $-V^{(k)}(x_0) \neq 0$, $V \in C^k$, $k > 2$

$V(x_0) = V_0$, $V'(x_0) = \dots = V^{(k-1)}(x_0) = 0$

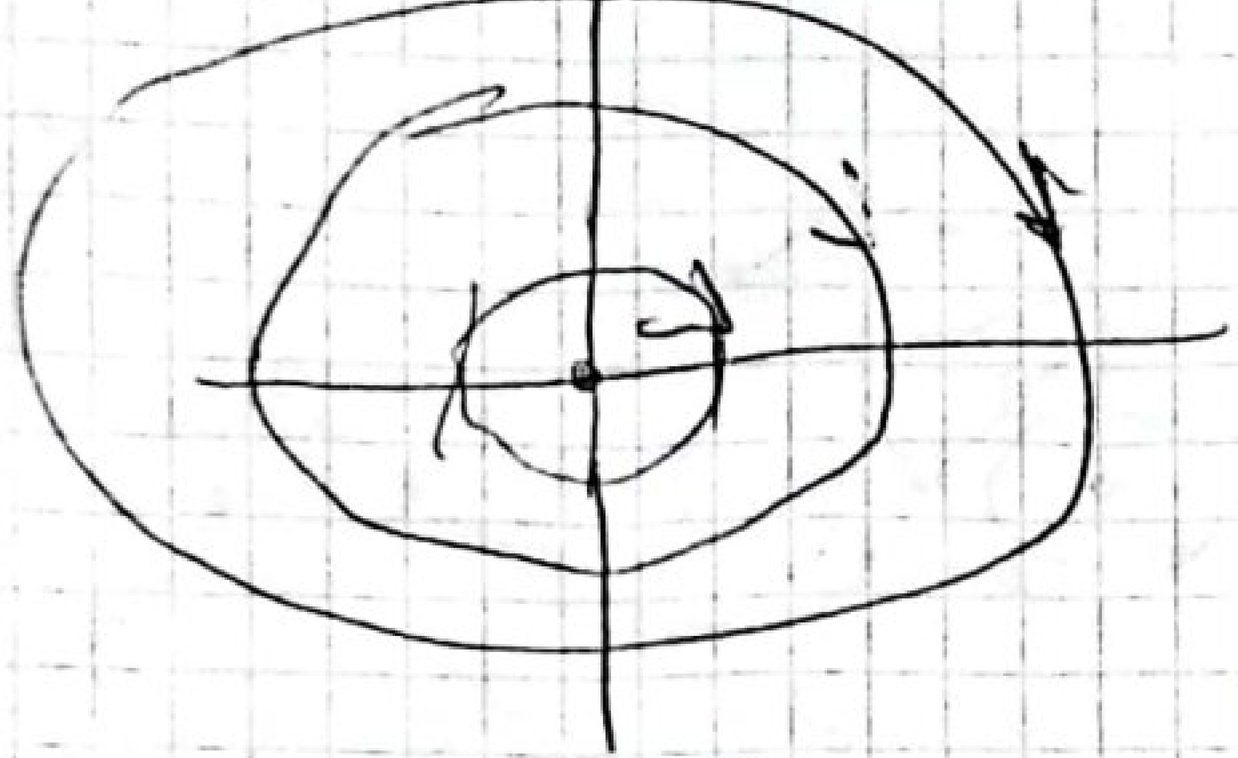
$\Rightarrow \quad \operatorname{tg} \varphi = 0 \quad \left(\dot{x} = \pm \left(\frac{1}{k!} V^{(k)} \Delta x^k + o(\Delta x^k) \right)^{\frac{1}{2}} \right)$

Examples $\ddot{x} = -x^3$, $U = +\int x^3 dx = +\frac{x^4}{4} + \cancel{c}_{\to 0}$ has no influence

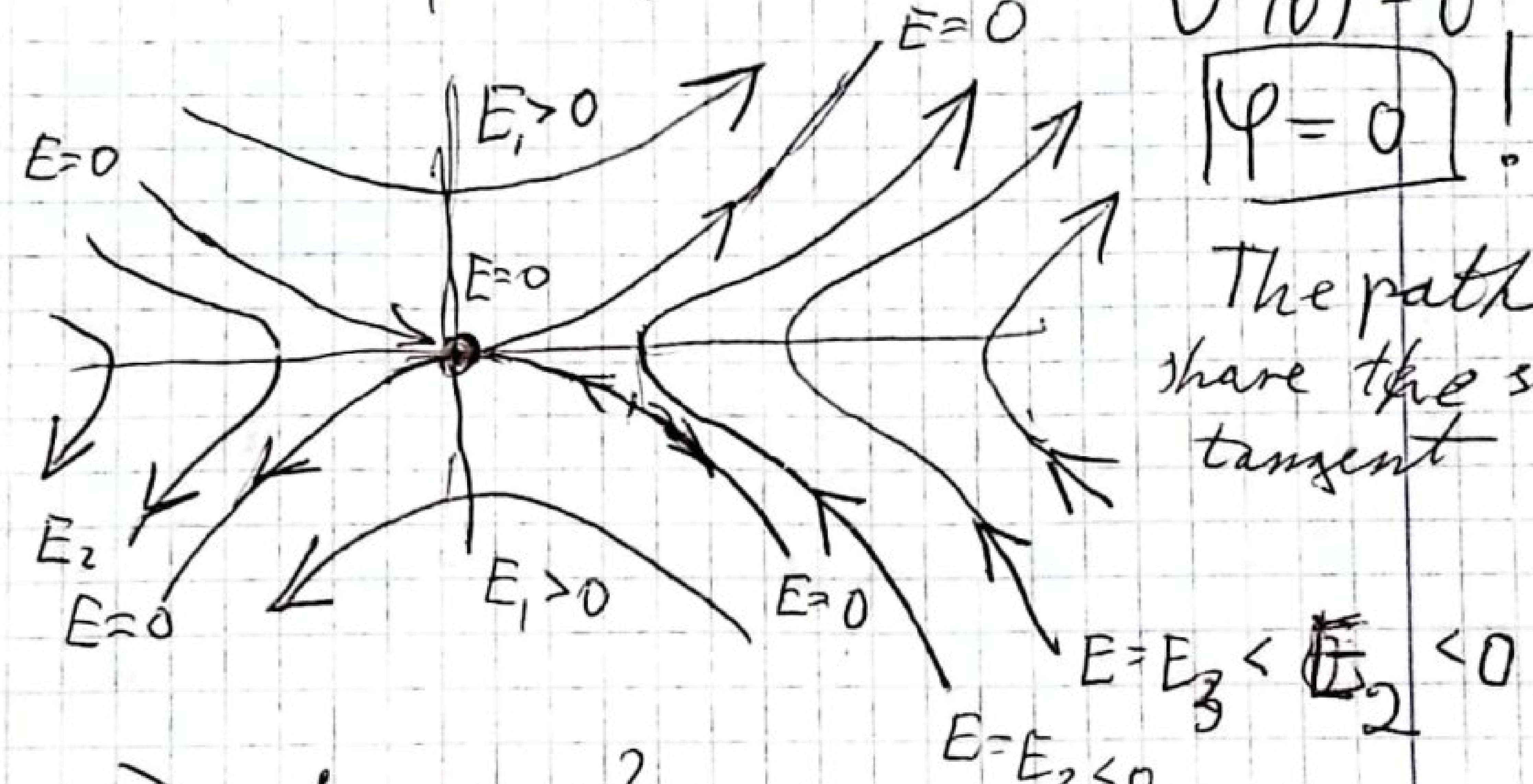
$$\frac{\dot{x}^2}{2} + \frac{x^4}{4} = \text{const} = E$$



Trivial



$$\ddot{x} = x^3, \quad U = -\frac{x^4}{4} < 0$$



$U''(0) = 0$

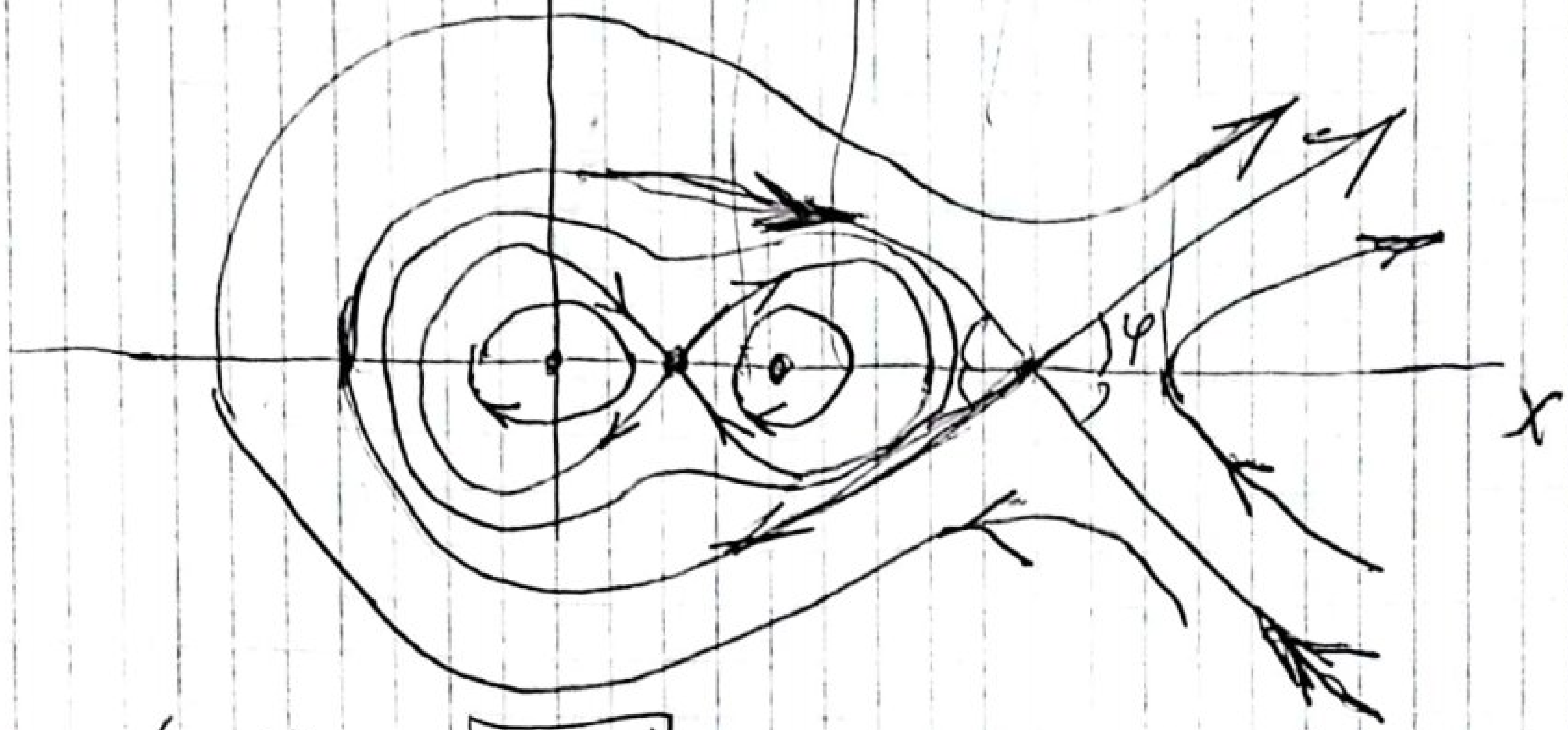
$\boxed{\varphi = 0}$!

The paths share the same tangent

$E = 0 \Rightarrow \dot{x} = \pm \frac{x^2}{2} \Rightarrow$

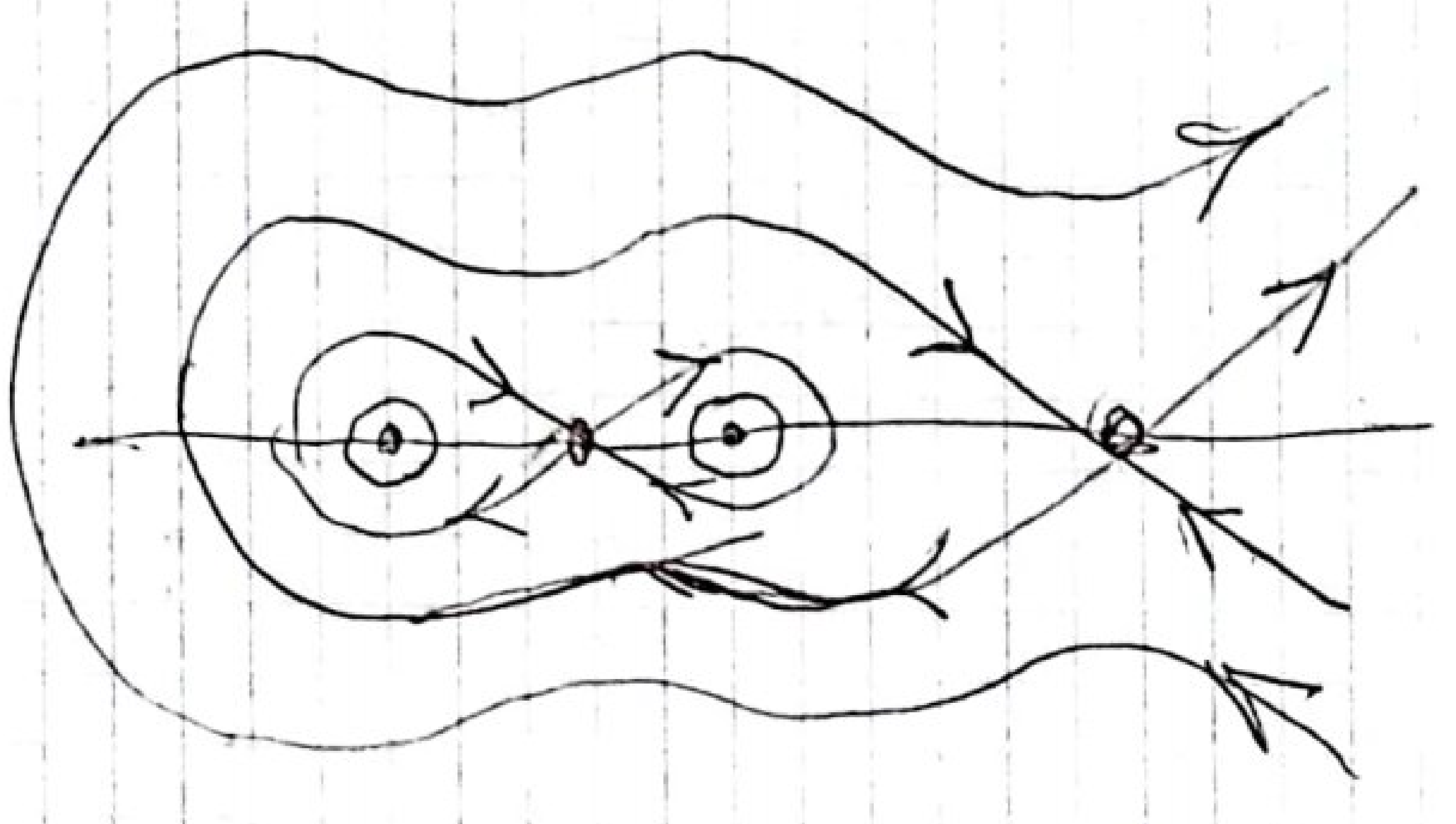
Solutions come from infinity in finite time and in finite time escape to infinity!

Example



$tg\,\varphi = \sqrt{|U''(x_0)|}$, one needs formula of $V(x)$

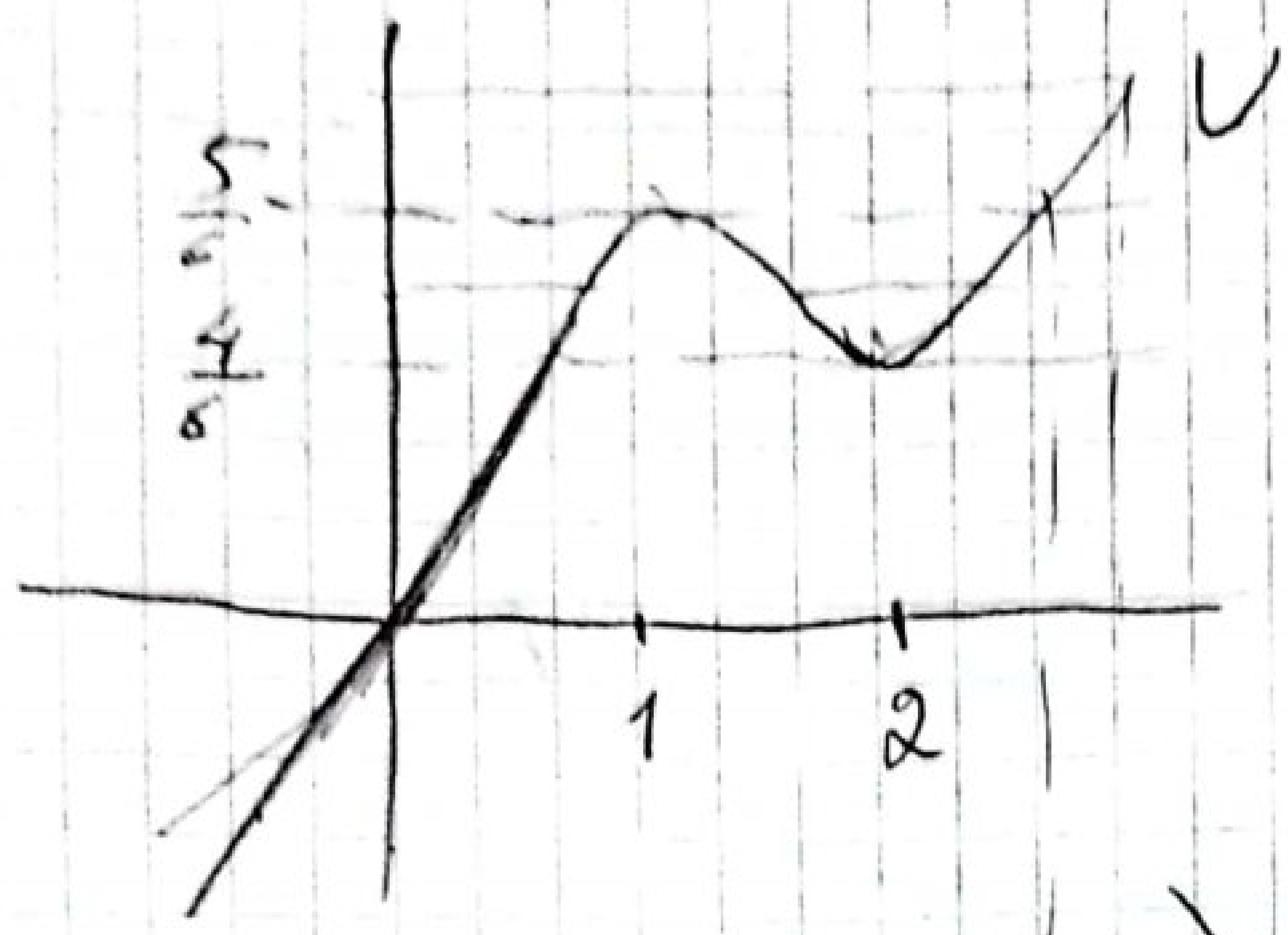
Once more:

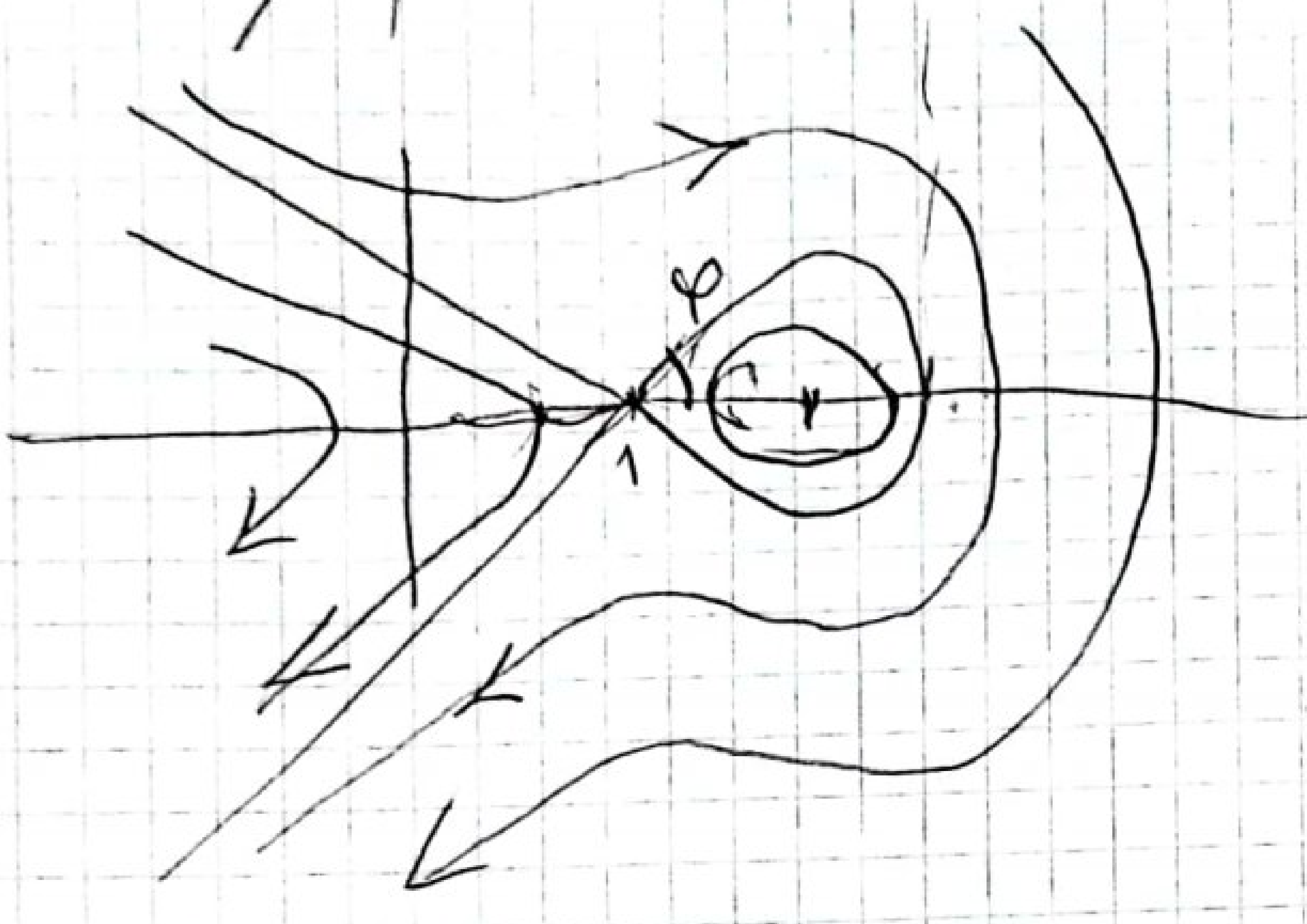


Example $\ddot{x} = -x^2+3x-2 = -(x-2)(x-1)$

$$U = \int (x^2-3x+2)\,dx = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x + c \to 0$$

$$U(1) = \frac{1}{3} - \frac{3}{2} + 2 = \frac{5}{6}, \quad U(2) = \frac{8}{3} - 6 + 4 = \frac{2}{3} = \frac{4}{6}$$





$$tg\,\varphi = \sqrt{|U''(1)|} = \sqrt{|(2x-3)|}\Big|_{x=1} = \sqrt{1} = 1$$

$$\varphi = 45^0$$

Linear systems

$\dot{x} = A(t)x + B(t)$ not homogeneous

$\dot{x} = A(t)x$ homogeneous

$x \in \mathbb{R}^n$, A, B : continuous, $A: \mathbb{R} \to \mathbb{R}^{n\times n}$, $B: \mathbb{R} \to \mathbb{R}^n$

Proposition
Each solution is extendable to $t \in (-\infty, \infty)$

Proposition
Obviously solutions of a homogeneous DE constitute a linear space

$$\dot{X}_1 = A(t)X_1, \quad \dot{X}_2 = A(t)X_2$$

$$\Rightarrow (\lambda X_1 + \mu X_2)^\circ = A(t)(\lambda X_1 + \mu X_2)$$

capital letter

Example $\left\{ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right.$

Solutions $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t}, \quad c_1, c_2 \in \mathbb{R}$

Proposition $\dot{x} = A(t)x + B(t)$, $\dot{x}_p = A(t)x_p + B(t)$

Each solution has the form

$$x(t) = x_h(t) + x_p(t)$$

where x_h satisfies $\dot{x}_h = A(t)x_h$

Moreover, let X, X_h – sets of all solutions of ~~the equat~~ $\dot{x} = A(t)x + B(t)$ and $\dot{x} = A(t)x$ respectively

Then $$X = X_h + x_p$$

Proof

$$\begin{array}{r} \dot{x} = A(t)x + B(t) \\ - \quad \dot{x}_p = A(t)x_p + B(t) \\ \hline (x - x_p)' = A(t)(x - x_p) \end{array}$$

QED.

Examples: $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{R}$$

Proposition $\dot{x} = A(t)x$, $x \in \mathbb{R}^n$

$$\Rightarrow \dim X_h = n$$

Proof! Init. cond.: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow$ solutions $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$

Then if $x(0) = \xi \in \mathbb{R}^n$, $\xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} \in \mathbb{R}^n$
the corresponding solution: $x(t) = \xi_1 \varphi_1(t) + \dots + \xi_n \varphi_n(t)$

QED

Moreover: $\varphi_1, \varphi_2, \dots, \varphi_n$ – basis

Indeed, suppose they linearly dependent

$$\lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \dots + \lambda_n \varphi_n = 0$$

$\Rightarrow \forall t \quad \lambda_1 \varphi_1(t) + \lambda_2 \varphi_2(t) + \dots + \lambda_n \varphi_n(t) = 0$

$t = 0 \Rightarrow \quad \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = 0 \qquad$ QED

<u>Definition</u> ~~Any~~ basis of X_h

is called fundamental solutions

(פתרונות יסודיים)

<u>Proposition</u> $\varphi_1, \dots, \varphi_n : \mathbb{R} \to \mathbb{R}^n$

are fundamental solutions $\Leftrightarrow$

they are linearly independent at each $t \in \mathbb{R}$

or even <u>some</u> $t_0 \in \mathbb{R}$

~~Tri~~

<u>Proof</u> Suppose linearly dependent

$\Rightarrow \lambda_1 \varphi_1 + \dots + \lambda \varphi_n = 0 \Rightarrow \lambda_1 \varphi_1(t_0) + \dots + \lambda_n \varphi_n(t_0) = \varphi(t_0) = 0$

But $x(t) \equiv 0$ – solution

and $\lambda_1 \varphi_1(t) + \dots + \lambda_n \varphi_n(t)$ – is a solution

$\varphi(t_0) = 0 \Rightarrow \varphi(t) \equiv 0$ (Theorem of $\forall \exists !$)

<u>Conclusion</u>: $\varphi_1, \dots, \varphi_n$ – fundamental solutions

$\Leftrightarrow \forall t \in \mathbb{R} \quad \det(\varphi_1(t), \dots, \varphi_n(t)) \neq 0$