Examples

Lecture 11 19/12-2021

1. Is $x(t) \equiv 0$ a stable solution of

$\ddot{x} - \pi\dot{x} + 7x = 0$ — NO

$\ddot{x} + 100x = 0$ Yes, But not as stable

$\ddot{x} + \pi^3\dot{x} + (\pi - e)x = 0$ Yes.

In all 3 cases the same is true with respect to any solution $x_*(t)$ of

Indeed, $\ddot{x} + b\dot{x} + cx = f(t)$, $x_*(t)$ solution

Denote $\Rightarrow$ $y = x(t) - x_*(t)$

$\Rightarrow$ $\ddot{y} + b\dot{y} + cy = 0$

$y \equiv 0 \Leftrightarrow x(t) = x_*(t)$

All solutions of $\ddot{x} + e^{37}\dot{x} + \pi^{20}x = e^t$ 145
are asymptotically stable

All solutions of $\ddot{x} - \dot{x} + \cos(e^2)x = \sin t$
are unstable

2, Does the equation
$$x^{(5)} + 7x^{(4)} - 2\dddot{x} + \pi\dot{x} + x = \operatorname{arctg} t^2$$
have stable solutions? NO

Once more $y = x - x_*$

3, Does the equation
$$x^{(6)} + \frac{1}{t^2+1}x^{(5)} - x = \operatorname{arctg} e^t$$
have as. stable solutions? – NO

$$y = x - x_*, \quad y^{(6)} + \frac{1}{t^2+1}y^{(5)} - y = 0$$

Abel-Liouville; solutions $y_1, y_2, \dots, y_6$

$$W[y_1, \dots, y_6](t) = \begin{vmatrix} y_1(t) & \cdots & y_6(t) \\ \vdots & & \\ y_1^{(5)}(t) & \cdots & y_6^{(5)}(t) \end{vmatrix}, \quad \dot{W} = -\frac{1}{t^2+1}W$$

$$\frac{\dot{W}}{W} = -\frac{1}{t^2+1}, \ W \neq 0 \Rightarrow \int \frac{\dot{W}}{W}\,dt = -\int \frac{dt}{t^2+1}$$

$\boxed{\text{But } \ln|W| = \operatorname{arccot} t + c, \quad W(t) = \tilde{c}\, e^{\operatorname{arccot} t}, \quad \tilde{c} = W(0)e^{-\frac{\pi}{2}}}$

$$\ln|W| = -\operatorname{arctg} t + C$$

$$W(t) = \pm e^C e^{-\operatorname{arctg} t} \qquad W(0) = \pm e^C$$

$$W(t) = W(0)e^{-\operatorname{arctg} t} \qquad \cancel{\ngtr} |W(t)| \geq |W(0)|e^{-\frac{\pi}{2}} > 0$$

Thus one of the columns $\begin{pmatrix} y_j \\ \dot{y}_j \\ \vdots \\ y_j^{(5)} \end{pmatrix} \not\to 0$

$\boxed{\text{Also true}}$

~~Thus~~

Example Check the stability of the constant solutions for

$$\ddot{x} + 2\dot{x} + \cos(\dot{x} - x) = 0$$

Solution If $x(t) = const \Rightarrow \dot{x} = \ddot{x} = 0$

$$\Rightarrow \cos(-x) = 0$$

$$x = \frac{\pi}{2} + \pi k, \ k \in \mathbb{Z}$$

Let $y_1 = x$, $y_2 = \dot{x}$

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -2y_2 - \cos(y_2 - y_1) \end{cases}$$

critical points:

$$\begin{pmatrix} y_1 = \frac{\pi}{2} + \pi k \\ y_2 = 0 \end{pmatrix}$$

Linearization $\dot{z} = Az$

$$A = \begin{pmatrix} 0 & 1 \\ -\sin(y_2 - y_1) & \sin(y_2 - y_1) - 2 \end{pmatrix}\Bigg|_{\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{\pi}{2} + \pi k \\ 0 \end{pmatrix}}$$

$$A = \begin{pmatrix} 0 & 1 \\ -\sin(-\frac{\pi}{2} - \pi k), & \sin(-\frac{\pi}{2} - \pi k) - 2 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 \\ (-1)^k, & (-1)^{k+1} - 2 \end{pmatrix}$$

$$p(\lambda) = \lambda^2 + (2 + (-1)^k)\lambda + (-1)^{k+1}$$

$\boxed{x = \frac{\pi}{2} + \pi k}$	$k+1 = 2l \Rightarrow k$ is an odd number, $\boxed{k \in 2\mathbb{Z}+1}$	
$\lambda^2 + \lambda + 1$	$\Rightarrow$ as. stability	
$k \in 2\mathbb{Z}$ $\lambda^2 + 3\lambda - 1$	unstable	

Criterion by Routh-Hurwitz

$\lambda^n + a_1\lambda^{n-1} + \dots + a_n$, $a_1,\dots,a_n \in \mathbb{R}$

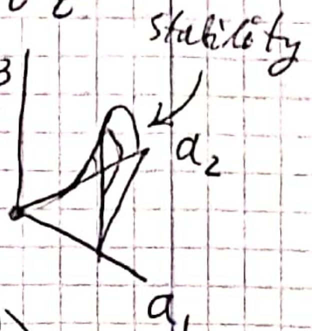
Is it Hurwitz? – Wikipedia
The complete criterion

Some determinants built of the coefficients are to be positive.

One case only! $n=3$

$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ is Hurwitz

$$\Leftrightarrow \underbrace{a_1, a_2 > 0, \ 0 <}_{\text{required}} \underbrace{a_3 < a_1 a_2}_{\text{new}}$$



There is no stability (roots in $\mathbb{C}_+$) if one of inequalities is ~~reversed~~ inverted "<" → ">" from continuity

Example Does equation

1. $\dddot{x} + \ddot{x} + \pi\dot{x} + 3x = e^t$

have a stable solution?

Answer: NO. Indeed, let $y = x - x_*$
x_* – any solution/concrete

$\Rightarrow \dddot{y} + \ddot{y} + \pi\dot{y} + 5y = 0$ $\quad y \equiv 0 \Leftrightarrow x \equiv x_*$

$p(\lambda) = \lambda^3 + \lambda^2 + \pi\lambda + 5$ ~ Routh-Hurwitz!

$1, \pi, 5 > 0$, $\quad 5 < 1\cdot\pi$ wrong

$5 > \pi \Rightarrow$ no stability at all

2. $\dddot{x} + \ddot{x} + \pi\dot{x} + 3x = e^t$ Answer: yes
$3 < \pi$

3. $\dddot{x} + \ddot{x} + \pi\dot{x} + \pi x = e^t$ – not clear according to R-H!

$\lambda^3 + \lambda^2 + \pi\lambda + \pi = (\lambda+1)(\lambda^2+\pi) \Rightarrow$ stability, not asymptotic.

9. $\dddot{x} + \ddot{x} - \dot{x} + x = \operatorname{arctg}(t^2+1)$

There are no stable solutions.

Example To find the stability region of parameters for the system

$$\begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = -3x_1 \\ \dot{x}_3 = -\alpha x_1 + 2x_2 - x_3 \end{cases}$$

$p(\lambda) = \det(A - \lambda I)$

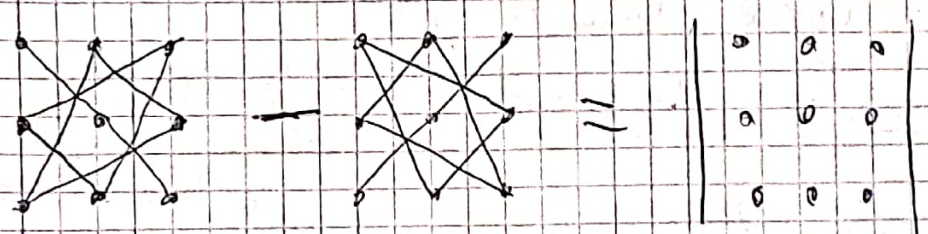
$$= \begin{vmatrix} -\lambda & 0 & 1 \\ -3 & -\lambda & 0 \\ -\alpha & 2 & -1-\alpha \end{vmatrix} = -(\lambda^3 + \lambda^2 + \alpha\lambda + 6)$$

$a_{1,2} > 0 \quad 0 < a_3 < a_1 a_2$

$\Rightarrow \alpha > 0, \quad 6 < \alpha \qquad \boxed{\alpha > 6}$

Answer

Calculation of the determinant 3x3



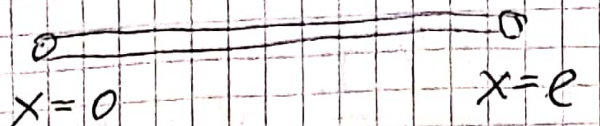
positive triangles negative

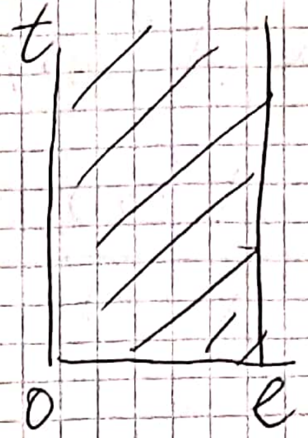
Boundary Value Problems

Sturm-Liouville theory

Motivation:

steel rod





$u(x,t)$ - temperature

$u(x,0) = f(x)$ - initial conditions

At the ends: $f(0) = f(l) = 0$

$u(0,t) = u(l,t) = 0$

The problem is to fill the "glass"

The heat (transfer) equation

$$\alpha^2 u''_{xx} = u'_t, \quad \alpha^2 \left[\frac{cm^2}{s}\right] = \begin{matrix} 0{,}00144 & \text{Water} \\ 0{,}12 & \text{Steel} \end{matrix}$$

Thermal diffusivity

Obviously $f \neq 0 \Rightarrow u \neq 0$

General case boundary conditions

$$\begin{cases} \gamma_1 u'_x(0,t) + \delta_1 u(0,t) = 0 \\ \gamma_2 u'_x(l,t) + \delta_2 u(l,t) = 0 \end{cases}$$

($u'_x(0,t) = u'_x(l,t) = 0$ no heat transfer at the ends)

The Fourier Method of the variables separation

Solution is searched for in the form

$$u(x,t) = X(x)\,T(t), \quad u(x,t) \not\equiv 0$$

substitute and get — since $u(x,0) \not\equiv 0 \Rightarrow X(x), T(t) \not\equiv 0$

$$\alpha^2 X''(x)\,T(t) = X(x)\,T'(t)$$

depends on x $\quad \dfrac{X''(x)}{X(x)} = \dfrac{1}{\alpha^2}\dfrac{T'(t)}{T(t)}$ depends on t

$$= -\lambda = \text{const} \in \mathbb{R}$$

$$u(0,t) \equiv 0 \Rightarrow X(0)\,\underset{\not\equiv 0}{T(t)} = 0 \Rightarrow X(0) = 0$$

$$u(\ell,t) \equiv 0 \Rightarrow X(\ell)\,T(t) \equiv 0 \Rightarrow X(\ell) = 0$$

For any $a \neq 0$ (const $\in \mathbb{R}$) one can redefine

$$u(x,t) = X(x)T(t) = \underbrace{\tfrac{1}{a}X(x)}\,\underbrace{aT(t)}$$
$$= \tilde{X}(x)\cdot\tilde{T}(t)$$

Thus one can assume $T(0) = 1$, and

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ \underbrace{X(0) = X(\ell) = 0}_{\text{boundary conditions}} \end{cases} \qquad \begin{cases} T'(t) + \alpha^2\lambda T(t) = 0 \\ T(0) = 1 \end{cases}$$

$$T(t) = e^{-\alpha^2\lambda t}$$

$$t = 0 \Rightarrow u(x,0) = X(x)\,T(0) = f(x)$$

not each $f(x)$ is admissible! $\quad X(x) = f(x)$, but f may not satisfy the equation $f'' + \lambda f = 0$, $f(0) = f(\ell) = 0$

Determine $X(x) \not\equiv 0$

1. $\lambda > 0 \Rightarrow X = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

$X(0) = 0 \Rightarrow c_1 = 0,\ X(l) = 0 \Rightarrow \sin(\sqrt{\lambda}\, l) = 0$

$\sqrt{\lambda}\, l = n\pi,\ n \in \mathbb{N}$

$$\underbrace{\lambda = \frac{n^2\pi^2}{l^2},\ X(x) = \sin\left(\frac{n\pi}{l}x\right)}$$

2. $\lambda = 0 \quad X''(x) = 0 \quad X(x) = c_1 x + c_2$

$X(0) = X(l) = 0 \Rightarrow c_2 = 0 \Rightarrow c_1 = 0 \Rightarrow X = 0$

$\Rightarrow u = 0$ no nontrivial solution

3. $\lambda < 0 \quad X = c_1 e^{\mu x} + c_2 e^{-\mu x},\ \mu = \sqrt{|\lambda|} > 0$

$$\begin{array}{ll} X(0) = 0 \Rightarrow & \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\mu l} + c_2 e^{-\mu l} = 0 \end{cases} \\ X(l) = 0 \Rightarrow & \end{array} \quad c_1, c_2 = ? \quad (c_1, c_2) \neq (0,0)$$

$$\underbrace{\begin{vmatrix} 1 & 1 \\ e^{\mu l} & e^{-\mu l} \end{vmatrix}} = 0 \text{ should be for } (c_1, c_2) \neq (0,0)$$

$$e^{\mu l} - e^{-\mu l} = e^{-\mu l}\left(e^{2\mu l} - 1\right) > 0$$

No solutions

The only solutions are

$n = 1, 2, \dots,\quad u_n = \gamma_n \sin\left(\frac{n\pi}{l}x\right) e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}$

That means that $f(x) = u(x, 0)$ should be of the form $\boxed{f(x) = \gamma \sin\left(\frac{n\pi}{l}x\right)}$

Let now the solution be of the form
$$\boxed{u = \sum_{j=1}^{\infty} \gamma_j u_j(x,t)}$$
$$u_j = \sin\left(\frac{n\pi}{\ell}x\right) e^{-\alpha^2 \frac{n^2\pi^2}{\ell^2} t}$$
Then
$$f(x) = \sum_{j=1}^{\infty} \gamma_j \sin\left(\frac{n\pi}{\ell}x\right)$$
It is the Fourier series
$$\boxed{\gamma_j = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi}{\ell}x\right)dx}$$

Boundary Value Problem

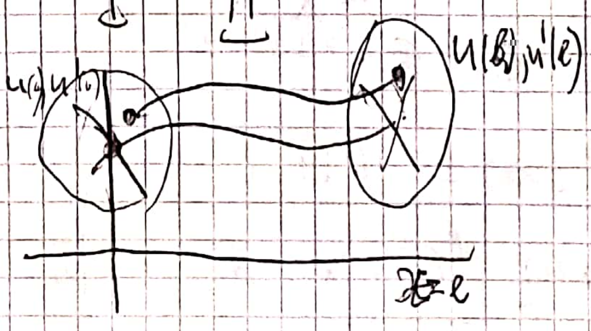
$$\begin{cases} u'' + a(x)u' + b(x)u = f(x) \\ \alpha_1 u'(0) + \beta_1 u(0) = \gamma_1 \\ \alpha_2 u'(\ell) + \beta_2 u(\ell) = \gamma_2 \end{cases}$$

$u: \mathbb{R} \to \mathbb{R}$
$u \in C^2[0,\ell]$
$f, a, b \in C$
Boundary Conditions

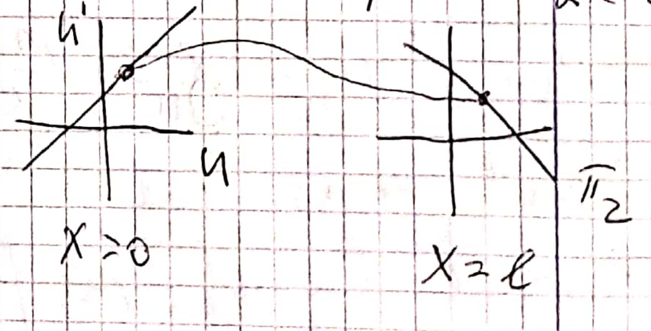
$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, $(\alpha_2, \beta_2) \neq (0,0)$, $(\alpha_1, \beta_1) \neq (0,0)$

It is alternative to the Cauchy Problem
In both cases we have 2 conditions

Cauchy problem
Initial conditions
$(u(0), u'(0)) = (\xi_1, \xi_2)$
I II



Boundary Value problem
I $\alpha_1 u'(0) + \beta_1 u(0) = \gamma_1$ (Π_1)
II $\alpha_2 u'(\ell) + \beta_2 u(\ell) = \gamma_2$ (Π_2)



Contrary to the Cauchy problem a solution for the boundary value problem can be absent, or there can be infinitely many solutions

Boundary conditions can be <u>not separated</u> when $x=0, l$ appear in both conditions. We only consider <u>separated</u> boundary conditions.

<u>Example</u> periodical boundary conditions $u'(0)=u'(l)$, $u(0)=u(l)$ are not separated. A similar theory can be developed as in the following.

<u>Definition</u> The boundary-value problem is said to have the standard form if it is in the form

$$\begin{cases} (u'p(x))' + q(x)\,u = f(x) & q\in C,\ f\in C \\ & p>0,\ p\in C^1 \\ \begin{cases} \alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_2 u'(l) + \beta_2 u(l) = 0 \end{cases} & \forall x\in[0,l]:\ p(x)>0 \\ & \alpha_1,\alpha_2,\beta_1,\beta_2 \in \mathbb{R} \\ & (\alpha_1,\beta_1),(\alpha_2,\beta_2)\neq(0,0) \end{cases}$$

<u>Proposition</u> The transfer to the standard form is always possible

Partial proof

1. Get rid of a $\quad p = ?$

$$p u'' + p a u' + p b u = p f \quad \text{multiply by } p$$

$$(p u')' + q u = p u'' + p' u' + q u$$

Require $p' = p a \Rightarrow \frac{p'}{p} = a$, $p(x) = e^{\int_0^x a(t)\,dt}$

Obviously $p(x) > 0$, $p \in C^1$

$q = p(x) b(x)$, $\quad \tilde{f}(x) = p(x) f(x)$
new function

2. We have obtained

$$\begin{cases} (p u')' + q u = f \\ \alpha_1 u'(0) + \beta_1 u(0) = \gamma_1 \\ \alpha_2 u'(l) + \beta_2 u(l) = \gamma_2 \end{cases}$$

$(\alpha_1, \beta_1) \neq (0,0)$
$(\alpha_2, \beta_2) \neq (0,0)$
Get rid of γ_1, γ_2

We need to find $v(x)$ satisfying

$\cancel{(p v')' + q v = 0}$ $\quad v \in C^2$

$$\begin{cases} \alpha_1 v'(0) + \beta_1 v(0) = \gamma_1 \\ \alpha_2 v'(l) + \beta_2 v(l) = \gamma_2 \end{cases}$$

then taking $\tilde{u} = u - v$ obtain

$$\begin{cases} (p \tilde{u}')' + q \tilde{u} = f(x) - (p v')' - q v \\ \alpha_1 \tilde{u}'(0) + \beta_1 \tilde{u}(0) = 0 \\ \alpha_2 \tilde{u}'(l) + \beta_2 \tilde{u}(l) = 0 \end{cases}$$

Such a function is simply found considering the cases $\beta_1 = \beta_2 = 0$; $\beta_1 = 0, \beta_2 \neq 0$; $\beta_1 \neq 0, \beta_2 = 0$; $\beta_1 \neq 0, \beta_2 \neq 0$ $\quad$ QED

Now we have the Standard BV Problem
boundary-value

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Denote $Lu = (pu')' + qu$

$$\mathcal{D}u = \begin{pmatrix} \alpha_1 u'(0) + \beta_1 u(0) \\ \alpha_2 u'(\ell) + \beta_2 u(\ell) \end{pmatrix} \in \mathbb{R}^2$$

The problem $\begin{cases} Lu = f \\ \mathcal{D}u = 0 \end{cases}$ $\quad p \in C^1, p > 0$; $q, f \in C$; $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \neq 0$

All numbers are real

$L: C^2[0,\ell] \to C[0,\ell]$, $\mathcal{D}: C^1[0,\ell] \to \mathbb{R}^2$

twice continuously differentiable; continuous; continuously differentiable

<u>Definition</u> The Standard BVP is called <u>regular</u>, if $\begin{cases} Lu = 0 \\ \mathcal{D}u = 0 \end{cases} \Leftrightarrow u = 0$
$(u(x) \equiv 0)$

Here and further we see the similarity between the standard BV problem and the standard linear equation $Ax = f$
$x \in \mathbb{R}^n$, $A: \mathbb{R}^n \to \mathbb{R}^n$

$Ax = 0 \Leftrightarrow x = 0$ means $\exists A^{-1}$, $\ker A = 0$

$Ax = f \Rightarrow x = A^{-1}f$ the unique solution

<u>Theorem</u> A regular BV problem has always a solution, and it is unique.

Uniqueness is trivial:

$$\begin{cases} Lu_1 = f \\ \mathcal{D}u_1 = 0 \end{cases} \quad \begin{cases} Lu_2 = f \\ \mathcal{D}u_2 = 0 \end{cases} \Rightarrow \begin{cases} L(u_1 - u_2) = 0 \\ \mathcal{D}(u_1 - u_2) = 0 \end{cases}$$

$$\Rightarrow u_1 - u_2 = 0$$

In order to prove the existence of the solution we will ~~build~~ construct the inverse operator using a Green function.

<u>Theorem</u> If the standard B.V problem is not regular then either there are infinitely many solutions or there is no solution at all

<u>Proof</u>: Suppose $\begin{cases} Lu_* = 0 \\ \mathcal{D}u_* = 0, \end{cases}$ $u_* \neq 0$

and there is a solution $\begin{cases} Lu = f \\ \mathcal{D}u = 0 \end{cases}$

Then also $\begin{cases} L(u+\alpha u_*) = f \\ \mathcal{D}(u+\alpha u_*) = 0 \end{cases}$ for any $\alpha \in \mathbb{R}$

If there is no solution then there is nothing to prove QED

ψ - psi

<u>Theorem</u> Let $\psi_1(x), \psi_2(x)$ be any two fundamental solutions of the linear ODE $Lu = 0$, then the standard B-V problem is regular

$$\Leftrightarrow \det \underbrace{\begin{pmatrix} \alpha_1 \psi_1'(0) + \beta_1 \psi_1(0), & \alpha_1 \psi_2'(0) + \beta_1 \psi_2(0) \\ \alpha_2 \psi_1'(l) + \beta_2 \psi_1(l), & \alpha_2 \psi_2'(l) + \beta_2 \psi_2(l) \end{pmatrix}}_{\text{matrix } G} \neq 0$$

Proof $Lu = 0 \Leftrightarrow u = c_1\psi_1 + c_2\psi_2$

Then $\mathcal{D}(c_1\psi_1 + c_2\psi_2) = 0$, $c_1, c_2 \in \mathbb{R}$

$$\underbrace{\left(\mathcal{D}\psi_1, \mathcal{D}\psi_2\right)}_{G}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$\det G \neq 0 \Rightarrow \begin{matrix} c_1 = 0 \\ c_2 = 0 \\ u = 0 \end{matrix}$, regular problem

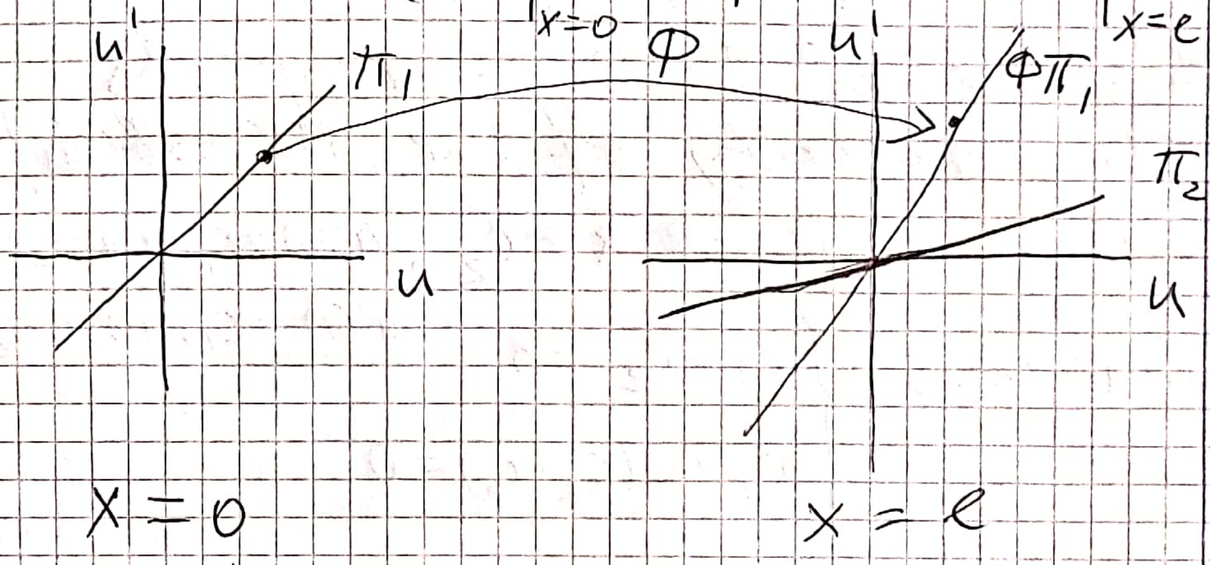
$\det G = 0 \Leftrightarrow \exists \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq 0$, $u = c_1\psi_1 + c_2\psi_2 \neq 0$

linearly independent

QED

Explanation ODE $Lu = 0$ defines a phase stream Φ

Each point $(u, u')\big|_{x=0}$ transfers to $(u, u')\big|_{x=\ell}$



π_1: $\alpha_1 u' + \beta_1 u = 0$ π_2: $\alpha_2 u' + \beta_2 u = 0$

Φ maps ~~a~~ direct lines to direct lines

$\Phi: 0 \mapsto 0$, $\Phi: \pi_1 \mapsto \Phi\pi_1$, Φ is linear

$\Phi\pi_1 \neq \pi_2 \Leftrightarrow$ The only intersection $\Phi\pi_1 \cap \pi_2$ is at 0

The problem is regular $\Leftrightarrow$ $\Phi\pi_1 \neq \pi_2 \Leftrightarrow \det G \neq 0$

Let $u = c_1\psi_1 + c_2\psi_2$ then

$$\left(\alpha_1\psi_1'(0)+\beta_1\psi_1(0),\ \alpha_1\psi_2'(0)+\beta_1\psi_2(0)\right)\begin{pmatrix}c_1\\c_2\end{pmatrix}=0$$

means that $(u,u')\big|_{x=0} \in \pi_1$

$$\left(\alpha_2\psi_1'(\ell)+\beta_2\psi_1(\ell),\ \alpha_2\psi_2'(\ell)+\beta_2\psi_2(\ell)\right)\begin{pmatrix}c_1\\c_2\end{pmatrix}=0$$

means that $(u,u')\big|_{x=\ell} \in \pi_2$

$\boxed{GC=0}$ QED.

In reality one never uses the formula

Example

$$\begin{cases} u''=e^x \\ u'(0)-u(0)=0 \\ u'(1)+\alpha u(1)=0 \end{cases}$$

Which value of $\alpha\in\mathbb{R}$ corresponds to the regularity of the BV problem?

$u''=0 \Rightarrow u=c_1x+c_2,\quad u'(0)=c_1,\ u(0)=c_2$
$u'(1)=c_1,\ u(1)=c_1+c_2$

$$\mathcal{D}u=0 \Leftrightarrow \begin{cases} c_1-c_2=0 \\ c_1+\alpha(c_1+c_2)=0 \end{cases}$$

$$\begin{vmatrix} 1 & -1 \\ 1+\alpha & \alpha \end{vmatrix} = \alpha+1+\alpha=2\alpha+1$$

Answer: $\alpha\neq-\frac{1}{2}$ regular BVP

$\alpha=-\frac{1}{2}$ irregular one

$$\begin{cases} u''=e^x \Rightarrow u=e^x+c_1x+c_2 \\ u'(0)-u(0)=0 \\ u'(1)-\frac{1}{2}u(1)=0 \end{cases} \quad \boxed{\alpha=-\frac{1}{2}}$$

The boundary conditions yield

$$\begin{cases} e^0 + c_1 - e^0 - c_2 = 0 \\ e + c_1 - \frac{1}{2}(e + c_1 + c_2) = 0 \end{cases} \qquad \begin{cases} c_1 = c_2 \\ \frac{1}{2}e + \frac{1}{2}c_1 - \frac{1}{2}c_2 = 0 \end{cases}$$

no solutions $\Leftarrow e = 0$

Example

$$\begin{cases} u'' + 4u = \sin x \\ u'(0) + u(0) = 0 \\ u\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

$u'' + 4u = 0 \Rightarrow u = c_1 \cos 2x + c_2 \sin 2x$

$u\left(\frac{\pi}{2}\right) = 0 \Rightarrow c_1 = 0,\ u = c_2 \sin 2x$

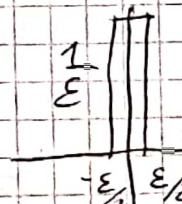
$\Rightarrow 0 = u'(0) + u(0) = 2c_2 + 0 \Rightarrow c_2 = 0$

The BVP is regular

~~Flun~~ The Green Function

$$\begin{cases} Lu = f \\ Du = 0 \end{cases} \qquad \begin{cases} (pu')' + qu = f \\ \alpha_1 u'(0) + \beta_1 u(0) = 0 \\ \alpha_2 u'(l) + \beta_2 u(l) = 0 \end{cases} \qquad \begin{matrix} p \in C^1,\ p > 0,\ u \in C^2 \\ q, f \in C \\ (\alpha_1, \beta_1), (\alpha_2, \beta_2) \neq 0 \end{matrix}$$

Let $\delta_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & |x| \le \frac{1}{2}\varepsilon \\ 0, & |x| > \frac{1}{2}\varepsilon \end{cases}$

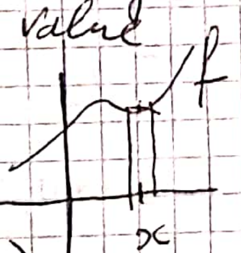


δ-sequence

(it should be smoothed)

Then $f(x) \simeq \int_0^l f(s)\,\delta_\varepsilon(x-s)\,ds = \bar{f}_\varepsilon(x)$ average value

$= \frac{1}{\varepsilon} \int_{x-\frac{\varepsilon}{2}}^{x+\frac{\varepsilon}{2}} f(s)\,ds$



In its turn $\bar{f}(x) \simeq \sum_i f(s_i)\,\delta_\varepsilon(x - s_i)\,\varepsilon = \bar{\bar{f}}_\varepsilon(x)$

integral sum

Let $\begin{cases} L g_\varepsilon^{(x,s)} = \delta_\varepsilon(x-s) \\ D g_\varepsilon^{(x,s)} = 0 \end{cases}$ $\quad \varepsilon \to 0$, $\; g_\varepsilon(x,s) \to g(x,s)$, $\; \bar{f}_\varepsilon \to f$

differentiation with respect to x

$$\bar{f}_\varepsilon(x) = \sum_i f(s_i)\,\delta_\varepsilon(x-s_i)\,\varepsilon$$

$$u_\varepsilon(x) = \sum_i f(s_i)\,g_\varepsilon(x,s_i)\,\varepsilon \quad \text{integral sum}$$

Obviously $\begin{cases} L u_\varepsilon = \bar{f}_\varepsilon \\ D u_\varepsilon = 0 \end{cases} \xrightarrow[\substack{u_\varepsilon \to u \\ \bar{f}_\varepsilon \to f}]{\varepsilon \to 0} \begin{cases} Lu = f \\ Du = 0 \end{cases}$

where $\boxed{u(x) = \int f(s)\, g(x,s)\, ds}$

the Green function

This calculation can be done rigorously but we will go by another way

solve qualitatively $\begin{cases} L g_\varepsilon(x,s) = \delta_\varepsilon(x-s) \\ D g_\varepsilon(x,s) = 0 \end{cases}$

$$L g_\varepsilon(x,s) = (p(x) g_\varepsilon')' + q(x) g_\varepsilon = \delta_\varepsilon(x-s)$$

1. $\varepsilon \to 0,\ x \neq s$ — $Lg(x,s) = 0 \Rightarrow g \in C^2$ in x for $x \neq s$

2. $\varepsilon \to 0,\ x = s$ Take $\int_{s-\varepsilon}^{s+\varepsilon} (\ )\, dx$

$$p(s+\varepsilon) g'(s+\varepsilon) - p(s-\varepsilon) g'(s-\varepsilon) + O(\varepsilon) = 1$$

$$p(s)\,g'(s+0,s) - p(s)\,g'(s-0,s) = 1, \quad x \searrow s, \ x \nearrow s$$

$$g'(s+0,s) - g'(s-0,s) = \frac{1}{p(s)}$$ derivative jump

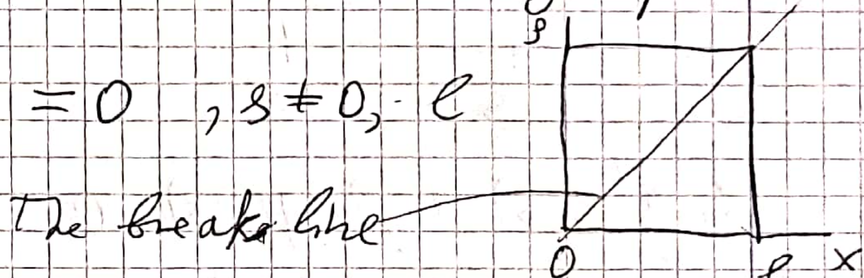
3. $\mathcal{D} g_\varepsilon(x,s) = 0 \Rightarrow \mathcal{D} g(x,s) = 0$

$s \neq 0, \ell$ where g is not differentiable

Formal Definition

A function $g(x,s) : \underset{I}{[0,\ell]} \times [0,\ell] \to \mathbb{R}$, is called the Green function for the standard B-V problem $\begin{cases} Lg = f \\ \mathcal{D}u = 0 \end{cases}$ if

1. $Lg(x,s) = 0$ for $x \neq s$, $g \in C[I \times I]$
   $g \in C^2$ for $x \neq s$
2. $g'_x(s+0,s) - g'_x(s-0,s) = \frac{1}{p(s)}$ derivative jump
3. $\mathcal{D} g(\cdot, s) = 0$, $s \neq 0, \ell$



Theorem 1 BV problem is regular $\Rightarrow \exists!$ Green function

Moreover $g(x,s) = g(s,x)$

We will directly construct it

Theorem 2 BVP is regular $\Rightarrow$ There is not more than one function satisfying $u(x) = \int_0^\ell f(s)\, g(x,s)\, ds$, $g \in C$ $\begin{cases} Lu = f \\ \mathcal{D}u = 0 \end{cases}$

I.E. if there are two functions

$g_1, g_2 \in C$: $u(x) = \int_0^l f(s)\, g_1(x,s)\, ds$

regular BVP $\Longrightarrow$ $\| \quad u(x) = \int_0^l f(s)\, g_2(x,s)\, ds$

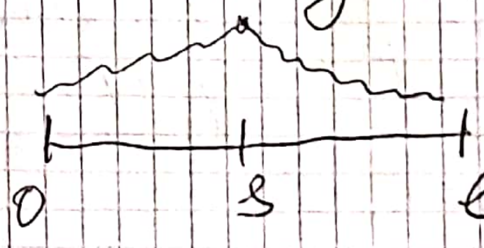
$\Rightarrow g_1 \equiv g_2$.

Theorem 3. The function which we have constructed in Theorem 1 satisfies the formula

$u(x) = \int_0^l f(s)\, g(x,s)\, ds$ which solves the BVP.

Proof of Theorem 1

1. $x \neq s \Rightarrow Lg(x,s) = 0$ $\quad g$ – solution, $s = \text{const}$



$\alpha_1 g'(0,s) + \beta_1 g(0,s) = 0$, $\quad x \in [0, s]$

Let $\varphi_1(x)$ satisfy $\varphi_1(0) = \alpha_1$, $\varphi_1'(0) = -\beta_1$, $L\varphi_1 = 0$

Then
if: $\begin{cases} \alpha_1 \varphi_1'(0) + \beta_1 \varphi_1(0) = -\alpha_1\beta_1 + \beta_1\alpha_1 = 0 \\ \alpha_1 \tilde{\varphi}_1'(0) + \beta_1 \tilde{\varphi}_1(0) = 0 \end{cases}$, $(\alpha_1, \beta_1) \neq (0,0)$

$\Rightarrow 0 = \begin{vmatrix} \varphi_1'(0) & \varphi_1(0) \\ \tilde{\varphi}_1'(0) & \tilde{\varphi}_1(0) \end{vmatrix} = W[\varphi_1, \tilde{\varphi}_1](0) \Rightarrow$

$\tilde{\varphi}_1$ is proportional to φ_1

In particular

$$g(x,\xi) = C_1 \varphi_1(x) \quad , 0 \le x \le \xi$$

Now let $x \in [\xi, l]$ and let

$\varphi_2'(l) = -\beta_2, \quad \varphi_2(l) = \alpha_2$

Then similarly get that

$$g(x,\xi) = C_2 \varphi_2(x), \quad \xi \le x \le l$$

Since C_1, C_2 depend on ξ get

$$g(x,\xi) = \begin{cases} C_1(\xi)\varphi_1(x) & x \in [0,\xi] \\ C_2(\xi)\varphi_2(x) & x \in [\xi, l] \end{cases}$$

(*) 1. $C_1(\xi)\varphi_1(\xi) - C_2(\xi)\varphi_2(\xi) = 0$ continuity

2. $-C_1(\xi)\varphi_1'(\xi) + C_2(\xi)\varphi_2'(\xi) = \frac{1}{p(\xi)}$ derivative jump

The determinant

$$\begin{vmatrix} \varphi_1(\xi), & -\varphi_2(\xi) \\ -\varphi_1'(\xi), & \varphi_2'(\xi) \end{vmatrix} = \begin{vmatrix} \varphi_1(\xi), & \varphi_2(\xi) \\ \varphi_1'(\xi), & \varphi_2'(\xi) \end{vmatrix} = W(\xi)$$

Suppose that $W(\xi) = 0 \Rightarrow W(x) = 0$

$\Rightarrow \varphi_2 = \mu\varphi_1$ (since $\varphi_1 \neq 0$), $\mu = $ const

$\Rightarrow \begin{cases} L\varphi_1 = 0 & \text{according to construction} \\ D\varphi_1 = 0 & \text{since } \varphi_2 = \mu\varphi_1 \end{cases}$

$\Rightarrow$ The BVP is not regular

contradiction

Now look at $Lu = 0$

$(pu')' + qu = \cancel{f}\,0$

$u'' + \frac{p'}{p} u' + \frac{q}{p} u = 0$, φ_1, φ_2 are solutions

$\Rightarrow W' = -\frac{p'(x)}{p(x)} W$ (Abel-Liouville)

$\ln|W(x)| = \ln \frac{1}{p(x)} + A$, $A = \text{const}$

$W(x) = B \frac{1}{p(x)}$, $B = e^{\pm A}, 0$

$\Rightarrow p(x)\, W(x) = B = \text{const}$

Solve (*): The Kramer rule

$$C_1 = \frac{1}{W(\xi)} \begin{vmatrix} 0 & -\varphi_2(\xi) \\ \frac{1}{p(\xi)} & \varphi_2'(\xi) \end{vmatrix}, \quad C_2 = \frac{1}{W(\xi)} \begin{vmatrix} \varphi_1(\xi) & 0 \\ -\varphi_1'(\xi) & \frac{1}{p(\xi)} \end{vmatrix}$$

$$C_1 = \frac{\varphi_2(\xi)}{p(\xi) W(\xi)}, \quad C_2 = \frac{\varphi_1(\xi)}{p(\xi) W(\xi)}$$

$$C_1 = \frac{1}{B} \varphi_2(\xi), \quad C_2 = \frac{1}{B} \varphi_1(\xi)$$

$$g(x,\xi) = \begin{cases} \frac{1}{B} \varphi_1(x) \varphi_2(\xi), & x \in [0, \xi] \\ \frac{1}{B} \varphi_1(\xi) \varphi_2(x), & x \in [\xi, \ell] \end{cases}$$

$$\boxed{g(x,\xi) = g(\xi, x)} \; !$$

QED

Example $\begin{cases} u'' + u = f(x) \\ u(0) = u(\frac{\pi}{2}) = 0 \end{cases}$

To find the Green function

$u'' + u = 0 \Rightarrow u = C_1 \cos x + C_2 \sin x$

$f = 0 \quad Lu = 0, \; Du = 0 \Rightarrow C_1 = 0, \; C_2 = 0$

Regular problem

$u(0) = 0 \Rightarrow \varphi_1 = \sin x$

$u(\frac{\pi}{2}) = 0 \Rightarrow \varphi_2 = \cos x$

$$g(x, s) = \begin{cases} C_1(s) \sin(x) & x \in [0, s] \\ C_2(s) \cos x & x \in [s, \frac{\pi}{2}] \end{cases}$$

Continuity: $C_1(s) \sin s = C_2(s) \cos s$

derivative jump: $-C_2(s) \sin s - C_1(s) \cos s = \frac{1}{1}, \; p = 1$

$$\begin{cases} C_1 \sin s - C_2 \cos s = 0 \\ C_1 \cos s + C_2 \sin s = 1 \end{cases}, \quad \begin{matrix} \text{determinant } \Delta \\ W = 1 \\ W = \Delta \end{matrix}$$

$$C_1 = -\frac{\cos s}{1}, \quad C_2 = -\frac{\sin s}{1}$$

$$g(x, s) = \begin{cases} -\cos s \sin x, & x \in [0, s] \\ -\sin s \cos x, & x \in [s, \frac{\pi}{2}] \end{cases}$$

$$u(x) = \int_0^{\frac{\pi}{2}} f(s) g(x, s) \, ds$$

Example: $f(x) = 1$

$$\begin{cases} u'' + u = 1 \\ u(0) = 0,\ u(\frac{\pi}{2}) = 0 \end{cases} \qquad u = 1 + C_1 \cos x + C_2 \sin x$$

Without the Green function

get $u = 1 - \cos x - \sin x$

$$u(x) = \int_0^{\frac{\pi}{2}} 1 \cdot g(x,s)\,ds = \int_0^{x} g(x,s)\,ds + \int_x^{\frac{\pi}{2}} g(x,s)\,ds$$

with $x \geq s$ in the first integral and $x \leq s$ in the second.

$$u(x) = \int_0^x (-\sin s \cos x)\,ds + \int_x^{\frac{\pi}{2}} (-\cos s \sin x)\,ds$$

$$= -\cos x \int_0^x \sin s\,ds - \sin x \int_x^{\frac{\pi}{2}} \cos s\,ds$$

$$= \cos x \cos s \Big|_0^x - \sin x \sin s \Big|_x^{\frac{\pi}{2}}$$

$$= \cos^2 x - \cos x - \sin s + \sin^2 x = 1 - \cos x - \sin x$$