Lecture 13 02/01-2021

(180a)

Now we need to prove/"show" that there are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \dots, \lambda_n \to \infty$ and the Fourier series

$$\begin{cases} Lu = f \\ \ell u = 0 \end{cases} \Longleftrightarrow u(x) = \int_0^\ell f(s)\, g(x,s)\, ds$$

Green function

$g(x,s) = g(s,x)$

Let the problem be regular

otherwise there is no Green function

~~Note that~~

$f = \sqrt{w(x)}\, \hat{f}(x)$

$\cancel{\sqrt{w(x)}\,u(x) = \int_0^\ell f(s)\sqrt{w(s)}}$

$\cancel{= \int_0^\ell \hat{f}(x)\sqrt{w(s)}\, g(x,s)\sqrt{w(x)}\, ds}$

$\begin{cases} Lu = 0 \\ \ell u = 0 \end{cases} \Rightarrow u = 0$

I. Assume $0 \notin \operatorname{Spec} L$

$u \in M,\ Lu = 0 \Rightarrow u = 0 \Rightarrow \exists\, g(x,s)$
the Green function

$$Lu = -\lambda w u, \quad u \in M = \{u \in C^2, \mathcal{D}u = 0\}$$

$$\Rightarrow u(x) = -\lambda \int_0^\ell w(s) u(s) g(x,s)\, ds$$

Denote $v(x) = \sqrt{w(x)}\, u(x)$, multiply by $\sqrt{w(x)}$

$$\sqrt{w(x)}u(x) = v(x) = -\lambda \int_0^\ell \sqrt{w(s)}u(s) \underbrace{\sqrt{w(x)}\, g(x,s) \sqrt{w(s)}}_{K(x,s) = K(s,x)}\, ds$$

$$v(x) = -\lambda \int_0^\ell v(s) K(x,s)\, ds = -\lambda A v$$

$K(x,s)$ is called kernel, $\int \ldots$

$v = -\lambda A v$ Operator by Fredholm
The Fredholm operator

$$A : L_2[0,\ell] \to C[0,\ell] \subset L_2[0,\ell]$$

A is a compact operator
(I.e. it is bounded and transfers each bounded set into a precompact set)

A is also self-adjoint

precompact sets: its closure is compact
(each sequence contains a convergent subsequence)

$\lambda \in \operatorname{Spec} L \Leftrightarrow -\frac{1}{\lambda} \in \operatorname{Spec} A$
$0 \notin \operatorname{Spec} L$

Proposition 1 A is self adjoint

$$\langle \varphi, A\psi \rangle = \int_0^\ell \varphi(s)\, \overline{\psi(s)}\, K(x,s)\, ds = \langle A\varphi, \psi \rangle \quad \text{QED}$$

Proposition 2 $A: L_2[0,\ell] \to C[0,\ell]$

$$|A\varphi(x_1) - A\varphi(x_2)|^2 = \cancel{\int_0^\ell (\varphi(x_1) - \varphi(x_2))} \left| \int_0^\ell \varphi(s) K(x_1,s)\, ds - \int_0^\ell \varphi(s) K(x_2,s)\, ds \right|^2$$

$$\leq \left(\int_0^\ell |\varphi(s)|\, |K(x_1,s) - K(x_2,s)|\, ds \right)^2 \xrightarrow[\|x_1-x_2\| \to 0]{} 0 \to \text{continuous}$$

$$\leq \int |\varphi(s)|^2 ds \int |\Delta K|^2 ds$$

bounded uniformly ~~continuous~~ in $I \times I$

$\forall \varepsilon > 0\ \exists \delta > 0 : |x_1 - x_2| \leq \delta,\ |s_1 - s_2| < \delta$

QED $\Rightarrow |K(x_1,s_1) - K(x_2,s_2)| < \varepsilon$

Proposition 3 A is compact

I.e. each set bounded in L_2 is mapped in a precompact set.

Proof: 1. $\|\varphi\|^2 = \int_0^\ell |\varphi(s)|^2 ds \leq M$, $A\varphi \in C$

Boundedness

$$\|A\varphi(x_1) - A\varphi(x_2)\|^2 = \left(\int_0^\ell |\varphi(s)|\, |K(x_1,s) - K(x_2,s)|\, ds \right)^2$$

$\langle a, b \rangle = \|a\| \|b\| \cos\alpha \leq \|a\| \|b\|$ Inequality by Cauchy

$$= \langle |\varphi|, |\Delta K| \rangle^2 \leq \langle |\varphi|, |\varphi| \rangle^2 \langle |\Delta K|, |\Delta K| \rangle^2$$

$$\leq M^2 \left(\int_0^\ell |K(x_1,s) - K(x_2,s)|^2 ds \right)^2 \rightrightarrows 0 \quad |x_1 - x_2| \to 0$$

~~uniformly~~ equi continuous

$$\|A\varphi\|^2 = \int_0^\ell |\varphi(s)|^2 |K(x,s)|^2 ds \leq M \cdot \int_0^\ell |K(x,s)|^2 ds \quad \text{bounded}$$

Arzela, Ascoli $\boxed{\text{QED}}$

Proposition 4 $\quad \text{Image}\, A\big|_{C[0,\ell]} = \sqrt{w}\, M$ 18/6

$$A: \underbrace{L_2[0,\ell] \to C[0,\ell]}_{\text{Proved}} \to \sqrt{w}\, M \subset C[0,\ell]$$

Proof
$$\begin{cases} Lu = f \\ u \in M \\ f \in C \end{cases} \iff u = \int_0^\ell f(s)\, g(x,s)\, ds$$

$$\sqrt{w}\, u = \int \frac{f(s)}{\sqrt{w(s)}} \underbrace{\sqrt{w(s)}\, g\, \sqrt{w(x)}}_{K}\, ds$$

1. $$\begin{cases} Lu = f \\ u \in M,\ f \in C \end{cases} \iff \sqrt{w}\, u = A \frac{f}{\sqrt{w}},\ u = \frac{1}{\sqrt{w}} A \frac{f}{\sqrt{w}}$$

$$\Rightarrow \forall \hat{f} \in C:\ A \frac{\sqrt{w}\,\hat{f}}{\sqrt{w}} \in \sqrt{w}\, M$$

$$\Rightarrow \text{Image}\, A \subset \sqrt{w}\, M \qquad \begin{pmatrix} = \sqrt{w}\, u, \\ Lu = \sqrt{w}\,\hat{f} \\ \exists u \in M \end{pmatrix}$$

2. Let now $f \in \sqrt{w}\, M \Rightarrow \hat{f} = \frac{f}{\sqrt{w}} \in M$

$$\Rightarrow \hat{f} = \frac{1}{\sqrt{w}} A \frac{L\frac{f}{\sqrt{w}}}{\sqrt{w}} \Rightarrow f = \sqrt{w}\,\hat{f} \in \text{Image}\, A$$

$$\text{Image}\, A \supset \sqrt{w}\, M$$

$$\Rightarrow \text{Image}\, A\big|_{C[0,\ell]} = \sqrt{w}\, M$$

$\boxed{\begin{array}{l} v \text{ is an eigenvector} \\ \text{of } A \Rightarrow \\ Av = \mu v \\ \Rightarrow v \in C[0,\ell] \\ \Rightarrow v \in \sqrt{w}\, M \end{array}}$

Proposition 5 $\sqrt{w}\, M$ is everywhere dense in $L_2[0,\ell]$

M is dense ($\mathcal{D}u = 0$ is not a restriction in L_2)

$\Rightarrow \sqrt{w}\, M$ is dense

Theorem (Hilbert)

A - compact and self-adjoint
$\Rightarrow$ its eigenvalues are a countable infinite set with the only density point which is 0,
That is: $\left|\frac{1}{\lambda_0}\right| \geq \left|\frac{1}{\lambda_1}\right| \geq \ldots, \quad \frac{1}{\lambda_n} \to 0$

The eigenvectors constitute a complete ortogonal system in the Image space of A

$$A : C[0,\ell] \to M\sqrt{w} = \text{Image } A$$

Since $\sqrt{w}$ is a continuous positive function

Image A is everywhere dense

$\Rightarrow$ It is a complete ortogonal system in L_2 (Basis) in L_2

$\Rightarrow \forall f \in L_2$

$$f = \sum_{k=1}^{\infty} \alpha_k v_k, \quad \alpha_k = \frac{\langle f, v_k\rangle}{\langle v_k, v_k\rangle}$$

Expansion with respect to the basis

$$\left\|f - \sum_{k=1}^{N} \alpha_k v_k\right\|_{L_2} \to 0, \; N \to \infty$$

In particular

$$\sqrt{w} f = \sum \alpha_k \underbrace{v_k}_{\sqrt{w} u_k}, \quad \alpha_k = \frac{\langle \sqrt{w} f, v_k\rangle}{\langle v_k, v_k\rangle}$$

$$\sqrt{w} f = \sum \alpha_k \sqrt{w} u_k$$

$$f = \sum \alpha_k u_k, \quad \alpha_k = \frac{\langle w f, u_k\rangle}{\langle w u_k, u_k\rangle}$$

Proposition 5

Prove that indeed the eigenvalues can be ordered as

$\lambda_1 < \lambda_2 < \dots$

There are only finite number of negative λ_k

$$\begin{cases} (pu')' + (\overbrace{q + \lambda w}^{q_\lambda})u = 0 \\ \pi_0: \dfrac{\alpha_1}{p(0)} p(0) u'(0) + \beta_1 u(0) = 0 \\ \pi_\ell: \dfrac{\alpha_2}{p(\ell)} p(\ell) u'(\ell) + \beta_2 u(\ell) = 0 \end{cases}$$

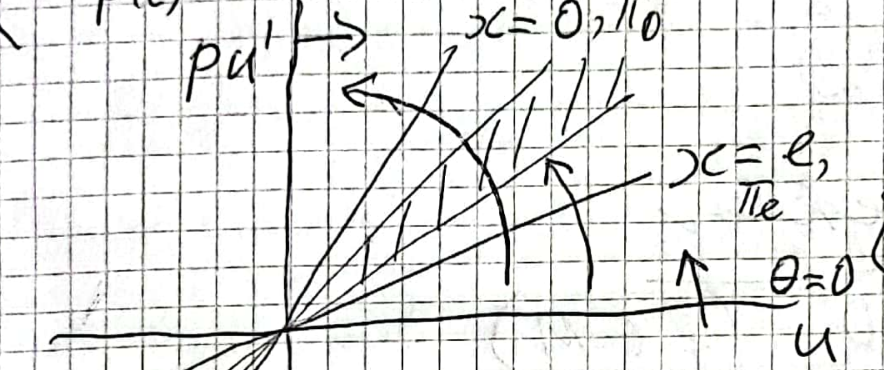
q, w bounded in $I = [0, \ell]$

$w \geq \min w > 0$

$$\boxed{\begin{matrix} x = 0 \Rightarrow \pi_0 \\ x = \ell \Rightarrow \pi_\ell \end{matrix} \quad \lambda = ?}$$

$\Rightarrow |q_\lambda| \gg 1$

$q_\lambda < 0, \lambda \to -\infty$



Prüfer coordinates

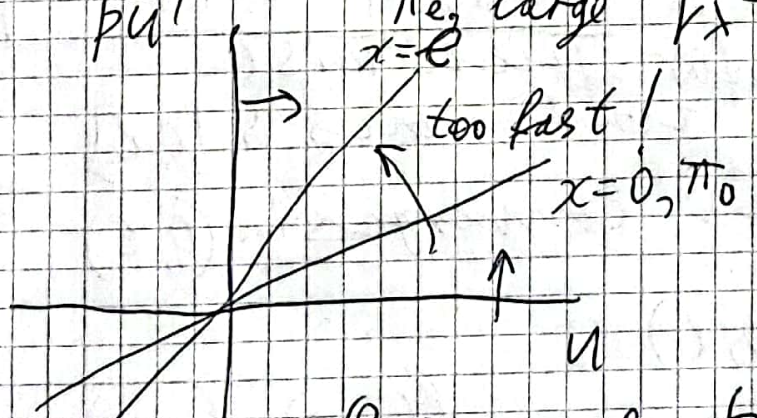
$$\begin{cases} u = \rho \cos\theta \\ p(x)u'(x) = \rho(x) \sin\theta(x) \end{cases}$$

One cannot come to ~~$x = \ell$~~ π_ℓ from π_0

The comparison Theorem by Sturm:

$$\boxed{\theta'(x) = -\frac{1}{p(x)} \sin^2\theta(x) - q_\lambda(x) \cos^2\theta(x)}$$

π_ℓ, large $q_\lambda < 0 \Leftarrow \lambda < 0, |\lambda| \gg 1$



$\Rightarrow$ finite number of negative eigenvalues

One comes to π_ℓ from π_0 but too fast, before $x = \ell$,

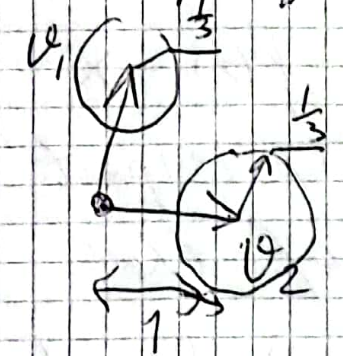
QED

General case: $0 \in \operatorname{Spec} L$
(There is no Green function)

Proposition There exists $\mathscr{x} \in \mathbb{R}$:
$$\mathscr{x} \notin \operatorname{Spec} L$$

Proof Assume the opposite: $\mathbb{R} \subset \operatorname{Spec} L$
$\Rightarrow$ We get continuum eigenvectors
($\frac{1}{\sqrt{w}}$ eigen functions)
which are ortogonal one to each other

It is known that $L_2[0, \ell]$ is a separable space (has a countable everywhere dense subset (polynomials with rational coefficients) P_K, $K \in \mathbb{N}$



eigenvectors v_α, $\alpha \in \mathbb{R}$, $\|v_\alpha\| = 1$

The ball of the radius $\frac{1}{3}$ around each v_α should contain a polynomial P_K, $K \in \mathbb{N}$

They do not intersect $\Rightarrow$ continuum polynomials

QED Contradiction

General case: ~~$\forall$~~ $0 \in \operatorname{Spec} L$ 18.4a

Propositions: $\exists\, \varkappa \in \mathbb{R},\ \varkappa \notin \operatorname{Spec} L$ L_2 is separable

Since they are ortogonal eigen functions

$$\tilde{\lambda} \qquad \lambda = \tilde{\lambda} + \varkappa \qquad \begin{cases} Lu + \lambda \omega u = 0 \\ Du = 0 \end{cases}$$

$$\begin{cases} (Lu + \varkappa u) + \tilde{\lambda} w u = 0 \\ Du = 0 \end{cases} \Longleftrightarrow \begin{cases} Lu + \lambda \omega u = 0 \\ Du = 0 \end{cases}$$

$$\tilde{\lambda}_1 < \tilde{\lambda}_2 < \dots$$

$$\Rightarrow \lambda_1 < \lambda_2 < \dots$$

QED

1. Theorem by Steklov

$f \in C^2,\ Df = 0 \quad (f \in M)$

$\Rightarrow$ The fourier series converges uniformly to f over $[0, l]$

$$\sum_{i=1}^{N} c_i u_{\lambda_i} \rightrightarrows f, \quad N \to \infty$$

$$\boxed{\max_{x \in [0,l]} \Big| \sum_{i=1}^{N} c_i u_{\lambda_i}(x) - f(x) \Big| \to 0}$$

2. Theorem $f(x), f'(x)$ - piece-wise continuous in $[0, l]$

$\Rightarrow$ The series converges in $(0, l)$

~~uni~~ and $\forall x \in (0, l)$

$$\lim_{n \to \infty} \sum_{i=1}^{n} c_i u_i(x) = \frac{f(x-0) + f(x+0)}{2}$$

3. Theorem $f(x), f'(x)$ are piece-wise continuous in $[0,l]$

If one of the boundary conditions is $u(0)=0$ (and/or $u(l)=0$) then also f satisfies $f(0)=0$ (and/or $f(l)=0$)

$\Rightarrow$ the convergence is over the whole segment $[0,l]$ to $\frac{1}{2}[f(x+0)+f(x-0)]$

Example To expand $f(x)=x$ in the Fourier series with respect to the eigenfunctions of the S-L problem

$$\begin{cases} u''+\lambda u=0 \\ u(0)=0 \\ u(l)=0 \end{cases} \qquad \begin{matrix} p=1, w=1 \\ q=0 \end{matrix}$$

From the motivation example we know that $u_k=\sin\left(\frac{\pi k}{l}x\right)$, $\lambda_k=\frac{\pi^2 k^2}{l^2}$, $k=1,2,\ldots$

$$f(x)=\sum_{k=1}^{\infty} c_k \sin\left(\frac{\pi k}{l}x\right), \quad c_k=\frac{\langle wf, u_k\rangle}{\langle wu_k, u_k\rangle}$$

$$\langle w u_k, u_k \rangle = \int_0^l \sin^2\left(\frac{\pi k}{l} x\right) dx$$

$$= \int_0^l \frac{1}{2}\left(1 - \cos\left(\frac{2\pi k}{l} x\right)\right) dx = \frac{l}{2} - \int_0^l \cos\left(\frac{2\pi k}{l} x\right) dx \to 0$$

period $\frac{l}{k}$

$$\boxed{c_k = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{\pi k}{l} x\right) dx}$$

$f(x) = x \qquad C_k = \frac{2}{l} \int_0^l x \sin\left(\frac{\pi k}{l} x\right) dx$

by parts

$$= -\frac{2}{l}\frac{l}{\pi k} x \cos\left(\frac{\pi k}{l} x\right)\Big|_0^l + \frac{2}{\pi k} \int_0^l \cos\left(\frac{\pi k}{l} x\right) dx$$

$$= -\frac{2l}{\pi k}(-1)^k + \frac{2}{\pi k}\frac{l}{\pi k} \sin\left(\frac{\pi k}{l} x\right)\Big|_0^l \to 0$$

$$c_k = \frac{2}{\pi k}(-1)^{k+1}$$

$$x = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2l}{\pi k} \sin\left(\frac{\pi k}{l} x\right), \quad x \neq l$$

$$x = l \qquad \Big(\underbrace{\textstyle\sum}_{\text{the series}}(l) = 0\Big)$$

Solution of ODEs by power series

(The first step to the Frobenius method)

$$a(t)\ddot{x} + b(t)\dot{x} + c(t)x = d(t), \quad x \in \mathbb{R}$$

$a(t_0) \neq 0$ To find solution approximately

Theorem Let a, b, c, d be analytic functions (i.e. power series around $t = t_0$) (Taylor series)

Then for each initial condition $x(t_0)$, $\dot{x}(t_0)$ there exists the unique solution $x(t) = \alpha_0 + \alpha_1(t-t_0) + \alpha_2(t-t_0)^2 + \dots$

$$x(t_0) = \alpha_0, \quad \dot{x}(t_0) = \alpha_1$$

A function can be smooth (C^∞) and not analytic

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}}, & t \neq 0 \\ 0, & t = 0 \end{cases} \qquad f^{(k)}(0) = 0 \quad \forall k = 0, 1, \dots$$

$f \in C^\infty$

Useful formulas:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \dots$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$

$$\sin t = \frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \dots$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

Example

$$\ddot{x} + \sin t \cdot \dot{x} - x \cos t = -t^2, \quad x(0)=1, \quad \dot{x}(0) = -1, \quad x \in \mathbb{R}$$

$$x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \dots$$
$$(\text{or } o(t^4) \text{ or } O(t^5))$$

1. To find $\alpha_0, \dots, \alpha_4$
2. $x(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + R(t)$, $R(t) = O(t^4)$

Evaluate in the form:

$$|t| \le \varepsilon \Rightarrow |R(t)| \le M t^4, \quad \varepsilon, M = ?$$

Solution

1. $x = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \dots$

$$\dot{x} = \alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2 + 4\alpha_4 t^3 + \dots$$

$$\ddot{x} = 2\alpha_2 + 6\alpha_3 t + 12\alpha_4 t^2 + \dots$$

$$2\alpha_2 + 6\alpha_3 t + 12\alpha_4 t^2 + \dots + \left(t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \dots\right)\left(\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2 + \dots\right)$$
$$- \left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \dots\right)\left(\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \dots\right) = -t^2$$

From the initial conditions

$$\alpha_0 = x(0) = 1, \quad \alpha_1 = \dot{x}(0) = -1$$

t^0: $2\alpha_2 - \alpha_0 = 0$, $\alpha_2 = \frac{1}{2}\alpha_0 = \frac{1}{2}$

t^1: $6\alpha_3 + \alpha_1 - \alpha_1 = 0$, $\alpha_3 = 0$

t^2: $12\alpha_4 + 2\alpha_2 + \frac{1}{2}\alpha_0 - \alpha_2 = -1$, $\alpha_4 = \frac{1}{12}\left(-1 - \alpha_2 - \frac{1}{2}\alpha_0\right)$
$= \frac{1}{12}\left(-1 - \frac{1}{2} + \frac{1}{2}\right) = -\frac{1}{24}$ $= -\frac{1}{12}$

$$x(t) = 1 - t + \frac{1}{2}t^2 - \frac{1}{12}t^4 + o(t^4)$$
$$(\text{or} \dots \text{ or } O(t^5))$$

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$\dddot{x}(0)=0$

$x(t) = 1 - t + \frac{1}{2}t^2 + R(t), \quad R(t) = O(t^4)$

$\Rightarrow \exists\, \varepsilon > 0, M > 0 ; \ \forall t : |t| \le \varepsilon, |R(t)| \le M t^4$

$\varepsilon = ?, \ M = ?$

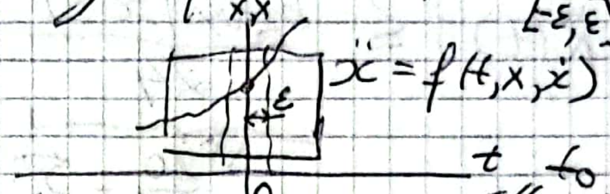
The Lagrange formula

$$R(t) = \frac{x^{(4)}(\theta t)}{4!} t^4, \quad \theta \in (0,1)$$

$\Rightarrow$ we need to evaluate $x^{(4)}$ in some vicinity of $t=0$, $M \ge \max\limits_{[\varepsilon,\varepsilon]} \frac{|x^{(4)}|}{24}$

$|t| \le \varepsilon$

Theorem $\exists!$



$\ddot{x} = f(t, x, \dot{x})$

Choose

$$\Omega = \left\{ (t, x, \dot{x}) \;\middle|\; \begin{array}{l} |t| \le 2 \\ |x| \le 2 \\ |\dot{x}| \le 2 \end{array} \right\}, \quad \underbrace{(\overset{t_0}{0}, 1, -1) \in \Omega}_{\text{the only condition}}$$

0. Find ε such that $|t| \le \varepsilon \Rightarrow (t, x, \dot{x}) \in \Omega$

1. $x(t) = \underbrace{x(t_0)}_{1} + \int\limits_{t_0}^{t} \dot{x}(t)\,dt, \quad |x(t)| \le 1 + 2|t|$

$|x| \le 2 \Rightarrow \boxed{|t| \le \frac{1}{2}}$

2. $\dot{x}(t) = \underbrace{\dot{x}(t_0)}_{-1} + \int\limits_{t_0}^{t} \ddot{x}(t)\,dt$

$|\dot{x}(t)| \le 1 + |t - t_0| \cdot \max\limits_{\Omega} |\ddot{x}| = 1 + |t| \max\limits_{\Omega} |\ddot{x}|$

$\{\ddot{x}\} = -\sin t\, \dot{x} + \cos t\, x - t^2$

$|\ddot{x}| \le 1 \cdot 2 + 1 \cdot 2 + \frac{1}{4} \le 5$ (or $+ 2^2$ also OK)

$|\ddot{x}| \le 5$

$1 + |t| \cdot 5 \le 2 \Rightarrow \boxed{|t| \le \frac{1}{5}}$

$\Rightarrow \varepsilon = \frac{1}{5}, \quad t \in [\frac{1}{5}, \frac{1}{5}] \Rightarrow \left. \begin{array}{l} |t| \le 2 \\ |x| \le 2 \\ |\dot{x}| \le 2 \end{array} \right] (t, x, \dot{x}) \in \Omega$

$$\ddot{x} = -\sin t\,\dot{x} + \cos t\,x - t^2$$

$$\dddot{x} = \cancel{-\cos t\,\dot{x}} - \sin t\,\ddot{x} + \cancel{\cos t\,\dot{x}} - \sin t\,x - 2t$$

$$|\dddot{x}| \le |\ddot{x}| + |x| + 2\cdot|t| \le 5 + 2 + \frac{2}{5} \le 8$$

$$x^{IV} = -\cos t\,\ddot{x} - \sin t\,\dddot{x} - \cos t\,x - \sin t\,\dot{x} - 2$$

$$|x^{IV}| \le |\ddot{x}| + |\dddot{x}| + |x| + |\dot{x}| + 2$$

$$\le 5 + 8 + 2 + 2 + 2 = 13 + 6 \le 20$$

$$|R(t)| \le \underbrace{\frac{\max|x^{IV}|}{24}}_{\le \frac{20}{24} = \frac{5}{6}} t^4, \quad |t| \le \frac{1}{5}$$

$$, \quad M = 5/6$$

$$\boxed{\begin{array}{l} x(t) = 1 - t + \frac{1}{2}t^2 + R(t), \quad R(t) = O(t^4) \\ |t| \le \frac{1}{5} \Rightarrow |R(t)| \le \frac{5}{6}t^4 \end{array}}$$