2021

נושאים מתקדמים במד"ר

Advanced Topics in ODE

דן דוד 204, יום א' 13:00-16:00

Dan David Classes

Zoom:
https://tau-ac-il.zoom.us/j/86389217582?
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שעות קבלה: יום ב' 16:10-17:00

ZOOM
(send email) 16:10-17:00 Monday ;Reception hours

http://www.tau.ac.il/~levant/applmath-1/

Literature:

Arnold, Ordinary Differential Equations (ODE)

Birkhoff, Rota ODE

Boyce, DiPrima Elementary DE and Boundary Value Problems

Coddington, Levinson Theory of ODE

Noone of these books covers the course.
There are some subjects of the course which are not covered by these books

Chosen problems for preparation to the exam
http://www.tau.ac.il/~levant/applmath-1/collection.pdf

I have copied all main materials to Moodle. (i.e. Exercises with solutions)

Lecture 1 10/10 - 2021

Introduction: Survey of the main notions

1. Big O, Little o $\left(\underset{o}{O} \Leftrightarrow \overset{\text{from}}{\text{"order"}} \right)$

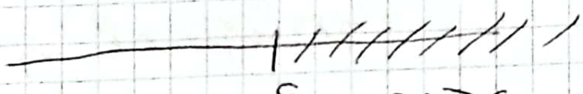
גדול קטן

(Paul Bachmann, Edmund Landau)
1894 1909

For simplicity let us restrict ourselves by the numeric functions

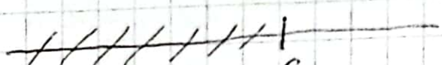
Let $A = \begin{cases} \infty \\ a \in \mathbb{R} \\ -\infty \end{cases}$, $V_\varepsilon(A)$ - ε vicinity of A ~~$\varepsilon > 0$~~

$V_\varepsilon(\infty)$



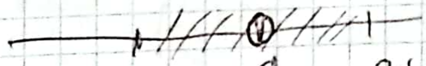
ε $x > \varepsilon$, $\varepsilon \in \mathbb{R}$

$V_\varepsilon(-\infty)$



$x < \varepsilon$ ε

$V_\varepsilon(a)$



, $\varepsilon > 0$

$a-\varepsilon$ a $a+\varepsilon$

$|x-a| < \varepsilon$, $x \neq a$

Let $f, g : \mathbb{R} \to \mathbb{R}$, $x \to A$ known from the context

(גדול) Big "Oh"

Definition: $f(x) = O(g(x))$ as $x \to A$

$\Leftrightarrow \exists c > 0 \ \exists \varepsilon \ (\varepsilon > 0 \text{ if } A = a)$:

$$(\forall x \in V_\varepsilon(A)) \longrightarrow |f(x)| \le c\,|g(x)|$$

$f(x)$ is in its size of the order of $g(x)$ and not larger

In particular in $V_\varepsilon(A)$

$$g(x)=0 \Rightarrow f(x)=0$$

little "oh" (ω, GP)

<u>Definition</u> $f(x) = o(g(x))$ as $x \to A$

$\forall \delta > 0 \; \exists \varepsilon$ ($\varepsilon > 0$ for $A = a$)

$$x \in V_\varepsilon(A) \to |f(x)| \le \delta |g(x)|$$

$f(x)$ is infinitesimally small compared with $g(x)$ as $x \to A$

Obviously the definitions are trivially extended to $f, g : M \to \mathbb{R}$ or $M \to \mathbb{C}$

where M is a topological space, $A \in M$

Examples: $x \to 0$, $x \in \mathbb{R}$

$x = o(1)$, $x = o\left(\frac{1}{x}\right)$ but $1 \neq O(x), o(x)$

$x = O(1)$, $x = O\left(\frac{1}{x}\right)$ $\quad \frac{1}{x} \neq o(x)$

$\neq O(x)$

Actually $x^2 \in o(x)$ is better notation

$x^2 \in O(e^x)$

~~Denote~~ $x \to \infty$

$$x = o(x^2), \quad 1 = o(x), \quad \ln x = o(x^\alpha) \quad \alpha > 0$$

$$x^\alpha = o(\ln x), \quad \alpha \le 0$$

~~Example. Continuity $f : \mathbb{R} \to \mathbb{R}$, $\Delta x \in \mathbb{R}$~~

~~$\Delta x \to 0$~~

~~$f(a + \Delta x) = f(a) +$~~

3

Example $d = f'(a)$, $f: \mathbb{R} \to \mathbb{R}$

$$\Leftrightarrow \quad f(a+\Delta x) = f(a) + d \cdot \Delta x + o(\Delta x) \qquad \Delta x \to 0$$

Indeed $\underbrace{\left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - d\right)}_{\alpha(\Delta x)} \Delta x = o(\Delta x)$

$\Delta x \to 0$
(naturally $\Delta x \neq 0$)
By the definition!

$$\Rightarrow \quad \alpha(\Delta x)\Delta x = o(\Delta x)$$

$$\forall \delta > 0 \ \exists \varepsilon > 0 : 0 < |\Delta x| < \varepsilon \Rightarrow \alpha(\Delta x) \leq \delta \cdot \Delta x$$

$$\frac{\alpha(\Delta x)}{\Delta x} \leq \delta$$

$$\lim_{\Delta x \to 0} \frac{\alpha(\Delta x)}{\Delta x} = 0 \quad \Leftarrow$$

What was needed to be proved i.e. Q.E.D, quod erat demonstrandum

Examples

$f(x) = O(1) \Leftrightarrow f(x)$ bounded in a vicinity of A

$f(x) = o(1) \Leftrightarrow \lim_{x \to A} f(x) = 0$

if $g(x) \neq 0$ in a vicinity of A

then $f(x) = o(g(x)) \Leftrightarrow \lim_{x \to A} \frac{f(x)}{g(x)} = 0$

Remark: $O(g(x))$, $o(g(x))$ – classes (sets) of functions

$$\boxed{\cancel{O(g(x)) \leq e^x} \text{ or } \cancel{x^2 \leq o(x)} \quad \text{SENSELESS!}}$$

Nevertheless! $\forall \alpha, \beta \in \mathbb{R}$

$$o(|x|^{\alpha})\, o(|x|^{\beta}) = o(|x|^{\alpha+\beta})$$

$$O(|x|^{\alpha}) \cdot O(|x|^{\beta}) = O(|x|^{\alpha+\beta})$$

set · set = set

(All products)

2. Greek letters

α alpha β beta

λ lambda

ε epsilon ζ zeta

ξ xi

3. Differential $k = f'(x)$

We have seen $\Leftrightarrow$ $f(x+\Delta x) = f(x) + k\,\Delta x + o(\Delta x)$

$$f(x+\Delta x) = f(x) + \underbrace{f'(x)\Delta x}_{df(x,\Delta x)} + o(\Delta x)$$

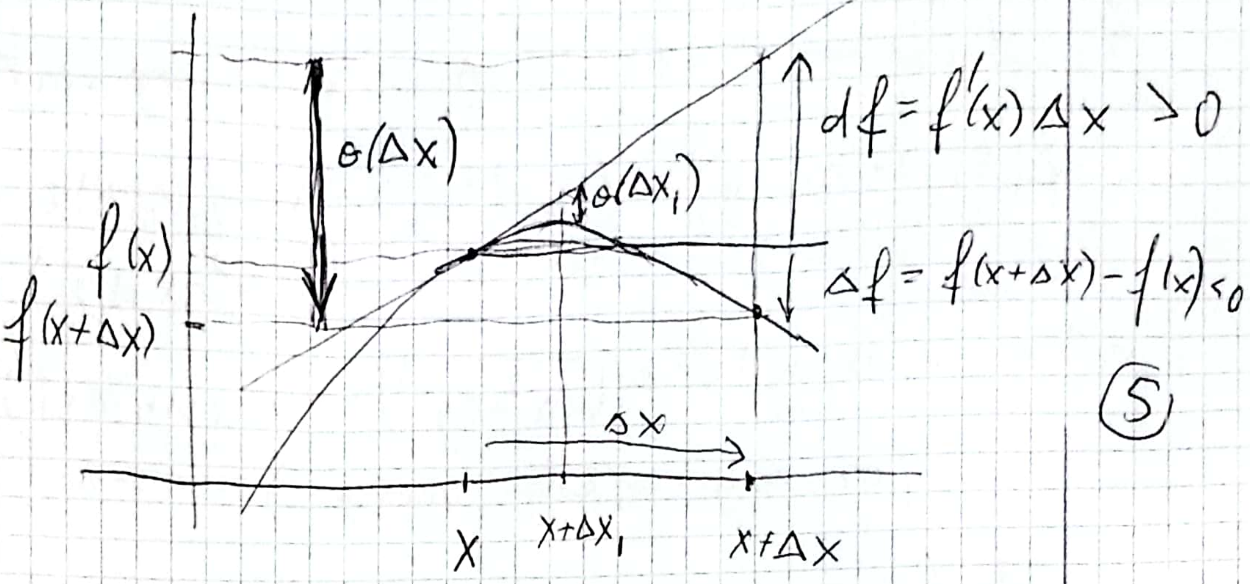
$df(x,\Delta x)$ is called the differential

Leibniz: $\underbrace{df = df(x) = df(x,\Delta x)}$

intentionally vague notation!

In particular $dx = 1 \cdot \Delta x = \Delta x$

$df(x,\Delta x)$ is a function of two arguments



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$$f(x+\Delta x) - f(x) = \underbrace{f'(x)\Delta x}_{df(x,\Delta x)} + o(\Delta x)$$

negative here

Let $f(x) = \sin x$, $x = t^2$, (מורכבת) composite function

$$df(t,\Delta t) = \underbrace{\cos t^2}_{\cos x} \cdot \underbrace{2t \cdot \Delta t}_{dx(t,\Delta t)} \overset{?}{=} df \overset{?}{=} \cos x \cdot \Delta x = df(x, \Delta x)$$

mockery by Leibniz !!

is called "the invariance of the form of the first differential"

but $2t \cdot \Delta t = dx(t, \Delta t) \neq dx(x, \Delta x) = \Delta x$

<u>functions of different arguments</u>, cannot be equal

besides: $\Delta x = (t+\Delta t)^2 - t^2 = 2t\Delta t + \Delta t^2$

for $x = t^2$

the difference

$$\Delta t^2 = o(\Delta t)$$

it disappears in the integration

which justifies the notation

4. differential of multiple arguments 6

$$f((x,y)+(\Delta x,\Delta y)) = f(x,y) + f'_x(x,y)\Delta x + f'_y(x,y)\Delta y + o(\Delta x, \Delta y)$$

where $o(\Delta x, \Delta y) \overset{def}{=} o(\sqrt{\Delta x^2 + \Delta y^2}) = o(\max(|\Delta x|, |\Delta y|))$

(or any other norm)

$$d_x f(x,y,\Delta x,\Delta y) = f'_x(x,y)\cdot\Delta x$$

$$d_y f(x,y,\Delta x,\Delta y) = f'_y(x,y)\Delta y$$

$$df(x,y,\Delta x,\Delta y) = d_x f + d_y f$$

in particular $d_x x = \Delta x = dx$, $d_y x = 0$

$d_x y = 0$, $d_y y = \Delta y = dy$

$$df = f'_x dx + f'_y dy$$

5. differentials of higher orders

one argument

$$d^2 f(x,\Delta x) = d(f'(x)\Delta x) = f''(x)\Delta x^2$$

Δx is considered as a const (partial differentiation)

$$d^k f(x,\Delta x) = f^{(k)}(x)\Delta x^k$$

6. multiple arguments

$$d^2 f(x,y,\Delta x,\Delta y) = d(f'_x \Delta x + f'_y \Delta y) =$$

$$= f''_{xx}\Delta x^2 + f''_{xy}\Delta x\Delta y + f''_{xy}\Delta x\Delta y + f''_{yy}\Delta y^2$$

$$d^2 f = f''_{xx}\Delta x^2 + 2 f''_{xy}\Delta x \Delta y + f''_{yy}\Delta y^2$$ 7

$$d^3 f = f'''_{xxx}\Delta x^3 + 3 f'''_{xxy}\Delta x^2 \Delta y + 3 f'''_{xyy}\Delta x \Delta y^2 + f'''_{yyy}\Delta y^3$$

$\dots$

7. Formula by Taylor

~~On~~ $f: \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \dots, x_n)$, $\Delta x = (\Delta x_1, \dots, \Delta x_n)$

$$\Delta f = f(x+\Delta x) - f(x) = \frac{df(x,\Delta x)}{1!} + \frac{d^2 f(x,\Delta x)}{2!} + \dots + \frac{d^k f(x,\Delta x)}{k!} + R$$

the Residual $R = \cancel{\dots} + o(\|\Delta x\|^k)$

k-times continuously differentiable $f \in C^k$

$\|\Delta x\| = \sqrt{\Delta x_1^2 + \dots + \Delta x_n^2}$ or $\|\Delta x\| = \max\limits_{i=1,\dots,n} |\Delta x_i|$

It is the Peano form for the residual

Peano $k=1 \Rightarrow$ differentiability definition

If $f \in C^k$, differentiable $k+1$ times

$$\Delta f = \frac{1}{1!} df + \dots + \frac{1}{k!} d^k f(x,\Delta x) + R$$

$$R(x,\Delta x) = \frac{1}{(k+1)!} d^{k+1} f(x + \theta \cdot \Delta x, \Delta x)$$

$\theta \in (0,1)$

The Lagrange form

$k = 0 \Rightarrow$ The Lagrange theorem

$\Delta f = f(x+\Delta x) - f(x) = f'(c)\Delta x$, $c = x + \theta\Delta x$

8. Vector Taylor formula

Peano – The same, $f: \mathbb{R}^m \to \mathbb{R}^n$

$$f(x+\Delta x) - f(x) = \Delta f(x, \Delta x) = \frac{1}{1!} df + \dots + \frac{1}{k!} d^k f + o(\|\Delta x\|^k)$$

$$d^k f = \begin{pmatrix} d^k f_1 \\ \vdots \\ d^k f_n \end{pmatrix}, \quad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Lagrange does not exist

$f \in C^k$, differentiable $k+1$ times

$$\Rightarrow \Delta f = \frac{1}{1!} df + \dots + \frac{1}{k!} d^k f + O(\|\Delta x\|^{k+1})$$

obviously $O(\|\Delta x\|^{k+1}) \subset o(\|\Delta x\|^k)$

9. Jacobi's Matrix

$$f: \mathbb{R}^m \to \mathbb{R}^n, \quad f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

$$df(x, \Delta x) = f'(x)\, dx(x, \Delta x)$$

$$\begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} = \underbrace{\begin{pmatrix} f'_{1x_1} & \dots & f'_{1x_m} \\ \vdots & & \vdots \\ f'_{nx_1} & \dots & f'_{nx_m} \end{pmatrix}}_{\text{Jacobi Matrix}} \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{pmatrix} = \underbrace{\begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_n \end{pmatrix}}_{\text{gradients}} \begin{pmatrix} dx_1 \\ \vdots \\ dx_m \end{pmatrix}$$

$$df = \underbrace{\nabla f(x)}_{f'(x)} dx \quad \text{short form}$$

משוואות דיפרנציאליות רגילות

Ordinary Diff. Equations

Normal form

$$\dot{x} = f(t, x)$$ $\dot{x} \overset{\text{def}}{=} \frac{d}{dt} x(t)$

t - time

We will use column vectors

$x = (x_1 \dots x_n)^T$ - column

T - Transpose

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, \dots, x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, \dots, x_n) \\ \dots \\ \dot{x}_n = f_n(t, x_1, x_2, \dots, x_n) \end{cases}$$ $x \in \mathbb{R}^n$

Example
$$\begin{cases} \dot{x}_1 = t\cos(x_1 - x_2) \\ \dot{x}_2 = x_1^2 - t x_2 \end{cases}$$

ODE of high. order is equivalent to a normal-form system

$$y^{(n)} = f(t, y, \dot{y}, \dots, y^{(n-1)}), \quad f: \mathbb{R}^{n+1} \to \mathbb{R}$$

Denote $x_0 = y,\ x_1 = \dot{y}, \dots,\ x_{n-1} = y^{(n-1)}$

$$\begin{cases} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_2 \\ \dots \\ \dot{x}_{n-2} = x_{n-1} \\ y^{(n)} = \dot{x}_{n-1} = f(t, x_0, x_1, \dots, x_{n-1}) \end{cases} \iff \dot{x} = F(t, x),\ x \in \mathbb{R}^n$$

Obviously that is one-to-one correspondence between the solutions.

The correspondence of the initial conditions

$$x(t_0) = \xi = \begin{pmatrix} \xi_0 \\ \vdots \\ \xi_{n-1} \end{pmatrix} \in \mathbb{R}^n$$

$$\Rightarrow \begin{pmatrix} y(t_0) \\ \dot{y}(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{n-1} \end{pmatrix}$$

Example: 1. $y^2 - \ddot{y}^3 + t = 0 \quad \begin{cases} \dot{x}_0 = x_1 \\ x_0^2 - \dot{x}_1^3 + t = 0 \end{cases}$

It is not the normal form

2. $\dddot{y} - ty^3 + \ddot{y} = 0 \quad \begin{cases} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = tx_0^3 - x_2 \end{cases}$

Theorem on the existence and uniqueness of solutions

משפט קיום ויחידות הפתרון

Autonomous ODE

does not contain t

$\dot{x} = f(x)$

Example $\dot{x}_1 = x_1 - x_2^2$

$\dot{x}_2 = \cos x_1$

Cauchy Problem

$$\begin{cases} \dot{x} = f(t, x), & x \in R^n \\ x(t_0) = \xi & \text{initial condition} \end{cases}$$

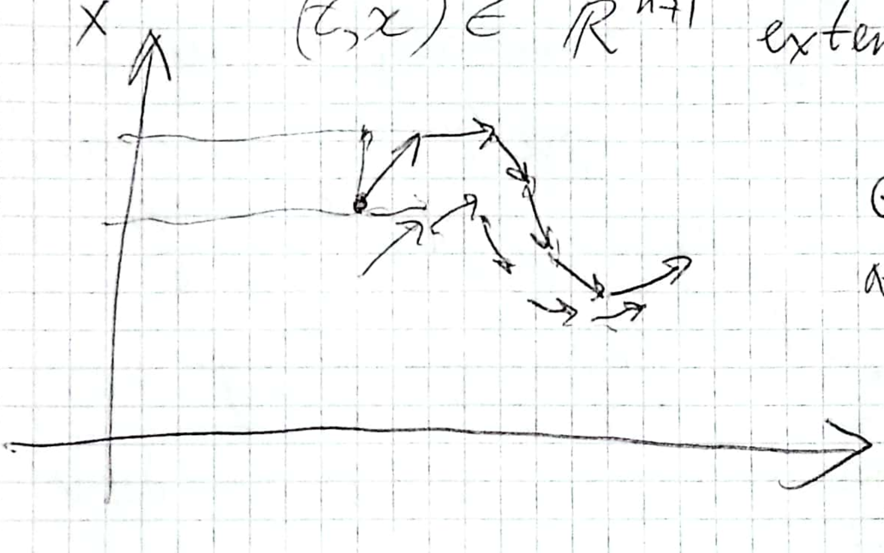
The problem: $x(t) = ?$

x is called the state, ~~sta~~ phase (מצב)

t, x — extended state, מצב מורחב

State space: R^n, $x \in R^n$

$(t, x) \in R^{n+1}$ extended state space (phase)



GPS
at each point

There is a velocity at each point

t also moves, $\dot{t} = 1$

In such a way $\dot{x} = f(t, x)$ turns into

Autonomous! $\begin{cases} \dot{x} = f(t, x) \\ \dot{t} = 1 \end{cases}$

Restriction
$t(t_0) = t_0$
initial condition

Peano

Theorem on the solution existence

$\Leftarrow$ continuity in (t, x): $f \in C$

There is
a local solution

Uniqueness is more complicated

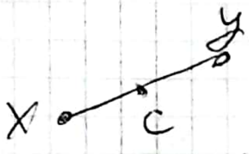
The Lipschitz property

$f: \mathbb{R}^m \to \mathbb{R}^n$ satisfies the Lipschitz condition in a region $\Omega \subset \mathbb{R}^m$ if there exists $L > 0$:

$$\forall x, y \in \Omega \quad \|f(x) - f(y)\| \le L\|x - y\|$$

Example Any continuously differentiable function is Lipschitz ~~over~~ in each ~~compact region~~ closed ball

Proof



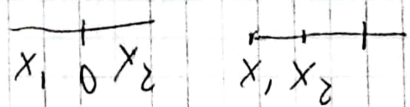
$$f(y) - f(x) = f'(c)(y - x)$$

$$L = \max_{c \in \Omega} |f'(c)| \qquad \text{Q.E.D.} \qquad \text{δ.e.N}$$

Example $y = |x|$, $x \in \mathbb{R}$

$$|x_1| - |x_2| \le |x_1 - x_2|, \quad L = 1$$

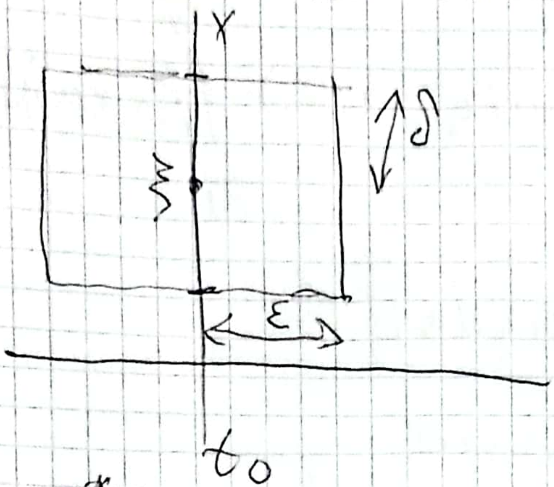
checking cases



Theorem ∃!

$$\begin{cases} \dot{x} = f(t,x) \\ x(t_0) = \xi \\ |f(t,x)| \le M \end{cases}$$

~~Lipschitz constant L~~



~~The unique~~ local solution is defined over the time interval $|t-t_0| \le \min\left(\varepsilon, \frac{\delta}{M}\right)$

if f is Lipschitzian in x

$$\|f(t,x_1) - f(t,x_2)\| \le L\,\|x_1 - x_2\|$$

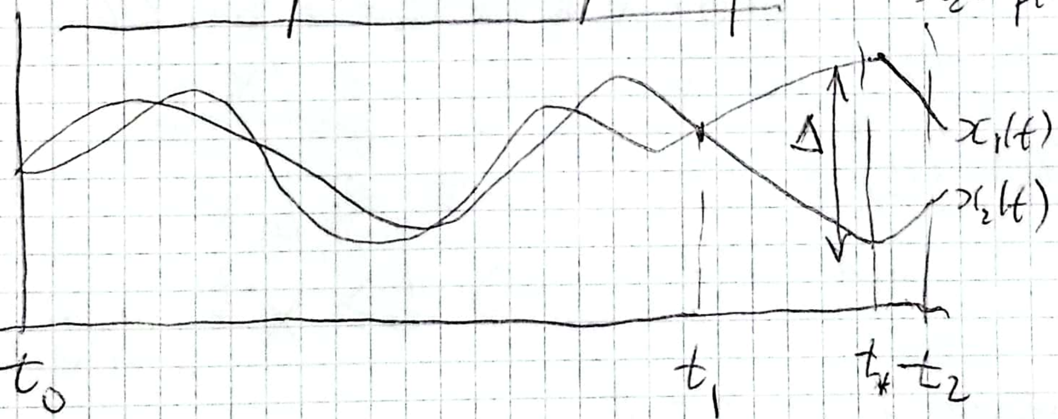
then the solution is also unique

The uniqueness proof

$\dot{x}_1 = f(t,x_1)$

$\dot{x}_2 = f(t,x_2)$

$x_1(t_0) = x_2(t_0)$



Suppose $x_1(t_2) \neq x_2(t_2) \Rightarrow t_1 = \sup\{t \in [t_0,t_2],\ x_1(t) = x_2(t)\}$

Move t_2 closer to t_1 so that $0 < t_2 - t_1 < \frac{1}{L}$

$0 < \Delta = \max\limits_{[t_1,t_2]} \|x_1(t) - x_2(t)\|$, $t_* = \arg\max(\text{—}\,\text{—})$

$$\Delta = \|x_1(t_*) - x_2(t_*)\| =$$

$$= \|\cancel{x_1(t_1)} + \int_{t_1}^{t_*} f(s, x_1(s))ds - \cancel{x_2(t_1)} - \int_{t_1}^{t_*} f(s, x_2(s))ds\|$$

$$= \| \int_{t_1}^{t_*} [f(s, x_1(s)) - f(s, x_2(s))] ds \| \leq$$

$$\leq \int_{t_1}^{t_*} \underbrace{\|f(s, x_1(s)) - f(s, x_2(s))\|}_{\leq L\|x_1(s) - x_2(s)\|} ds \leq$$

$$\leq \Delta \cdot L(t_* - t_1) \leq \Delta L \underbrace{(t_2 - t_1)}_{< \frac{1}{L}} < \Delta$$

$$\Rightarrow \Delta < \Delta$$

<u>contradiction</u>

<u>Example</u> $\dot{x} = x^{\frac{1}{3}}$, $x = 0$ ~~2nd~~ solution

$f(x) = x^{\frac{1}{3}}$ is not Lipschitz in the vicinity of $x = 0$

$f(x) - f(0) = x^{\frac{1}{3}}$ $\quad x^{\frac{1}{3}} \leq Lx$

Let $x > 0$ $\quad \Rightarrow L \geq x^{-\frac{2}{3}} \xrightarrow[x \to 0+]{} \infty$

Solve the ODE

$$\dot{x}(t)\, x(t)^{-\frac{1}{3}} = 1, \quad x \neq 0$$

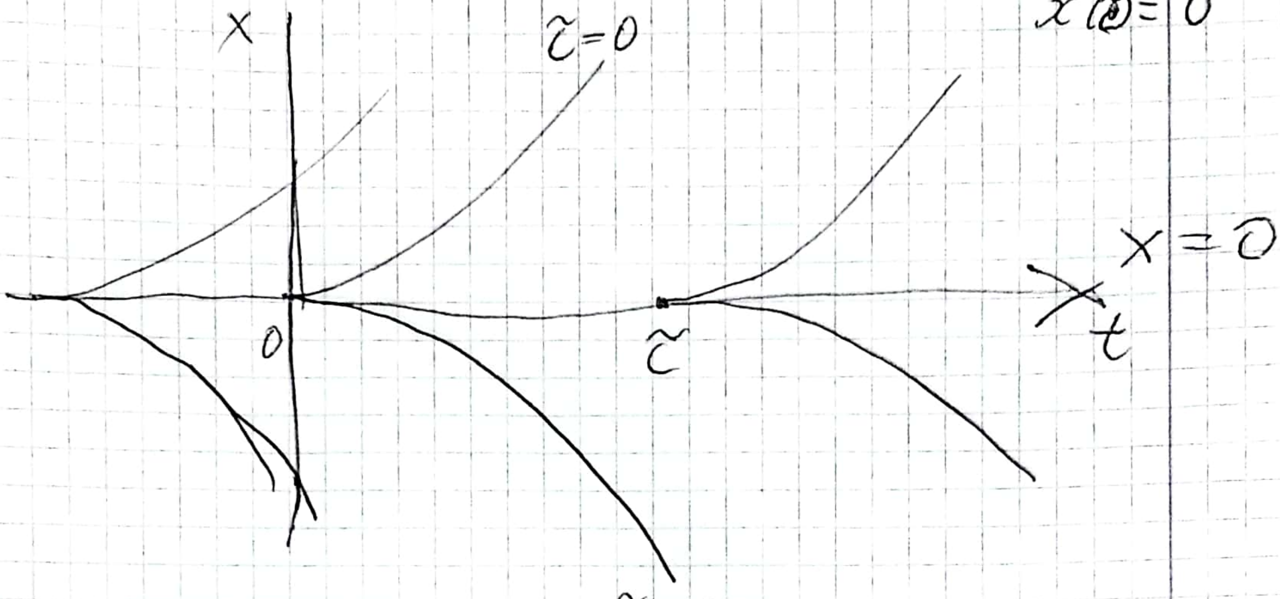
$$\int x^{-\frac{1}{3}} dx = \int \dot{x}(t)\, x(t)^{-\frac{1}{3}} dt = t + c_1$$

$$= \frac{3}{2} x^{\frac{2}{3}} + c_2$$

$$x^{\frac{2}{3}} = \frac{2}{3}(t + c)$$

$$x(t) = \pm\left(\frac{2}{3}\right)^{\frac{3}{2}}\left(t - \tilde{\tau}\right)^{\frac{3}{2}}, \quad \tilde{\tau} = -c, \quad t \geq \tilde{\tau}$$

$x(t) = 0$ - another solution, $t \in \mathbb{R}$, $x(0) = 0$



For any $\tilde{\tau} \in \mathbb{R}$:

$$x(t) = \begin{cases} 0 & t \leq \tilde{\tau} \\ \left[\begin{array}{l} \left(\frac{2}{3}\right)^{\frac{3}{2}}\left(t-\tilde{\tau}\right)^{\frac{3}{2}} \\ \text{or } 0 \\ -\left(\frac{2}{3}\right)^{\frac{3}{2}}\left(t-\tilde{\tau}\right)^{\frac{3}{2}} \end{array}\right] & t > \tilde{\tau} \end{cases}$$

~~Infinity~~ infinite number of solutions which can split from zero at any moment $\tilde{\tau}$