Lecture 2 17/10-2021

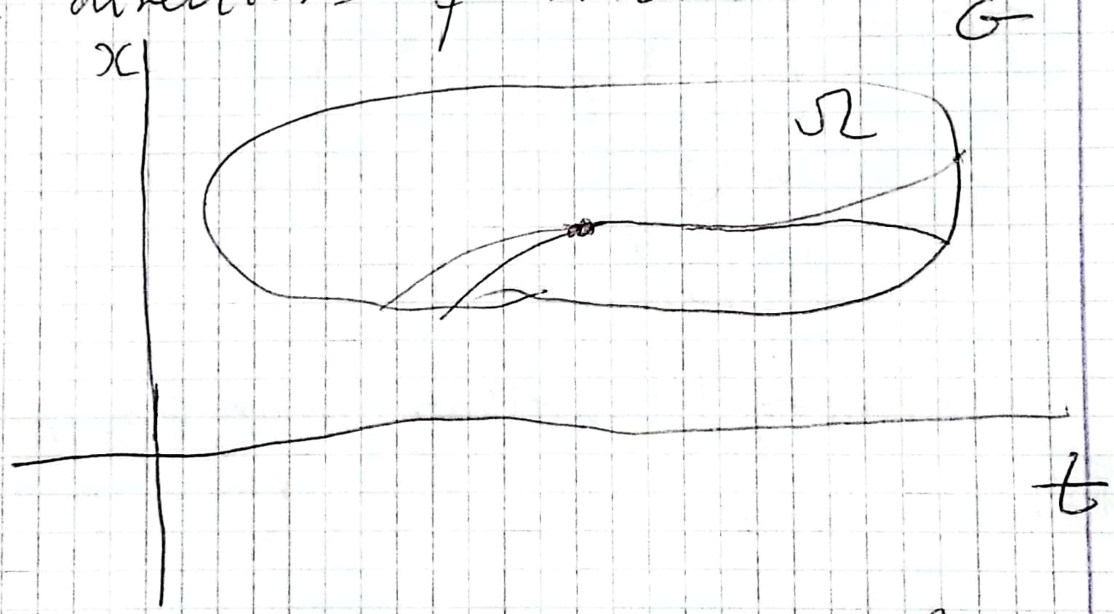
Extension of solutions till the boundary of a compact region

Suppose $\dot{x} = f(t, x)$, $(t, x) \in G$ open region

$\oint x(t_0) = \xi \in \Omega \subset G$ compact

~~At~~ each point of G the conditions of $\exists$ Theorem hold (f, continuous in vicinity)

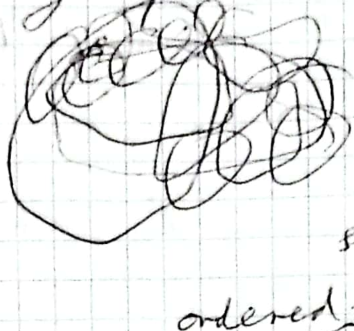
Theorem 1, Then each solution of the Cauchy problem is extended till the boundary of Ω in the both directions of time



~~Theorem~~ 2. If also the local Lipschitz property holds in G then the extension is unique

Recall: compactness in complete metric spaces

1. Ω is called compact if each its cover by open sets has a finite subcover

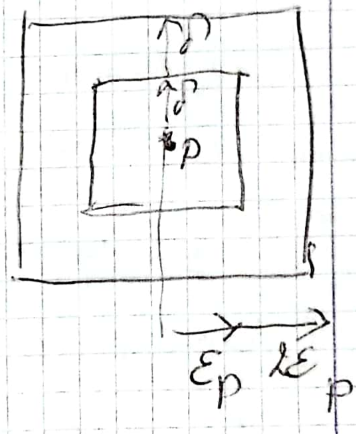


— one can always choose a finite subcover: finite set of open sets covering Ω

2. Each infinite ordered subset (sequence) of points from Ω has a subsequence converging to some point of Ω
(if the limit point ~~does no~~ is not required to belong to Ω $\Rightarrow$ called precompact set)

3. Ω is a bounded closed set

Proof of the theorem



Choose a vicinity

$$p = (t_*, x_*) \in \Omega$$

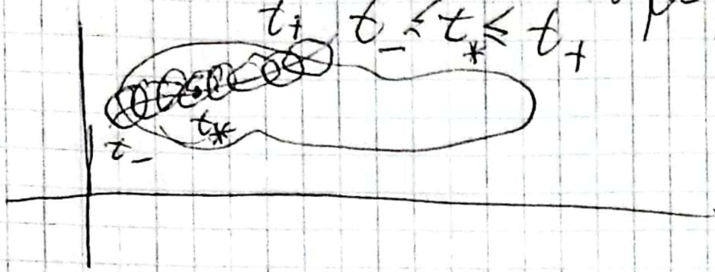
for each p ~~solution starting~~ ~~point~~ from $|t - t_*| < \varepsilon$, $\|x - x_*\| < \delta$

exists a solution defined over the interval $\tau_p = \min\left\{\varepsilon_p, \frac{\delta}{M_p}\right\}$, M_p — local $\max \|f\|$

$\Rightarrow$ finite subcover $\Rightarrow$ $\tau = \min_i \tau_{p_i}$

$\Rightarrow$ At each point one can extend the solution by τ forwards or backwards

Ω – bounded $\Rightarrow$ finite number of steps



$t_- \le t_* \le t_+$

$$t_+ = \sup \left\{ t \ge t_* : x([t_*, t]) \subset \Omega \right\}$$

$$t_- = \inf \left\{ t \le t_* : x([t, t_*]) \subset \Omega \right\}$$

2. Uniqueness – exactly like in the previous lecture, $L = \max_i \{ L_{D_i} \}$, $D_i, i \in N$

QED

Autonomous case! Theorem

Each solution is extendable in each direction of time till $\pm$ infinity or till the boundary

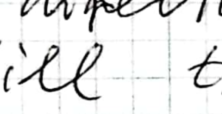
Solution of an autonomous system can be periodic, or constant, etc.

The peculiarity $\quad \dot{x} = f(x)$, $x(t_0) = \xi \in \mathbb{R}^n$

Lipschitz property $\|f(x_1) - f(x_2)\| \le L \|x_1 - x_2\|$

Each solution $x_1(t)$, $x_1(t_0) = \xi$

can be transformed into $x_0(t) = x_1(t - t_0)$

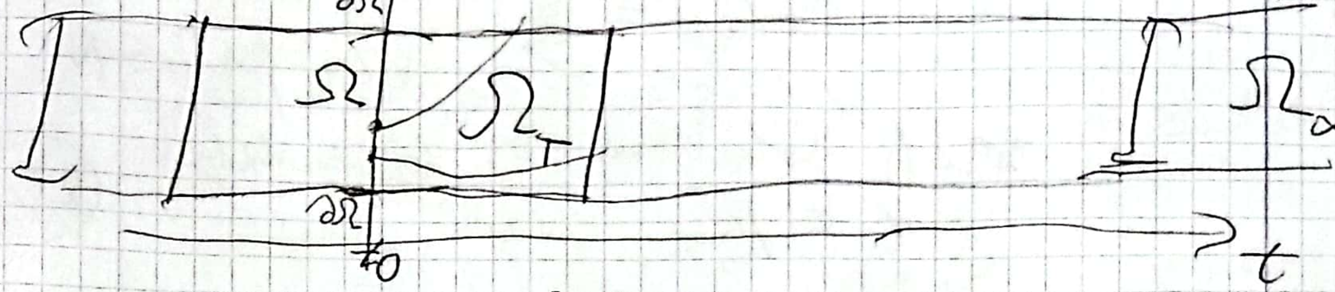


$x_0(0) = \xi$

Only one solution path passes through one point

Proof of the theorem Ω compact

~~Let~~



$T > 0$, Ω_T: $\{(t,x) : x \in \Omega, |t - t_0| \leq T\}$

$$\Omega_\infty = \{(t,x),\ x \in \Omega,\ t \in \mathbb{R}\}$$

Each solution can be extended till the boundary of Ω_T (compact)

If $\exists T > 0$ ~~these~~ solution leaves Ω_T through the boundary of Ω then it has been extended till $\partial\Omega$

$\partial\Omega$ – boundary of Ω (standard notation)

If $\forall T \geq 0$ solution is extendable till it leaves Ω_T at $t = t_0 + T$ then it is extended till $+\infty$ in time

$t = -\infty$ – similarly. QED

Each set $\Omega = \partial\Omega \cup \mathrm{Int}\,\Omega$
(not only compact) Interior

$\partial\Omega = \{$points $x \in \Omega$ such that each its vicinity contains points not belonging to $\Omega\}$

$\mathrm{Int}\,\Omega = \{$points $x \in \Omega$ for which there is an open vicinity $\subset \Omega$ nonempty$\}$

Important conclusion

Each solution is defined by its initial conditions in the case when the conditions of the $\exists!$ Theorem hold (continuity + Lipshitz property).

1. It means that in the case of the standard system/ODE

$$\dot{x} = f(t, x), \quad x \in \mathbb{R}^n$$

The general solution (the phase stream) alway contains ~~exactly~~ $\boxed{n}$ free parameters which actually determine the initial value $x(t_0)$

$F \in C$, Lipschitz with respect to $y, \dots, y^{(n-1)}$

2. In the case of a high-order DE

$$y^{(n)} = F(t, y, \dot{y}, \dots, y^{(n-1)}), \quad y \in \mathbb{R}$$

the general solution is also defined by $\boxed{n}$ free scalar parameters ~~defining~~ actually determining the initial values

$$y(t_0), \dot{y}(t_0), \ddot{y}(t_0), \dots, y^{(n-1)}(t_0)$$

Example: $\dot{x} = a(t,x)$, $a: \mathbb{R}^{n+1} \to \mathbb{R}^n$

$x \in \mathbb{R}^n$

$x(t_0) = \xi$ $\quad \forall (t,x)\ \|a\| \leq M$

$\Rightarrow$ each solution is extendable to $t \in \mathbb{R}$ (20)

Proof: ~~Exactly similar~~

$\forall T$: $\|x(t)\| \leq MT + \|\xi\|$ (Lagrange theorem in each component)

$t \in [t_0 - T, t_0 + T]$

$\Omega_T \overset{def}{=} \left\{ \begin{array}{l} t, x \in \mathbb{R}^{n+1} \\ \|x\| \leq MT + \|\xi\| + 1 \\ t \in [t_0 - T, t_0 + T] \end{array} \right.$

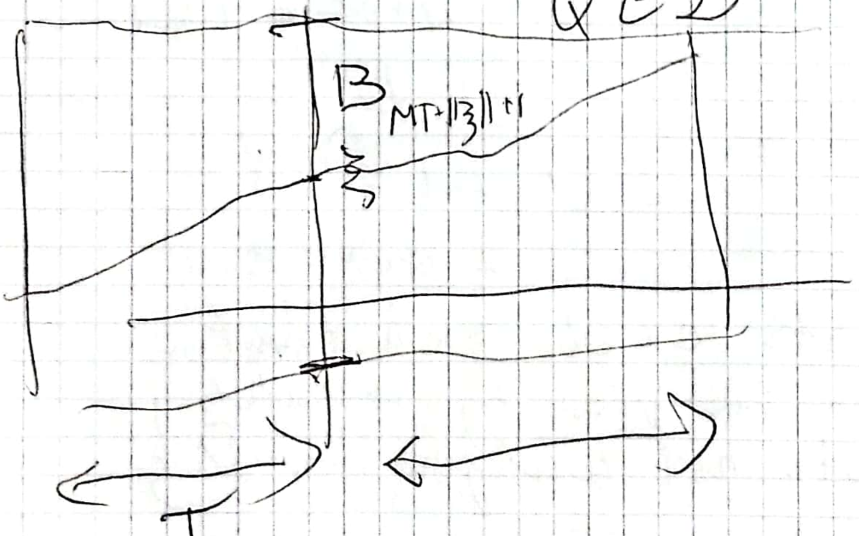
no solution can leave Ω_T through the boundary of the ball

$B_{MT + \|\xi\| + 1} = \{x, \|x\| \leq MT + \|\xi\| + 1\}$

But it is extendable till ~~the~~ $\partial \Omega_T$

$\Rightarrow$ it is extendable till $t_0 + T$, $t_0 - T$

QED



cylinder $\delta' \delta\varepsilon$

$x(t_0)=\xi$

Example: $\dot{x} = f(t,x)$, $f: \mathbb{R}^{n+1} \to \mathbb{R}$ continuous $a \in C$

$\exists\, \alpha(t): \ \forall t, x: \ \|f(t,x)\| \le \cancel{\alpha\|x\| +}\, \beta(t)$

$\le \alpha(t)\|x\| +$ continuous, $\beta \ge 0$

$\alpha \ge 0$

Then each solution can be extended till $\pm$ infinity

$$\|f(t,x)\| \le \alpha(t)\|x\| + \beta(t)$$

$\alpha, \beta: \mathbb{R} \to \mathbb{R}$ nonnegative continuous

Proof: over $[t_0 - T, t_0 + T]$

any solution satisfies

$$\|x(t)\| = \left\|\xi + \int_{t_0}^{t} \dot{x}(s)\, ds\right\| \quad (t \ge t_0)$$

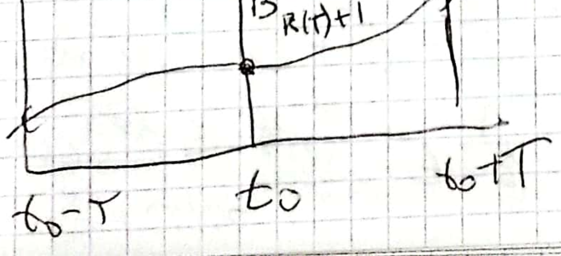
$$\|x(t)\| \le \|\xi\| + \int_{t_0}^{t} \alpha(t)\|x(s)\|\, ds + \int_{t_0}^{t} \beta(s)\, ds$$

$$\|x(t)\| \le \underbrace{\|\xi\| + \max_{[t_0-T,\, t_0+T]} \beta(s)\,(t-t_0)}_{b(t)} + \int_{t_0}^{t} \underbrace{\max_{[t_0-T,\, t_0+T]} \alpha(s)}_{a(t)} \|x(s)\|\, ds$$

$\Rightarrow$ ~~$\|x(t)\| \le \|\xi\| + \max_{s\in[t_0-T,\, t_0+T]} \beta(s)$~~

$$\|x(t)\| \le \underbrace{b(T)\, e^{a(T)|t-t_0|}}_{R(t)}$$ Lemma (Grönwall-Bellman)

Further the proof is the same!



QED.

Lemma Grönwall-Bellman

(Without Proof) $u(t) \le \gamma(t) + \int_{t_0}^{t} \omega(s) u(s) ds$

$t_0 \in [t_a, t_b]$, $u, \gamma, \omega \in C[t_a, t_b]$ (continuous) 22

γ - monotonously grows

Then $u(t) \le \gamma(t) \exp\left(\int_{t_0}^{t} \omega(s) ds\right)$

Corollary $\dot{x} = A(t)x + B(t)$

$x \in \mathbb{R}^n$, A, B – continuous

$A: \mathbb{R} \to \mathbb{R}^{n \times n}$, $B: \mathbb{R} \to \mathbb{R}^n$

Each solution is extendable to $(-\infty, \infty)$

Moreover

$$\left\| \left(A(t)x_1 + B(t)\right) - \left(A(t)x_2 + B(t)\right) \right\| \le \underbrace{\|A(t)\|}_{\text{locally bounded (from continuity)}} \|x_1 - x_2\|$$

Lipschitz property!

$\Rightarrow$ the solution is also unique

Example $\dot{x} = x^2$ does not satisfy the above conditions

$x(t) \equiv 0$ solution

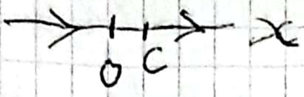
$\frac{\dot{x}}{x^2} = 1$, $x \neq 0$

$-\left(\frac{1}{x}\right)^{\cdot} = 1 \quad \int_0^t \Rightarrow \quad x(t)^{-1} = -t + C$ $C > 0$

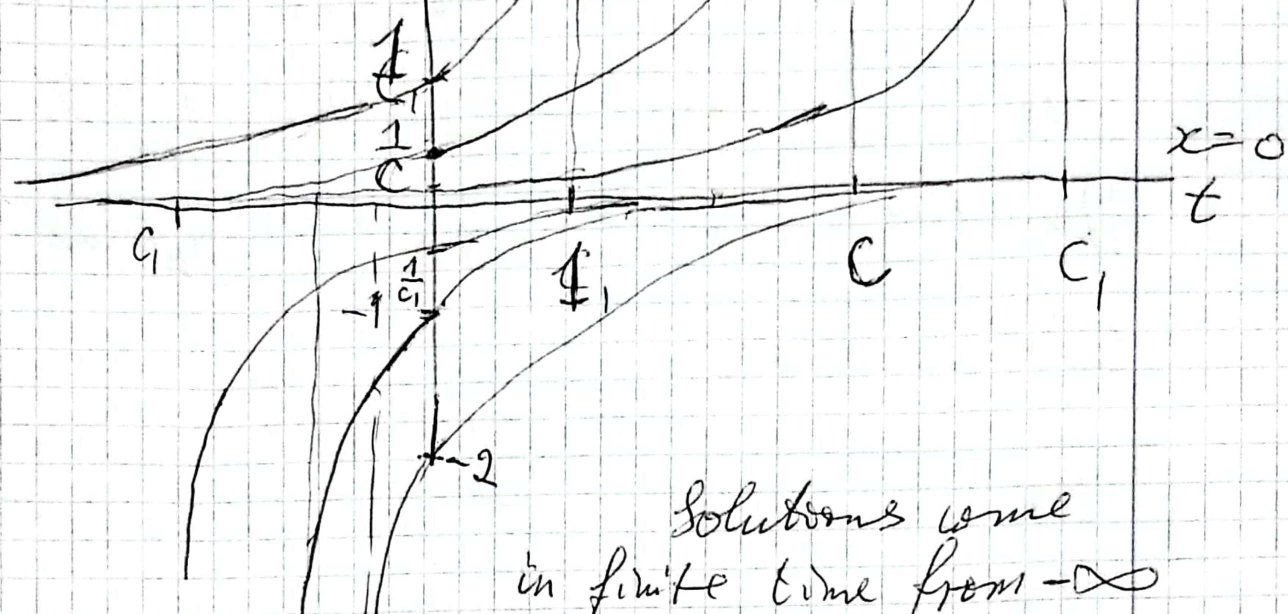
Solutions: $x(t) = \frac{1}{C - t}$, $t \le C$, $C \in \mathbb{R}$

$x(t) \equiv 0$ $\quad x(0) = C$

$\dot{x} = x^2$



$x(0) = c$



Solutions come in finite time from $-\infty$ and escape in finite time to ∞

Each solution can be extended ~~t~~ $(\frac{1}{c}, \infty)$ for $c < 0$

and $(-\infty, \frac{1}{c})$ for $c > 0$

$x(t) \equiv 0$ for $c = 0$ $t \in \mathbb{R}$

It is called $\boxed{\text{Finite-Time Escape}}$

__Example__ Can $y_*(t) = t^2 + t^5$, $y_{**} = t^2$ satisfy the equation

a) $y^{(4)} = f_1(t, y, \dot{y}, \ddot{y}, \dddot{y})$

f_1 locally Lipschitzian in $y, \dot{y}, \ddot{y}, \dddot{y}$ and continuous

b) $f_2 \in C$ $y^{(5)} = f_2(t, y, \dot{y}, \ldots, y^{(5)})$

locally Lipschitzian in $y, \ldots, y^{(5)}$

a) Impossible, Theorem $\exists!$ holds

for ~~y~~ The initial conditions at $t=0$ (24)

are the same for y_*, y_{**}:

$y_{**}=t^2$, $y_{**}(0)=0$, $\dot{y}_{**}(0)=0$, $\ddot{y}_{**}=2$

$y_* = t^2+t^5$ $y_*(0)=0$, $\dot{y}_*(0)=0$, $\ddot{y}_*=2$

b) possible

y_*: $y_*(0)=0, \underbrace{\dots}_{\dot{y}_*(0)=0,\ \ddot{y}_*(0)=2,\ \dddot{y}_*(0)=0,}\ y_*^{(4)}(0)=0$, $y_*^{(5)}(0)=120$

y_{**}: $y_{**}(0)=0$, $\dot{y}_{**}(0)=0$, $\ddot{y}_{**}=2$, $y_{**}^{(i)}(0)=0$ $i>2$

Example: $y^{(6)}=0$

Solutions: $y=c_0+c_1t+\dots+c_5t^5$

Example Can the equation

$$y^{(4)}=y+\arctan(\dot{y}+\sin t)$$

have the solution $y=\cos t+(t-1)^5$?

Answer: Impossible, $\exists!$ Theorem holds here, for $\arctan(\cdot)$ is cont. differentiable

$y=\cos t$ is a solution, since

$$(\cos t)^{(4)}=\cos t+\arctan(-\sin t+\sin t)$$

but $\cos t$ and $\cos t+(t-1)^5$ have the same initial conditions at $t=1$

Example: Can equation

$$\dot{x} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

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$A: \mathbb{R} \to \mathbb{R}^{2\times 2}$, continuous in t,

have the solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \sin t \\ t \end{pmatrix}$

Answer: Impossible! $\exists!$ Theorem holds

$x(t) \equiv 0$ – solution

but $x(0) = 0$, $x(t) \not\equiv 0$ $\Big| \Rightarrow$ impossible contradiction!

Dependence of solutions on parameters and the right-hand side

~~Theorem~~ $\begin{cases} \dot{x} = f(t, x), & f \in C^k \\ x(t_0) = \xi & \end{cases}$ (k times continuously differentiable in t, x)

We know that the local solution exists and is unique

$$x(t) = \varphi(t, t_0, \xi)$$

Theorem $\varphi \in C^k$ in t, t_0, ξ !

Proof: Arnold

It is easy to see that it is also true for solutions extended to a segment in time (26)

parameters:

Theorem

$$\begin{cases} \dot{x} = f(t, x, p) \\ x(t_0) = \xi \end{cases}$$

$x \in \mathbb{R}^n$, $p \in \mathbb{R}^m$, $f \in C^k$ in t, x, p, $k = 0, 1, \ldots$

$\Rightarrow$ $x(t) = \Phi(t, t_0, \xi, p)$, $\Phi \in C^k$ as well

Proof:

$$\begin{cases} \dot{x} = f(t, x, p) \\ \dot{p} = 0 \\ x(t_0) = \xi \\ p(t_0) = p_0 \end{cases}$$

QED.

Dependence on the right-hand side

$$\dot{x} = f(t, x) + w(t, x), \quad \|w(t,x)\| \le \varepsilon$$

$$x(t_0) = \xi + \eta \qquad \|\eta\| \le \varepsilon$$

$$t \in [t_\alpha, t_\beta], \quad t_0 \in [t_\alpha, t_\beta]$$

Solutions exist

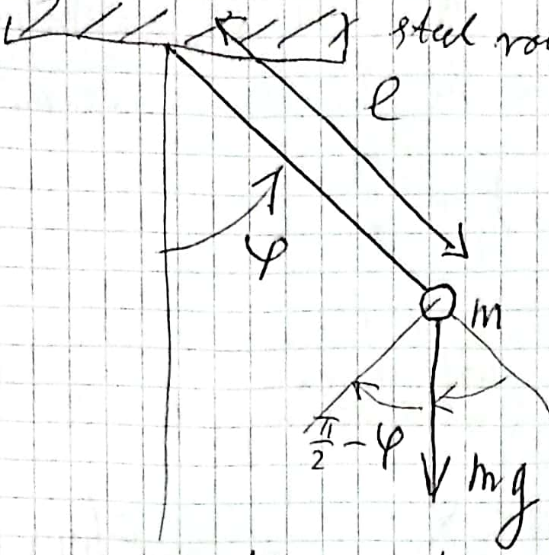
Let x_* be the unique solution of the equation for $\varepsilon = 0$

Then $x(t) \underset{\varepsilon \to 0}{\rightrightarrows} x_*$ uniformly converge as

$$\lim_{\varepsilon \to 0} \max_{t \in [t_\alpha, t_\beta]} \|x(t) - x_*(t)\| = 0$$

Without proof

Example Pendulum



steel rod, load mass m, $g = 9.81\,\frac{m}{s^2}$

$$m\ell^2\ddot{\varphi} = -mg\ell\sin\varphi$$

Inertion moment — rotation moment

no friction

$$\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi$$

$$\sqrt{\frac{\ell}{g}}\,\frac{d}{dt} = \frac{d}{d\tau} \Rightarrow \tau = t\cdot\sqrt{\frac{g}{\ell}}$$

time transformation

$$\dot{\varphi} = \sqrt{\frac{g}{\ell}}\,\varphi'_{\tau}$$

$$\ddot{\varphi} = \frac{g}{\ell}\,\varphi''_{\tau\tau}$$

$$\boxed{\varphi''_{\tau\tau} = -\sin\varphi}$$

$\ddot{x} = -\sin x$ is called mathematical pendulum (Arnold)

Equilibria: $x = k\pi$, $k \in \mathbb{Z}$ integers

Add the disturbance in initial conditions

$$\begin{cases} \ddot{x} = -\sin x \\ x(0) = a, \ \dot{x}(0) = a, \ a \approx 0 \end{cases}$$

$$x(t,a) = \Phi(t,a), \quad \Phi(t,0) \equiv 0, \quad \Phi \in C^\infty$$

$$x(t,a) = \Phi(t,0) + \Phi'_a(t,0)\,a + \frac{1}{2}\Phi''_{aa}(t,\theta a)\,a^2$$

The Taylor formula for each t: $\theta = \theta(t)$, $\theta \in (0,1)$

Let $|t| \le T$, $|a| \le a_M$ $\rightarrow$ compact

$\Rightarrow$ Φ''_{aa} bounded on the compact

Thus $x(t,a) = 0 + \varphi'_a(t,0)\cdot a + O(a^2)$ (28)

$|\varphi''_{aa}| \le M$, $|\frac{1}{2}\varphi''_{aa}(t,\theta a)a^2| \le \frac{1}{2}Ma^2$

Let us find x in the first approximation

$$z(t) = \varphi'_a(t,0) = ?$$

The system
$$\begin{cases} \ddot{x} = -\sin x \\ x(0) = a, \ \dot{x}(0) = a \end{cases}$$
$x(0)=\varphi(0,a)$, $\dot{x}(0) = \varphi'_t(0,a)$

differentiate with respect ~~according~~ to a

$$\frac{\partial^3}{\partial a \partial^2 t}\varphi = \ddot{x}'_a = -\cos x \cdot x'_a$$

chain rule

ჯაჭვის წესი

$$\varphi'_a(0,a) = x'_a(0) = 1, \quad \dot{x}'_a(0) = 1$$
$$\dot{x}'_a(0) = \varphi''_{ta}(0,a)$$

Substitute $a=0 \Rightarrow z(t) = \varphi'_a(t,0)$

$$x\big|_{a=0} = x(t,0) = \varphi(t,0) \equiv 0 \qquad \dot{z}(t) = \varphi''_{at}(t,0)$$
$$\ddot{z}(t) = \varphi'''_{att}(t,0)$$

$$\Rightarrow \begin{cases} \ddot{z} = -\cos 0 \cdot z = -z \\ z(0) = 1, \ \dot{z}(0) = 1 \end{cases}$$

The solution is (we will see it later)

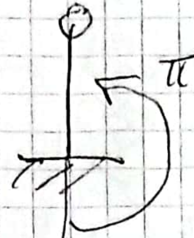
$$\left.\begin{matrix} z = C_1\cos t + C_2 \sin t \\ z(0) = 1, \ \dot{z}(0) = 1 \end{matrix}\right| \Rightarrow z = \cos t + \sin t$$

Approximation: $x(t) = (\cos t + \sin t)a + O(a^2)$

$O(a^2)$: $o(a)$ is also OK

The approximation $x(t) \approx a(\cos t + \sin t)$ stays true only over $|t| \le T$ and a small enough

To see it better take another initial condition



$$\begin{cases} \ddot{x} = -\sin x \\ x(0) = \pi + a, \ \dot{x}(0) = a^2 \end{cases}$$

~~zero velocity~~

Now $\varphi(t,0) \equiv \pi$, $z(t) = \varphi'_a(t,0)$

Once more

$$x(t) = \varphi(t,a) = \varphi(t,0) + \varphi'_a(t,0)a + O(a^2)$$

$$x(t) = \pi + z(t)\cdot a + O(a^2), \quad z(t) = \varphi'_a(t,0) = ?$$

$$\begin{cases} \ddot{x}'_a = -\cos x \; x'_a \\ x'_a(0) = 1 \quad \dot{x}'_a(0) = 2a \end{cases}$$

substitute $a = 0$

$$\begin{cases} \ddot{z} = -\cos \pi \cdot z = z \\ z(0) = 1, \ \dot{z}(0) = 0 \end{cases}$$

The solution is $\begin{cases} z(t) = c_1 \cosh t + c_2 \sinh t \\ z(0) = 1, \ \dot{z}(0) = 0 \end{cases}$

$\Rightarrow z(t) = \cosh t$, $\quad \cosh t = \frac{e^t + e^{-t}}{2}$, $\quad \sinh t = \frac{e^t - e^{-t}}{2}$

$x(t) = \pi + a\cosh t + O(a^2)$ unbounded rotation with acceleration!

It is only right over bounded time interval for a small enough