Black-Box Control
in Theory and Applications
RMIT University, 19.08.2016

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Control problems

The task is to make a process behave as we want.

Mathematical control appears only when we succeed to quantify the problem.

Mathematical control theory usually requires a mathematical model of the process.

Contr. problems which cannot be addressed here

Control of war and peace, geopolitics,
Long-term climate control,
Public opinion control

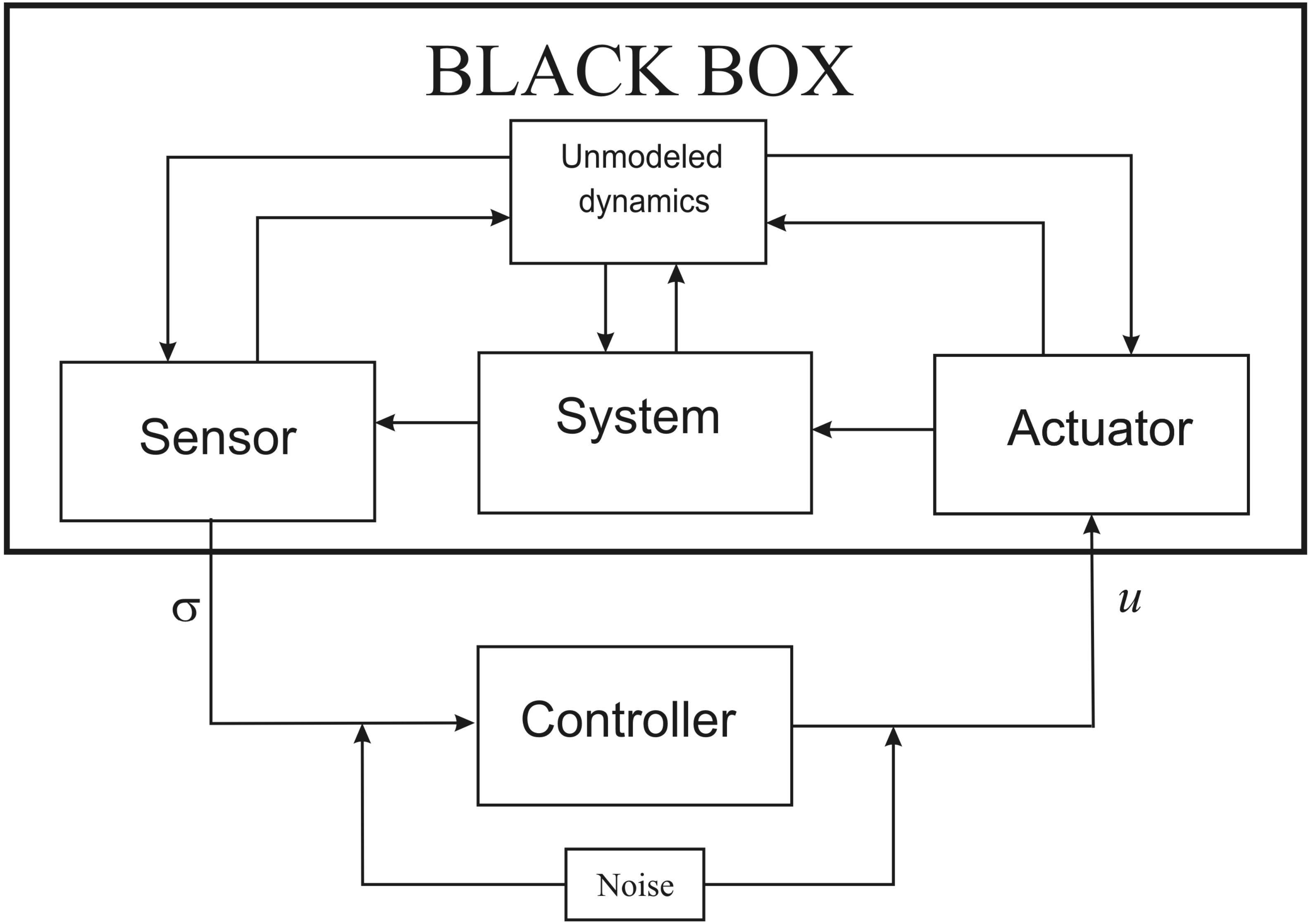
Contr. problems which maybe can be addressed

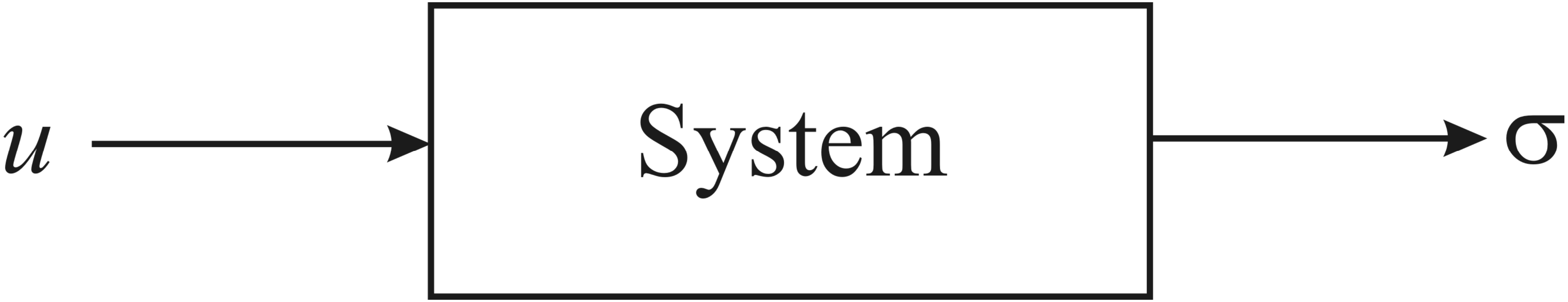
Finances: Macro-economic control by state bank, Taxes control, etc
Short-term climate control (?)

Contr. problems which are addressed

Air condition, auto-pilots, keeping bicycle balance, targeting, tracking, orientation, hormonal levels in blood, etc.

General Control Problem as Black-Box control





Tracking deviation: $\sigma = y - y_c(t)$

The goal: $\sigma = 0$

We need some PSEUDO-MODEL

Main "principles"

System **model** is a mathematical model which adequately describes the input-output relations. Whatever it means … No model is exact.

The control goal is to make the output σ satisfy some requirements by a proper choice of the control u in real time.

Any solution of the problem should be feasible and robust.

Models & approaches to "Black Box"

1. Sliding-Mode Control (here):

$$\frac{d^r}{dt^r}\sigma = h(t) + g(t)u \,,$$

$$r \in \mathbb{N},\, h \in [-C, C], \quad g \in [K_m, K_M]$$

2. Model-free control (Fliess, Join, Lafont, et al)

"Ultra-local model"

$$\frac{d^r}{dt^r}\sigma = F + Ku \,, \quad r = 1,2 \,,\, F, K = const$$

PID (proportional, integral, derivative) control

Some names and notation

$$\frac{d^r}{dt^r}\sigma = h(t) + g(t)u$$

r is called the relative degree

$$\frac{d^r}{dt^r}\sigma = \sigma^{(r)}, \quad \frac{d}{dt}\sigma = \dot{\sigma}$$

$$x \in [a,b] \Leftrightarrow a \le x \le b$$

$\sigma = O(\varepsilon)$ of the order of ε,

i.e. roughly proportional

In order to control a Black Box

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

one should at least identify r.

r is called the Practical Relative Degree (PRD)

In the framework by Fliess $r = 1, 2$

We also want some nice features:

smooth / Lipschitzian bounded control

Start with control of a smooth system

$$\dot{x} = f(t, x, u), \quad \sigma = \sigma(t, x)$$

$$x \in \mathbf{R}^n, \quad u, \sigma \in \mathbf{R}$$

Systems non-affine in control

$$\dot{x} = f(t,x,u), \quad x \in \mathbf{R}^n,$$

output $\sigma(t,x)$ (tracking error), input $u \in \mathbf{R}^l$

The goal: $\sigma \equiv 0$

Nonlinearity in control and its discontinuity $\Rightarrow$

$v = \dot{u}$ is taken as a new control,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} f(t,x,u) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} v$$

The new system is affine in control,
$u(t)$ is differentiable.

From that moment the system is

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t,x)$$

$$a, b, \sigma \in C^{\infty}, \quad x \in \mathbf{R}^{n}, \quad u, \sigma(t,x) \in \mathbf{R}$$

Relative Degree (RD)

$$\dot{x} = a(t,x) + b(t,x)u,\ x \in \mathbf{R}^n,\ \sigma,\ u \in \mathbf{R}$$

Informally: RD is the number r of the first total derivative of σ where the control explicitly appears with a not-vanishing coefficient.

$$\sigma^{(r)} = h(t,x) + g(t,x)u,\ g \neq 0$$

Newton law: $\ddot{x} = \frac{1}{m}F$, RD=2

In my practice the relative degrees

$r = 2, 3, 4, 5$

mechanical systems, Newton law, integrators

But the solution will be valid for any r.

Increasing the relative degree

Black-Box Control problem: $\sigma \to 0$

$$\sigma^{(r)} = h(t,x) + g(t,x)u$$

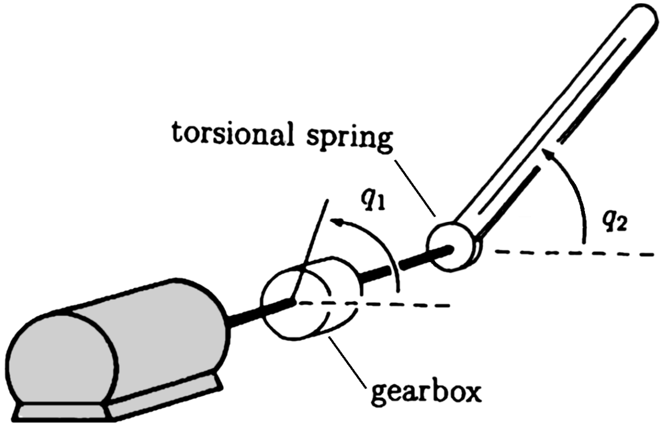
$$\sigma^{(r+1)} = \tilde{h}(t,x) + \tilde{g}(t,x)u + g(t,x)\dot{u}$$

$$v = \dot{u}, \quad \sigma^{(r+1)} \in [-C_1, C_1] + [K_m, K_M]v$$

Remark: *u* is to be kept bounded …

Any relative degree is possible

(example by Isidori)



$$J_1\ddot{q}_1+F_1\dot{q}_1-\frac{K}{N}\left(q_2-\frac{q_1}{N}\right)=u,$$

$$J_2\ddot{q}_2+F_2\dot{q}_2+K\left(q_2-\frac{q_1}{N}\right)+mgd\cos q_2=0$$

The output : q_2, The input: u. The relative degree $r = 4$

$$\dot{u}=v$$

The output : q_2, The input: v. The relative degree $r = 4+1=5$

Any relative degree can be got in such a way.

Inevitable BAD subproblem

$$\dot{z}_0 = z_1,\ \dot{z}_1 = z_2,\ \ldots,\ \dot{z}_{r-2} = z_{r-1},$$

$$\dot{z}_{r-1} = u,\quad \text{output: } y = z_0$$

The goal: $\sigma = y(t) - f(t) = 0$

$\sigma^{(r)} = f^{(r)}(t) + u$ **compare** $\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$

Let $|f^{(r)}(t)| \leq C$

If $\sigma \equiv 0$ then $z_i = f^{(i)}(t)$, $i = 0, 1, \ldots, r-1$

Exact differentiation is included!

Main idea

Black-Box Control problem: $\sigma \to 0$

$$\sigma^{(r)} = h(t, x(t)) + g(t, x(t))u$$

is replaced with

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

Assumptions $h \in [-C, C], g \in [K_m, K_M]$

Solution method

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

$$u = \alpha U_r(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$$

or $$\sigma^{(r+1)} \in [-C_1, C_1] + [K_m, K_M]\dot{u}$$

$$\dot{u} = \alpha_1 U_{r+1}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r)})$$

Continuous control cannot solve the problem

U_r, U_{r+1} are discontinuous, but bounded

Sliding mode (SM)

(not a math. definition)

Any system motion mode existing due to high-frequency (theoretically infinite-frequency) control switching is called SM.

*r*th-order sliding mode (*r*-SM)

(not a math. definition)

r-SM is a SM keeping $\sigma \equiv 0$ for RD = *r* by means of high(infinite)-frequency switching of *u*.

Some abbreviations till now

SM - sliding mode,

r-SM – *r*th order SM

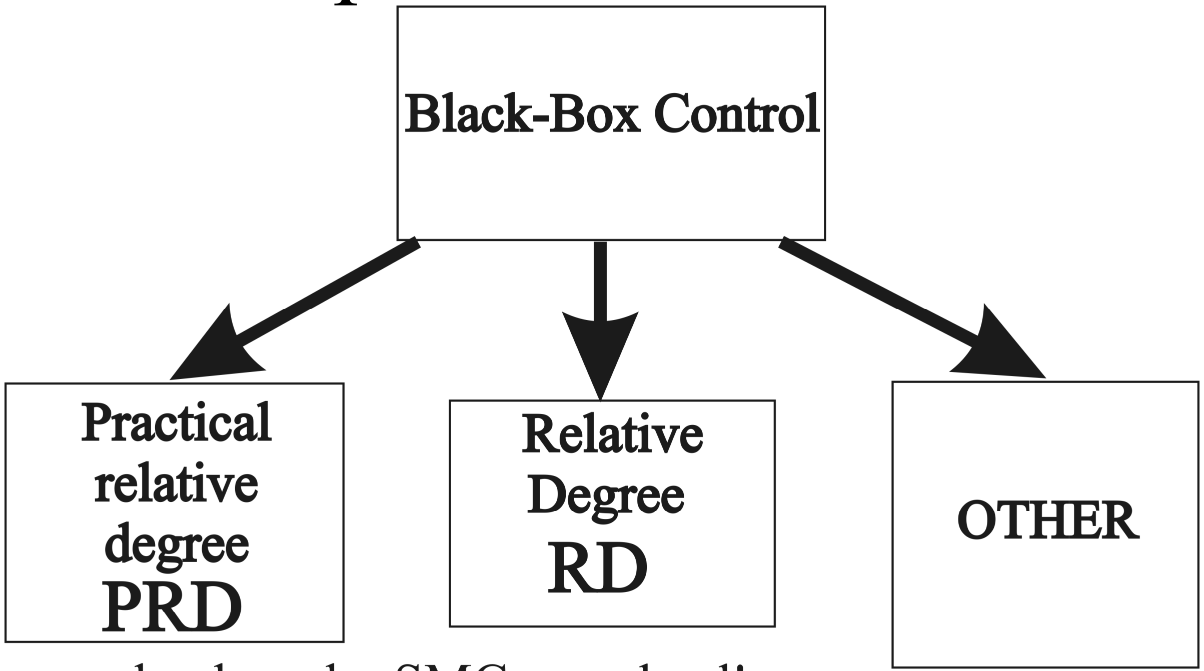
SMC – sliding mode control

RD – relative degree

PRD – practical relative degree

Preliminary conclusions

SMC theoretically "almost" solves the classical Black-Box control problem.



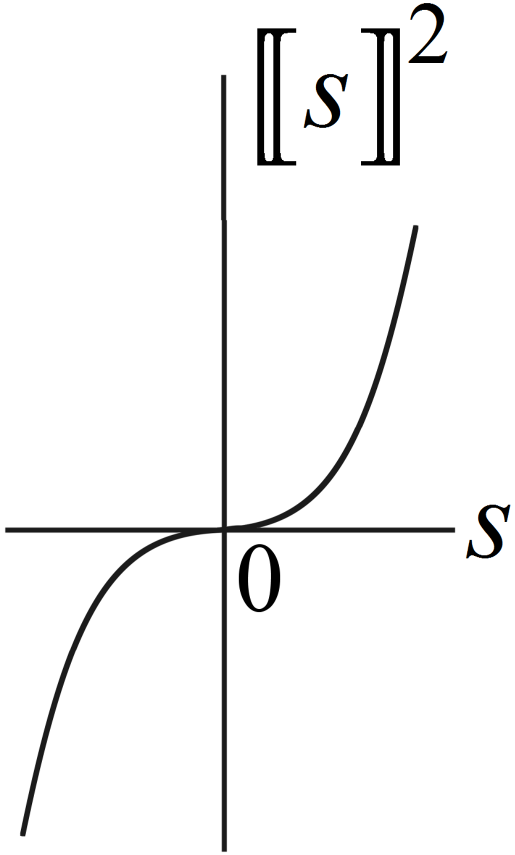
~ solved - by SMC - solved!

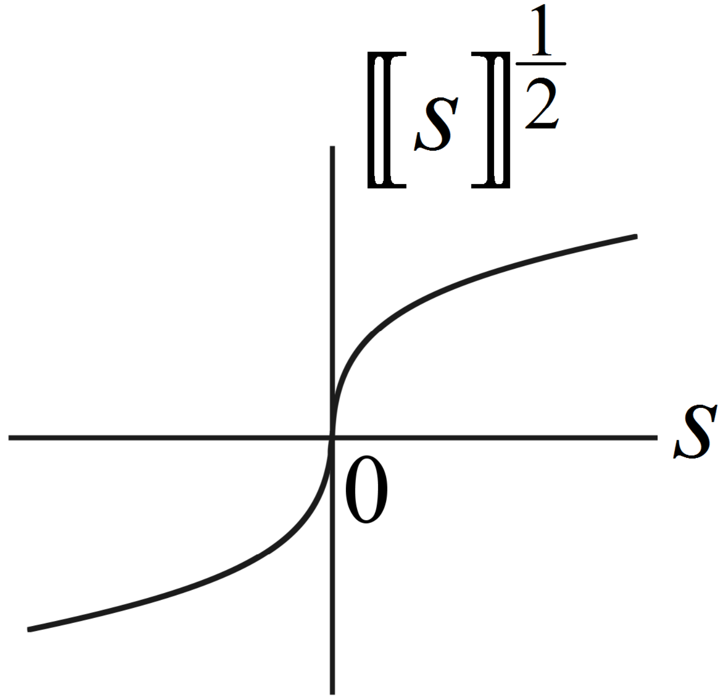
It includes exact robust differentiation of any order and robustness to small sampling/model noises, delays and disturbances (also singular).

Special power functions

(standard notation)

$$\lfloor s \rceil^{\gamma} = [\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s$$





The following controllers exactly robustly and in finite time provide for

$$\sigma \equiv 0$$

for the simplest model

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

Simplest *r*-SM controllers

(Ding, Levant, Li, Automatica 2016)

$$\llbracket s \rrbracket^{\gamma} \triangleq |s|^{\gamma}\operatorname{sign} s, \qquad \forall\, d > 0\,,\ \exists \beta_0,\dots,\beta_{n-2} > 0$$

Relay-polynomial homogeneous *r*-SMC

$$u = -\alpha\ \operatorname{sign}\left[\llbracket \sigma^{(r-1)} \rrbracket^{\frac{d}{1}} + \beta_{n-2} \llbracket \sigma^{(r-2)} \rrbracket^{\frac{d}{2}} + \cdots + \beta_0 \llbracket \sigma \rrbracket^{\frac{d}{r}} \right]$$

Quasi-continuous polynomial homogeneous *r*-SMC

$$u = -\alpha\ \frac{\llbracket \sigma^{(r-1)} \rrbracket^{\frac{d}{1}} + \beta_{n-2} \llbracket \sigma^{(r-2)} \rrbracket^{\frac{d}{2}} + \cdots + \beta_0 \llbracket \sigma \rrbracket^{\frac{d}{r}}}{|\sigma^{(r-1)}|^{\frac{d}{1}} + \beta_{n-2} |\sigma^{(r-2)}|^{\frac{d}{2}} + \cdots + \beta_0 |\sigma|^{\frac{d}{r}}}$$

Quasi-continuous control

$$u = U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$$

is called quasi-continuous (quasi-smooth), provided it remains a continuous (smooth) function whenever

$$(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \neq (0, 0, ..., 0)$$

Example: $u = -\alpha \dfrac{\left\lfloor \sigma^{(r-1)} \right\rceil^{\frac{d}{1}} + \beta_{n-2} \left\lfloor \sigma^{(r-2)} \right\rceil^{\frac{d}{2}} + \cdots + \beta_0 \lfloor \sigma \rceil^{\frac{d}{r}}}{\left|\sigma^{(r-1)}\right|^{\frac{d}{1}} + \beta_{n-2} \left|\sigma^{(r-2)}\right|^{\frac{d}{2}} + \cdots + \beta_0 |\sigma|^{\frac{d}{r}}}$

$d > kr \Rightarrow$ quasi k-smooth

List of controllers, *d* = *r*

$r = 1,2,3,4,5$

1. $u = -\alpha \ \operatorname{sign} \sigma,$
2. $u = -\alpha \ \operatorname{sign}(\lfloor \dot{\sigma} \rceil^{2} + \sigma),$
3. $u = -\alpha \ \operatorname{sign}(\ddot{\sigma}^{3} + \lfloor \dot{\sigma} \rceil^{\frac{3}{2}} + \sigma),$
4. $u = -\alpha \operatorname{sign}(\lfloor \dddot{\sigma} \rceil^{4} + 2\lfloor \ddot{\sigma} \rceil^{2} + 2\lfloor \dot{\sigma} \rceil^{\frac{4}{3}} + \sigma),$
5. $u = -\alpha \operatorname{sign}(\lfloor \sigma^{(4)} \rceil^{5} + 6\lfloor \dddot{\sigma} \rceil^{\frac{5}{2}} + 5\lfloor \ddot{\sigma} \rceil^{\frac{5}{3}} + 3\lfloor \dot{\sigma} \rceil^{\frac{5}{4}} + \sigma).$

α is to be taken sufficiently large.

Quasi-continuous controllers, *d* = *r*

$$1.\ u = -\alpha\ \operatorname{sign}\sigma,$$

$$2.\ u = -\alpha\ \frac{\llbracket\dot\sigma\rrbracket^{2} + \sigma}{\dot\sigma^{2} + |\sigma|},$$

$$3.\ u = -\alpha\ \frac{\ddot\sigma^{3} + \llbracket\dot\sigma\rrbracket^{\frac{3}{2}} + \sigma}{|\ddot\sigma|^{3} + |\dot\sigma|^{\frac{3}{2}} + |\sigma|},$$

$$4.\ u = -\alpha\frac{\llbracket\dddot\sigma\rrbracket^{4} + 2\llbracket\ddot\sigma\rrbracket^{2} + 2\llbracket\dot\sigma\rrbracket^{\frac{4}{3}} + \sigma}{\dddot\sigma^{4} + 2\ddot\sigma^{2} + 2\dot\sigma^{\frac{4}{3}} + |\sigma|},$$

$$5.\ u = -\alpha\frac{\left\llbracket\sigma^{(4)}\right\rrbracket^{5} + 6\llbracket\dddot\sigma\rrbracket^{\frac{5}{2}} + 5\llbracket\ddot\sigma\rrbracket^{\frac{5}{3}} + 3\llbracket\dot\sigma\rrbracket^{\frac{5}{4}} + \sigma}{\sigma^{(4)}|^{5} + 6|\dddot\sigma|^{\frac{5}{2}} + 5|\ddot\sigma|^{\frac{5}{3}} + 3|\dot\sigma|^{\frac{5}{4}} + |\sigma|}.$$

Another family (Levant 2005)

quasi-continuous controller $r = 2$

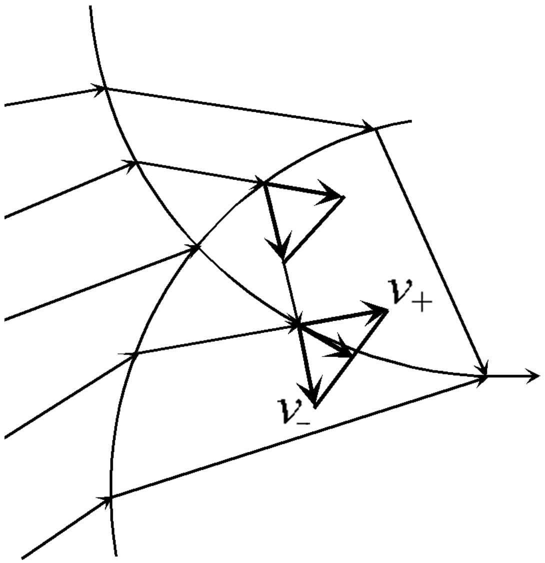
$$u = -\alpha \frac{\dot{\sigma} + |\sigma|^{1/2}\operatorname{sign}\sigma}{|\dot{\sigma}| + |\sigma|^{1/2}}$$

quasi-continuous controller $r = 3$

$$u = -\alpha \frac{\ddot{\sigma} + 2\dfrac{(\dot{\sigma} + |\sigma|^{2/3}\operatorname{sign}\sigma)}{(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}}}{|\ddot{\sigma}| + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}}$$

Discontinuous Differential Equations

Filippov Definition



$$\dot{x} = f(x) \;\Leftrightarrow\; \dot{x} \in F(x)$$

$x(t)$ is an absolutely continuous function

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu N=0} \text{convex\_closure } f\,(O_\varepsilon(x) \setminus N)$$

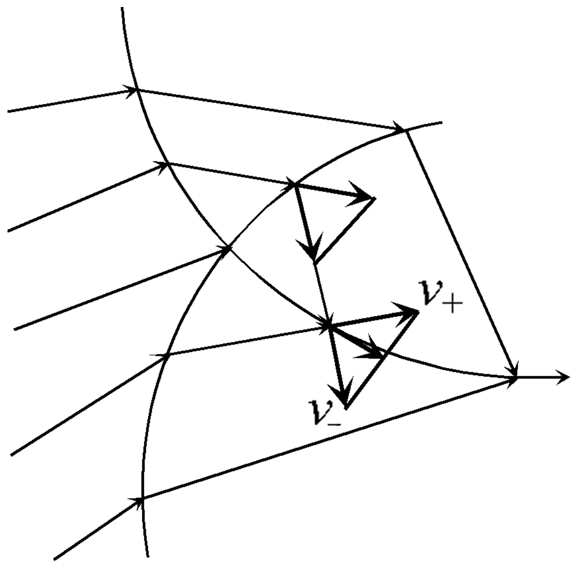
Filippov DI: $F(x)$ is non-empty, convex, compact, upper-semicontinuous.

Theorem (Filippov 1960-1970): $\Rightarrow$ Solutions exist for Filippov DIs, and for any locally bounded Lebesgue-measurable $f(x)$.

Non-autonomous case: $\dot{t} = 1$ is added.

Discontinuous Differential Equations

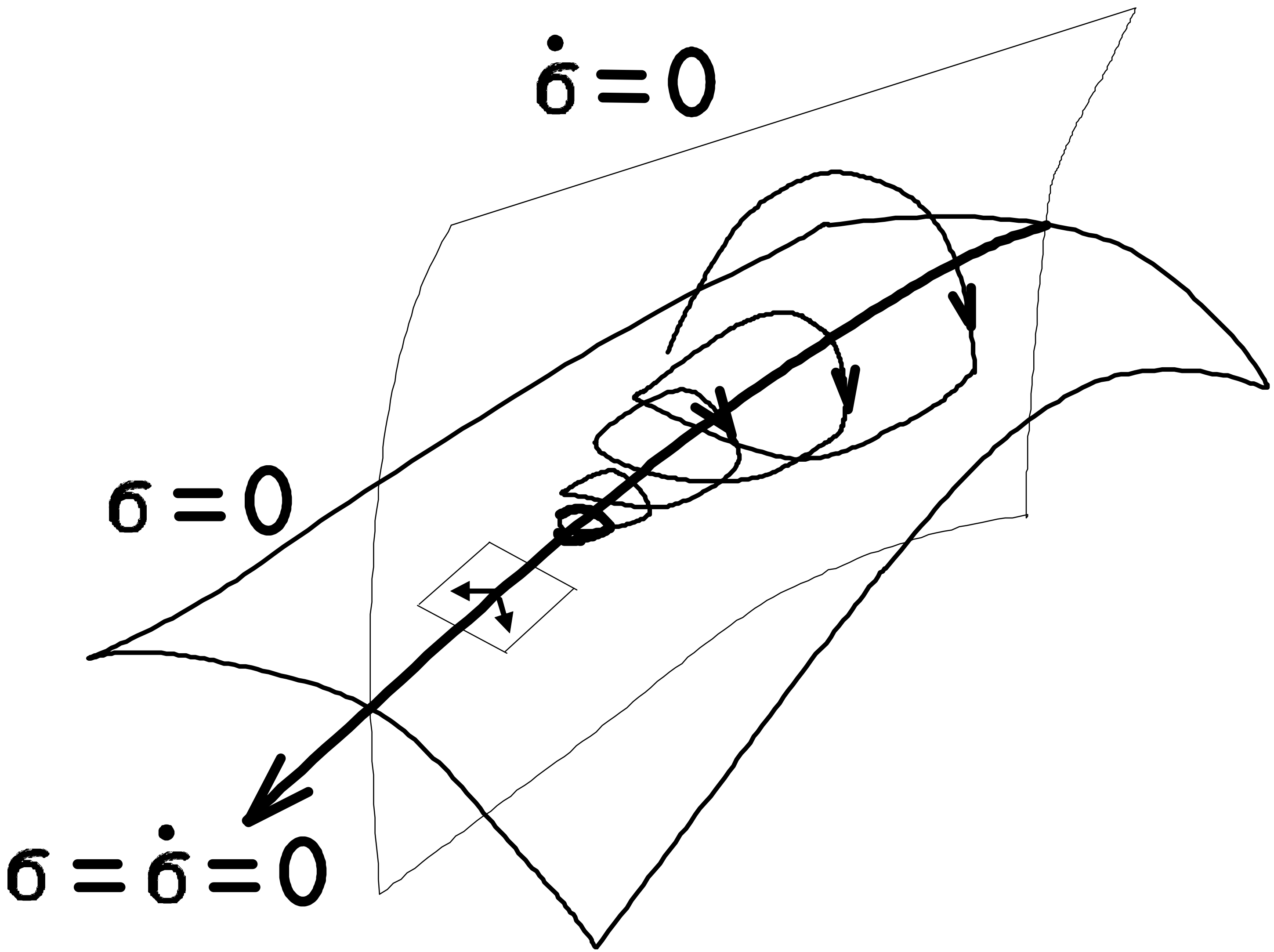
Filippov Definition



When switching imperfections (delays, sampling errors, etc) tend to zero **usual solutions**

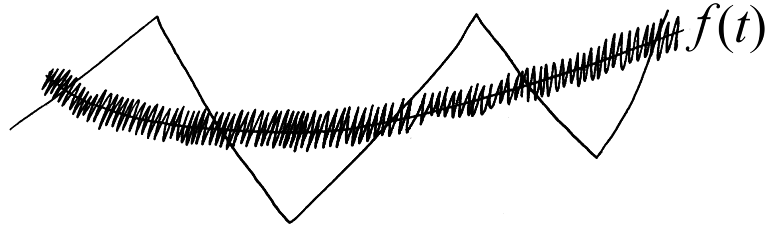
uniformly converge to Filippov solutions

2-sliding mode



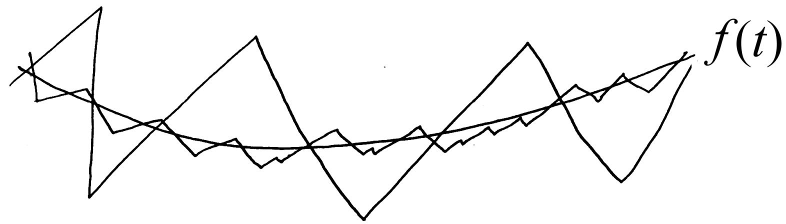
Robust differentiation problem

Unbounded derivatives



Bounded 1st derivatives

$$|\dot{\hat{f}}| \le L$$



Bounded 2nd derivatives

$$|\ddot{\hat{f}}| \le L$$

Arzela Theorem: Bounded functions with bounded derivative of the order k constitute a compact set in C.

"Solution": Take the closest function $\hat{f}(t)$!

Landau-Kolmogorov inequalities

Landau: $k = 1$, $\mathbb{R}_+$, 1912; Kolmogorov: $k > 1$, 1935

There exist such constants

$\beta_{jk} \geq 1$, $k = 1,2,\ldots$, $j = 0,1,\ldots,k+1$, $\beta_{0k} = \beta_{k+1,k} = 1$

that for any function φ

$\varphi : \mathbb{R} \to \mathbb{R}$ (or $\varphi : \mathbb{R}_+ \to \mathbb{R}$), $\forall t: \quad |\varphi(t)| \leq \varepsilon$;

$\varphi^{(k)}(t)$ is a Lipschitz function, i.e. a.e. $|\varphi^{(k+1)}(t)| \leq L$

implies

$$\forall t: \quad |\varphi^{(j)}(t)| \leq \beta_{jk} L^{\frac{j}{k+1}} \varepsilon^{\frac{k+1-j}{k+1}}$$

β_{jk} cannot be decreased and are realizable.

Kolmogorov constants

$$\varphi : \mathbb{R} \to \mathbb{R} \quad |\varphi(t)| \le \varepsilon; |\varphi^{(k+1)}(t)| \le L$$

$$\forall t: \quad |\varphi^{(j)}(t)| \le \beta_{jk} L^{j/(k+1)} \varepsilon^{(k+1-j)/(k+1)}$$

$$1 \le \beta_{jk} < \pi / 2$$

k	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
1	1.41421					
2	1.04004	1.44225				
3	1.08096	1.09545	1.48017			
4	1.04426	1.11665	1.11942	1.49631		
5	1.04298	1.08001	1.14520	1.14280	1.50892	
6	1.03451	1.07289	1.10472	1.16471	1.15137	1.51748

kth-Order Differentiation Problem

Parameters of the problem: $k \in \mathbb{N}, L > 0$

Measured input: $f(t) = f_0(t) + \eta(t), \quad |\,\eta\,| < \varepsilon$

f_0 ,η, ε are unknown,

$\eta(t)$ - Lebesgue-measurable function,

known: $\quad |f_0^{(k+1)}(t)| \le L$

(or |Lipschitz constant of $f_0^{(k)}| \le L$)

The goal:

real-time estimation of $\dot{f}_0(t), \ddot{f}_0(t), ..., f_0^{(k)}(t)$

Best worst differentiation error

Suppose both $f(t)$, $f_0(t)$ satisfy: $|f_0^{(k+1)}(t)|, |f_0^{(k+1)}(t)| \le L$

Then $|\eta^{(k+1)}(t)| \le 2L$, $|\eta(t)| \le \varepsilon$, $\forall \eta$ is possible

The worst possible error in the jth derivative is not less than $\sup|\eta^{(j)}(t)| \le \beta_{jk} 2^{\frac{j}{k+1}} L^{\frac{j}{k+1}} \varepsilon^{\frac{k+1-j}{k+1}}$

In particular for $n = j = 1$ get $\beta_{11} = \sqrt{2}$

$$\sup|\dot{f}(t) - \dot{f}_0(t)| \le 2L^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}$$

Differentiator (Levant 1998, 2003)

$$\dot{z} = D_k(z, f(t), L), \quad |f^{(k+1)}| \le L$$

$$\dot{z}_0 = -\lambda_k L^{\frac{1}{k+1}} \lfloor z_0 - f(t) \rceil^{\frac{k}{k+1}} + z_1,$$

$$\dot{z}_1 = -\lambda_{k-1} L^{\frac{1}{k}} \lfloor z_1 - \dot{z}_0 \rceil^{\frac{k-1}{k}} + z_2,$$

$$\ldots$$

$$\dot{z}_{k-1} = -\lambda_1 L^{\frac{1}{2}} \lfloor z_{k-1} - \dot{z}_{k-2} \rceil^{\frac{1}{2}} + z_k,$$

$$\dot{z}_k = -\lambda_0 L \ \mathrm{sign}(z_k - \dot{z}_{k-1}), \quad z_i - f^{(i)} \to 0.$$

$\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$, …

The differentiation accuracy

$\varepsilon = 0$ (**no noise**) $\Rightarrow$ in a finite time $z_i \equiv f^{(i)}$, $i = 0,\ldots,k$

In the presence of the noise with the magnitude ε, and sampling with the step τ: $\exists \mu_j \geq 1$

$$| z_j - f_0^{(j)} | \leq \mu_j L \rho^{k+1-j}, \quad \rho = \max\left(\tau^{k+1-j}, \left(\frac{\varepsilon}{L}\right)^{\frac{k+1-j}{k+1}}\right),$$

The asymptotics with respect to noise cannot be improved! (Kolmogorov, ≈ 1935)

$$\tau = 0 \Rightarrow | z_j - f^{(j)} | \leq \mu_{kj} L^{\frac{j}{k+1}} \varepsilon^{\frac{k+1-j}{k+1}}, \mu_{kj} \geq 2^{\frac{j}{k+1}}$$

In particular the *k*th derivative has the **worst-case** accuracy

$$|z_k - f^{(k)}| \le \mu_{kk} L^{\frac{k}{k+1}} \varepsilon^{\frac{1}{k+1}}$$

For $k = 5, 6, \ldots: \quad \mu_{kk} \ge 3$

$k = 5,\ L = 1,\ \varepsilon = 10^{-6}$, error of $f^{(5)} > 0.3$

Digital round up: $\varepsilon = 5 \cdot 10^{-16}$

$k = 5:$ error ~ 0.01; $k = 6:$ error ~ 0.02

It is bad, but it cannot be improved!

Universal controller for any RD *r*

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

$$u = -\alpha\Psi_r(z),$$

$$z = D_{r-1}(z, \sigma, L)$$

$L \geq C + \alpha K_M$, α is sufficiently large

Accuracy: |noise| $\leq \varepsilon$, sampling step $\leq \tau$

$$|\sigma^{(j)}| \leq \nu_j \rho^{n+1-j}, \quad \rho = \max(\tau^r, |\varepsilon|^{\frac{1}{n+1}}),$$

$\tau = \varepsilon = 0 \Rightarrow \sigma \equiv 0$ in finite time

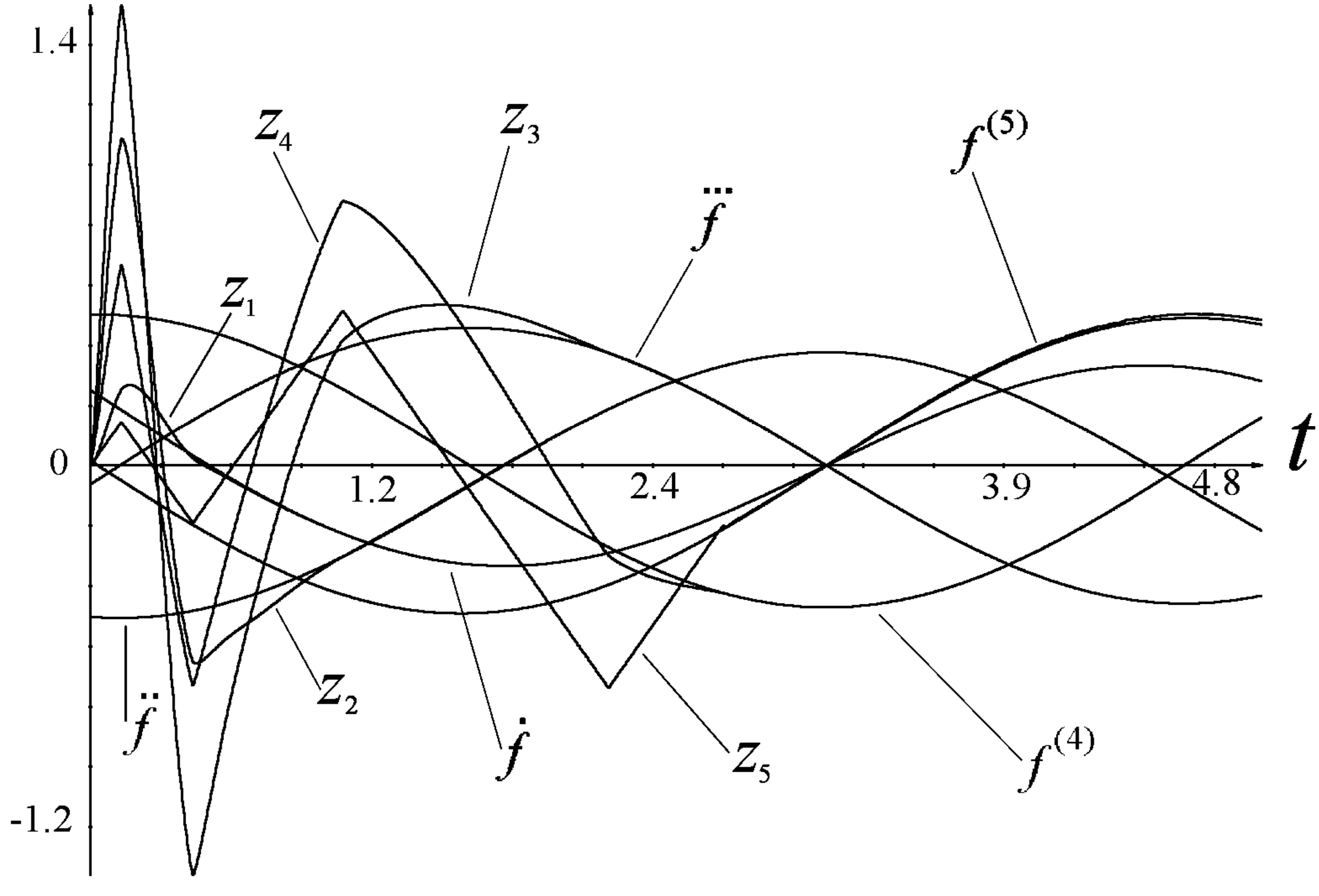
EXAMPLES

5th-order differentiator, $|f^{(6)}| \le L$.

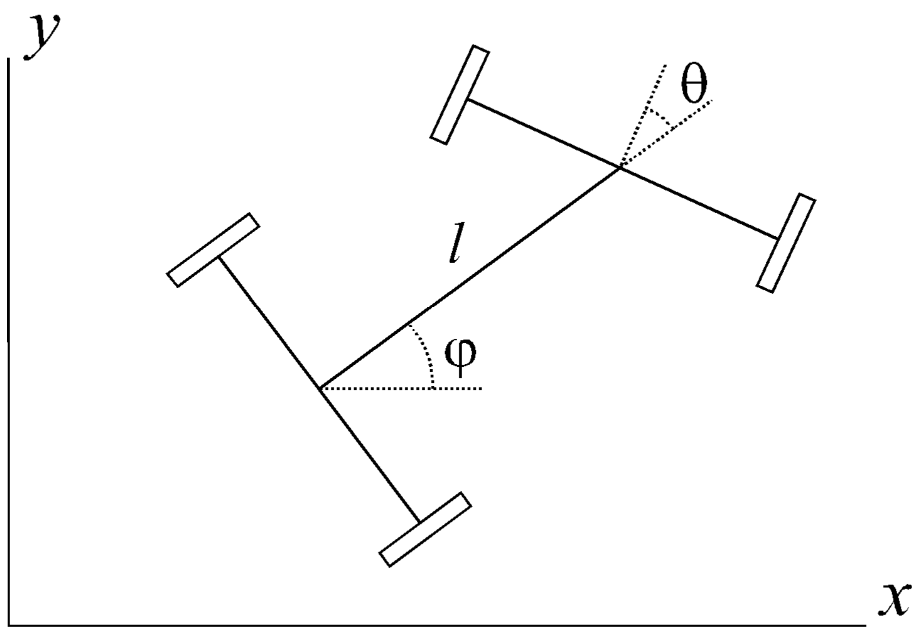
$$
5\left[\begin{array}{l}
\dot{z}_0 = -12L^{\frac{1}{6}} \left[\!\left[z_0 - f(t) \right]\!\right]^{\frac{5}{6}} + z_1, \\
4\left[\begin{array}{l}
\dot{z}_1 = -8L^{\frac{1}{5}} \left[\!\left[z_1 - \dot{z}_0 \right]\!\right]^{\frac{4}{5}} + z_2, \\
3\left[\begin{array}{l}
\dot{z}_2 = -5L^{\frac{1}{4}} \left[\!\left[z_2 - \dot{z}_1 \right]\!\right]^{\frac{3}{4}} + z_3, \\
2\left[\begin{array}{l}
\dot{z}_3 = -3L^{\frac{1}{3}} \left[\!\left[z_3 - \dot{z}_2 \right]\!\right]^{\frac{2}{3}} + z_4, \\
1\left[\begin{array}{l}
\dot{z}_4 = -1.5L^{\frac{1}{2}} \left[\!\left[z_4 - \dot{z}_3 \right]\!\right]^{\frac{1}{2}} + z_5, \\
\dot{z}_5 = -1.1L\,\mathrm{sign}(z_5 - \dot{z}_4)
\end{array}\right.
\end{array}\right.
\end{array}\right.
\end{array}\right.
\end{array}\right.
$$

5th-order differentiation

$$f(t) = \sin 0.5t + \cos 0.5t, \quad L = 1$$



Example: car control



$\dot{x} = V \cos \varphi$, $\dot{y} = V \sin \varphi$,

$\dot{\varphi} = (V/l) \tan \theta$,

$\dot{\theta} = u$

RD = 3

x, y are measured.

The task: real-time tracking $y = g(x)$

$V = const = 10$ m/s = 36 km/h, $l = 5$ m,

$x = y = \varphi = \theta = 0$ at $t = 0$

Solution: $\sigma = y - g(x)$, $r = 3$

3-sliding controller (N°3), $\alpha = 2$, $L = 100$

3-sliding car control

$\sigma = y - g(x)$.

Simulation: $g(x) = 10 \sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$.

$$u = 0, \quad 0 \le t < 1,$$

The controller:

$$u = -2\frac{s_2 + 2\dfrac{(s_1 + |s|^{2/3} \operatorname{sign} s)}{(|s_1| + |s|^{2/3})^{1/2}}}{|s_2| + 2(|s_1| + |s|^{2/3})^{1/2}}, \quad t \ge 1$$

Differentiator:

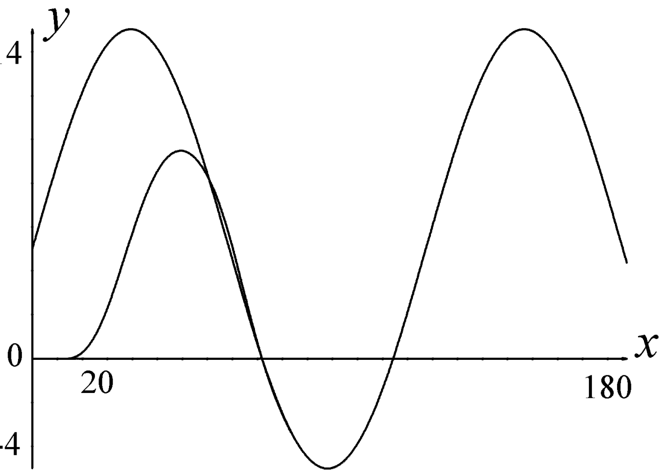
$\dot{s} = D_2(s, \sigma, 100)$, $L = 100$:

$$\dot{s}_0 = -9.28 [\![s_0 - \sigma]\!]^{\frac{2}{3}} + s_1,$$

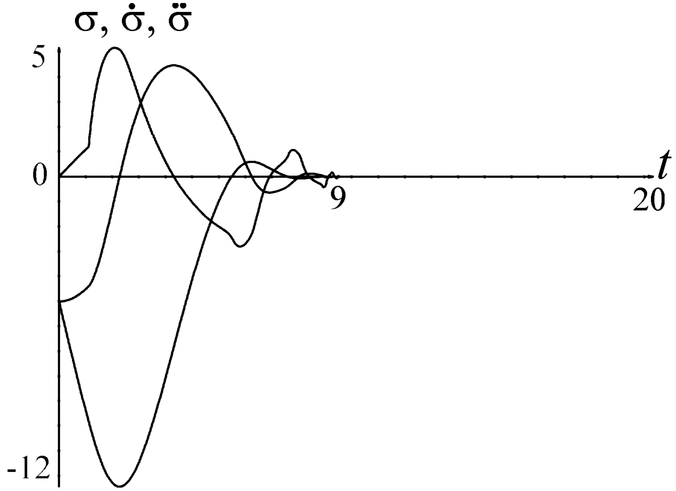
$$\dot{s}_1 = -15 [\![s_1 - \dot{s}_0]\!]^{\frac{1}{2}} + s_2,$$

$$\dot{s}_2 = -110 \operatorname{sign}(s_2 - \dot{s}_1)$$

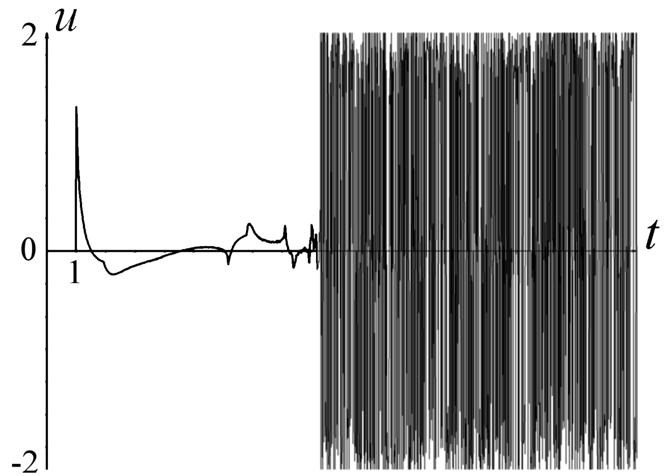
3-sliding car control



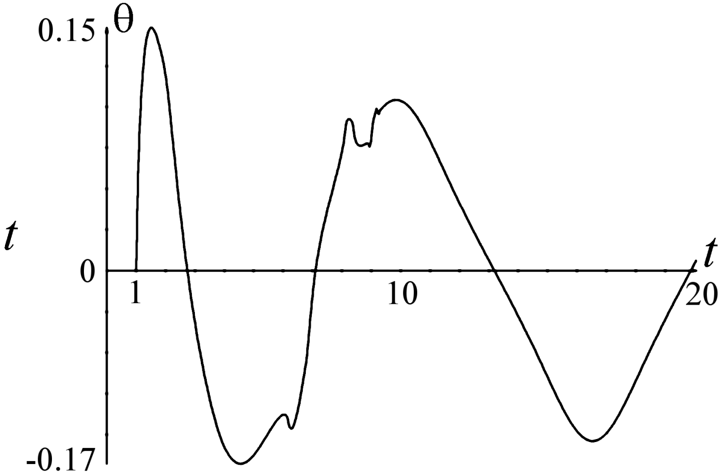
a. Car trajectory



b. 3-sliding deviations



c. Steering-angle derivative



d. Steering angle

$$\tau = 10^{-4} \Rightarrow \quad |\sigma| \le 5.4\cdot 10^{-7},\ |\dot\sigma| \le 2.5\cdot 10^{-4},\ |\ddot\sigma| \le 0.04$$

$$\tau = 10^{-5} \Rightarrow \quad |\sigma| \le 5.6\cdot 10^{-10},\ |\dot\sigma| \le 1.4\cdot 10^{-5},\ |\ddot\sigma| \le 0.004$$

Practical Relative Degree
PRD

NO MODEL AT ALL

Practical Relative Degree Definition

Nothing is known on the system.

$r \in \mathbb{N}$ is called the PRD, if $\exists \lambda_\sigma = 1$ or -1:

$\exists \varepsilon, \delta_t, \alpha_M, \alpha_m, L, L_m > 0, \alpha_m \le \alpha_M, L_m \le L$,:

1. For any (measurable) $u(t)$, $|u-u_0| \le U_M$:

$$\text{Output: } \tilde{\sigma} = \sigma + \eta,\ |\eta| \le \varepsilon,$$

$$\sigma^{(r-1)} \in \mathrm{Lip}(L)$$

2. For $\omega = \lambda_\sigma \sigma$:

If $\forall t \ge t_0$

$$\alpha_M \ge u(t) - u_0 \ge \alpha_m \quad (-\alpha_M \le u(t) - u_0 \le -\alpha_m),$$

then $\forall t \ge t_0 + \delta_t$:

$$\omega^{(r)} \ge L_m \quad (\omega^{(r)} \le -L_m)$$

Naming

u_0 is the *reference input*,

in the following $u_0 = 0$

λ_σ is the *influence direction* parameter,

in the following $\lambda_\sigma = 1$

δ_t is the *delay* parameter

ε is the *approximation* parameter.

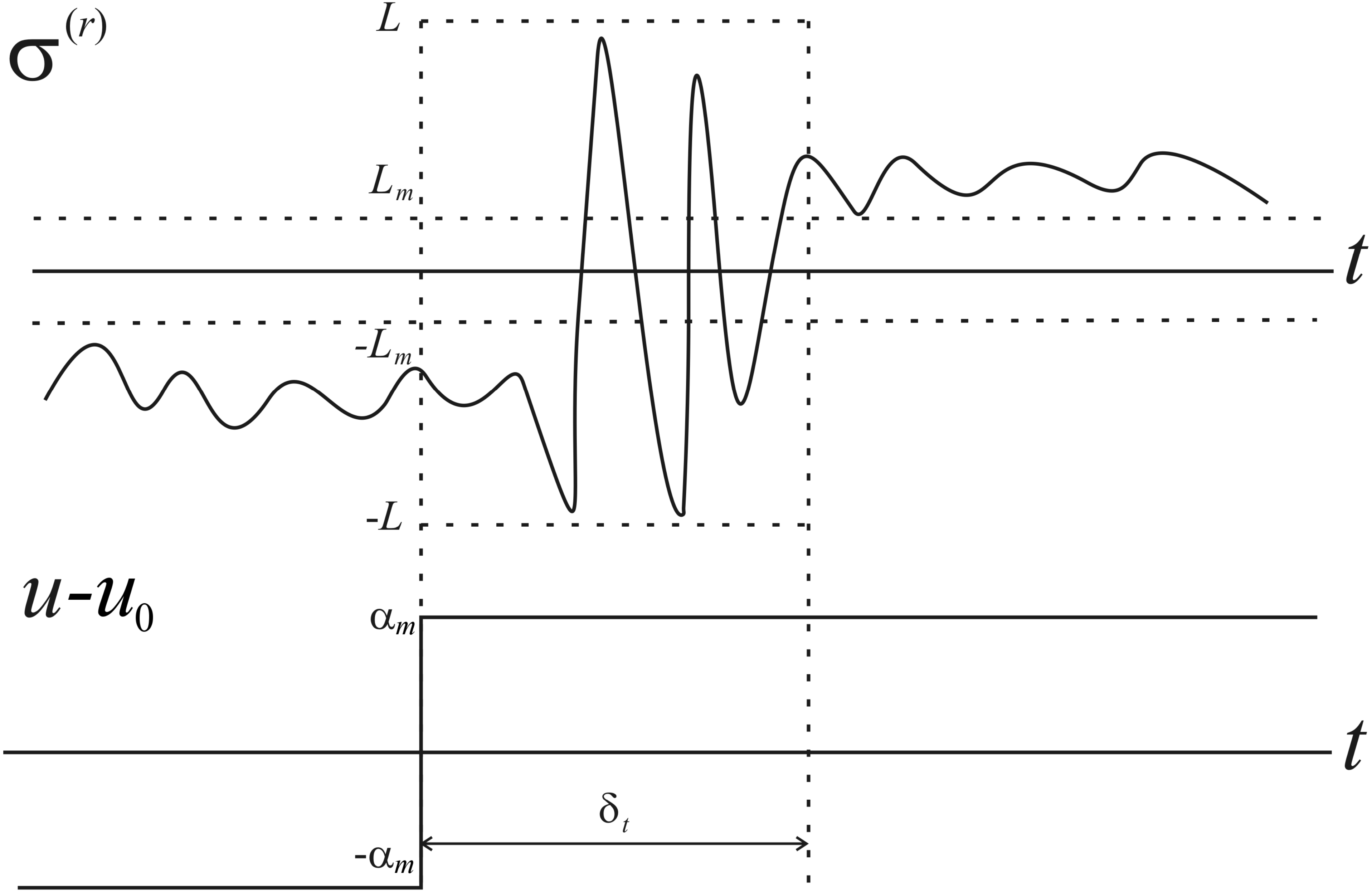
Local Practical Relative Degree Definition

$\exists\ t_1, t_2, T, t_1 < t_2, \delta_t < T$, such that

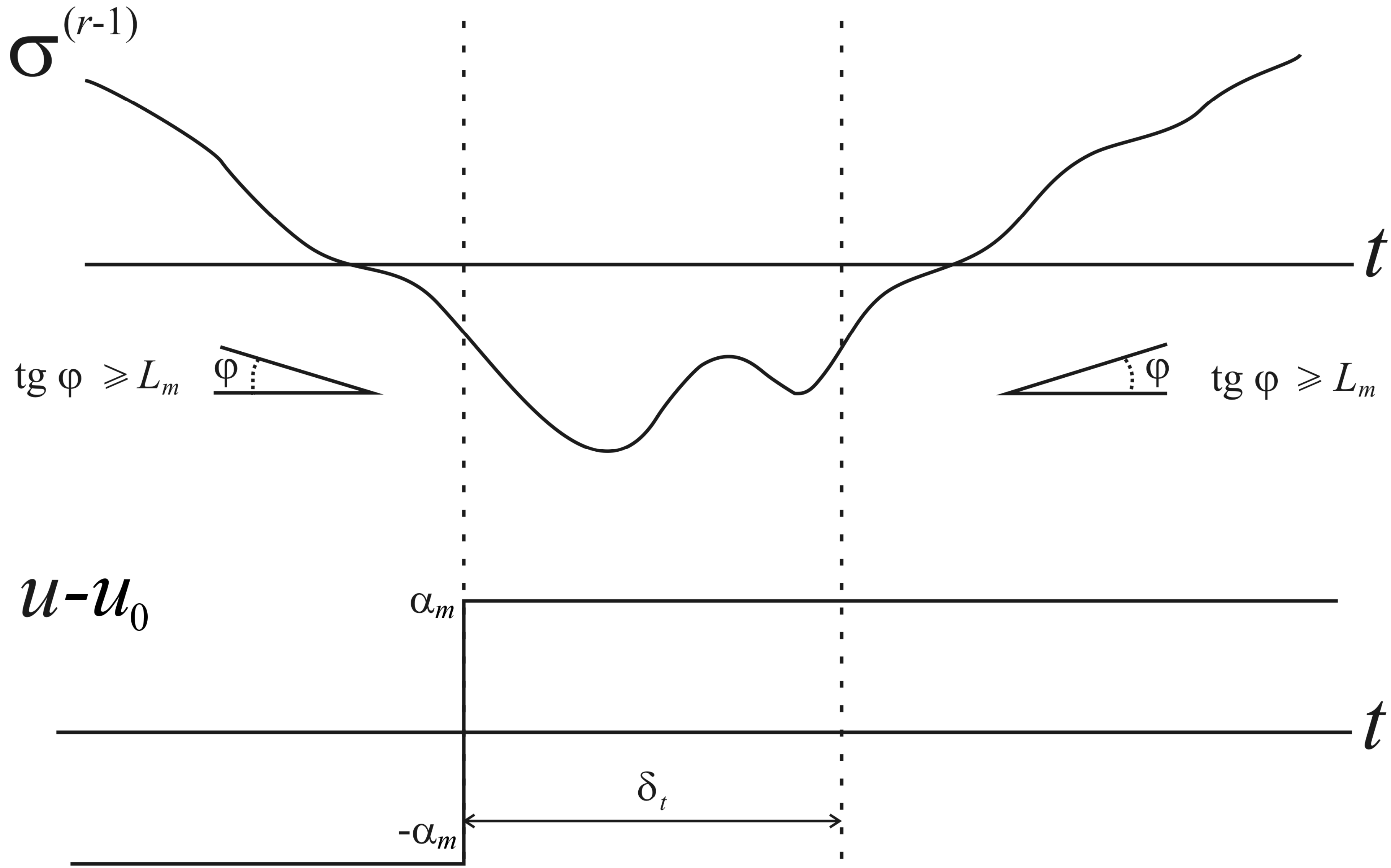
requirement 1 is true over the time interval $[t_1, t_2 + T]$;

requirement 2 is true for each $t_0 \in [t_1, t_2]$ over $[t_0, t_0 + T]$.

Graphical interpretation - 1



Graphical interpretation – 2



Remarks

The function σ does not necessarily need to have any real meaning. It can be just an output of some smoothing filter, in particular, of a tracking differentiator.

Local practical relative degree is used for temporary output regulation.

Keeping $\sigma \equiv 0$ is not possible under these conditions.

Control

$$u = -\alpha \Psi_r(z), \;\; \dot{z} = D_{r-1}(z, \sigma, L),$$
$$\alpha_m \le \alpha \le \alpha_M$$

Differentiator parameters λ_i are properly chosen

Theorem. $\exists$ β_1, …, $\beta_{r\text{-}1}$ (coefficients of the r-SM homogeneous controller):

Accuracy: $\sigma = O(\max[\varepsilon, \delta_t^{\,r}])$

Important remark

The differentiator is the just a dynamic part of the controller. Its outputs do not have any physical meaning, because PRD $\neq$ RD.

Without the differentiator the controller loses its robustness. An attempt to replace the differentiator with any sensors or other observers is very dangerous!

Continuous controller

based on quasi-continuous controller

$$u = -\alpha\Phi(\|z\|_h)\Psi_r(z)$$

(SM regularization)

$$\Phi(\| z \|_h) = \begin{cases} 1 \text{ with } \| z \|_h > \gamma \max[\varepsilon, \delta_t^r], \\ \dfrac{1}{\max[\varepsilon, \delta_t^r]} \| z \|_h^2 \text{ with } \| z \|_h \le \gamma \max[\varepsilon, \delta_t^r], \end{cases}$$

$$\| z \|_h^2 = z_0^{2/r} + z_1^{2/(r-1)} + \ldots + z_{r-1}^2$$

The accuracy is the same.

Simulation

Perturbed car model

$\dot{x} = V\cos\phi, \dot{y} = V\sin\phi,$

$\ddot{\phi} = -4\text{sign}(\phi-\varphi)-6\dot{\phi}, \quad \Rightarrow \quad$ Rel. degree does not exist!

$\dot{\varphi} = \frac{V}{\Delta}\tan\theta, \qquad \dot{\theta} = \zeta_1,$

Actuator: input u, output ζ_1

$$\ddot{\zeta}_1 = -100(2(\zeta_1 - u) + 0.01\dot{\zeta}_1)^3 - 100(\zeta_1 - u) - 2\dot{\zeta}_1,$$

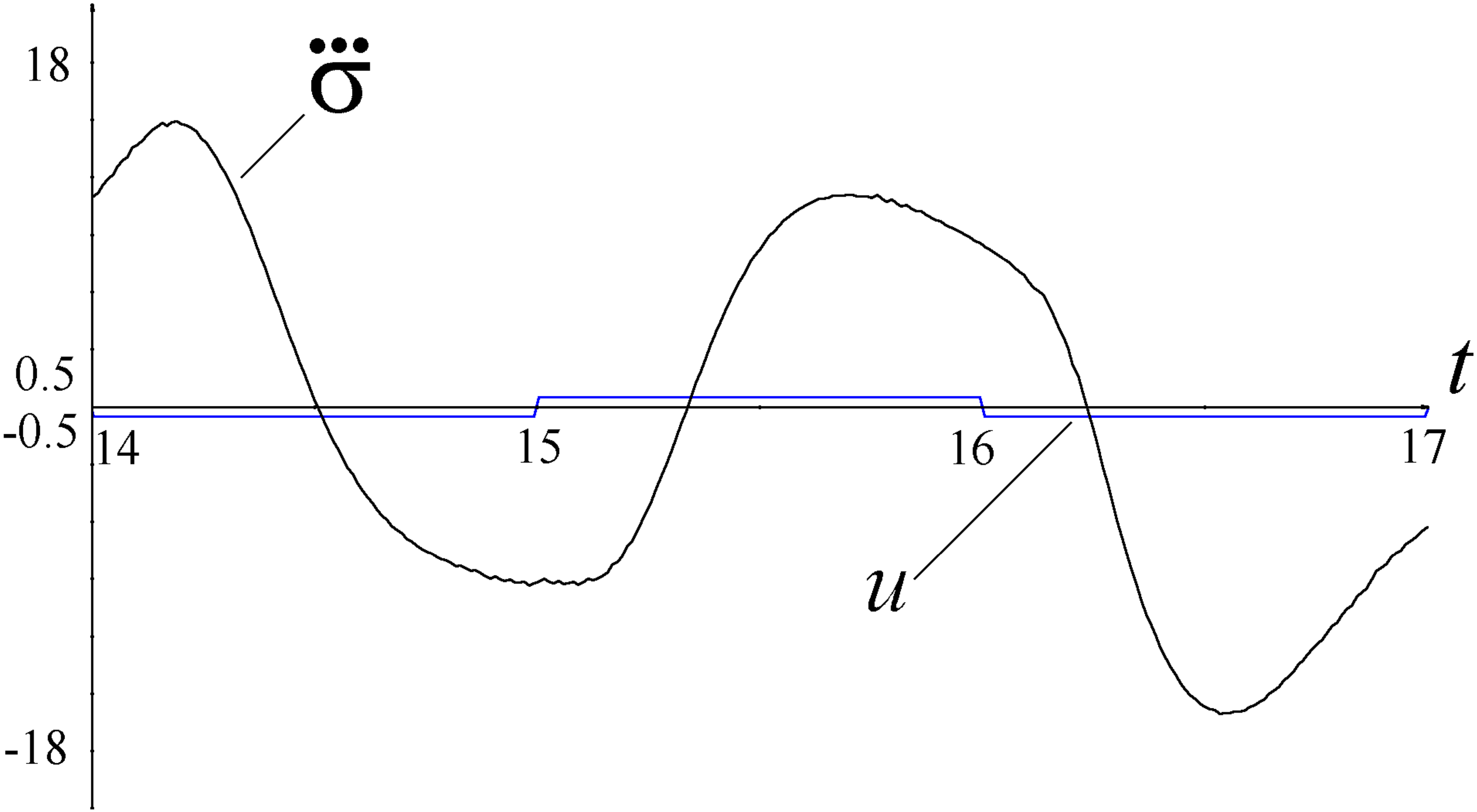
Sensor: $\tilde{\sigma} = \zeta_2 + 0.01\dot{\zeta}_2 - g(x) + \eta(t)$, η is a noise, $|\eta| \le 0.01$.

$$\dddot{\zeta}_2 = -100(\zeta_2 - y) - 2\dot{\zeta}_2 - 0.02\ddot{\zeta}_2,$$

$\zeta_2 = -10$, $\dot{\zeta}_2 = 2000$, $\ddot{\zeta}_2 = -80000$, $\zeta_1 = \dot{\zeta}_1 = \phi = \dot{\phi} = 0$ at $t = 0$,

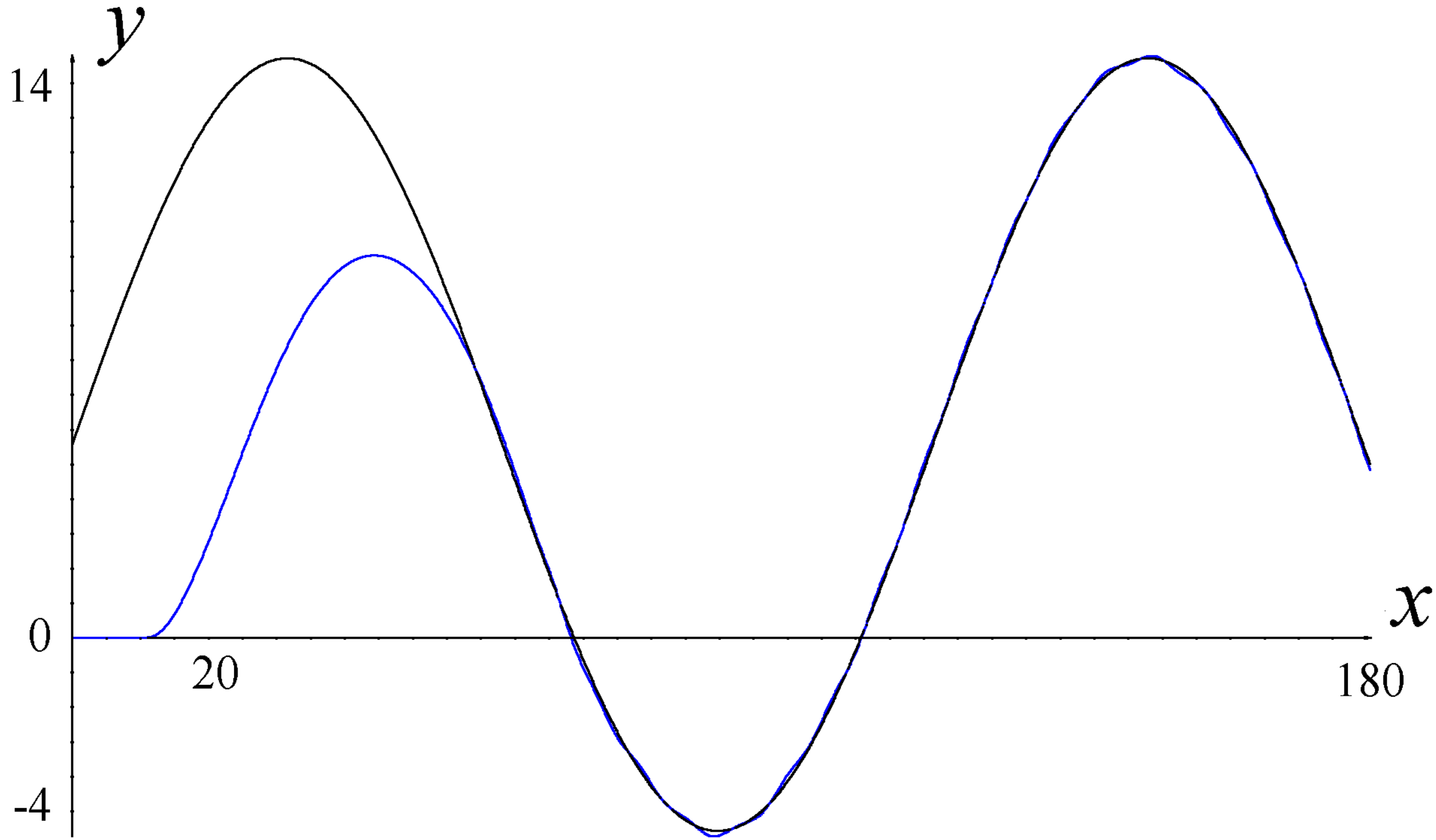
If the system were smooth the new RD were 10

Practical rel. degree = 3



Differentiator of the order 3 is used with $L = 100$.

System performance



$|\underline{\sigma}| \leq 0.16$

APPLICATION

Blood Glucose Control

High-Order Sliding-Mode Control of Blood Glucose Concentration via Practical Relative Degree Identification

CDC-ECC 2011

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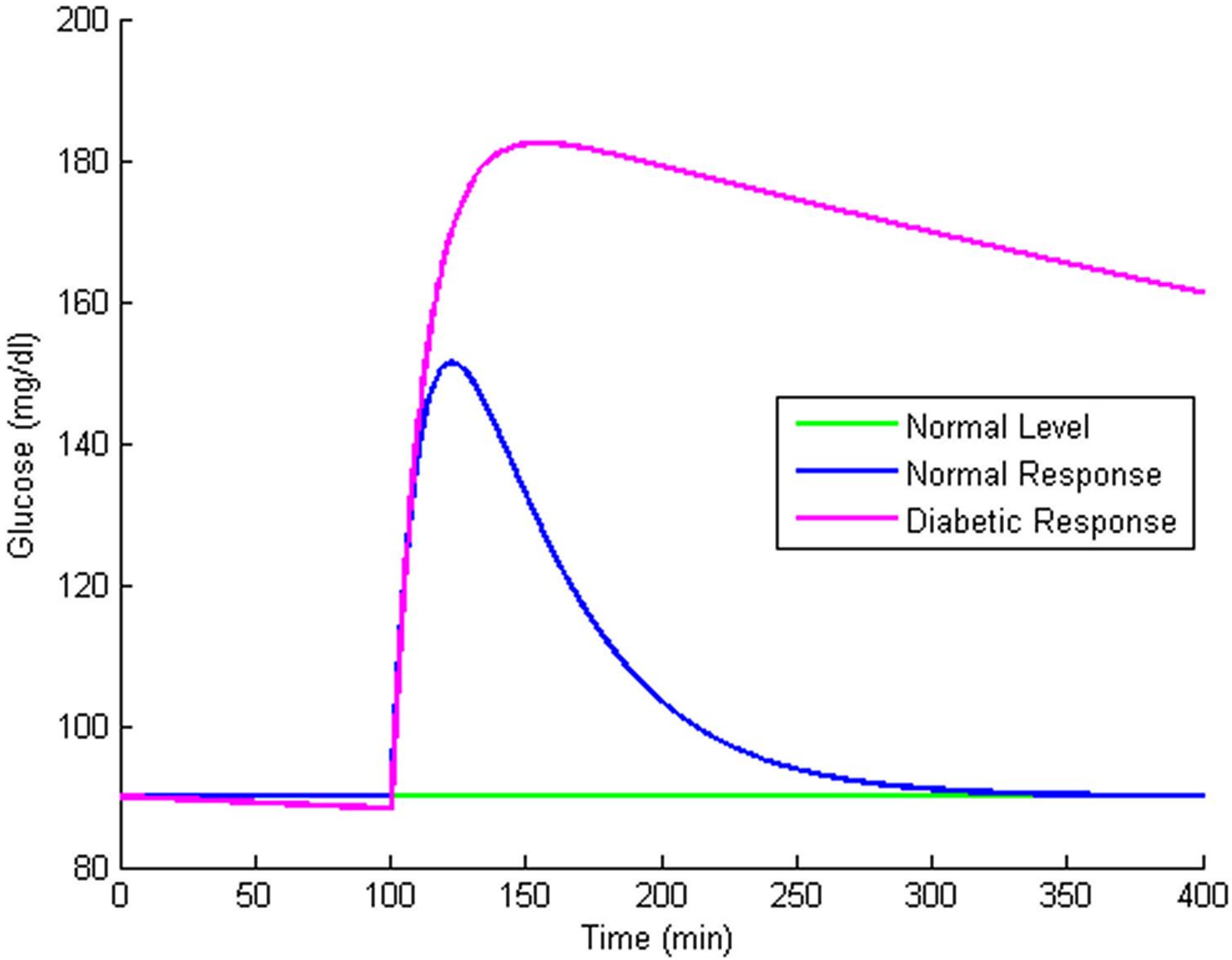
Universidad Nacional Autónoma de México
Centro Médico Nacional Siglo XXI, IMSS
Tel Aviv University
University of Alabama in Huntsville

SLIDING MODE LAB
UNAM

Body reaction to glucose concentration increase



Different models

Model	RD	No. States
Bergman	3	3
Candas-Radziuk	3	4
Cobelli	3	7
Hovorka	5	8
Dalla Man	5	8
Sorensen	5	18

- Output: blood glucose
- Input: insulin

The simplest model

BeM Bergman et al., 1979

$$\dot{B}_1 = -p_1[B_1 - G_b] - B_1B_2 + \dot{D}(t)$$
$$\dot{B}_2 = -p_2B_2 + p_3[B_3 - I_b]$$
$$\dot{B}_3 = -n[B_3 - I_b] + \gamma[B_1 - h]^+u(t)$$

Table: Variables

Variable	Description
B_1	Blood glucose concentration (Output)
B_2	Effect of insulin on glucose uptake
B_3	Blood insulin concentration

Sorensen model

$$\dot{S}_1 = \frac{1}{V_H^G}(-Q_H^G S_1 + Q_L^G S_2 + S_7 - F_{RBGU})$$

$$\dot{S}_2 = \frac{2}{V_L^G}(Q_A^G S_1 + Q_G^G S_6 - Q_L^G S_2 + f_{HGP} S_8 - f_{HGU} S_3)$$

$$\dot{S}_3 = \frac{1}{\tau_1}(2\tanh(0.55 S_4^N) - S_3)$$

$$\dot{S}_4 = \frac{2}{V_L^I}(Q_A^I S_5 + Q_G^I S_{10} - Q_L^I S_4 - F_{LIC})$$

$$\dot{S}_5 = \frac{1}{V_H^I}(Q_L^I S_4 - Q_H^I S_5 + S_9 + u(t))$$

$$\dot{S}_6 = \frac{Q_G^G}{V_G^G}(S_1 - S_6) + \frac{1}{V_G^G}(F_{MEAL} - R_{GGU})$$

$$\dot{S}_7 = Q_K^G \dot{G}_K + G_P^G \dot{G}_{PV} + Q_B^G \dot{G}_{BV}$$

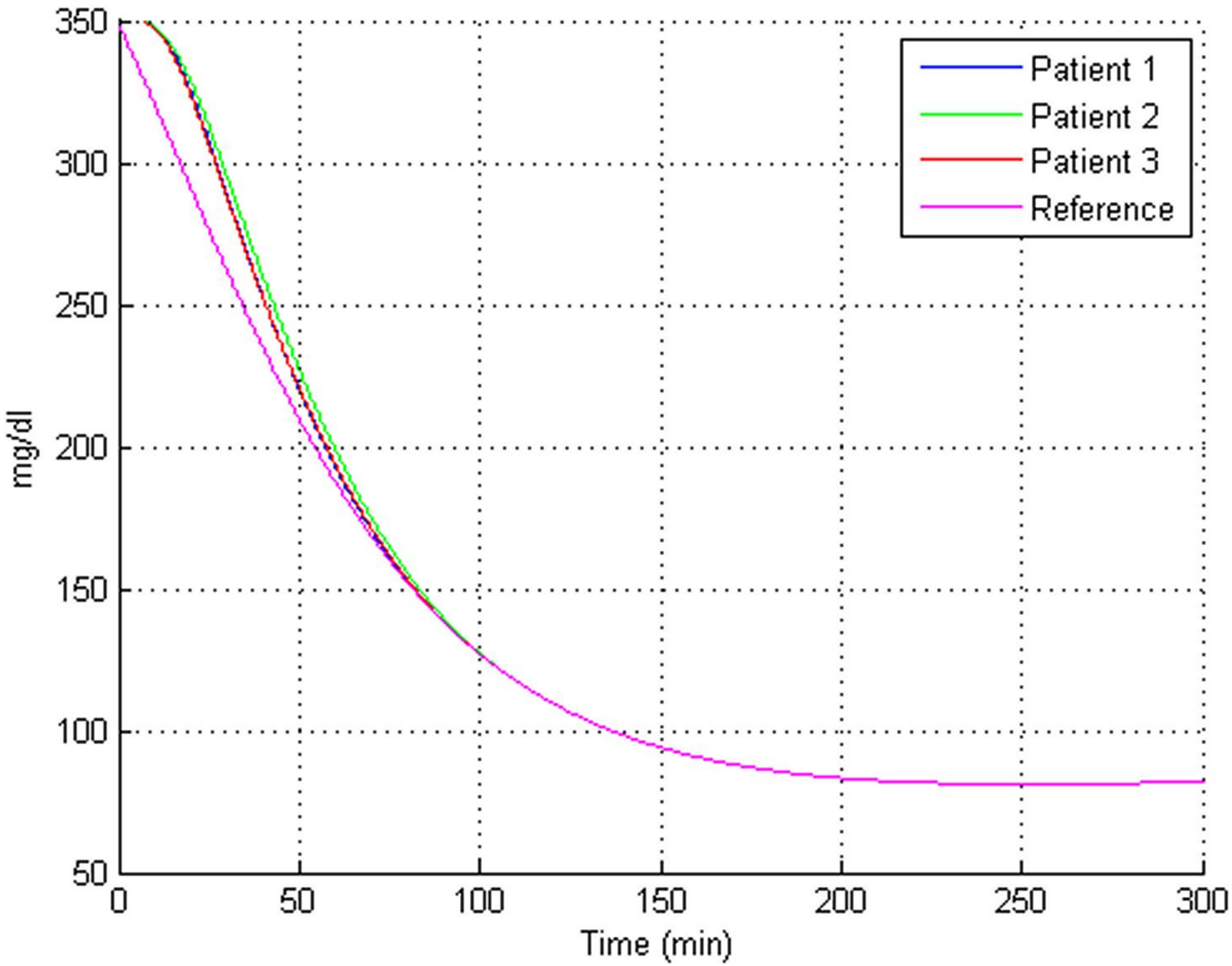
$$\dot{S}_8 = \frac{1}{\tau_1}(1.21 - 1.14\tanh[1.66(S_4^N - 0.89)] - S_8)$$

$$\dot{S}_9 = Q_B^I \dot{I}_B + Q_K^I \dot{I}_K + Q_P^I \dot{I}_{PV}$$

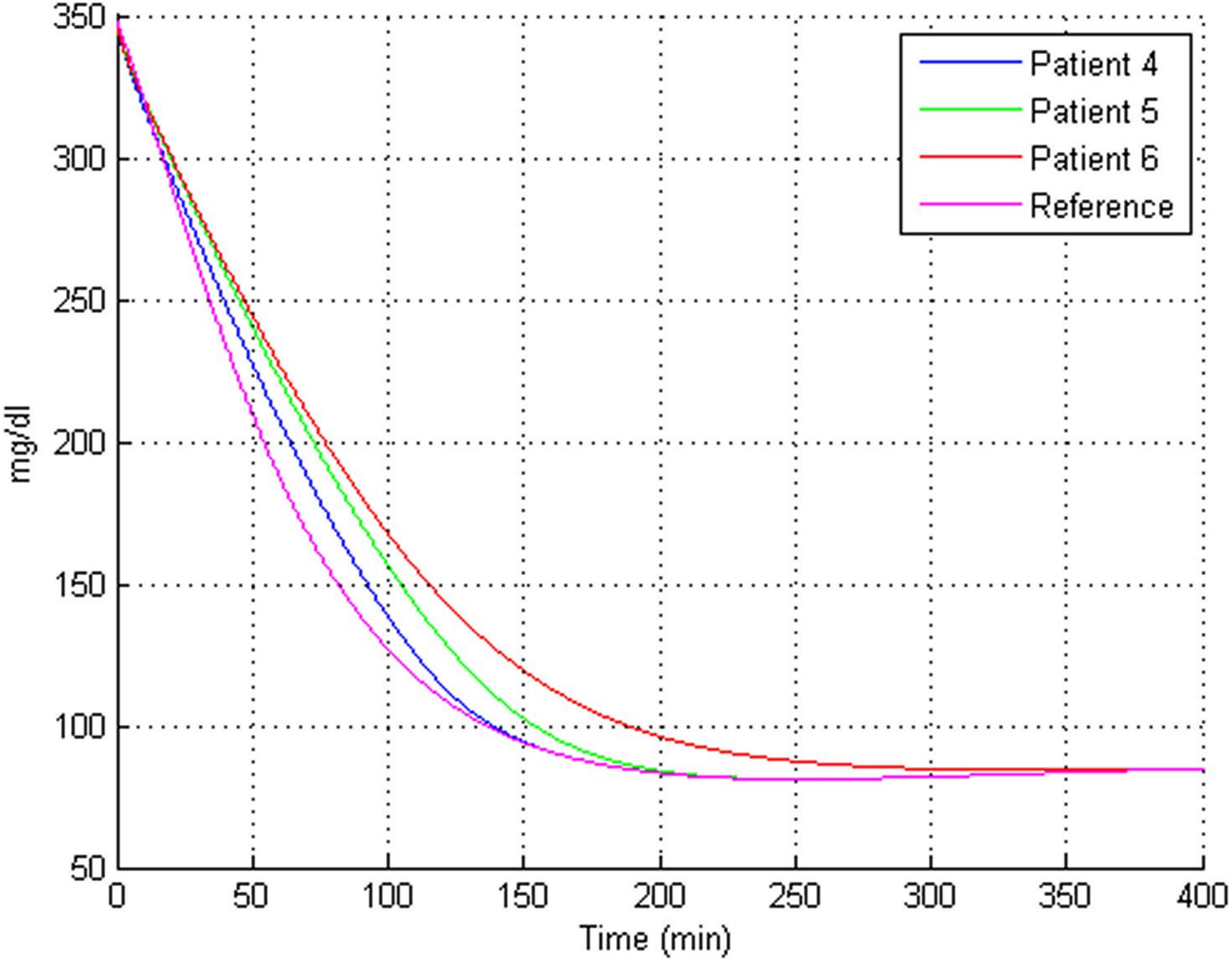
$$\dot{S}_{10} = \frac{Q_G^I}{V_G^I}(S_5 - S_{10})$$

$$\dot{S}_{11} = \frac{1}{V_C}(F_{PCR} - F_{MCC} S_{11}^N)$$

3-sliding QC control (BeM)

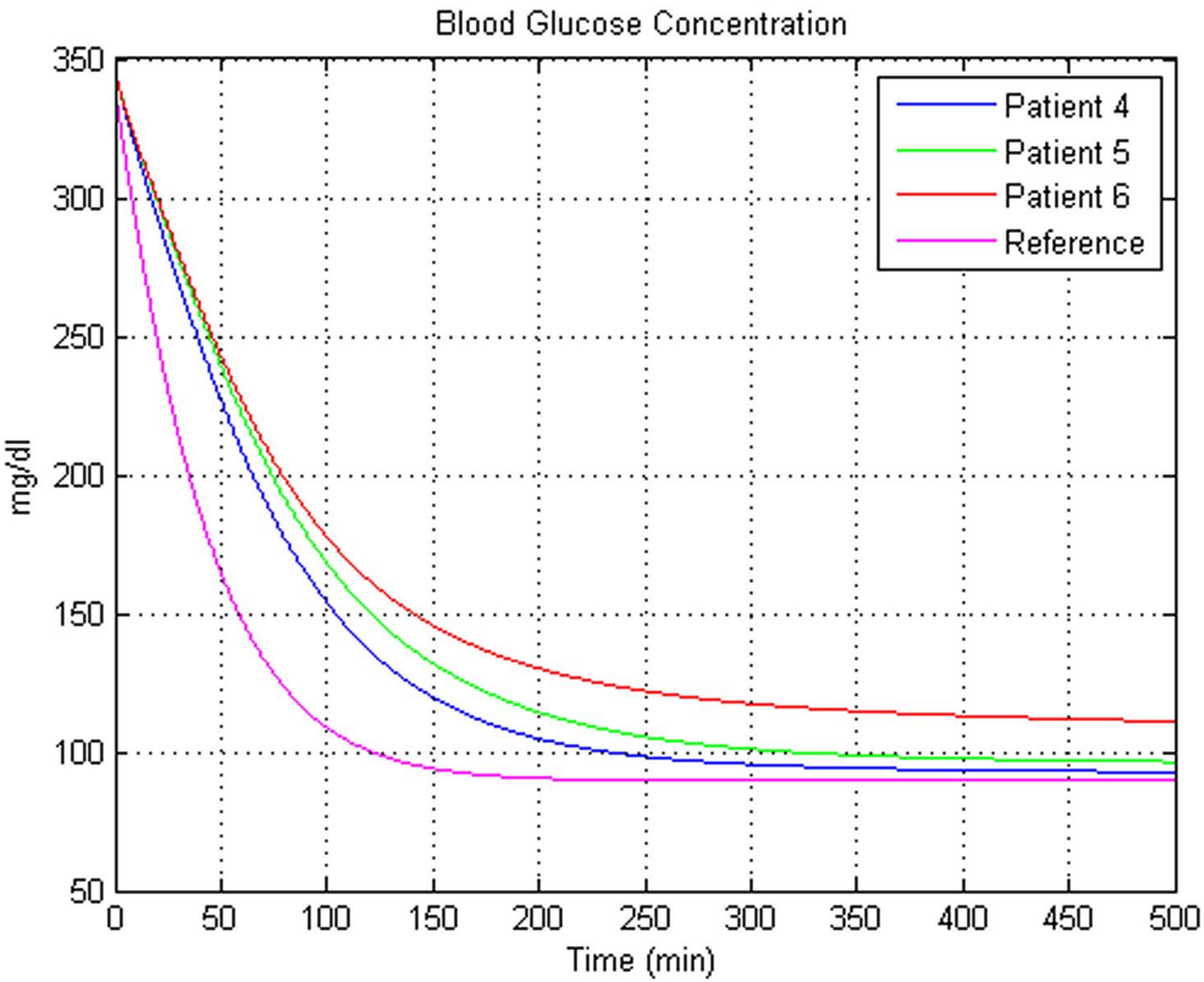


3-sliding QC control (SoM)



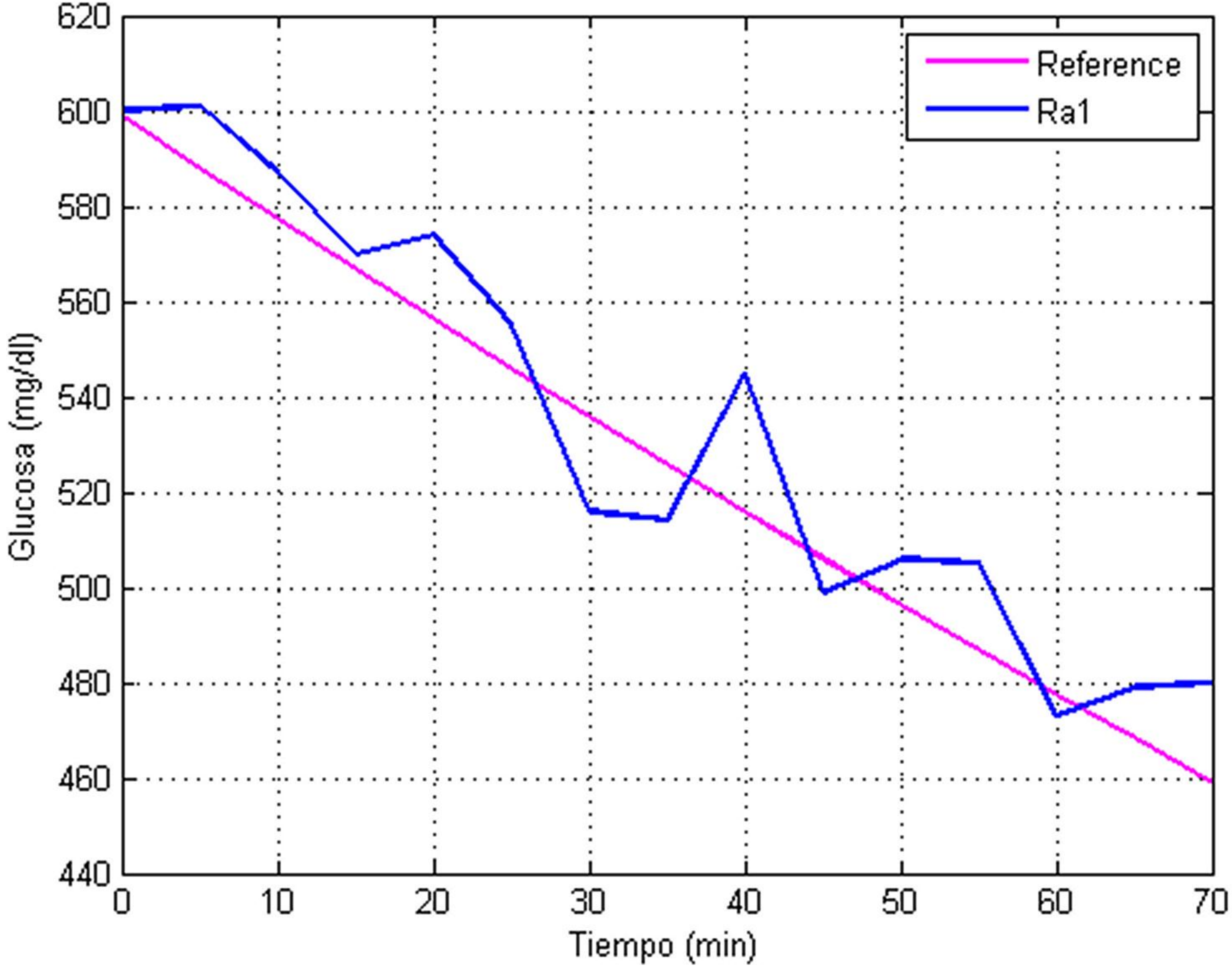
The same parameters

PID control (SoM)



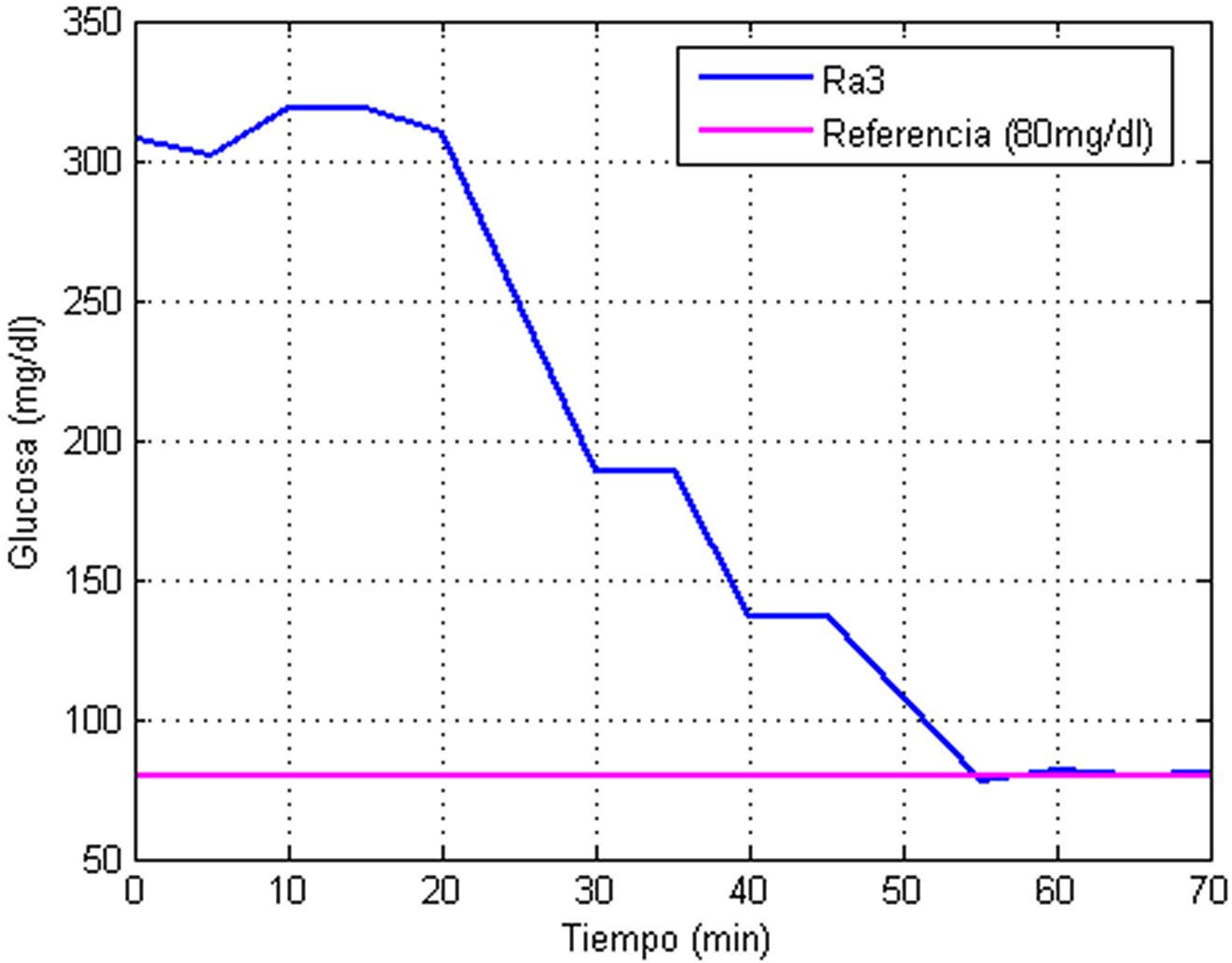
Experiments on rats

Rat 1

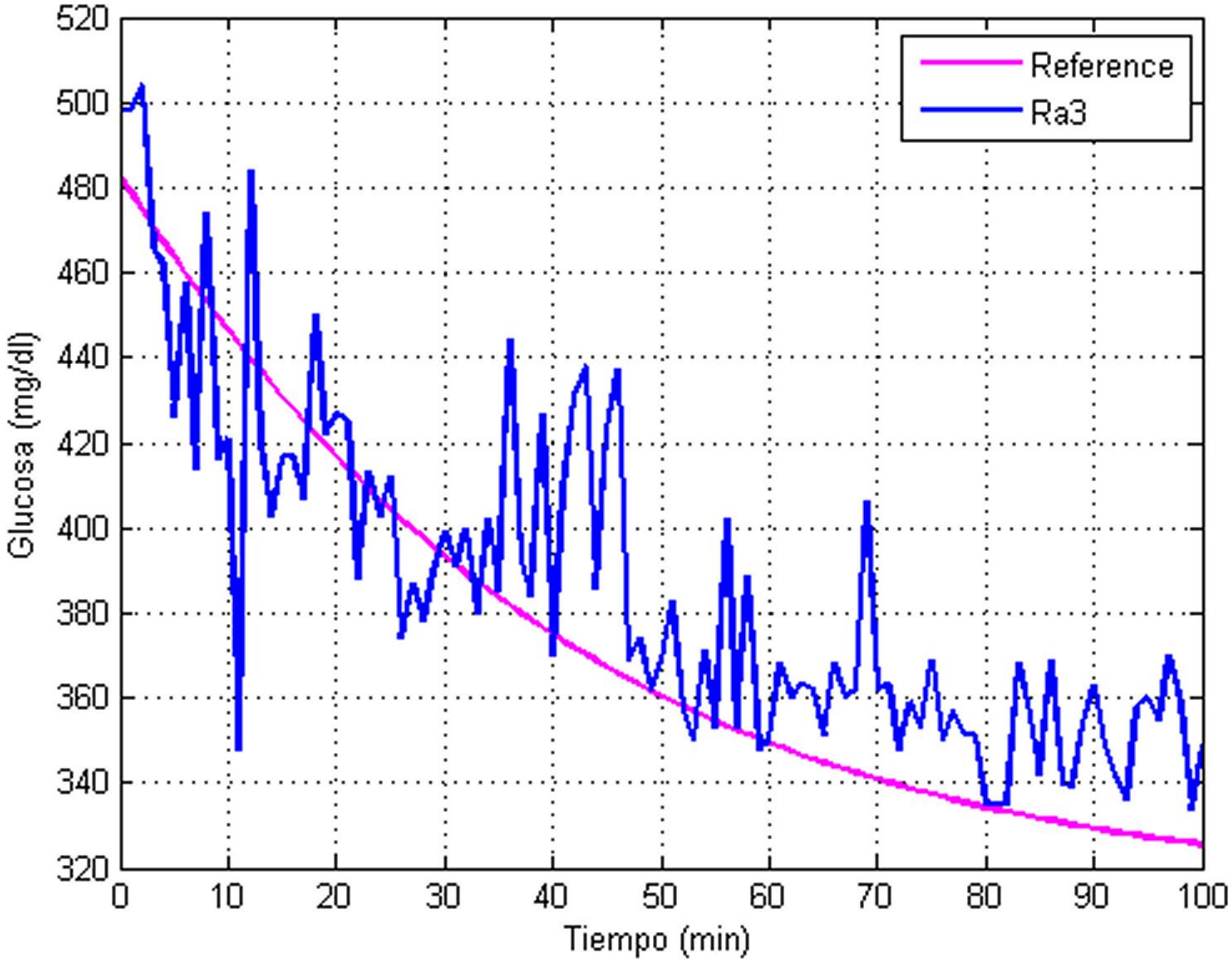


620
600
580
560
540
520
500
480
460
440
Glucosa (mg/dl)
0
10
20
30
40
50
60
70
Tiempo (min)
Reference
Ra1

Rat 2


Ra3
Referencia (80mg/dl)
Glucosa (mg/dl)
Tiempo (min)
350
300
250
200
150
100
50
0
10
20
30
40
50
60
70

Rat 3



Reference
Ra3
Glucosa (mg/dl)
Tiempo (min)
520
500
480
460
440
420
400
380
360
340
320
0
10
20
30
40
50
60
70
80
90
100

Conclusions

In practice the system relative degree is a design parameter.

Systems of uncertain nature can be effectively controlled, provided their practical relative degree is identified.

A system can have a few generalized PRDs! That is why the considered control is universal.

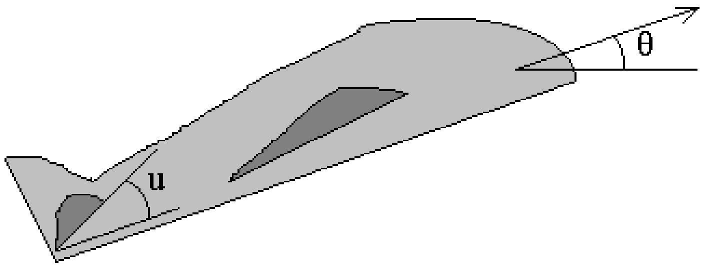
Hypothesis

Humans (and animals) have universal controllers embodied for PRD $\leq$ 2 (3?).

Thank you very much!

Applications

Pitch Control



Problem statement. A non-linear process is given by a set of 42 linear approximations

$$\frac{d}{dt}(x,\theta,q)^t = G(x,\theta,q)^t + Hu, \; q = \dot{\theta},$$

$$x \in \mathbf{R}^3, \; \theta, \; q, \; u \in \mathbf{R},$$

x_1, x_2 -velocities, x_3 - altitude

The Task: $\theta \rightarrow \theta_c(t)$, $\theta_c(t)$ is given in real time.

G and *H* are not known properly

Sampling Frequency: 64 Hz, Measurement noises

Actuator: delay and discretization.

dθ/dt does not depend explicitly on *u* (relative degree 2)

Primary Statement:

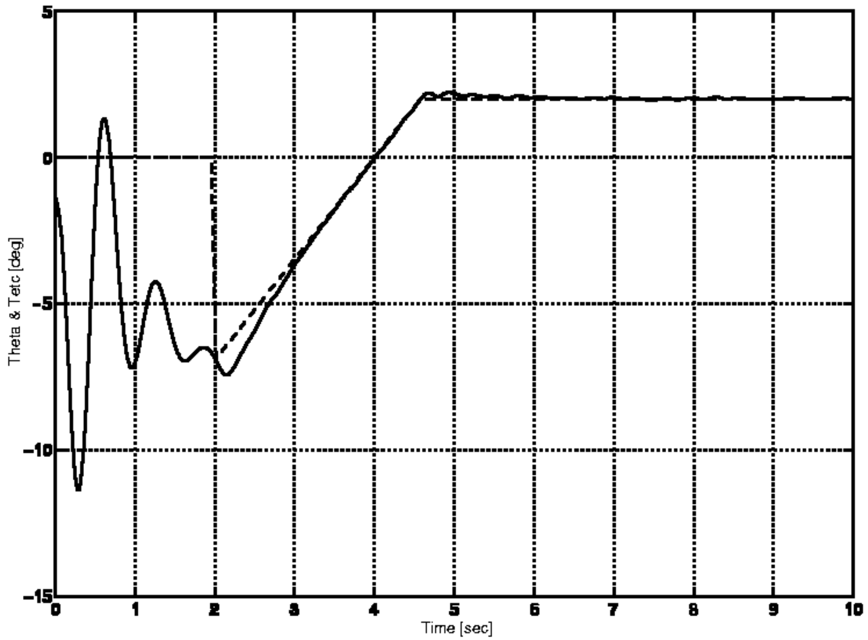
Available: θ, θ_c, Dynamic Pressure and Mach.

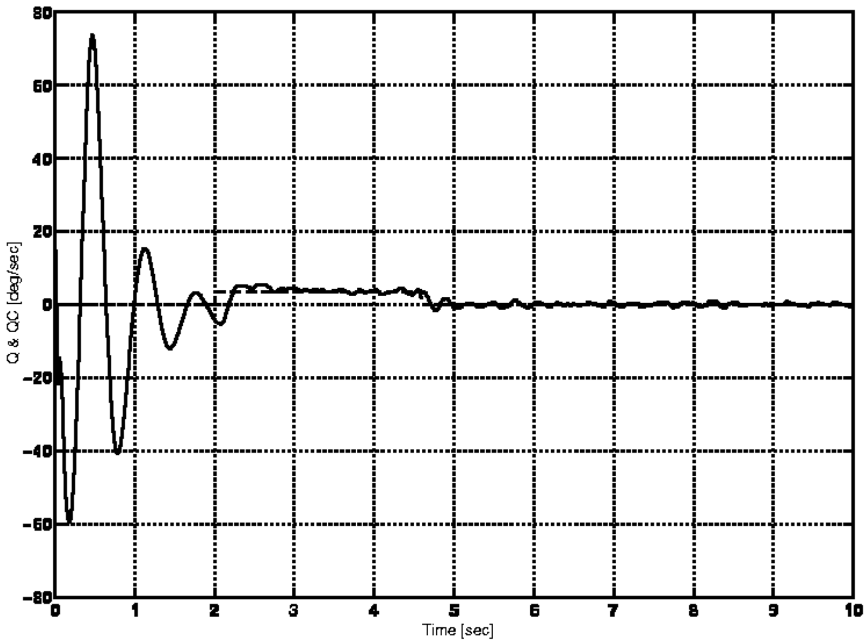
Main Statement: also $\dot{\theta}$, $\dot{\theta}_c$ are measured

The idea: keeping $5(\theta - \theta_c) + (\dot{\theta} - \dot{\theta}_c) = 0$ in 2-sliding mode

(asymptotic 3-sliding)

Flight Experiments



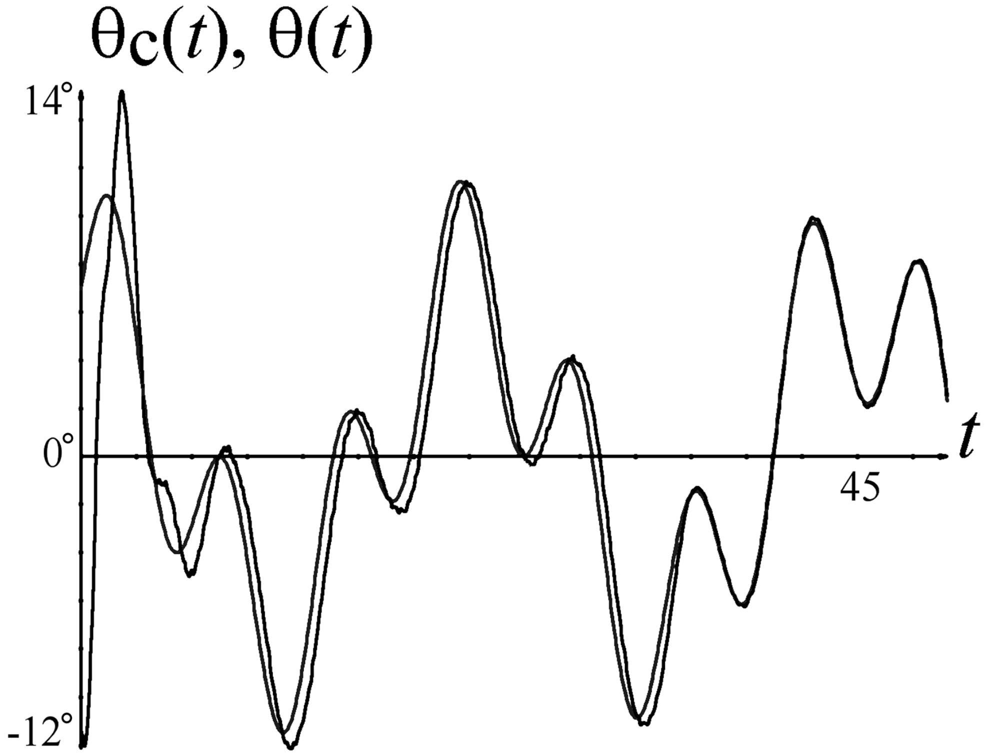


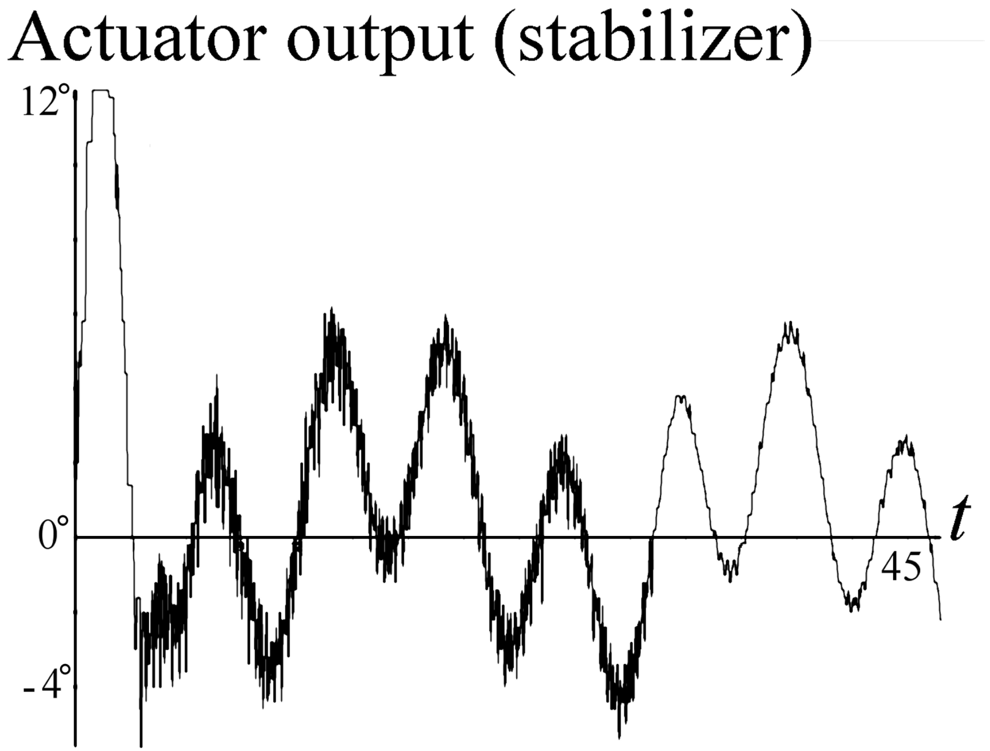
$\theta_c(t), \theta(t)$

$\dot{\theta}_c = q_c(t), \dot{\theta} = q(t)$

Example: practical pitch control

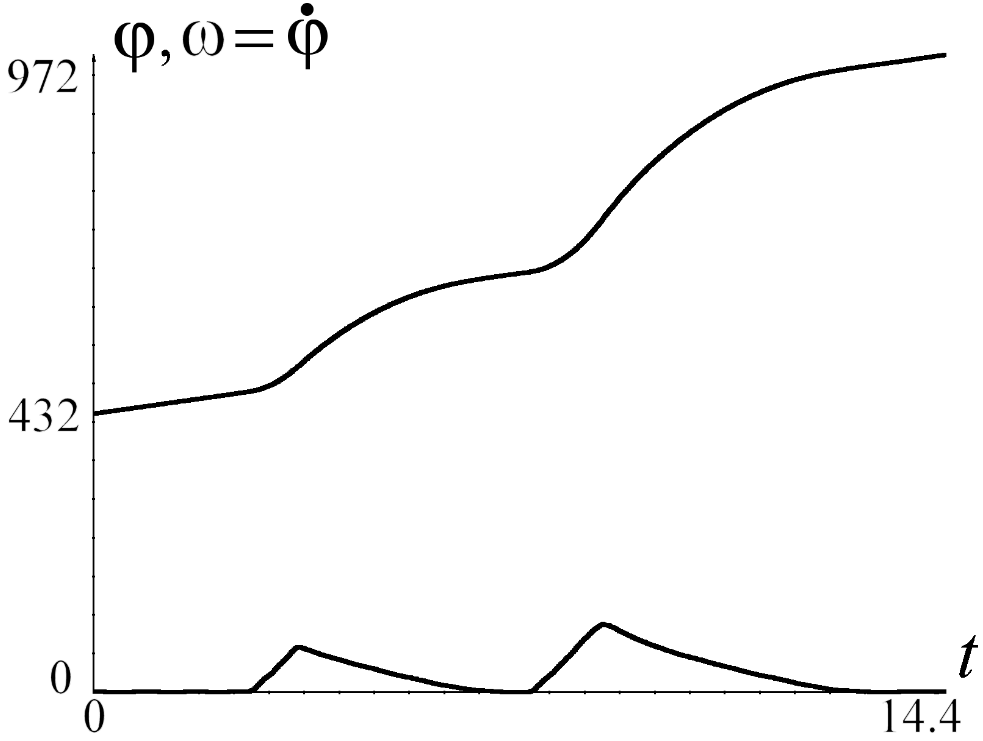
Actuator (server-stepper) output, Levant et al., 2000



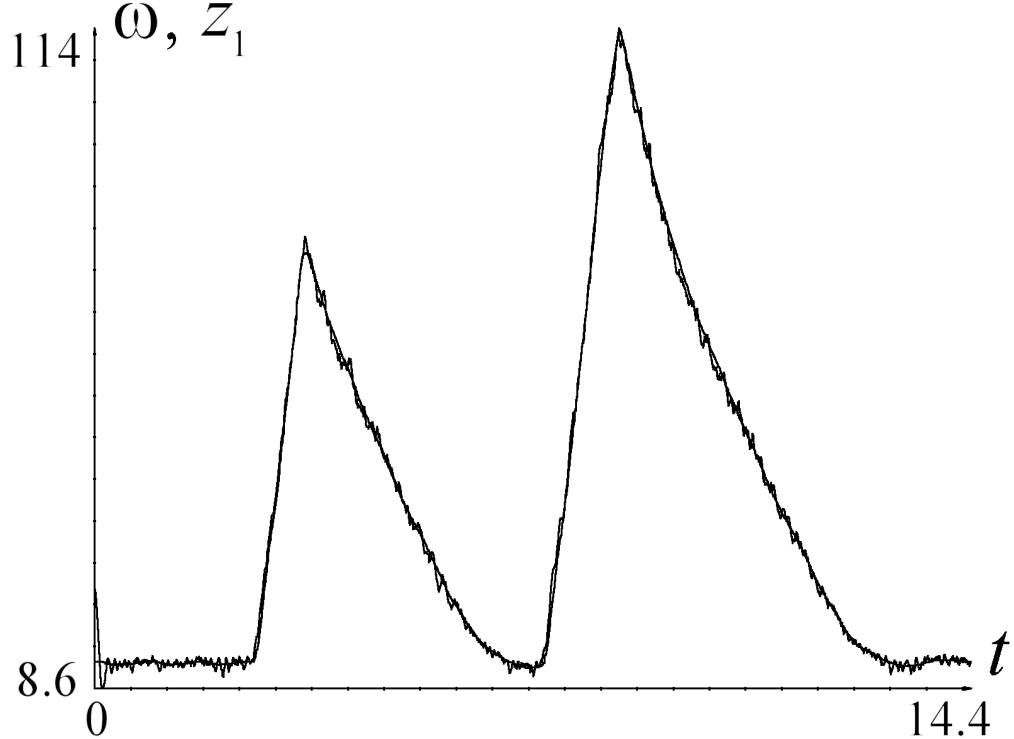


Switch from Linear (H_∞) control to 3-SM control

On-line calculation of the angular motor velocity and acceleration (data from Volvo Ltd)



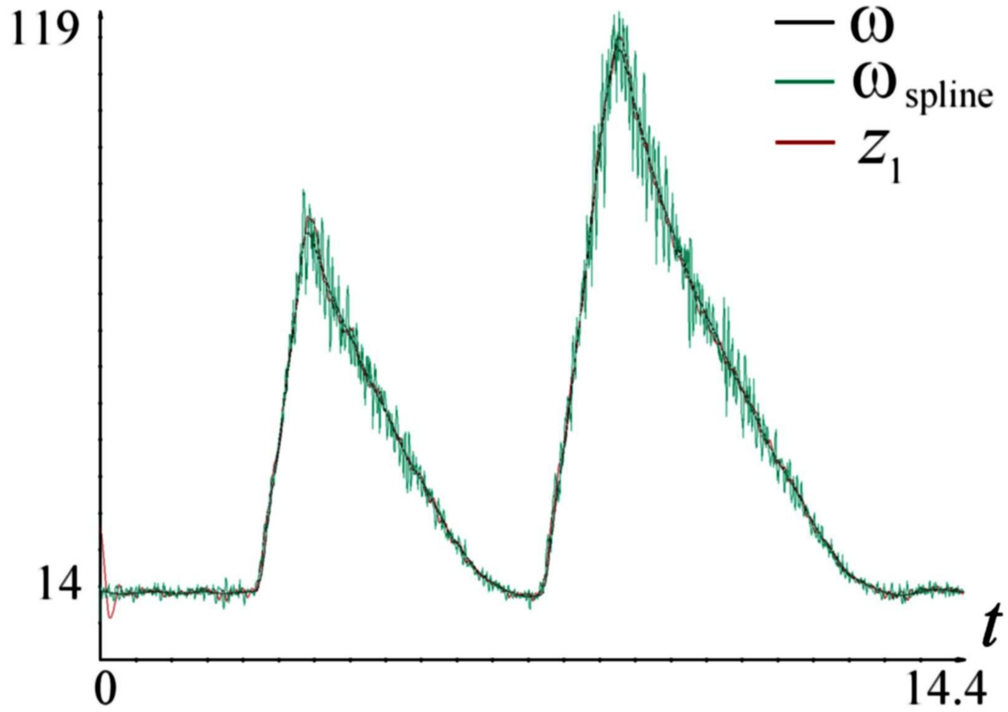
Experimental data, $\tau = 0.004$



1st derivative.
2nd order differentiation, $L = 625$

On-line 2nd order differentiation

Volvo: comparison with optimal spline approximation



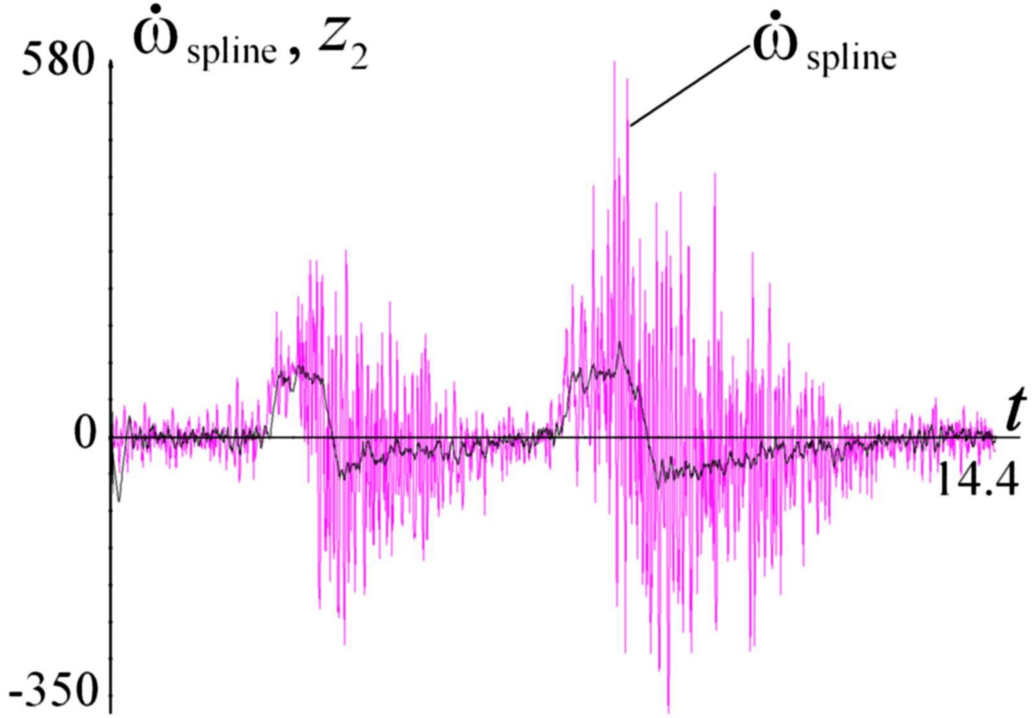
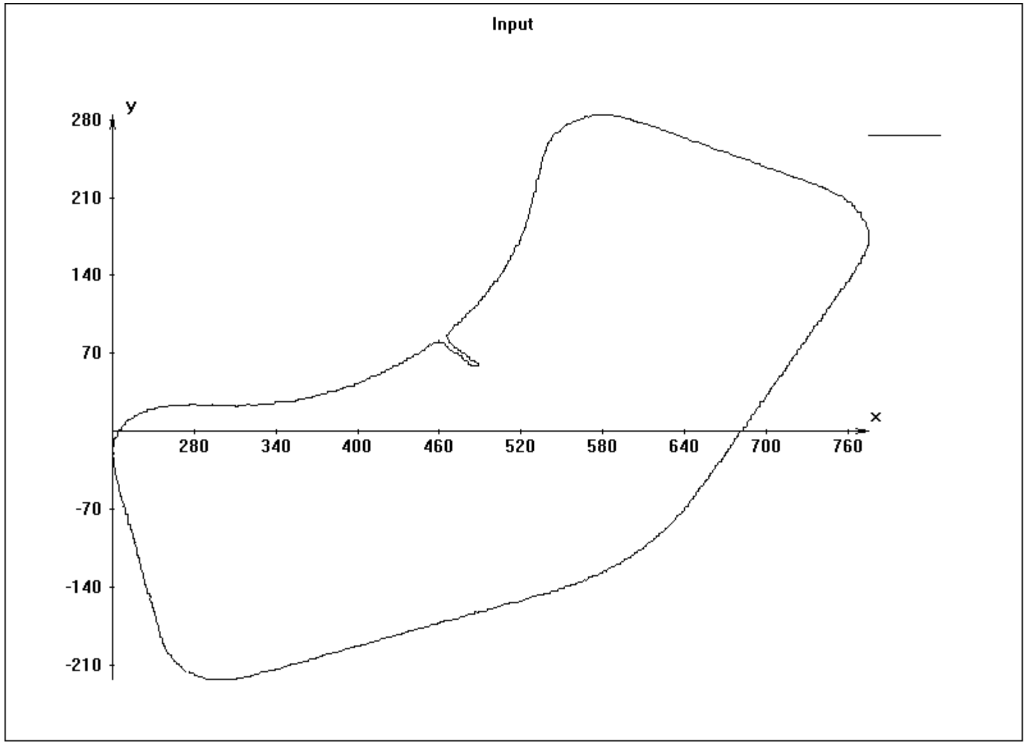
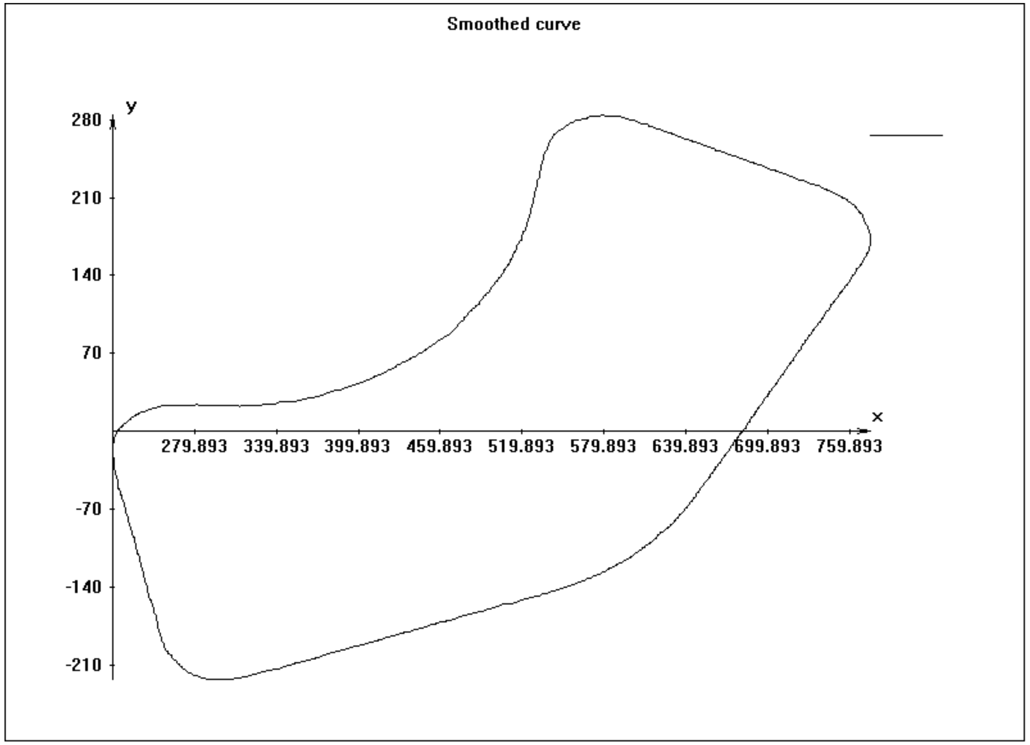
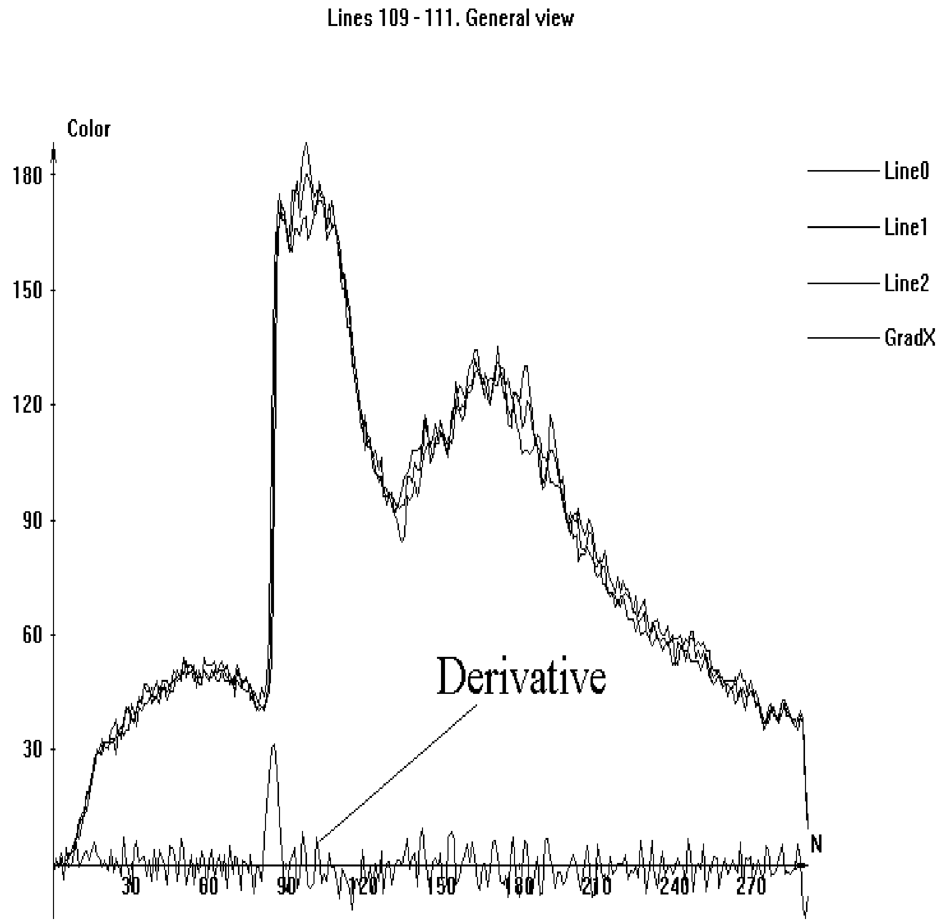


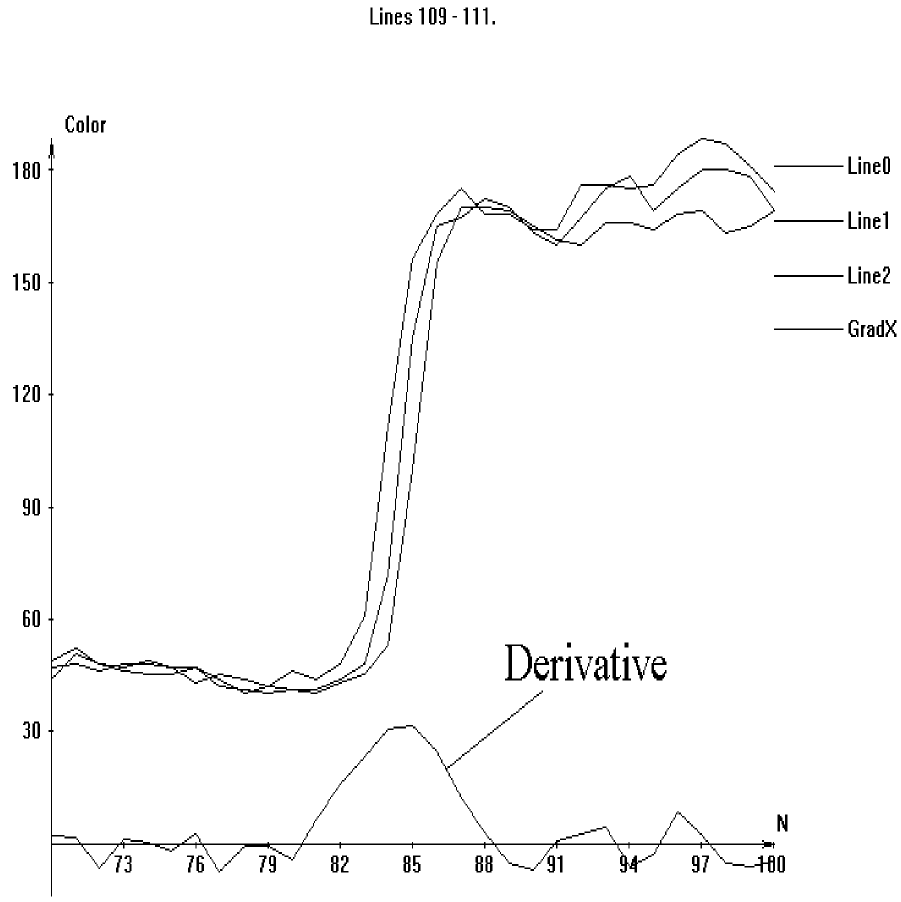
Image Processing: Crack Elimination





Edge Detection





3 successive lines of a grey image zoom

Edge Detection