Lecture 10, 12/12-2021

Proof plan for Theorem 1

1. There exists a Lyapunov function $\underbrace{}_{V_L(x)}$ for $\dot{x} = Ax$, $x \in \mathbb{R}^n$

2. The same function $V(x) = V_L(x - \xi)$ serves as ~~the~~ a Lyapunov function for the original equation

Proof 1 $A \in \mathbb{R}^{n\times n}$, Spec $A \subset \mathbb{C}_-$

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

~~The~~ Lyapunov function exists of the form $V_L(x) = x^T H x$ pos. definite

symmetric $H = H^T$, $A, H \in \mathbb{R}^{n\times n}$

$$\dot{V}_L = \frac{d}{dt}(x^T H x) = (Ax)^T H x + x^T H A x$$ 132

$$= x^T(A^T H + H A) x$$

Correspondingly one needs

The Lyapunov equation

$$A^T H + H A = \underbrace{Q < 0}_{\text{negative definite}}, \quad Q^T = Q$$

(means $-x^T Q x$ positive def)

Lemma (Lyapunov) If $\operatorname{Spec} A \subset \mathbb{C}_-$

Then for any $Q \in \mathbb{R}^{n\times n}$ (symmetric)

exists (unique!) symmetric H and only one.

Formula: $H = -\int_0^\infty e^{A^T t} Q e^{At}\, dt$ symmetric, converges for $\mathbb{C}_- \supset \operatorname{Spec} A$

Proof

$$A^T H + H A = -\int_0^\infty \left(A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A\right) dt$$

$$= -\int_0^\infty \left(\frac{d}{dt} e^{A^T t} Q e^{At}\right) dt = \cancel{Q+A}$$

$$= -e^{A^T t} Q e^{At}\Big|_0^\infty = \underset{\operatorname{Spec} A \subset \mathbb{C}_-}{0} + Q$$

It is not convenient though

Uniqueness:

$$\begin{array}{ccc} \text{Symmetric Matrices} & \xrightarrow{\pi} & \text{Symm. } M \\ \cup & & \cup \\ H & \xmapsto[\pi]{A^T H + H A} & Q \\ H & \xleftarrow[\pi^{-1}]{\text{Int. formula}} & Q \end{array}$$

π: $H \to A^T H + H A$

It is a linear mapping surjection and bijection

$\Rightarrow$ no kernel

Q.E.D

Examples

1.
$$\begin{cases} \dot{x} = \boxed{x - y} + x^2 + y^2 \sin t \\ \dot{y} = \boxed{x + y} - y^2 \\ \dot{z} = 0 \end{cases}$$

$$p(\lambda) = \begin{vmatrix} 1-\lambda, & -1 \\ 1, & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2, \quad \lambda_{1,2} = +1 \pm i$$

unstable focal point

2.
$$\begin{cases} \dot{x} = -4y - x^3 \\ \dot{y} = 3x - y^3 \end{cases} \qquad \begin{vmatrix} -\lambda & -4 \\ 3 & -\lambda \end{vmatrix} = \lambda^2 + 12$$

The first approximation (linearization) does not help here

Lyapunov function

$$V(x, y) = 3x^2 + 4y^2$$

$$\dot{V} = 6x(-4y - x^3) + 8y(3x - y^3)$$

$$\dot{V} = -6x^4 - 8y^4 \leq 0 \quad \text{neg. definite}$$

radially unbounded

$\Rightarrow$ asymptotically stable

Fixed-time convergence to ball

moreover: converges in finite time to any ball from infinity!

3.
$$\begin{cases} \dot{x} = -4y + x^3 \\ \dot{y} = 3x + y^3 \end{cases} \qquad p(\lambda) = \lambda^2 + 12$$

$$\dot{V} = 6x^4 + 8y^4 \geq 0 \quad \text{unstable}$$

(Chetaev)

In fact explosion

finite time escape (to infinity)

<u>Additional proof of the Lyapunov lemma</u> (Arnold). It provides an easy way of constructing a Lyapunov function.

Consider $\dot{x} = Ax$, $\operatorname{Spec} A \subset \mathbb{C}_-$
$x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$

Searching for a Lyapunov function.
$V = x^T H x$, $H = H^T \in \mathbb{R}^{n\times n}$

Assume that there exists a basis of the eigenvectors

$e_1, e_2, e_3, \dots, e_n \in \mathbb{C}^n$

$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n \in \mathbb{C}_-$

Change of coordinates: $x = Ez$, $E = (e_1 \dots e_n)$

$z = E^{-1}x$, $\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

<u>Proposition:</u> $V = \bar{z}^T z = \sum_{i=1}^n |z_i|^2 \geq 0$ is the Lyapunov function.

<u>Proof</u> As we have already seen once

Let $\dot{x} = Ax$ $\quad E\dot{z} = AEz = E\Lambda z$

$\mathbb{R}^n \subset \mathbb{C}^n$ $\quad \dot{z} = \Lambda z$ $\quad \dot{z}_i = \lambda_i z_i, \; i = 1, \dots, n$

$(\Rightarrow \dot{\bar{z}}_i = \bar{\lambda}_i \bar{z}_i)$

$$\dot{V} = \left(\sum_{i=1}^n \bar{z}_i z_i\right)^{\cdot} = \sum \left(\bar{\lambda}_i \bar{z}_i z_i + \bar{z}_i \lambda_i z_i\right) =$$

$$= \sum_i (\bar{\lambda}_i + \lambda_i)|z_i|^2 = 2\sum \operatorname{Re}\lambda_i |z_i|^2 \leq 0$$

$\bar{x}^T = x^*$

We need $V = x^T H x$, $H = H^T = H^*_{\text{adjoint}} \in \mathbb{R}^{n\times n}$

$$V = \bar{z}^T z = \underbrace{\bar{x}^T (E^{-1})^* E^{-1}}_{\tilde{H}} x, \quad \tilde{H} = \tilde{H}^* \in \mathbb{C}^{n\times n}$$

2. Let $\Delta x = x - \xi$

$V(x) = \Delta x^T H \Delta x$

$\dot{x} = \Delta\dot{x} = A\Delta x + o(\Delta x)$

$\dot{V} = \Delta\dot{x}^T H \Delta x + \Delta x^T H \Delta\dot{x} =$

$= [\Delta x^T A^T + o(\Delta x)^T] H \Delta x + \Delta x^T H [A\Delta x + o(\Delta x)]$

$= \Delta x^T [A^T H + HA] \Delta x + \underbrace{o(\Delta x)^T H \Delta x + \Delta x^T H o(\Delta x)}_{\text{little } o}$

$\dot{V} = \Delta x^T \underbrace{Q}_{\text{negative definite}} \Delta x + o(\|\Delta x\|^2)$

Denote $e = \dfrac{\Delta x}{\|\Delta x\|}$, $\|e\| = 1$

$\Delta x = 0 \Rightarrow \dot{V} = 0$, Let $\Delta x \neq 0$

$\dot{V} = \|\Delta x\|^2 (\underbrace{e^T Q e}_{\text{scalar}} + o(1)) < 0$ for Δx small enough

$e^T Q e \leq \max\limits_{\|x\|=1} x^T Q x < 0$

Now the Lyapunov theorem on the asymptotic stability finishes the proof. QED

Remark: Only local asymptotic stability is proved!

Let $x \in \mathbb{R}^n$, $x^* = x^T$

Then $V = (x_1 \dots x_n) \underbrace{\begin{pmatrix} \tilde{h}_{11} & \dots & \tilde{h}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{h}_{1n} & \dots & \tilde{h}_{nn} \end{pmatrix}}_{\tilde{H}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x^T \tilde{H} x$

$$\overline{V} = V = x^T \overline{\tilde{H}} x$$

$$\Rightarrow V = \frac{1}{2}\left(x^T \overline{\tilde{H}} x + x^T \tilde{H} x\right) = x^T \underbrace{Re \tilde{H}}_{\text{symmetric}} x$$

$$\boxed{H = Re \tilde{H}}$$

QED

The case of the Jordan form is considered in the book by Arnold. We don't need it.

<u>Remark</u> $A^{-1} = \frac{1}{\det A} \left(A_{ij}\right)^T$, $A_{ij} = (-1)^{i+j} \det(\dots)$

Algebraic complement

In particular

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<u>Example</u> Find a Lyapunov function

$$\begin{cases} \dot{x}_1 = 3x_1 - 5x_2 \\ \dot{x}_2 = 6x_1 - 8x_2 \end{cases} \qquad p(\lambda) = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2)$$

$\lambda = -2 \quad 5x_1 - 5x_2 = 0 \quad x_1 = x_2 \quad e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -3 \quad 6x_1 - 5x_2 = 0 \quad 6x_1 = 5x_2 \quad e_2 = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

$$x = E z = \begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix} z \qquad x, z \in \mathbb{R}^2$$

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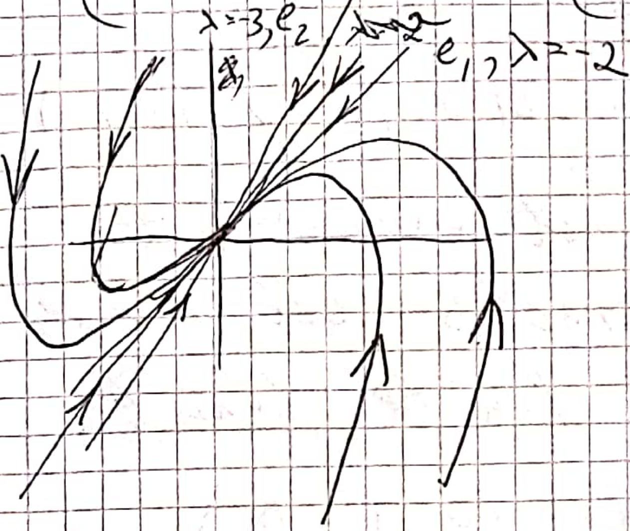
$$z = E^{-1} x = \frac{1}{1} \begin{pmatrix} 6 & -5 \\ -1 & 1 \end{pmatrix} x = \begin{pmatrix} 6x_1 - 5x_2 \\ -x_1 + x_2 \end{pmatrix}$$

$$V = |z_1|^2 + |z_2|^2 = z_1^2 + z_2^2$$

$$V = (6x_1 - 5x_2)^2 + (x_1 - x_2)^2$$



Stable nodal point

Stable Knote

$$\dot V = 2(6x_1 - 5x_2)(6\dot x_1 - 5\dot x_2) + 2(x_1 - x_2)(\dot x_1 - \dot x_2) =$$

$$= 2(6x_1 - 5x_2)(6(3x_1 - 5x_2) - 5(6x_1 - 8x_2)) + 2(x_1 - x_2)((3x_1 - 5x_2) - (6x_1 - 8x_2))$$

$$\le 0$$

We know the answer in advance

$$\dot V = (z_1^2 + z_2^2)' = 2z_1(-2z_1) + 2z_2(-3z_2)$$

$$= 2(-2z_1^2 - 3z_2^2) = -2(2(6x_1 - 5x_2)^2 + 3(x_1 - x_2)^2) \le 0$$

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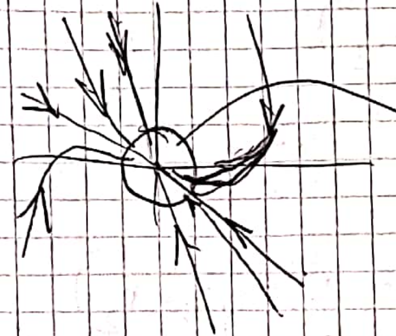
<u>Example</u>

$$\begin{cases} \dot{x} = y + \xi, \quad \cancel{\xi = 0.01 \sin(x^2 + 10^5 t)} \\ \dot{y} = -2x - 3y \end{cases}$$

$$\xi = 0.01 \sin(x^2 - y^3 + 10^6 t)$$

~~What is~~ To find a (it is not unique) compact invariant region which attracts all phase trajectories in finite time?

If $\xi \equiv 0 \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $p(\lambda) = \lambda^2 + 3\lambda + 2$, $\lambda_{1,2} = -1, -2$



Region of attraction

$$p(\lambda) = (\lambda + 1)(\lambda + 2)$$

$\lambda = -1 \quad x + y = 0 \quad e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}$

$\lambda = -2 \quad 2x + y = 0 \quad e_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

New coordinates: $\begin{pmatrix} x \\ y \end{pmatrix} = E \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ -x_1 - 2y_1 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = E^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ -x - y \end{pmatrix}$$

$$E^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$$

In new coordinates obtain $\begin{cases}\dot{x} = \dots + \xi \\ \dot{y} = \dots \quad \dots \xi\end{cases}$

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$$\begin{cases}\dot{x}_1 = -x_1 + 2\xi \\ \dot{y}_1 = -2y_1 - \xi\end{cases}$$

$\dot{x}_1 = (2x+y)^{\cdot} = 2\dot{x} + \dot{y} = \underbrace{-(2x+y)}_{\text{has to be!}} + 2\xi$

$\dot{y}_1 = (-x-y)^{\cdot} = -\dot{x} - \dot{y} = \underbrace{-2(-x-y)}_{\text{is to be}} - \xi$

$V = x_1^2 + y_1^2 = (2x+y)^2 + (x+y)^2$ according to the Arnold formula

$$\dot{V} = 2x_1\underbrace{(-x_1 + 2\xi)}_{\dot{x}_1} + 2y_1\underbrace{(-2y_1 - \xi)}_{\dot{y}_1}$$

$$\dot{V} = -2x_1^2 - 4y_1^2 + 2\xi(2x_1 - y_1)$$

Express $\dot{V}$ in the terms of V

Then from the inequality $\dot{V} < 0$ we will find some region $\{V \le C\}$ which provides for $\dot{V} < 0$,

$$V = x_1^2 + y_1^2 \Rightarrow |x_1| \le V^{\frac{1}{2}}, |y_1| \le V^{\frac{1}{2}}$$

$$|\xi(2x_1 - y_1)| \le \underset{2+1}{3} V^{\frac{1}{2}} \cdot 0{,}01$$

$$|2x_1 - y_1| \le 2|x_1| + |y_1| \le 3V^{\frac{1}{2}}$$

$$-2x_1^2 - 4y_1^2 \le -2(x_1^2 + y_1^2) = -2V$$

$$\dot{V} \le -2V + 0{,}06\, V^{\frac{1}{2}} = -2V^{\frac{1}{2}}(V^{\frac{1}{2}} - 0{,}03)$$

$$V^{\frac{1}{2}} \ge 0{,}03 \Rightarrow \dot{V} \le 0$$

$V \ge 0.0009 \Rightarrow \dot{V} \le 0$ Exclude $\dot{V} = 0$

$V \ge 0{,}001$ ensures $\dot{V} < 0$

Moreover $V \geq 0{,}001$ implies that

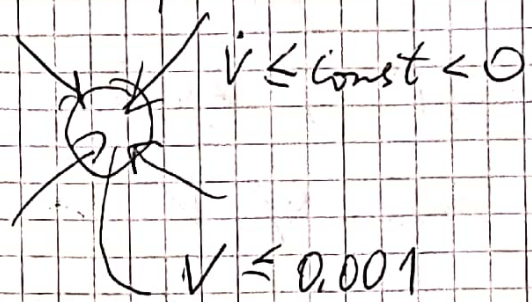
$$\dot V \leq -2V^{\frac12}\left(V^{\frac12} - 0.03\right) \leq -2\cdot 0{,}001^{\frac12}\cdot\left(0.001^{\frac12} - 0{,}03\right)$$

$$\leq -2\cdot 0{,}033\cdot 0{,}003 < 0$$

Thus in finite time the function V becomes less than $0{,}001$

<u>The answer:</u> $\{(x,y) \mid V \leq 0.001\}$

$$\{(x,y) \mid (2x+y)^2 + (x+y)^2 \leq 0.001\}$$



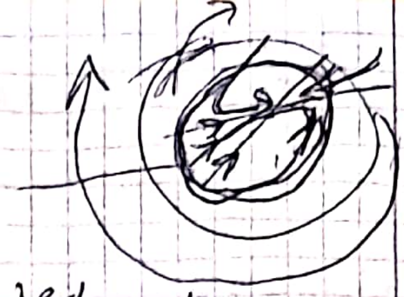
The attraction region can be here always <u>increased</u>

<u>Example</u>
$$\begin{cases} \dot x = y + x^3 \\ \dot y = -2x - 3y + x^3 \end{cases}$$

To find a compact invariant region of the assymptotic attraction to the origin.

The system is asymptotically stable in the first (linear) approximation. But it only means the local asymptotic stability. We are asked to find ~~the~~ an actual region of attraction

Once more:



The attraction region

It can always be reduced

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$p(\lambda) = \lambda^2 + 3\lambda + 2$

$e_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ e_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad \begin{cases} x = x_1 + y_1 \\ y = -x_1 - 2y_1 \end{cases}$ $\quad \begin{cases} x_1 = 2x + y \\ y_1 = -x - y \end{cases}$

$\lambda_1 = -1,\ \lambda_2 = -2$

This time $\xi = x^3$, We will keep it in such a form for a while

$$\begin{cases} \dot{x}_1 = -x_1 + 3x^3 \\ \dot{y}_1 = -2y_1 - 2x^3 \end{cases}$$

$\dot{x}_1 = 2\dot{x} + \dot{y} =$

$= \underbrace{-(2x+y)}_{\text{has to be}} + (2+1)x^3$

$\dot{y}_1 = -\dot{x} - \dot{y} = \ldots - 2x^3$

$V = x_1^2 + y_1^2$

$\frac{1}{2}\dot{V} = x_1\dot{x}_1 + y_1\dot{y}_1 = x_1(-x_1 + 3x^3) + y_1(-2y_1 - 2x^3)$

$= -x_1^2 - 2y_1^2 + x^3(3x_1 - 2y_1) \le -x_1^2 - y_1^2 + x^3(3x_1 - 2y_1)$

$\frac{1}{2}\dot{V} \le -V + x^3(3x_1 - 2y_1)$

$|x^3| = |(x_1 + y_1)^3| \le |(2V^{\frac{1}{2}})^3| = 8V^{\frac{3}{2}}$

$|x^3(3x_1 - 2y_1)| \le 8V^{\frac{3}{2}}(3V^{\frac{1}{2}} + 2V^{\frac{1}{2}}) = 40V^2$

$\frac{1}{2}\dot{V} \le -V + 40V^2 = +V\left(V - \frac{1}{40}\right)40$

$V \le -\frac{1}{40} \Rightarrow \dot{V} \le 0,\quad \dot{V} = 0 \Rightarrow (x,y) = 0$ or $V = \frac{1}{40}$

$V \le \frac{1}{50} \Rightarrow \frac{1}{2}\dot{V} \le 40V\left(\frac{1}{50} - \frac{1}{40}\right) \le 0$

$\dot{V} = 0 \Leftrightarrow V = 0 \Leftrightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix}$

Thus according to the Lyapunov theorem there is asymptotic convergence of solutions to the origin in the invariant region $V \le \frac{1}{50}$

<u>Answer:</u> $\left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, (2x+y)^2 + (x+y)^2 \le \frac{1}{50} \right\}$

<u>Clarification</u>

A region Ω (i.e. a set in $\mathbb{R}^n$) is called invariant (forward invariant) with respect to the differential equation $\dot{x} = f(t,x)$, $x \in \mathbb{R}^n$, if $x(t_0) \in \Omega$ implies $x(t) \in \Omega$ for any $t \ge t_0$. (backward invariant: $t \le t_0$)

<u>Trivial definition</u>

Dynamic system and ordinary differential equation are actually <u>synonimic</u>.

(There are also dynamic systems with control, infinite-dimensional dynamic systems. Formal definitions...) Not in this course

Stable polynomials

A polynomial is called stable (Hurwitz) if all its roots $\lambda_1, \dots, \lambda_n$ belong to $\mathbb{C}_-$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}_-$

$Re\, \lambda_j < 0$

$$p_n(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n, \quad a_1, \dots, a_n \in \mathbb{R}$$

__Theorem__ 1. If $p_n(\lambda)$ is a Hurwitz polynomial then $a_1, a_2, \dots, a_n > 0$. 2. If one of the coefficients is negative then there are roots in $\mathbb{C}_+$ ($Re\,\lambda > 0$)
3. If all coefficients are nonnegative and there are zeros, then there is a root in $\mathbb{C}_+ \cup \{\lambda = \beta i, \beta \in \mathbb{R}\}$

__Proof__ There are real roots $-\gamma_1, \dots, -\gamma_k$, $\gamma_j > 0$ and conjugate pairs $-\alpha_e \pm \beta_e i$, $\alpha_e > 0, \beta_e > 0$, $e = 1, 2, \dots, s$

$$p(\lambda) = (\lambda + \gamma_1) \dots (\lambda + \gamma_k)(\lambda^2 + 2\alpha_1 \lambda + \alpha_1^2 + \beta_1^2) \dots (\lambda^2 + 2\alpha_s \lambda + \alpha_s^2 + \beta_s^2)$$

Obviously all coefficients are positive (formally one can prove by induction)

$$p(\lambda) = \prod_{j=1}^{n} (\lambda - \lambda_j) \Rightarrow \text{coefficients}$$

continuously depend on the roots $\Rightarrow$ 2, 3.

QED

Proposition $n = 2$

$p(\lambda) = \lambda^2 + b\lambda + c$, $b, c \in \mathbb{R}$,

is Hurwitz $\Leftrightarrow$ $b > 0$ and $c > 0$.

Proof One direction is already proved. Prove $b > 0$, $c > 0 \Rightarrow$ Hurwitz

$$\lambda_{1,2} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

1. $b^2 - 4c > 0 \Rightarrow \sqrt{b^2 - 4c} < b \Rightarrow \lambda_{1,2} < 0$
2. $b^2 - 4c = 0$ $\quad p(\lambda) = \lambda^2 + 2\sqrt{c}\,\lambda + c = (\lambda + \sqrt{c})^2$

   roots $-\sqrt{c}, -\sqrt{c} < 0$
3. $b^2 - 4c < 0 \Rightarrow \lambda_{1,2} = -\frac{b}{2} \pm i\sqrt{4c - b^2}$

QED.