Lecture 12 26/12-2021

~~Funct~~ Green Function for BV problem

$$\begin{cases} Lu = f \\ Du = 0 \end{cases} \qquad \begin{cases} (pu')' + qu = f \\ \alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_2 u'(l) + \beta_2 u(l) = 0 \end{cases} \qquad \begin{matrix} p \in C^1, \ p > 0 \\ q, f \in C \\ (\alpha_1, \beta_1) \neq (0,0) \\ (\alpha_2, \beta_2) \neq (0,0) \end{matrix}$$

regular: $f = 0 \Rightarrow u = 0$ $\qquad I = [0, l]$

Green function! $\quad g(x,s)$, $g \in C[I \times I]$

1. $Lg = 0$, $x \neq s$ $\quad g \in C^2$

2. $p(s)\left(g'(s+0, s) - g'(s-0, s)\right) = 1$ $\quad x = s$

3. $Dg = 0$ $\quad s \neq 0, l$

$$\boxed{u(x) = \int_0^l f(s)\, g(x,s)\, ds} \qquad (*)$$

<u>Theorem 1</u> For any regular BV problem!!! $\exists!$ Green function which satisfies 1–3, also $g(x,s) = g(s,x)$.

<u>Theorem 2</u> There can be only one continuous function $g(x,s)$ providing for the solution $(*)$ of BV problem for any $f \in C$

<u>Theorem 3</u> Function g which is constructed in theorem 1 satisfies $(*)$ (provides solution $(*)$ to the BV problem).

Theorem 1 was proved.

Proof of Theorem 2

Suppose that for any $f \in C$

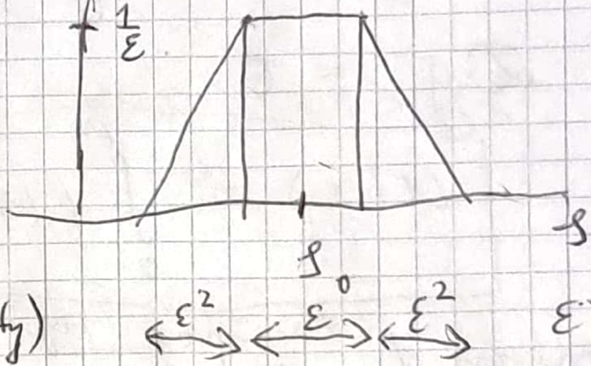
$u_1 = \int_0^l f(s)\, g_1(x,s)\, ds$, $u_2 = \int_0^l f(s)\, g_2(x,s)\, ds$

are solutions for the regular BV problem

$$\Rightarrow \quad u_1 = u_2, \quad I(x) = \int_0^l f(s) \underbrace{\left[g_1(x,s) - g_2(x,s)\right]}_{\text{Denote: } \Delta g(x,s) \in C} ds \equiv 0$$

Let $\Delta g(x_0, s_0) \neq 0$, $(x_0, s_0) \in (0,l) \times (0,l)$

Fix x_0. Let $f_\varepsilon(s)$



Obviously

$\Delta g(x,s) = \Delta g(x_0,s_0) + o(1)$

as $(x,s) \to (x_0,s_0)$ (continuity)

$\varepsilon \to 0$

$$I_\varepsilon(x_0) = \int_0^l f_\varepsilon(s) \Delta g(x_0, s)\, ds = \int_{|s-s_0| \le \frac{\varepsilon}{2}} + \int_{|s-s_0| \ge \frac{\varepsilon}{2}}$$

$$I_\varepsilon(x_0) = \frac{1}{\varepsilon} \int_{|s-s_0| \le \frac{\varepsilon}{2}} \left(\Delta g(x_0,s_0) + o(1)\right) ds + I_{R\varepsilon}$$

Let $|g| \le M$ (bounded since continuous)

$$|I_{R\varepsilon}| \le 2 \cdot \underbrace{\frac{1}{2} \frac{1}{\varepsilon} \varepsilon^2}_{\text{triangle area}} \cdot M = O(\varepsilon)$$

$$0 = I_\varepsilon(x_0) = \Delta g(x_0, s_0) + o(1) + O(\varepsilon)$$

$$\varepsilon \to 0 \Rightarrow \quad \Delta g(x_0, s_0) = 0$$

QED

Proof of Theorem 3: Function satisfying the definition built in Theorem 1 (it is symmetrical) satisfies the integral formula (for which it was aimed!).

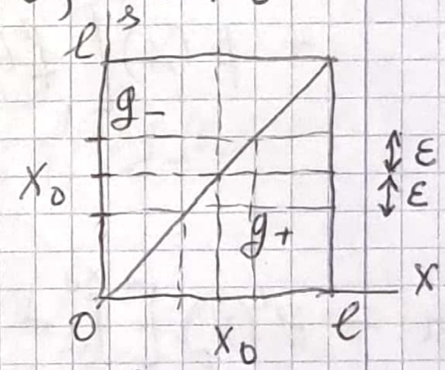
$$g(x,s) = \begin{cases} g_+(x,s), & x \geq s, \ (x,s) \in I \times I \\ g_-(x,s), & x \leq s, \ (x,s) \in I \times I \end{cases}$$

$I = [0, \ell]$, $g_+, g_- \in C^2[I \times I]$ from the construction

Continuity: $g_+(s,s) = g_-(s,s)$

Jump of the derivative: $g_+'(s,s) - g_-'(s,s) = \frac{1}{p(s)}$ differentiation with respect to x

Boundary conditions: $\begin{cases} \alpha_1 g_-'(0,s) + \beta_1 g_-(0,s) = 0, & s \neq 0 \\ \alpha_2 g_+'(\ell,s) + \beta_2 g_+(\ell,s) = 0, & s \neq \ell \end{cases}$



1. Prove that $Lu = f$

$$u(x) = \int_0^\ell f(s)\, g(x,s)\, ds$$

$$L\Big|_{x=x_0} \int_0^\ell f(s)\, g(x,s)\, ds = L\Big|_{x=x_0} \Big(\int_{|x_0-s| \leq \varepsilon} + \int_{|x_0-s| \geq \varepsilon} \Big)$$

$$= L\Big|_{x=x_0} \int_{|x_0-s| \leq \varepsilon} + \int_{|x_0-s| \leq \varepsilon} f(s) \underbrace{L\Big|_{x=x_0} g(x,s)}_{0}\, ds$$

$\cancel{\frac{d}{dx}\Big|_{x=x_0} p(x)}$

$$L\Big|_{x=x_0} u = \frac{d}{dx}\Big|_{x=x_0}\left(p(x)\frac{d}{dx}\int_{|x_0-s|\le\varepsilon} f(s)\, g(x,s)\,ds\right) + O(\varepsilon)$$

$$Lu = (p^{*}u')' + qu$$

Remark It follows from $\frac{d}{dt}\int_{t_0}^{t}\varphi(s)ds = \varphi(t)$ that

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,y)\,dy = F(x,B(x))B'(x) - F(x,A(x))A'(x) + \int_{A(x)}^{B(x)} F'_x(x,y)\,dy$$

$$L\Big|_{x=x_0} u = \frac{d}{dx}\Big|_{x=x_0}\left(p(x)\frac{d}{dx}\left[\int_{x_0-\varepsilon}^{x} f(s)\, g_+(x,s)\,ds + \int_{x}^{x_0+s} f(s) g_-(x,s)\,ds\right]\right) + O(\varepsilon)$$

$$= \frac{d}{dx}\Big|_{x=x_0}\left(p(x)\left[\int_{x_0-\varepsilon}^{x} f(s)\, g'_+(x,s)\,ds + \int_{x}^{x_0+\varepsilon} f(s) g'_-(x,s)\,ds\right] + p(x)\underbrace{\left[f(x) g_+(x,x) - f(x) g_-(x,x)\right]}_{0 \text{ (continuity)}}\right) + O(\varepsilon)$$

$$L\Big|_{x=x_0} u = p'(x_0)\underbrace{\left[\int_{x_0-\varepsilon}^{x_0}\dots + \int_{x_0}^{x_0+\varepsilon}\dots\right]}_{O(\varepsilon),\ g'_+, g'_- \text{ bounded}} + \underbrace{p(x_0)\left[f(x_0) g'_+(x_0,x_0) - f(x_0) g'_-(x_0,x_0)\right]}_{\text{derivative jump} \quad f(x_0)},$$

$$+ p(x_0)\underbrace{\left[\int_{x_0-\varepsilon}^{x_0} f(s) g''_+(x_0,s)\,ds + \int_{x_0}^{x_0+\varepsilon} f(s) g''_+(x_0,s)\,ds\right]}_{O(\varepsilon),\ g''_+, g''_- \text{ bounded}} + O(\varepsilon)$$

Thus ~~$L\big|_{x=x_0} g(x,s) =$~~ for any $\varepsilon > 0$, $\varepsilon \to 0$

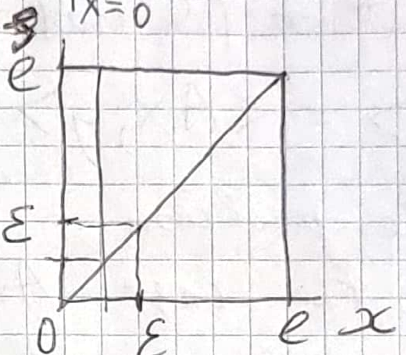
$$L\Big|_{x=x_0} u(x) = f(x_0) + O(\varepsilon)$$

$$\Rightarrow \quad Lu\Big|_{x=x_0} = f(x_0) \qquad \boxed{Lu = f}$$

2. Boundary conditions

$$\alpha_1 u'(0) + \beta_1 u(0) = \alpha_1 \frac{d}{dx}\Big|_{x=0} \int_0^l f(s) g(x,s)\, ds + \beta_1 \int_0^l f(s) \underbrace{g(0,s)}_{= g_-(0,s)}\, ds$$

Let $0 \le x \le \varepsilon \ll l$



$$= \alpha_1 \frac{d}{dx}\Big|_{x=0} \left[\int_0^{\varepsilon} f(s) g(x,s)\, ds \right] + \overbrace{\alpha_1 \frac{d}{dx}\Big|_{x=0} \int_{\varepsilon}^{l} f(s) g_-(x,s)\, ds + \beta_1 \int_{\varepsilon}^{l} f(s) g_-(0,s)\, ds}^{= \int_{\varepsilon}^{l} f(s) \underbrace{\left[\alpha_1 g_-'(0,s) + \beta_1 g_-(0,s)\right]}_{0} ds}$$

$$+ \underbrace{\beta_1 \int_0^{\varepsilon} f(s) g_-(0,s)\, ds}_{O(\varepsilon)}$$

g_- is bounded

$$= \alpha_1 \frac{d}{dx}\Big|_{x=0} \left[\int_0^x f(s) g_+(x,s)\, ds + \int_x^{\varepsilon} f(s) g_-(x,s)\, ds \right] + O(\varepsilon)$$

$$= \alpha_1 \underbrace{\left[f(0) g_+(0,0) - f(0) g_-(0,0) \right]}_{\text{continuity} \quad 0} + \alpha_1 \left[\cancel{\int_0^0 f(s) g_+'(0,s)\, ds} + \int_0^{\varepsilon} f(s) g_-'(0,s)\, ds \right] + O(\varepsilon)$$

0

$$= O(\varepsilon), \quad \forall \varepsilon \qquad \boxed{QED}$$

Sturm-Liouville problem

~~What if~~ Whereas the boundary-value problem was similar to the problem $Ax=f$, $x=?$ $x\in\mathbb{R}^n$,

The S-L problem is similar to the problem of finding the eigenvectors and eigenvalues.

Moreover, it is similar to the case of the self-adjoint operator

$A=A^*$ (in matrix language $A^*=\bar{A}^T$)

$\langle x, Ay\rangle = \langle Ax, y\rangle \quad \forall x,y\in\mathbb{C}^n$

It is the problem $Ax-\lambda x=0$

and in the case $A=A^*$ get $\lambda=\bar{\lambda}\in\mathbb{R}$

There is a basis of eigenvectors. $x=\sum_{i=1}^{n} c_i v_i$

S-L Problem

$$\begin{cases} Lu+\lambda w u=0 \\ \mathcal{D}u=0 \end{cases}$$

~~$(p(x)u')'+p\ldots$~~

$$\begin{cases} (p(x)u')'+q(x)u+\lambda w(x)u=0 \\ \alpha_1 u'(0)+\beta_1 u(0)=0 \\ \alpha_2 u'(\ell)+\beta_2 u(\ell)=0 \end{cases}$$

$u:[0,\ell]\to\mathbb{C}$

$\lambda\in\mathbb{C}$

$u\in C^2$

$w, p>0$ (real-valued positive), $p\in C^1$

$w, q\in C$ (real-valued continuous)

$(\alpha_1,\beta_1), (\alpha_2,\beta_2)\in\mathbb{R}^2$, $\neq(0,0)$

$\lambda\in\mathbb{C}$ is called the eigenvalue, $u\neq 0$

$u\neq 0$ is called eigenfunction

Introduce $M = \{u \in C^2[0,\ell],\ Du = 0\}$
$u : [0,\ell] \to \mathbb{C}$

Then $L : M \longrightarrow C[0,\ell]$

Scalar (inner) product

$$\langle f, g\rangle = \int_0^\ell f(x)\,\overline{g}(x)\,dx$$

(defined up to proportionality)

Denote by u_λ the eigenfunction corresponding to λ. We will prove

I. Eigenvalues are real numbers.

One can also choose real-valued eigenfunctions.

Eigenfunctions $u_{\lambda_1}, u_{\lambda_2}, \lambda_1 \neq \lambda_2$, are ortogonal with the weight w:

$$\langle w u_{\lambda_1}, u_{\lambda_2}\rangle = \int_0^\ell w(x)\, u_{\lambda_1}(x)\, \overline{u}_{\lambda_2}(x) = 0,$$

there is only one eigenfunction for each eigenvalue.

II There are infinitely many eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_2 < \dots,\quad \lambda_n \longrightarrow \infty$$

If $\forall i\ \langle w h, u_{\lambda_i}\rangle = 0 \Rightarrow h = 0$ (in L_2)

It means that $\{u_{\lambda_i}\}$ is a complete system of functions (in $L_2[0,\ell]$)

(finite linear combinations of u_{λ_i} constitute an everywhere dense set in $L_2[0,\ell]$)

III There exists Fourier Series

$\forall f \in L_2$: $f = \sum_{i=1} \alpha_i u_{\lambda_i}$, $\alpha_i = \dfrac{\langle wf, u_{\lambda_i}\rangle}{\langle w u_{\lambda_i}, u_{\lambda_i}\rangle}$

$L_2[0,\ell]$ - all functions $f:[0,\ell]\to\mathbb{C}$ for which $\int_0^\ell f(x)\bar{f}(x)\,dx < \infty$ - The Lebesgue integral

This knowledge was not required as the prerequisite. Correspondingly many proofs are incomplete.

<u>Definition</u> Let $\hat{L}u = \hat{a}(x)\frac{d^2}{dx^2}u + \hat{b}(x)\frac{d}{dx}u + \hat{c}(x)u$

The B-V problem $\begin{cases}\hat{L}u = f \\ \mathcal{D}u = 0\end{cases}$ $f, \hat{a}, \hat{b}, \hat{c} \in C$, $(\alpha_1,\beta_1)\neq 0$, $(\alpha_2,\beta_2)\neq 0$

$f, \hat{a}, \hat{b}, \hat{c}, \alpha_1, \beta_1, \alpha_2, \beta_2$ - real-valued.

is called self-adjoint if

$$\langle \hat{L}u, v\rangle = \langle u, \hat{L}v\rangle \text{ for any } u, v \in M$$

<u>Proposition</u> The B-V problem $\begin{cases}\hat{L}u = f \\ \mathcal{D}u = 0\end{cases}$

is self-adjoint iff it is in the standard form

I.e. $\exists p, q$: $\hat{L}u = (pu')' + au' + qu$, $a \in C$

$\forall u, v \in M$ $\langle \hat{L}u, v\rangle = \langle u, \hat{L}v\rangle \Leftrightarrow a = 0$.

$u = 0$ means $u(x) \equiv 0$, $a = 0$ means $a(x) \equiv 0$

<u>Proof</u> First show that L is self adjoint

$$\langle Lu, v\rangle - \langle u, Lv\rangle$$

$$= \int_0^\ell \left[\frac{d}{dx}(pu')\bar{v} + \cancel{qu\bar{v}} - \frac{d}{dx}(p\bar{v}')u - \cancel{q\bar{v}u}\right]dx$$

Integration by parts

$$= \left[pu'\bar{v} - p\bar{v}'u\right]\Big|_0^\ell - \underbrace{\int_0^\ell \left[pu'\bar{v}' - p\bar{v}'u'\right]dx}_{0}$$

$$= p(\ell)\underbrace{\left(u'(\ell)\bar{v}(\ell) - \bar{v}'(\ell)u(\ell)\right)}_{-W(\ell)} - p(0)\underbrace{\left(u'(0)\bar{v}(0) - \bar{v}'(0)u(0)\right)}_{-W(0)}$$

$$-W(\ell) = \begin{vmatrix} u'(\ell) & \bar{v}'(\ell) \\ u(\ell) & \bar{v}(\ell) \end{vmatrix}$$ Wronskian $W(x) = W[u, \bar{v}](x)$

$u, \bar{v}$ are not solutions of an homogeneous ODE

$\Rightarrow$ their Wronskian can be zero inspite of $u, \bar{v}$ being linearly independent

Boundary conditions:

$$\begin{cases} \alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_1 \bar{v}'(0) + \beta_1 \bar{v}(0) = 0 \end{cases} \quad (\alpha_1, \beta_1) \neq (0, 0) \quad \Rightarrow W(0) = 0$$

The system determinant

$$\alpha_2 u'(\ell) + \beta_2 u(\ell) = 0 \qquad (\alpha_2, \beta_2) \neq (0, 0) \quad \Rightarrow W(\ell) = 0$$
$$\alpha_2 \bar{v}'(\ell) + \beta_2 \bar{v}(\ell) = 0$$

$$\Rightarrow \langle Lu, v\rangle = \langle u, Lv\rangle$$

Now prove that $\overset{\forall u,v \in M}{} \langle \hat{L}u, v\rangle = \langle u, \hat{L}v\rangle$

$\Rightarrow \hat{L} = L, \ a = 0$

The same calculation now shows

$$\langle \hat{L}u, v\rangle - \langle u, \hat{L}v\rangle =$$

$$= \langle Lu + au', v\rangle - \langle u, Lv + av'\rangle$$

$$= \langle au', v\rangle - \langle u, av'\rangle$$

So $\forall u,v \in M \quad \int_0^\ell (a u'\bar{v} - a u \bar{v}')\,dx = 0$

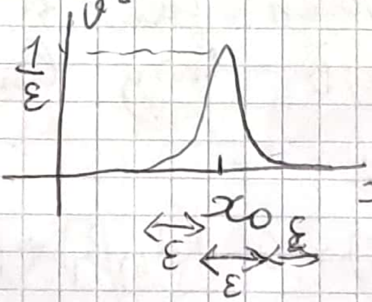
Let $u, v: \mathbb{R} \to \mathbb{R}$ ~~$u, v \in$~~

$$\int_0^\ell a(x)\left[u'(x)v(x) - u(x)v'(x)\right]dx = 0$$

Prove: $a(x) \equiv 0$, Choose $u = xv$

Then $\left(\frac{u}{v}\right)' = x' = 1$, $u'v - uv' = \begin{cases} 0 & v = 0 \\ \underbrace{\left(\frac{u}{v}\right)'}_{=1} v^2, & v \neq 0 \end{cases}$

Choose v



$$\int_0^\ell v(x)\,dx = 1 + O(\varepsilon)$$

Then for any continuous $a(x)$ obtain

$$0 = \int_0^\ell a(x)\left[u'(x)v(x) - u(x)v'(x)\right]dx \overset{\varepsilon \to 0}{=} a(x_0) + O(\varepsilon)$$

$$\Rightarrow a(x_0) = 0$$

QED

<u>Remark.</u> Nowhere $p > 0$ is required!!

Transformation into the standard form usually changes the operator L!

Example Which BV problems are self adjoint?

$$\begin{cases} \sin x\, u'' + \cos x\, u' + u = e^x \\ u'(0) - u(0) = 0,\ u'(\pi) = 0 \end{cases} \sim \begin{cases} (\sin u')' + u = e^x \\ \dots \end{cases}$$

self-adjoint

$$\begin{cases} u'' + \cos x\, u' + u = 1 \\ 2u'(0) + u(0) = 0 \\ u'(1) = 0 \end{cases}$$

not self adjoint $(p(x) \overset{!}{=} 1)$

$$\begin{cases} (e^x u')' + \arctan x\, u = \ln(x+1) \\ u(0) = 0 \\ u'(1) = 0 \end{cases}$$

self adjoint (standard form)

The following proofs are mostly ~~very~~ similar to the standard linear algebra

Proposition Multiplicity of each eigenvalue ~~is 1~~ of a S-L problem is 1

$$\begin{cases} Lu + \lambda w u = 0 \\ Du = 0 \end{cases} \qquad \begin{cases} Lv + \lambda w v = 0 \\ Dv = 0 \end{cases}$$

$u, v \neq 0 \Rightarrow$ they are proportional

Proof: u, v are solutions of the same ODE linear and homogeneous

$$\begin{cases} \alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_1 v'(0) + \beta_1 v(0) = 0 \end{cases} \Rightarrow \begin{vmatrix} u'(0) & v'(0) \\ u(0) & v(0) \end{vmatrix} = W[v, u](0) = 0$$

$(\alpha_1, \beta_1) \neq (0, 0)$

$\Rightarrow$ proportionality of columns QED

Obviously $\langle wf, g\rangle = \langle f, wg\rangle$

Proposition ~~One can al~~

All eigenvalues are real numbers

Proof $\begin{cases} L u_\lambda + \lambda w u_\lambda = 0, & u_\lambda \neq 0 \\ D u_\lambda = 0 \end{cases}$

$$\langle L u_\lambda, u_\lambda\rangle = \langle -\lambda w u_\lambda, u_\lambda\rangle = -\lambda \underbrace{\langle w u_\lambda, u_\lambda\rangle}_{>0}$$

$$\|$$

$$\langle u_\lambda, L u_\lambda\rangle = \langle u_\lambda, -\lambda w u_\lambda\rangle = -\bar\lambda \langle w u_\lambda, u_\lambda\rangle$$

$$\Rightarrow \lambda = \bar\lambda \quad \text{QED}$$

Proposition Eigenvalues $\lambda \neq \mu$, u_λ, u_μ

$$\Rightarrow \langle w u_\lambda, u_\mu\rangle = 0$$

Proof

$$\langle L u_\lambda, u_\mu\rangle = \langle u_\lambda, L u_\mu\rangle$$

$$\langle -w\lambda u_\lambda, u_\mu\rangle \qquad \langle u_\lambda, -\mu w u_\mu\rangle$$

$$-\lambda\langle w u_\lambda, u_\mu\rangle = -\mu\langle w u_\lambda, u_\mu\rangle$$

$$\underset{\neq 0}{(\lambda-\mu)}\langle w u_\lambda, u_\mu\rangle = 0 \Rightarrow \langle w u_\lambda, u_\mu\rangle = 0$$

QED

Proposition One can choose real-valued eigenfunctions

$$\begin{cases} Lu + \lambda w u = 0 \\ Du = 0 \end{cases} \Leftrightarrow \begin{cases} L\bar u + \lambda w \bar u = 0 \\ D\bar u = 0 \end{cases} \Rightarrow \tfrac{1}{2}(u+\bar u) \text{ and } \frac{u-\bar u}{2i} \text{ eigen functions}$$

one $\neq 0$ QED.

Example To find eigenvalues and eigen functions, One can use the graphic method

$$\begin{cases} u'' + \overbrace{u + \lambda u}^{(\lambda+1)u} = 0 \\ u'(0) = 0 \\ u'(1) + 2u(1) = 0 \end{cases}$$

S-L problem
$\Rightarrow$ self adjoint
$\Rightarrow$ eigen functions and eigen values are real

1. $\lambda + 1 > 0$ $\quad u = C_1 \cos(\sqrt{\lambda+1}\, x) + C_2 \sin(\sqrt{\lambda+1}\, x)$

$u' = -C_1\sqrt{\lambda+1}\sin(\sqrt{\lambda+1}\,x) + C_2\sqrt{\lambda+1}\cos(\sqrt{\lambda+1}\,x)$

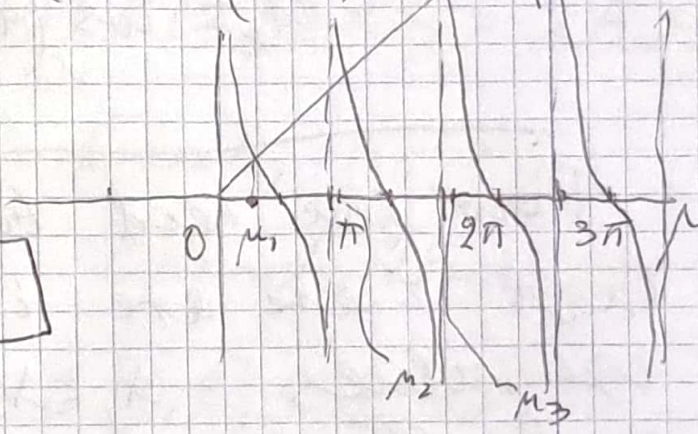
$u'(0) = 0 \Rightarrow C_2 = 0,\ u = C_1 \cos(\sqrt{\lambda+1}\,x),\ C_1 \neq 0$

$u'(1) + 2u(1) = 0 \Rightarrow -C_1\sqrt{(\lambda+1)}\sin(\sqrt{\lambda+1}\,\cancel{x}) + 2C_1\cos(\sqrt{\lambda+1}\,\cancel{x}) = 0$

$\sqrt{\lambda+1} = 2\,\mathrm{ctg}(\sqrt{\lambda+1}\,\cancel{x})$

$\sqrt{\lambda+1} = \mu,\quad \mu = 2\,\mathrm{ctg}\,\mu$

$\sqrt{\lambda_k+1} = \mu_k,\ \boxed{\lambda_k = \mu_k^2 - 1}$



$\boxed{u_k = \cos(\mu_k x)}$

2. $\lambda + 1 = 0$ $\quad u'' = 0 \quad u = c_1 x + c_2$

$u'(0) = 0 \Rightarrow c_1 = 0,\ 0 = u'(1) + 2u(1) = 2c_2 \Rightarrow c_2 = 0$

$u = 0$ is not an ~~prop~~ eigen function

3. $\lambda + 1 < 0$ $\quad \lambda + 1 = -\mu^2,\ \mu > 0$

$u'' - \mu^2 u = 0 \quad u = C_1 e^{-\mu x} + C_2 e^{\mu x}$

$u' = -C_1 \mu e^{-\mu x} + C_2 \mu e^{\mu x}$

Boundary conditions

$$\begin{aligned} u'(0) &= \\ u'(1)+2u(1) &= \end{aligned} \begin{cases} -C_1\mu + C_2\mu = 0 \\ C_1(-\mu+2)e^{-\mu} + C_2(\mu+2)e^{\mu} = 0 \end{cases}$$

We need $u \neq 0 \Rightarrow (C_1, C_2) \neq 0 \Rightarrow$ determinant $= 0$

$$\begin{vmatrix} -1, & 1 \\ (-\mu+2)e^{-\mu}, & (\mu+2)e^{\mu} \end{vmatrix} = -(\mu+2)e^{\mu} - (2-\mu)e^{-\mu} < 0$$

since $|(\mu+2)e^{\mu}| > |2-\mu|e^{-\mu}$

no solutions

<u>Answer</u> $\lambda_k = \mu_k^2 - 1$, $k = 1, 2, \ldots$ from the graph

$u_k = \cos(\mu_k x)$ - eigenfunctions

Now we need to prove / "show" that there are infinitely many eigenvalues $\lambda_1 < \lambda_2 < \ldots$, $\lambda_n \to \infty$ and the Fourier series

$$\begin{cases} Lu = f \\ \ell u = 0 \end{cases} \Leftrightarrow u(x) = \int_0^{\ell} f(s)\, \underbrace{g(x,s)}_{\text{Green function}}\, ds$$

$g(x,s) = g(s,x)$

Let the problem be regular

Note that $f = \sqrt{w(x)}\, \hat{f}(x)$

$$\sqrt{w(x)}\, u(x) = \int_0^{\ell} \cancel{f(s)\sqrt{w(s)}}$$

$$= \int_0^{\ell} \hat{f}(x) \sqrt{w(s)}\, g(x,s) \sqrt{w(x)}\, ds$$