Proper discretization of homogeneous differentiators

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Abstract

Homogeneous sliding-mode-based differentiators (HD) are known to provide for the high-accuracy robust estimation of derivatives in the presence of sampling noises and discrete measurements, provided that the differentiator dynamics evolve in continuous time. The popular one-step Euler discrete-time implementation is proved to cause differentiation accuracy deterioration, if the differentiation order exceeds 1. A novel discrete-time realization of the HD is proposed, which preserves the ultimate accuracy of the continuous-time HD also with discrete measurements.

Key words: Sliding mode control; Observation; Finite-time stability

1 Introduction

Sliding-modes (SMs) are used to control uncertain systems by keeping some functions (sliding variables) at zero due to high-frequency control switching. SMs are established in finite time, are accurate and robust [15,33]. Possible dangerous vibrations (chattering effect) constitute their main drawback [8,15,19,33].

Standard SMs [15,33] require the sliding-variable relative degree to be 1. High-order sliding modes (HOSMs) [10,23–26,28,29,31] remove this restriction, placing the switching in the higher sliding-variable derivatives. Artificially increasing the relative degree one can remove the high-energy chattering [10,28,31]. Their high accuracy is due to the local homogeneity features [26].

One of the main applications of sliding-mode control is the robust finite-time-exact differentiation and observation [11,12,22,24,25,31,33,35]. HOSM-based homogeneous differentiator (HD) [25] estimates n derivatives of a signal, provided the absolute value of its $(n+1)$th derivative has a known bound. Contrary to the popular linear [4] and nonlinear [34] high-gain observers, having been robust with respect to noises, HDs also produce exact finite-time derivative estimations in the absence of noises. Such differentiators have found a lot of theoretical and practical applications [6,9,10,14,20,21,30–32].

The HD accuracy originates from the homogeneity of the error dynamics [25]. It is asymptotically optimal in the presence of infinitesimal input noises [24], and the accuracy of its ith derivative is of the order τ^{n-i+1}, with the sampling interval τ, if the noise is absent.

The recent HD modifications (for example [2,13]) contain additional higher order terms and feature faster convergence. The asymptotic accuracy is the same [1], if the local homogeneous error dynamics is preserved. It is usually worsened, if the local homogeneity is lost [16].

The above features were proved under the assumption that the system evolves in continuous time between the sampling time instants. Unfortunately, in practice the differentiator is a hybrid computer-based discrete dynamic system with a sampled continuous-time input. It obviously requires special study and design.

A natural approach is to make the discrete-time HD [25] emulate the corresponding continuous-time HD. Since the system is discontinuous, the Euler method is taken with the integration step much less than the sampling period, which makes the integration step choice difficult. Hence, only one Euler integration step is usually applied at each sampling interval. The corresponding asymptotic accuracy is calculated in this paper for the cases of constant and variable sampling intervals. In particular, the accuracy is proved to be proportional to the sampling interval in the first case, whereas it is worth in the second case. Thus, the high accuracy of the continuous-time HD [25] is lost, if the HD order is higher than 1.

We propose a novel discretization scheme of the differentiators [25]. Terms of higher-order with respect to the sampling intervals are added to the original Euler inte-

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gration scheme. Similar terms are notably introduced in [7] for the proper analysis of discrete dynamics. The proposed scheme preserves the computational simplicity of the one-step Euler scheme and provides for the homogeneous discrete error dynamics. Thus, the novel scheme restores the asymptotic accuracy of its continuous-time counterpart. Simulation demonstrates the calculated accuracies and the advantages of the proposed scheme.

Notation

A sum or multiplication of two sets is understood in the Minkowski sense, e.g., $AB = \{ab|\ a \in A, b \in B\}$. $d(x, A)$ is the Euclidean distance from $x \in \mathbf{R}^m$ to $A \subset \mathbf{R}^m$, $d(x, A) = \inf\{\|x - a\| \mid a \in A\}$. Following [17], $A^\varepsilon = \{x \in \mathbf{R}^m \mid d(x, A) \le \varepsilon\}$; $\mathrm{co}(A)$ is the convex closure of A. Denote $f(A) = \{f(x)|x \in A\}$, and $F(A) = \bigcup_{x\in A} F(x)$ for any function f and set-valued function F.

The distance $d_{\mathrm{Hs}}(A, B)$ between non-empty bounded sets A, B is taken in the Hausdorff metric, $d_{\mathrm{Hs}}(A, B) = \max\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\}$.

A set-valued function $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^m$, is called continuous, if $\lim\limits_{x\to y} d_{\mathrm{Hs}}(F(x), F(y)) = 0$, and upper-semicontinuous, if $\lim\limits_{x\to y} (\sup\{d(z, F(y)) \mid z \in F(x)\}) = 0$.

Let $p_i = \deg x_i$, $p_i > 0$, be homogeneous degrees (weights) of the coordinates $x_1, ..., x_m$. Then $\|x\|_h = \left(|x_1|^{p/p_1} + ... + |x_n|^{p/p_m}\right)^{1/p}$ is called the homogeneous norm, $p \ge \max\{p_i \mid i = 1, 2, ..., m\}$.

2 Preliminaries: HOSM-based differentiation and discretization problem

The main idea of the differentiation based on control methods is to construct a dynamic system, tracking an input function with no knowledge of its derivatives. Let the input be $f(t) = f_0(t) + \eta(t)$, $f : \mathbf{R} \to \mathbf{R}$, where $\eta(t)$ is a Lebesgue-measurable bounded noise, $|\eta| \le \varepsilon$, $\varepsilon \ge 0$ is unknown. The function $f_0(t)$ is an n-times differentiable unknown function to be restored together with its n derivatives. The last derivative $f_0^{(n)}$ is known to have a Lipschitz constant $L > 0$, which means that $f_0^{(n+1)}(t) \in [-L, L]$ almost everywhere.

Note that the considered noise restrictions actually imply that the "worst-case" bounded noises are considered. That approach significantly differs from stochastic noise restrictions, or the requirement that the noises be "highly fluctuating" functions [18] with infinitesimally-small integrals over any finite time interval.

A general differentiator mostly has the form

$$\begin{aligned} &\dot{z}_i = \varphi_i(z_0 - f) + z_{i+1}, \quad i = 0, 1, ..., n-1,\\ &\dot{z}_n = \varphi_n(z_0 - f), \end{aligned} \tag{1}$$

where φ_i is a scalar function of scalar argument [4,2,13,25]. The system is understood in the Filippov sense [17] to allow discontinuities of φ_i. Subtracting $f_0^{(i+1)}$ from both sides, denoting $\sigma_i = z_i - f_0^{(i)}$ and using $f_0^{(n+1)}(t) \in [-L, L]$, with $\eta = 0$ obtain

$$\begin{aligned} &\dot{\sigma}_i = \varphi_i(\sigma_0) + \sigma_{i+1}, \qquad i = 0, 1, ..., n-1,\\ &\dot{\sigma}_n \in \varphi_n(\sigma_0) + [-L, L], \end{aligned} \tag{2}$$

which is a differential inclusion in the error space $\sigma_0, \sigma_1, ..., \sigma_n$. Here and further, for notational simplicity, the equality is considered as an inclusion with the corresponding set having only one element. Solutions of a differential inclusion are defined as absolutely continuous functions satisfying the inclusion almost everywhere.

Inclusion (2) becomes homogeneous and finite-time-stable with properly chosen functions φ_i. The homogeneity means that some positive number (called *the weight* or *the homogeneity degree* [5]) is assigned to each coordinate σ_i, $\deg\ \sigma_i = m_i$, $m_i > 0$. Also the time t gets its weight $\deg\ t = p$ (called the *minus* homogeneity degree of the inclusion [26]), which means that the transformation

$$(t, \sigma_0, \sigma_1, ..., \sigma_n) \mapsto (\kappa^p t, \kappa^{m_0}\sigma_0, \kappa^{m_1}\sigma_1, ..., \kappa^{m_n}\sigma_n) \tag{3}$$

preserves the trajectories of (2) with any positive κ. Recall that a function of $\sigma_0, \sigma_1, ..., \sigma_n$ is said to have the homogeneity degree (weight) q, if the same transformation of the arguments is equivalent to the multiplication of the function by κ^q.

Since (2) is finite-time stable, the inclusion homogeneity degree is to be negative [26]. It is easy to see that all weights can be proportionally changed, thus in the following assume that the homogeneity degree is -1, i.e., $\deg t = 1$. Due to the segment present in the last nth equation of (2) the only possible weight of $\dot{\sigma}_n$ is 0, thus $\deg \sigma_n = 1$, and $\deg \sigma_i = n - i + 1$, $i = 0, ..., n$ [26].

The recursive form of the nth-order homogeneous HOSM differentiator [25] is

$$\begin{aligned} &\dot{z}_0 = -\tilde{\lambda}_n L^{\frac{1}{n+1}} |z_0 - f_0|^{\frac{n}{n+1}} \operatorname{sign}(z_0 - f_0) + z_1,\\ &\dot{z}_1 = -\tilde{\lambda}_{n-1} L^{\frac{1}{n}} |z_1 - \dot{z}_0|^{\frac{n-1}{n}} \operatorname{sign}(z_1 - \dot{z}_0) + z_2,\\ &...\\ &\dot{z}_n = -\tilde{\lambda}_0 L \operatorname{sign}(z_n - \dot{z}_{n-1}). \end{aligned} \tag{4}$$

Here z_i, $i = 0, 1, ...n$, is the estimation of $f_0^{(i)}$, and parameters $\tilde{\lambda}_i$ of differentiator (4) are chosen in advance for each n. An infinite sequence of parameters $\tilde{\lambda}_i$ can be built, which is valid for all n [25]. In particular, one can choose $\tilde{\lambda}_0 = 1.1$, $\tilde{\lambda}_1 = 1.5$, $\tilde{\lambda}_2 = 2$, $\tilde{\lambda}_3 = 3$, $\tilde{\lambda}_4 = 5$, $\tilde{\lambda}_5 = 8$ [27], which correspond to the differentiators of the order n, $n \leq 5$.

In the absence of noises the equalities $z_i = f_0^{(i)}$ are established in finite time. In the presence of a sampling noise with the maximal magnitude ε the accuracy $|z_i - f_0^{(i)}| = O(\varepsilon^{i/(n+1)})$ is obtained, and these asymptotics cannot be improved [25].

Extracting $\dot{z}_i$ from (4) obtain the standard form (1) with

$$\varphi_i(z_0 - f_0) = -\lambda_{n-i} L^{\frac{i+1}{n+1}} |z_0 - f_0|^{\frac{n-i}{n+1}} \operatorname{sign}(z_0 - f_0), \quad (5)$$

and the new coefficients $\lambda_0, \lambda_1, ..., \lambda_k > 0$, calculated from (4). That is,

$$\begin{aligned}
&\dot{z}_0 = -\lambda_n L^{\frac{1}{n+1}} |z_0 - f_0|^{\frac{n}{n+1}} \operatorname{sign}(z_0 - f_0) + z_1, \\
&\dot{z}_1 = -\lambda_{n-1} L^{\frac{2}{n+1}} |z_0 - f_0|^{\frac{n-1}{n+1}} \operatorname{sign}(z_0 - f_0) + z_2, \\
&... \\
&\dot{z}_n = -\lambda_0 L \operatorname{sign}(z_0 - f_0),
\end{aligned} \quad (6)$$

with $\lambda_0 = \tilde{\lambda}_0$, $\lambda_n = \tilde{\lambda}_n$, and $\lambda_j = \tilde{\lambda}_j \lambda_{j+1}^{j/(j+1)}$, $j = n-1, n-2, ..., 1$. It is always assumed in the following that system (2), (5) is finite-time stable.

In the case of discrete-time measurements of the input function f differentiator (1) takes the form

$$\begin{aligned}
&\dot{z}_i = \varphi_i(z_0(t_k) - f(t_k)) + z_{i+1}, \; i = 0, 1, ..., n-1, \\
&\dot{z}_n = \varphi_n(z_0(t_k) - f(t_k)), \qquad\quad t \in [t_k, t_{k+1}).
\end{aligned} \quad (7)$$

Note that, whereas the measurements are discrete, the integration is performed in continuous time. It is proved in [25] that the accuracy of system (7) is defined by the homogeneity weights of its coordinates. In particular, the relations $|z_i - f_0^{(i)}| = O(\tau^{n-i+1})$, $\tau = \sup_k\{t_{k+1} - t_k\}$, hold in the absence of noises.

The computer-based realization of (7) requires its discrete-time integration. Obviously, in order to get the specified accuracy, the integration step should be much smaller than the measurement step, which is troublesome. Hence, the integration step and the measurement step are usually taken the same. Unfortunately, it is further shown that the accuracy can significantly degrade in that case.

Consider the computationally inexpensive and easily implemented one-step Euler integration of the nth order differentiator (7). Note that the cases of constant and variable sampling intervals differ significantly. The asymptotic accuracy is further shown to be worse in the latter case. Nevertheless, in both cases the discrete-time solutions uniformly converge to the continuous-time solutions over any compact time interval [17], as the maximal sampling interval tends to zero.

Applying the one-step Euler method to (7) get

$$\begin{aligned}
&z_i(t_{k+1}) = z_i(t_k) + \tau_k \varphi_i(z_0(t_k) - f(t_k)) + \tau_k z_{i+1}(t_k), \\
&z_n(t_{k+1}) = z_n(t_k) + \tau_k \varphi_n(z_0(t_k) - f(t_k)),
\end{aligned} \quad (8)$$

where $i = 0, ..., n-1$, $\tau_k = t_{k+1} - t_k$, $\tau_k > 0$. In the case of constant sampling intervals get $\tau_k = \tau$, $k = 0, 1, ...$, with some fixed $\tau > 0$.

Subtracting $f_0^{(i)}(t_{k+1})$ from both sides of (8), utilizing that $\eta \in [-\varepsilon, \varepsilon]$, $f_0^{(n)}(t)$ is an absolutely continuous function, and using the Taylor expansion [3] of $f_0^{(i)}(t_{k+1}) = f_0^{(i)}(t_k + \tau_k)$ with the Lebesgue-integral remainder form, obtain that

$$\begin{aligned}
&\sigma_i(t_{k+1}) \in \sigma_i(t_k) + \tau_k \varphi_i(\sigma_0(t_k) + [-\varepsilon, \varepsilon]) + \\
&\begin{cases}
\tau_k \sigma_{i+1}(t_k) - \frac{\tau_k^2}{2} f_0^{(i+2)}(\xi_{ik}), & i \in [0, n-2] \\
\tau_k \sigma_n(t_k) + \frac{\tau_k^2}{2}[-L, L], & i = n-1 \\
\tau_k[-L, L], & i = n,
\end{cases}
\end{aligned} \quad (9)$$

where $i = 0, 1, ..., n$ and $\xi_{ik} \in [t_k, t_{k+1}]$.

It is easy to see that in the absence of the terms $f_0^{(i+2)}(\xi_{ik})$, (i.e., with $n = 1$) inclusion (9) becomes homogeneous with respect to the transformation

$$(t_k, \varepsilon, \sigma_0, ..., \sigma_n) \mapsto (\kappa t_k, \kappa^{n+1}\varepsilon, \kappa^{n+1}\sigma_0, ..., \kappa\sigma_n),$$

so that trajectories of the system with parameters τ_k, ε are bijectively transferred onto trajectories with parameters $\kappa\tau_k$, $\kappa^{n+1}\varepsilon$. In such a case the discrete-time differentiator (8) would preserve the asymptotic properties of the original continuous-time HD [25]. Unfortunately, the terms $f_0^{(i+2)}(\xi_{ik})$ appear with $n \geq 2$. This means that the one-step Euler method destroys the homogeneity of the continuous-time differentiator for $n \geq 2$.

In the sequel the accuracy of system (8) is evaluated, and a novel discretization method is proposed, preserving the ultimate accuracy $\sigma_i = O(\tau^{n-i+1})$ also with $n \geq 2$. The proofs are based on a general auxiliary lemma and presented in Section 5.

3 Accuracy of one-step Euler differentiator discretization

The accuracy of the discretized HD (5), (8) is studied separately for variable and constant sampling periods.

3.1 Asymptotic accuracy in the case of variable sampling intervals

Theorem 1 *Let the input $f(t)$ of the discrete-time differentiator* (5), (8) *consist of an n-smooth function $f_0(t)$, $n \geq 1$, and a Lebesgue-measurable additive noise not exceeding ε in its absolute value. Suppose that $|f_0^{(j)}| \leq D_j$, $j = 2, 3, ..., n$, and $f_0^{(n)}(t)$ is a Lipschitzian function with the Lipschitz constant $L > 0$, $D_{n+1} = L$. Also let the sampling intervals be bounded from above by τ, $0 < \tau_k \leq \tau$, $k = 0, 1, 2, ...$.*

Then, the accuracy $|z_i - f_0^{(i)}| \leq \mu_i \rho^{n-i+1}$, $i = 0, 1, ..., n$,

$$\rho = \max_{j=2,3,...,n+1} \left\{ \left(\frac{\tau}{2} D_j \right)^{\frac{1}{n-j+2}}, \tau, \varepsilon^{\frac{1}{n+1}} \right\}, \qquad (10)$$

is established in finite-time and kept forever. The coefficients μ_i only depend on the differentiator parameters $\lambda_0, \ldots, \lambda_n, L$.

The proof is provided in the Appendixes. Clearly, only the first argument of the maximum (10) is significant, as τ tends to zero in the absence of noise, and $\sigma_i = O(\tau^{(n-i+1)/n})$. The input function derivatives gradually change in the steady state. As a result, the local-in-time accuracy is determined by the local derivative bounds instead of the larger global bounds D_j, $j = 2, 3, ..., n$. Thus, in practice, if the input derivatives are large and τ is fixed, different arguments of the formula (10) can play the main role.

3.2 Asymptotic accuracy in the case of constant sampling intervals

The one-Euler-step discrete differentiator (8) with constant sampling intervals takes the form

$$\begin{aligned} z_i(t_{k+1}) &= z_i(t_k) + \tau\varphi_i(z_0(t_k) - f(t_k)) + \tau z_{i+1}(t_k), \\ z_n(t_{k+1}) &= z_n(t_k) + \tau\varphi_n(z_0(t_k) - f(t_k)), \end{aligned} \qquad (11)$$

where $i = 0, 1, ..., n-1$.

Theorem 2 *Let the input $f(t)$ of the discrete-time differentiator* (11) *be as in Theorem* 1*, and $\tau > 0$ be the constant sampling interval. Then, the accuracies*

$$|z_i - f_0^{(i)}| \leq \mu_i \max\{\varepsilon^{\frac{n-i+1}{n+1}}, \tau^{n-i+1}\} + i\,\tau D_{i+1}, \qquad (12)$$

$i = 0, 1, ..., n$, are established in finite time and kept forever. The constants $\mu_i > 0$ only depend on the differentiator parameters $\lambda_0, \ldots, \lambda_n, L$.

The proof is provided in the Appendixes.

Remark 1 *Note that in the absence of noises all accuracies are proportional to τ except the tracking accuracy $\sigma_0 = |z_0 - f_0|$, which is of the order τ^{n+1}. With sufficiently small τ the local upper bounds of errors $|z_i - f_0^{(i)}|$, $i = 1, 2, ..., n-1$, obviously follow the evolution of the absolute value $|f_0^{(i+1)}|$ of the **next** derivative. The estimation accuracies of f_0 and $f_0^{(n)}$ do not depend on the unknown input f_0 and its derivatives, since in the first case D_1 is multiplied by $i = 0$, and in the second case $D_{n+1} = L$, L being the differentiator parameter.*

Remark 2 *In the case when $n = 1$, Theorems* 1, 2 *provide for the same well-known standard accuracy $z_0 - f_0 = O(\max\{\varepsilon, \tau^2\})$, $z_1 - \dot{f}_0 = O(\max\{\varepsilon^{1/2}, \tau\})$ of the 1st-order HD* [25] *valid both for variable and constant sampling intervals* (also see Remark 3, Section 4).

4 Proper discretization of homogeneous SM-based differentiators

The proposed nth-order homogeneous discrete-time differentiator (HDD) has the form

$$\begin{aligned} z_i(t_{k+1}) = z_i(t_k) + \tau_k\varphi_i(z_0(t_k) - f(t_k)) + \\ \textstyle\sum_{j=1}^{n-i} \frac{\tau_k^j}{j!} z_{j+i}(t_k), \\ z_n(t_{k+1}) = z_n(t_k) + \tau_k\varphi_n(z_0(t_k) - f(t_k)), \end{aligned} \qquad (13)$$

where $z(t_k) = (z_0(t_k), ..., z_n(t_k))$ is the vector of derivative estimations and $i = 0, 1, ..., n-1$. For example, the 3rd-order HDD takes the form

$$\begin{aligned} z_0(t_{k+1}) &= z_0(t_k) + \tau_k\varphi_0(z_0(t_k) - f(t_k)) + \\ &\quad \tau_k z_1(t_k) + \tfrac{\tau_k^2}{2!} z_2(t_k) + \tfrac{\tau_k^3}{3!} z_3(t_k), \\ z_1(t_{k+1}) &= z_1(t_k) + \tau_k\varphi_1(z_0(t_k) - f(t_k)) + \\ &\quad \tau_k z_2(t_k) + \tfrac{\tau_k^2}{2!} z_3(t_k), \\ z_2(t_{k+1}) &= z_2(t_k) + \tau_k\varphi_2(z_0(t_k) - f(t_k)) + \\ &\quad \tau_k z_3(t_k), \\ z_3(t_{k+1}) &= z_3(t_k) + \tau_k\varphi_3(z_0(t_k) - f(t_k)). \end{aligned} \qquad (14)$$

The proposed discrete-time differentiator (13) formally is just another approximation of the standard continuous-time differentiator (1). Also note that the 1st-order discrete-time differentiator remains intact.

Theorem 3 *Let the input $f(t)$ of the discrete-time differentiator* (13) *be as in Theorems* 1, 2*. Also let the sampling intervals be bounded by the constant τ, $0 < \tau_k \leq \tau$,*

$k = 0, 1, 2, ...$ *Then, the accuracy* $|z_i - f_0^{(i)}| \le \mu_i \rho^{n-i+1}$, $i = 0, 1, ..., n$, $\rho = \max\{\tau, \varepsilon^{1/(n+1)}\}$, *is kept after a finite-time transient. The coefficients* μ_i *only depend on the differentiator parameters* $\lambda_0, \dots, \lambda_n, L$.

The proof is provided in the Appendixes.

Remark 3 *Since the additional terms only appear in* (13) *with* $n \ge 2$, *also this Theorem confirms the standard accuracy of the 1st-order HD* [24,25].

5 Asymptotic accuracy of disturbed homogeneous inclusions

The proofs of the presented Theorems are based on Lemmas 1, 2 formulated in this Section and substantially generalizing the corresponding results from [26]. They represent a stand-alone result, and the reader who is only interested in the implementation of the differentiators can skip this section.

Consider the differential inclusion

$$\dot{s} \in F(s), \tag{15}$$

where $s \in \mathbf{R}^m$. The inclusion is assumed to be a Filippov differential inclusion [17], i.e., $F(s) \subset \mathbf{R}^m$ is an upper-semicontinuous non-empty compact convex set-valued function. Let also inclusion (15) be finite-time stable and homogeneous of the degree -1. The latter means that the time-coordinate transformation $G_\kappa : (t, s) \mapsto (\kappa t, d_\kappa s)$, $\kappa > 0$, with the homogeneity dilation $d_\kappa : (s_1, ..., s_m) \mapsto (\kappa^{p_1} s_1, ..., \kappa^{p_m} s_m)$ and the weights $p_1, p_2, ..., p_m > 0$ preserves the differential inclusion. In other words, $F(s) = \kappa\, d_\kappa^{-1} F(d_\kappa s)$. It is easy to prove that the upper-semicontinuity and compactness features of F imply that $p_i \ge 1$, $i = 1, 2, ..., m$.

The disturbances considered in the sequel are of a special form. Let $\rho \ge 0$ be a parameter determining the disturbance intensity. Assume that $\tilde{\Gamma}(s, \rho) = (\tilde{\Gamma}_1(s, \rho), ..., \tilde{\Gamma}_m(s, \rho))^T$ be a column of numeric sets $\tilde{\Gamma}_i$, $i = 1, 2, ..., m$, satisfying the following conditions.

D1 $\tilde{\Gamma}_i(s, \rho)$ are set-valued functions with non-empty compact segment values, $s \in \mathbf{R}^m$, $\rho \ge 0$.
D2 $\tilde{\Gamma}_i$ are homogeneous with respect to the transformation $(\rho, s) \mapsto (\kappa\rho, d_\kappa s)$ with $\deg \tilde{\Gamma}_i = \deg s_i = p_i$, i.e., $\tilde{\Gamma}_i(d_\kappa s, \kappa\rho) = \kappa^{p_i}\, \tilde{\Gamma}_i(s, \rho)$.
D3 $\tilde{\Gamma}_i(s, \rho)$ monotonously increase with respect to the parameter ρ in the sense that for any s the inequality $0 \le \rho \le \hat{\rho}$ implies $\tilde{\Gamma}_i(s, \rho) \subset \tilde{\Gamma}_i(s, \hat{\rho})$.
D4 The disturbances vanish with $\rho = 0$, i.e., $\tilde{\Gamma}_i(s, 0) = \{0\}$. Moreover, there exists some $r > 0$, such that $\tilde{\Gamma}_i(s, \rho)$ are uniformly Hausdorff continuous in ρ at $\rho = 0$ for any $\|s\| \le r$. By that we mean that for any $\varepsilon > 0$ there exists $\hat{\rho} > 0$, such that if $0 \le \rho \le \hat{\rho}$, $||s|| \le r$, then the Hausdorff distance of the set from the origin satisfies $d_{Hs}(\tilde{\Gamma}_i(s, \rho), \{0\}) \le \varepsilon$.

Obviously, due to the homogeneity of $\tilde{\Gamma}_i$ condition D4 is satisfied for any $r > 0$. An example of such a function is provided by $\tilde{\Gamma}_i = |s_i|^{1/2} \rho^{p_i/2}[-1, 1]$.

Recall that by definition $\tilde{\Gamma}_i(A, \rho) = \bigcup\limits_{s \in A} \tilde{\Gamma}_i(s, \rho)$ for any $A \subset \mathbf{R}^m$. Obviously, also $\tilde{\Gamma}_i(A, \rho)$ are monotonous and uniformly continuous at $\rho = 0$. The first property means that if $0 \le \rho \le \hat{\rho}$ then $\tilde{\Gamma}_i(A, \rho) \subset \tilde{\Gamma}_i(A, \hat{\rho})$. The second means that for any $r > 0$, $\varepsilon > 0$ there exists $\hat{\rho} > 0$, such that if $d_{Hs}(A, \{0\}) \le r$ and $0 \le \rho \le \hat{\rho}$ then $d_{Hs}(\tilde{\Gamma}_i(A, \rho), \{0\}) \le \varepsilon$.

With any $\rho \ge 0$ the function $\tilde{\Gamma}_i$ maps bounded sets to bounded sets. Indeed, let $B_r \subset \mathbf{R}^m$ be any m-dimensional ball with radius r centered at the origin, and such that $A \subset B_r$. Fix some $\varepsilon > 0$. Due to the uniform continuity of $\tilde{\Gamma}_i$ at $\rho = 0$ there exists $\hat{\rho} > 0$, such that if $0 \le \rho \le \hat{\rho}$ then $\tilde{\Gamma}_i(B_r, \rho) \subset B_\varepsilon$. Hence, for any $\nu > \hat{\rho}$, letting $\kappa = \nu/\hat{\rho}$, obtain that $\tilde{\Gamma}_i(A, \nu) \subset \tilde{\Gamma}_i(d_\kappa B_r, \kappa\hat{\rho}) = \kappa^{p_i} \tilde{\Gamma}_i(B_r, \hat{\rho}) \subset B_{\kappa^{p_i}\varepsilon}$.

Define the column $\Gamma(A, \rho) = (\tilde{\Gamma}_1(A, \rho), ..., \tilde{\Gamma}_m(A, \rho))^T$ and consider the disturbed differential inclusion

$$\dot{s} \in F(s(t - \rho[0, 1]) + \Gamma(s(t - \rho[0, 1]), \rho)), \tag{16}$$

where $t \ge 0$ and $\rho \ge 0$. Realization of the time delayed process (16) requires some initialization

$$s(t) = \xi(t), \quad t \in [-\rho_\xi, 0], \quad \xi(0) = s_\xi. \tag{17}$$

Here $\xi : [-\rho_\xi, 0] \to \mathbf{R}^m$, $(\xi, s_\xi, \rho_\xi) \in X_{s_\xi, \rho_\xi}$, is a continuous function of the initial values, s_ξ is its initial value at $t = 0$, ρ_ξ is the length of its domain time segment, and X_{s_ξ, ρ_ξ} is the set of all possible initial conditions with fixed s_ξ, ρ_ξ. Respectively, $X = \bigcup\limits_{s \in \mathbf{R}^m, \rho \in \mathbf{R}} X_{s,\rho}$ is the set of all initial conditions.

The homogeneity of the differential inclusion (15) imposes a few natural consistency conditions on the set X. In particular, since any solution is transferred to another solution under the homogeneity transformation $G_\kappa : (t, \rho, s) \mapsto (\kappa t, \kappa\rho, d_\kappa s)$, then also its initial condition should be transferred to the new initial condition. Also with bounded s_ξ and small ρ_ξ, the maximal homogeneous norm of the function $\xi(t) - s_\xi$, $t \in [-\rho_\xi, 0]$, should be small as well.

Let the parameter $\varpi > 0$ formally measure the dependence of the initial conditions on the disturbance. The following procedure provides for a reasonably large family $X(\varpi)$, $\varpi > 0$, which is sufficient all over the paper. Consider solutions of the simple Filippov differential inclusion

$$\dot{\zeta}_i \in \varpi \left(\|\zeta\|_h^{p_i - 1} + \rho^{p_i - 1} \right) [-1, 1], \ i = 1, ..., m, \quad (18)$$

defined over $0 \leq t \leq \rho_\xi$, with initial condition $\zeta(0) = s_\xi$. It is formally assumed here that $\omega^0 = 1$ for any $\omega \geq 0$. The inclusion is homogeneous with respect to the transformation $(t, \rho, \zeta) \mapsto (\kappa t, \kappa \rho, d_\kappa \zeta)$. Inverting the time, i.e., defining $\xi(t) = \zeta(-t)$ for all possible solutions ζ, obtain the set X.

Obviously, solutions of (15) with appropriate initial values are also solutions of (16), (17). Other solutions appear in the sequel of the paper. In the following Lemma, the existence of a global attractor of (16), (17) is proved.

Lemma 1 *There are such constants $\mu_i > 0$ that with any $\rho > 0$ all solutions of the disturbed differential inclusion* (16), (17) *after a finite-time transient enter the region $|s_i(t)| \leq \mu_i \rho^{p_i}$ to stay there forever.*

The proof is provided in the Appendixes. Note that the convergence time of the original continuous-time differential inclusion (15) does not exceed the homogeneous norm of the initial condition with some constant coefficient. Obviously, with any compact set of initial errors the convergence time of the disturbed inclusion (16) is practically the same, provided the disturbance parameter ρ is small enough.

Lemma 2 *Let the set-valued functions $\Lambda_j(A, \rho)$, $j = 1, 2, ..., l$, be constructed in the same manner as $\Gamma(A, \rho)$. Then the statement of Lemma* 1 (*with possibly different coefficients μ_i*) *is true for the differential inclusion*

$$\begin{aligned} \dot{s} \in F(s(t - \rho[0, 1])) + \Lambda_1(s(t - \rho[0, 1]), \rho) + ... \\ + \Lambda_l(s(t - \rho[0, 1]), \rho)), \end{aligned} \quad (19)$$

with initial conditions (17).

The proof of the Lemma is straightforward.

6 Simulation results

The discrete-time approximation of the 3rd-order homogeneous differentiator (HD) (6) with $\lambda_0 = 1.1$, $\lambda_1 = 3.0$, $\lambda_2 = 4.1$, $\lambda_3 = 3$ (respectively, (4) with $\tilde{\lambda}_0 = 1.1$, $\tilde{\lambda}_1 = 1.5$, $\tilde{\lambda}_2 = 2$, $\tilde{\lambda}_3 = 3$) is considered. The time is measured in some fixed unified time units. In the constant sampling intervals' case the sampling interval $\tau_k = \tau$ takes values in $[0.001, 0.01]$ with the increment 0.0001. In the case of variable sampling intervals the intervals τ_k are generated by a random-value generator with uniform probability density function in the range $(0, \tau)$, whereas τ remains constant during each run, but varies in the range $[0.001, 0.01]$ from run to run, also with the increment 0.0001.

The steady state accuracies are calculated using the norm $\| \cdot \|_{\infty, [a, b]}$, which is simply the maximal absolute value of the considered sampled variable over a finite time interval $[a, b]$.

6.1 One-step Euler approximation

Consider the noise-free input function $f_0(t) = \sin(2t) + 5\cos(t) + \cos(3t)$ for both the constant and variable sampling intervals, $L = 102$. Note that here and further the concrete final accuracies are easily extracted from the demonstrated below logarithmic plots. Let $Y_j =$

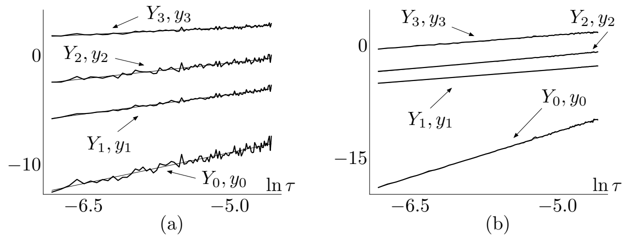

Fig. 1. Asymptotics of the Euler-based discrete-time differentiator: (a) with random intervals; (b) constant intervals.

$\ln \|z_j - f_0^{(j)}\|_{\infty, [8, 10]}$, $j = 0, ..., 3$, be the logarithmic errors. The straight lines $y_j = \alpha_j \ln \tau + \beta_j$, approximate Y_j. Their slopes α_j are equal to the corresponding asymptotic orders. The values of β_j determine the proportionality coefficients, depending on the differentiator parameters and the upper bounds of the input function derivatives.

Compare the asymptotic accuracies of the the one-step Euler-approximation discrete HD with variable and constant sampling intervals. The accuracy orders $\{1.8, 1.1, 0.9, 0.4\}$ are obtained with variable intervals, which correspond to $\{Y_0, Y_1, Y_2, Y_3\}$, respectively (see Fig. 1b). According to Theorem 1 the respective worst-case accuracy orders are $\{1.3, 1.0, 0.6, 0.3\}$. Hence, the simulation results obey the theory. In the constant sampling intervals' case the accuracy orders $\{4, 1, 1, 1\}$ are obtained, which correspond to $\{Y_0, Y_1, Y_2, Y_3\}$, respectively (see Fig. 1a). That exactly corresponds to Theorem 2 ($n = 3$, $|z_0 - f_0| = O(\tau^4)$), whereas other accuracies are of the order $O(\tau)$. As it is expected, the accuracy improves in comparison with the variable-sampling-intervals' case.

Theorems 1 and 2 state that the accuracies of the Euler-integration-based discrete-time differentiator depend on the upper bounds of the input function derivatives.

Thus, the performance of the nth-order differentiator can significantly degrade for input functions with unbounded derivatives of the orders $1, ..., n$. Demonstrate this phenomenon and compare it with the HDD performance.

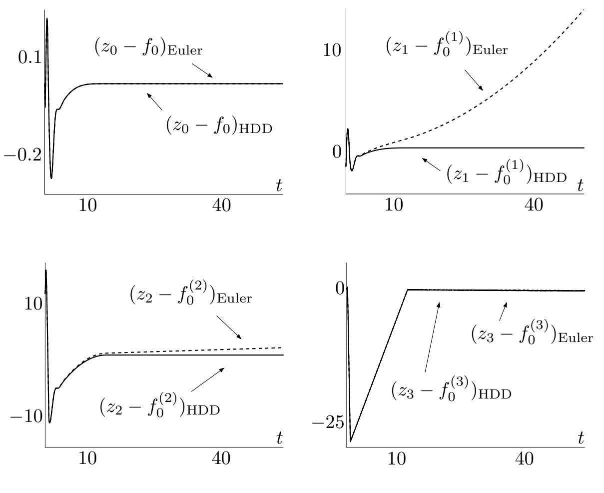

Fig. 2. Convergence of the novel 3rd-order HDD (solid lines) vs. the Euler-based HD (dashed lines); the input is $t^4 - 5t^2 + 2t$ (i.e. with unbounded derivatives), $\tau_k = \tau = 10^{-3}$.

Consider the 3rd-order one-Euler-step discrete-time differentiator (11) with constant sampling intervals, and the input function $f_0(t) = t^4 - 5t^2 + 2t$, $t \in [0, 50]$, $L = 24$. The errors $z_i - f_0^{(i)}$, $i = 0, 1, 2, 3$, obtained with $\tau = 10^{-3}$ are shown by the dashed lines in Fig. 2. It is seen that, according to Theorem 2 and Remark 1, the absolute errors $|z_i - f_0^{(i)}|$, $i = 1, 2$, increase with time respectively, following $|\ddot{f}_0|$ and $|f_0^{(3)}|$. In agreement with Remark 1 the maximal errors $|z_0 - f_0| = O(\tau^4)$ and $|z_3 - f_0^{(3)}| = O(\tau)$ are independent of the input and its derivatives. Note that the relative errors $(z_i - f_0^{(i)})/\|f_0^{(i)}\|_{\infty,[0,50]}$, $i = 0, 1, 2, 3$, still remain small. According to Theorem 1 the error divergence is naturally also observed with variable sampling intervals.

6.2 *Homogeneous discrete-time differentiator (HDD)*

Now apply the HDD (14) with the same parameters as before, and $L = 24$. The same polynomial input function $f_0(t) = t^4 - 5t^2 + 2t$, $t \in [0, 50]$, is considered. Respectively, the errors $z_i - f_0^{(i)}$, $i = 0, 1, 2, 3$, converge to approximate zero, according to Theorem 3 (see Fig. 2, solid lines). This time the accuracies do not depend on the unbounded derivatives of f_0 and remain the same at all times. The asymptotic accuracies are shown in Fig. 3. The slopes of the straight lines are as predicted in Theorem 3, i.e., $\sigma_i = O(\tau^{n-i+1})$. Indeed, the slopes of $\{y_0, y_1, y_2, y_3\}$ correspond to the accuracy orders $\{3.9, 2.9, 1.9, 0.9\}$, which are reasonably close to the

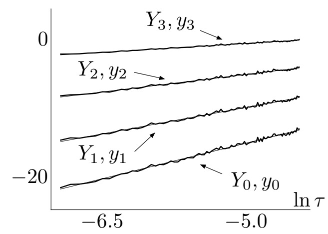

Fig. 3. Asymptotics of the novel 3rd-order HDD with random sampling intervals and the input $t^4 - 5t^2 + 2t$.

predicted orders $\{4, 3, 2, 1\}$. Thus, the simulation results agree with the theory.

7 Conclusions

The discrete-time implementation of the nth-order homogeneous differentiators [25] has been analyzed. It is shown that whereas the accuracy of the 1st-order differentiator is preserved, the simplistic one-step Euler discretization destroys the higher-order differentiators' accuracies. The accuracy differs in the cases of constant and variable sampling intervals. In the latter case the accuracy is worse, and even depends on the upper bounds of the input function derivatives of the orders $2, \ldots, n$.

A novel discretization scheme is proposed, which totally removes the sensitivity to the sampling interval variation and to the upper bounds of the input function derivatives up to the order n. The scheme is easily implemented and does not require any noticeable increase of the calculation complexity. The well-known ultimate accuracy of the continuous-time differentiator with discrete measurements is restored.

The proofs of the presented results are based on general Lemmas 1, 2, which significantly generalize results of [26] on the asymptotic accuracy of disturbed finite-time stable homogeneous differential inclusions.

A Appendix. Proof of Theorem 1

Describe solutions of (9) by piecewise-linear continuous functions $s(t) = (s_0(t), ..., s_n(t))$, defined by the equations

$$s(t) = s(t_k) + (t - t_k)v_k, \quad v_k \in F(s(t_k)), \tag{A.1}$$

where $t_k \le t \le t_{k+1}$ and $s_i(t_0) = \sigma_i(t_0)$, $i = 0, 1, ..., n$. The vector $v_k \in F(s(t_k))$ is taken arbitrarily from the set

$$F(s(t_k)) = \varphi_i(s_0(t_k) + [-\varepsilon, \varepsilon]) + \begin{cases} s_{i+1}(t_k) + \bar{\rho}^{n-i}[-1, 1], & i \in [0, n-1], \\ [-L, L], & i = n, \end{cases} \tag{A.2}$$

where $i = 0, 1, ..., n$. With $\bar{\rho}$ large enough obtain that each solution of (9) satisfies (A.1), (A.2) in the sense that for each interval $[t_k, t_{k+1}]$ there exists $v_k \in F(s(t_k))$, such that the same values $s(t_k)$ are got at the same sampling times.

Define

$$\bar{\rho} = \max\left\{\left(\frac{\tau}{2}D_2\right)^{1/n}, \left(\frac{\tau}{2}D_3\right)^{1/(n-1)}, ..., \frac{\tau}{2}D_{n+1}\right\}.$$

By construction the function $s(t)$ has a bounded derivative $|\dot{s}(t)|$ with $t \neq t_k$. Therefore [17], each solution of the discrete system (A.1), (A.2) satisfies the differential inclusion

$$\dot{s}_i(t) \in \varphi_i(s_0(t_k) + [-\varepsilon, \varepsilon]) + \begin{cases} s_{i+1}(t_k) + \bar{\rho}^{n-i}[-1, 1], & i \in [0, n-1] \\ [-L, L], & i = n. \end{cases} \tag{A.3}$$

where $i = 0, 1, ..., n$, and $t \in (t_k, t_{k+1})$. Consider now the disturbed continuous-time inclusion

$$\dot{s}_i \in \varphi_i(s_0(t - \tau[0, 1]) + [-\varepsilon, \varepsilon]) + \begin{cases} s_{i+1}(t - \tau[0, 1]) + \bar{\rho}^{n-i}[-1, 1], & i \in [0, n-1], \\ [-L, L], & i = n, \end{cases} \tag{A.4}$$

where $i = 0, 1, ..., n$. Obviously, each solution of inclusion (A.3) satisfies inclusion (A.4) almost everywhere. Indeed, for each $t \in [t_k, t_{k+1}]$ there exists $\varpi \in [0, 1]$, such that $t - \tau\varpi = t_k$.

Define $\rho = \max\left\{\bar{\rho}, \tau, \epsilon^{1/(n+1)}\right\}$, and rewrite (A.4) as

$$\dot{s}_i \in \varphi_i(s_0(t - \rho[0, 1]) + \rho^{n+1}[-1, 1]) + \begin{cases} s_{i+1}(t - \rho[0, 1]) + \rho^{n-i}[-1, 1], & i \in [0, n-1], \\ [-L, L], & i = n, \end{cases} \tag{A.5}$$

where $i = 0, 1, ..., n$. Due to the choice of ρ every solution of inclusion (A.4) satisfies inclusion (A.5) almost everywhere [17]. Denote by $F(s)$ the right-hand side of (A.5) with $\rho = 0$. Now the Theorem follows from Lemma 1 with the homogeneity weights $\deg s_i = n - i + 1$ and $\tilde{\Gamma}_i(s, \rho) = \rho^{n-i+1}[-1, 1]$, $i = 0, 1, ..., n$. □

B Appendix. Proof of Theorem 2

Introduce the variables $(\gamma_0, ..., \gamma_{n+1})$ defined at time t_k as the divided differences $\gamma_0(t_k) = f_0(t_k)$, $\gamma_1(t_k) = (\gamma_0(t_{k+1}) - \gamma_0(t_k))/\tau$, ..., $\gamma_{n+1}(t_k) = (\gamma_n(t_{k+1}) - \gamma_n(t_k))/\tau$. It is well-known that $\gamma_i(t_k) = f_0^{(i)}(\xi_k)$, $\xi_k \in [t_k, t_{k+i}]$, $i = 0, ..., n$, and $\gamma_{n+1}(t_k) \in [-L, L]$. Letting $s_i = z_i - \gamma_i$, obtain from (11) that

$$\begin{aligned} s_i(t_{k+1}) &\in s_i(t_k) + \tau\varphi_i(s_0(t_k) + [-\varepsilon, \varepsilon]) + \tau s_{i+1}(t_k), \\ s_n(t_{k+1}) &\in s_n(t_k) + \tau\varphi_n(s_0(t_k) + [-\varepsilon, \varepsilon]) + \tau[-L, L], \end{aligned} \tag{B.1}$$

where $i = 0, 1, ...n - 1$. Note that this coordinate transformation does not affect the differentiator dynamics. With $\rho = \max\{\tau, \varepsilon^{1/(n+1)}\}$, similarly to the proof of Theorem 1 obtain that any solution of (B.1) almost everywhere satisfies the disturbed differential inclusion

$$\begin{aligned} \dot{s}_i &\in \varphi_i(s_0(t - \rho[0, 1]) + \rho^{n+1}[-1, 1]) + s_{i+1}(t - \rho[0, 1]), \\ \dot{s}_n &\in \varphi_n(s_0(t - \rho[0, 1]) + \rho^{n+1}[-1, 1]) + [-L, L], \end{aligned} \tag{B.2}$$

where $i = 0, 1, ...n - 1$.

Denoting by $F(s)$ the right-hand side of (B.2) with $\rho = 0$, and using Lemma 1 with the homogeneity weights $\deg s_i = n - i + 1$, $i = 0, 1, ..., n$, and $\tilde{\Gamma}_0(s, \rho) = \rho^{n+1}[-1, 1]$, $\tilde{\Gamma}_i(s, \rho) = 0$ with $i = 1, ..., n$, obtain that the accuracy $|s_i| \leq \mu_i \max\{\varepsilon^{(n-i+1)/(n+1)}, \tau^{(n-i+1)}\}$ is established in finite time. The positive constants μ_i depend only on the differentiator parameters. Now, $|z_i(t_k) - f_0^{(i)}(t_k)| \leq |z_i(t_k) - \gamma_i(t_k)| + |\gamma_i(t_k) - f_0^{(i)}(t_k)| \leq \mu_i \max\{\varepsilon^{(n-i+1)/(n+1)}, \tau^{(n-i+1)}\} + (t_{k+i} - t_k)\sup|f_0^{(i+1)}|$, and the needed asymptotics are obtained. □

C Appendix. Proof of Theorem 3

Similarly to Section 2, subtracting $f_0^{(i)}(t_{k+1})$ from both sides of (13) and using $\eta \in [-\varepsilon, \varepsilon]$, obtain

$$\begin{aligned} \sigma_i(t_{k+1}) &\in \sigma_i(t_k) + \tau_k\varphi_i(\sigma_0(t_k) + [-\varepsilon, \varepsilon]) + \\ &\qquad \textstyle\sum_{j=1}^{n-i} \frac{\tau_k^j}{j!}\sigma_{j+i}(t_k) + \frac{\tau_k^{n-i+1}}{(n-i+1)!}[-L, L], \\ \sigma_n(t_{k+1}) &\in \sigma_n(t_k) + \tau_k\varphi_n(\sigma_0(t_k) + [-\varepsilon, \varepsilon]) + \tau_k[-L, L], \end{aligned} \tag{C.1}$$

where $i = 0, 1, ...n - 1$. Consider a piecewise-linear continuous function $s(t) = (s_0(t), ..., s_n(t))$ defined by

$$s(t) = s(t_k) + (t - t_k)v_k, \tag{C.2}$$

where $t_k \leq t \leq t_{k+1}$ and $s_i(t_0) = \sigma_i(t_0)$, $i = 0, 1, ..., n$. The vector $v_k \in F(s(t_k), \tau)$ is taken from the set

$$F(s(t_k), \tau) = \begin{cases} \varphi_i(s_0(t_k) + [-\varepsilon, \varepsilon]) + s_{i+1}(t_k) + \\ \quad \sum_{j=2}^{n-i} \frac{\tau^{j-1}}{j!} s_{j+i}(t_k)[-1, 1] + \frac{\tau^{n-i}}{(n-i+1)!}[-L, L], \\ \varphi_n(s_0(t_k) + [-\varepsilon, \varepsilon]) + [-L, L], \end{cases} \tag{C.3}$$

where $i = 0, 1, ..., n-1$. Each solution of (C.1) satisfies (C.2), (C.3) in the sense that for each interval $[t_k, t_{k+1}]$ there exists $v_k \in F(s(t_k), \tau)$ such that the same values $s(t_k)$ are got at the same sampling instants.

Similarly to the proof of Theorem 1 obtain that any solution of system (C.2), (C.3) almost everywhere satisfies the differential inclusion

$$\begin{aligned}\dot{s}_i &\in \varphi_i(s_0(t-\rho[0,1]) + \rho^{n+1}[-1,1]) + s_{i+1}(t-\rho[0,1]) + \\ &\quad \textstyle\sum_{j=2}^{n-i} \frac{\rho^{j-1}}{j!} s_{j+i}(t-\rho[0,1])[-1,1] + \frac{\rho^{n-i}}{(n-i+1)!}[-L,L], \\ \dot{s}_n &\in \varphi_n(s_0(t-\rho[0,1]) + \rho^{n+1}[-1,1]) + [-L,L],\end{aligned} \tag{C.4}$$

where $i = 0, 1, ..., n-1$. Denote by $F(s)$ the right-hand side of (C.4) with $\rho = 0$, and obtain that it can be rewritten in the form (19). Each element of every vector Λ_ν is either a fixed set or a set of the form $\frac{\rho^{j-1}}{j!} s_{j+i}(t - \rho[0,1])[-1,1]$. Now Lemma 2 implies the statement of the Theorem. $\square$

D Appendix. Proof of Lemma 1

Due to the finite-time stability of (15) solutions starting at time 0 from a closed ball B_0 centered at the origin, converge to the origin at some time T. The points, lying on their corresponding graphs over the time interval $[0, T]$, constitute a compact set [17]. Obviously, this set is confined within another closed ball B_1, $B_0 \subset B_1$.

Fix some value $\hat{\rho}$ of the parameter ρ in (16), (17). Assume that if $\xi(0) \in B_0$ then $\xi(t) \in B_1$ for all $t \in [-\hat{\rho}, 0]$. Such $\hat{\rho}$ always exists due to (18). Take any solution $s(t)$ of (16), (17). It is easy to see that there exists a closed ball B_2, $B_1 \subset B_2$ such that for any $t \geq 0$ the argument of F satisfies $s(t - \hat{\rho}[0,1]) + \Gamma(s(t - \hat{\rho}[0,1]), \hat{\rho}) \subset B_2$ whenever $s(t) \in B_1$. Moreover, the claim is true for any ρ, $0 \leq \rho \leq \hat{\rho}$.

Denote the right-hand side of (16) by $F^*(s, \rho)$. Due to the boundedness of F over B_2 and the continuity feature of Γ, obtain that for every $\delta > 0$ there exists $\tilde{\rho} > 0$, such that if $0 \leq \rho \leq \tilde{\rho}$, then $F^*(s, \rho) \subset \mathrm{co}F(\{s\}^\delta)$. Therefore [17], with small enough $\rho = \rho_0$ all solutions of (16), (17), $\xi(0) \in B_0$ converge to some small compact vicinity of the origin $W_0 \subset B_0$ at the time T. Similarly, during the time interval $[0, T]$ any solution of (16), (17) with $\xi(0) \in W_0$ does not leave some larger neighborhood of the origin still contained in B_0. Denote by W the compact set comprising all solutions of (16), (17) with $\xi(0) \in W_0$ over the time interval $[0, T]$. Obviously, $W_0 \subset W \subset B_0$, and W is an invariant attractor of (16), (17).

Due to the continuity property of the Hausdorff metric there exists $\kappa, 0 < \kappa < 1$, such that $W \subset d_\kappa B_0 \subset B_0$. Therefore, any solution of (16), (17) with $\xi(0) \in B_0$ is localized in $d_\kappa B_0$ at the time T (the *contractivity* feature [26]). Due to the homogeneity property of (16) and of the initial conditions with respect to $\hat{G}_\kappa : (t, \rho, s) \mapsto (\kappa t, \kappa\rho, d_\kappa s)$ obtain that the *contractivity* property is preserved under the transformation $\hat{G}_{\kappa^{-1}}$ with the disturbance ρ_0 being enlarged to $\kappa^{-1}\rho_0$. Due to the monotonicity of the set-valued functions Γ_i with respect to the parameter ρ, obtain that the *contractivity* property is preserved also with the disturbance $\rho_0 \leq \kappa^{-1}\rho_0$. Therefore, any solution of (16), (17) with $\rho \leq \rho_0$ converges to the global attractor W in finite-time.

Finally, let W satisfy $|s_i| \leq a_i$ for the chosen $\rho = \rho_0$. Now, applying the transformation $\hat{G}_\kappa$ with $\kappa = \rho/\rho_0$, and taking $\mu_i = a_i/\rho_0^{p_i}$, achieve the needed asymptotics. $\square$

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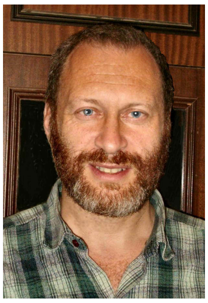

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