Chattering Analysis

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***Abstract*—A formal mathematical definition of chattering is proposed. Chattering phenomena are classified into three types. In particular, the first type is harmless and cannot be avoided. Chattering properties of various control approaches are considered. The dangerous second and third types of chattering phenomena are proved to be removable by proper use of high-order sliding-modes (HOSM). Fast stable actuators and sensors only generate the first type of chattering in HOSM systems and practically never affect the sliding motion. Computer simulation confirms the theoretical results.**

***Index Terms*—Chattering effect, high-order sliding mode (HOSM), homogeneity, variable structure systems.**

I. Introduction

UNCERTAINTY of the mathematical model remains one of the main challenges of modern control theory. High-gain feedback and sliding-mode control [10], [32], [33] are often applied in that case. Also finite-time convergent nonlinear control systems [1] might be considered as belonging to this class, since they feature infinite feedback derivatives at the operational point. In reality sampling noises, unaccounted-for fast dynamics of sensors and actuators, delays, discretization and hysteresis effects might cause dangerous vibrations in such systems. In sliding-mode control systems these effects are well-known as the chattering effect. The idea of the sliding-mode control approach is to react immediately to any deviation of the system from some properly chosen constraint, steering it back by a sufficiently energetic effort. Sliding mode is accurate and insensitive to disturbances, and much research was devoted to the analysis of chattering and its avoidance [2]–[4], [6]–[12], [14], [15], [17], [19], [21]–[34].

Sliding-mode control chattering is caused by the high, theoretically infinite, frequency of control switching and reveals itself as high-frequency dangerous vibrations of the whole system. Unfortunately, the generally accepted understanding of the chattering effect ends at this point. There is no formal mathematical definition of chattering, which would enable the estimation of its threat to a system. Indeed, any real system inevitably undergoes infinitesimal high-frequency vibrations, most of which are harmless. Therefore, some kinds of chattering are negligible. On the other hand, as a result of high control gains, even classical linear control systems may undergo harmful vibrations in the presence of small measurement noises. This paper is probably the first attempt to formalize the chattering notion mathematically. "Mathematical chattering" is considered in finite-dimensional dynamic systems. That approach does not deal with complicated physical phenomena associated with real-system chattering.

The author agrees in advance that the proposed concept is arguable, and welcomes any alternative approaches and definitions. While the presented results have been proved mathematically, the practical interpretation of the theorems might be debatable.

In the paper, it is shown for the first time that the well-known chattering attenuation procedure [3], [4], [11], [12], [19], [25], [29]–[31] indeed removes dangerous types of chattering. Moreover, it is proved that the inclusion of fast stable sensors and actuators into the mathematical model of the closed system preserves real (approximate) sliding modes and only produces non-harmful chattering of the plant. Therefore such sensors and actuators can be ignored at initial design stages. Simulation confirms the theoretical results.

In order to avoid complicated notation, some of the variables have different meaning in the different parts of the paper.

II. Definitions of Chattering

A. Chattering of a Single Signal

The notion of chattering inevitably depends on the time and coordinate scales. For example, consider the temperature measured at some fixed place in London. It obviously does not fluctuate much in one hour, but if the time is measured in years, then the chattering is very apparent. At the same time, compared with the temperature on Mercury, these vibrations are negligible. It is also obvious that any linear function of time does not chatter. Thus, the chattering of a signal is to be considered with respect to some nominal signal, which is known from the context.

Consider an absolutely continuous scalar signal $\xi(t) \in \mathbf{R}$, $t \in [0,T]$. Also let $\bar{\xi}$ be an absolutely continuous nominal signal, such that ξ is considered as its disturbance. Let $\Delta\xi = \xi - \bar{\xi}$, and introduce virtual dry (Coloumb) friction, which is a force of constant magnitude k directed against the motion vector $\Delta\dot{\xi}(t)$. Its work ("heat release") during an infinitesimal time increment dt equals $-k\,\mathrm{sign}\,(\Delta\dot{\xi})\Delta\dot{\xi}dt = -k|\Delta\dot{\xi}|dt$. Define the L_1-chattering of the signal $\xi(t)$ with respect to $\xi(t)$ as the energy required to overcome such friction with $k=1$, i.e.

$$L_1 - \mathrm{chat}(\xi, \bar{\xi}; 0, T) = \int_0^T |\dot{\xi}(t) - \dot{\bar{\xi}}|dt.$$

In other words, L_1-chattering is the distance between $\dot{\xi}$ and $\dot{\bar{\xi}}$ in the L_1-metric, or the variation of the signal difference $\Delta\xi$. Similarly, considering virtual viscous friction proportional to $\Delta\dot{\xi}$, obtain

$$L_2 - \mathrm{chat}(\xi, \bar{\xi}; 0, T) = \left[\int_0^T \left|\dot{\xi}(t) - \dot{\bar{\xi}}\right|^2 dt\right]^{1/2}$$

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where mathematical instinct requires the power 1/2 to appear. Other power models of friction produce L_p-chattering, $p > 1$, which is defined in the obvious way. If the nominal signal $\bar{\xi}$ is not defined, the linear signal $\xi(0) + t(\xi(T) - \xi(0))/T$ is naturally used for the comparison. The three last arguments of the chattering function may be omitted in the sequel, if they are known from the context.

Let $x(t) \in \mathbf{R}^n$, $t \in [0,T]$, be an absolutely continuous vector function, and $M(t,x)$ be some positive-definite continuous symmetric matrix with the determinant separated from 0. The chattering of the trajectory $x(t)$ with respect to $\bar{x}(t)$ is defined as

$$L_p - \mathrm{chat}(x, \bar{x}, 0, T) = \left\{ \int_0^T \left[\left(\dot{x}^t(t) - \dot{\bar{x}}^t \right) M(t,x) \left(\dot{x}(t) - \dot{\bar{x}} \right) \right]^{p/2} dt \right\}^{1/p}.$$

The matrix M is introduced here to take into account a local metric. Note that with $M = I$ the L_1-chattering is the length of the curve $x(t) - \bar{x}(t)$.

B. Chattering Family

The notions introduced depend on the time scale and the space coordinates. The following notions are free of this drawback.

Consider a family of absolutely continuous trajectories (signals) $x(t,\varepsilon) \in \mathbf{R}^n$, $t \in [0,T]$, $\varepsilon \in \mathbf{R}^l$. The family *chattering parameters* ε_i measure some imperfections and tend to zero. Define the nominal trajectory (signal) as the limit trajectory (signal) $\bar{x}(t) = \lim_{\varepsilon \to 0} x(t,\varepsilon), t \in [0,T]$. Chattering is not defined in the case when the limit trajectory $\bar{x}(t)$ does not exist or is not absolutely continuous.

L_p-chattering is classified as **infinitesimal**, if the "heat release" is infinitesimal, i.e.

$$\lim_{\varepsilon \to 0} L_p - \mathrm{chat}(x, \bar{x}; 0, T) = 0.$$

L_p-chattering is classified as **bounded** if

$$\lim_{\varepsilon \to 0} L_p - \mathrm{chat}(x, \bar{x}; 0, T) > 0.$$

L_p-chattering is classified as **unbounded** if the "heat release" is not bounded, i.e.

$$\overline{\lim_{\varepsilon \to 0}} L_p - \mathrm{chat}(x, \bar{x}; 0, T) = \infty.$$

The last two chattering types are to be considered as potentially destructive.

Proposition 1: If the chattering is infinitesimal in the L_p-sense, then it is infinitesimal also in the L_q-sense, $p > q \geq 1$. Similarly, if it is unbounded (bounded) in the L_q-sense, $p > q \geq 1$, it is unbounded (bounded or unbounded) also in the L_p-sense.

Proof: The proof, indeed, follows from the inequality [5]

$$\left(\frac{1}{T} \int_0^T |f(t)|^q dt \right)^{1/q} \leq \left(\frac{1}{T} \int_0^T |f(t)|^p dt \right)^{1/p}$$

which holds for any function $f \in L_p$.

Obviously, if L_1-chattering is infinitesimal, the length of the trajectory $x(t,\varepsilon)$ tends to the length of $\bar{x}(t)$. The chattering is bounded or unbounded iff the length of $x(t,\varepsilon)$ is respectively bounded or unbounded when $\varepsilon \to 0$. ■

Proposition 2: Let $x(t,\varepsilon)$ uniformly tend to $\bar{x}(t)$ with $\varepsilon \to 0$. Then the above classification of chattering is invariant with respect to smooth transformations of time and coordinates, and to the choice of a continuous positive-definite symmetric matrix M

Proof: Indeed, it follows from the uniform convergence that the trajectories are confined to a compact region. The proposition now follows from the boundedness from above and from below of the norm of the Jacobi matrix of the transformation. ■

Proposition 3: Let $x(t,\varepsilon)$ uniformly tend to $\bar{x}(t)$ with $\varepsilon \to 0$. Then the chattering is infinitesimal, iff the chattering of all coordinates of $x(t,\varepsilon)$ is infinitesimal. The chattering is unbounded iff the projection to some subset of the coordinates has unbounded chattering. The chattering is bounded iff it is not unbounded, and the projection to some subset of the coordinates has bounded chattering.

Proof: This is a simple consequence of Proposition 2. ■

Suppose now that the mathematical model of a closed-loop control system is decoupled into two subsystems

$$\dot{x} = X_\varepsilon(t,x,y), \quad \dot{y} = Y_\varepsilon(t,x,y)$$

where ε is a chattering parameter. Consider any local chattering family of that system. Then, similarly to Proposition 2, the above classification of the chattering of the vector coordinate x does not depend on any smooth state coordinate transformation of the form $\tilde{x} = \tilde{x}(t,x), \tilde{y} = \tilde{y}(t,x,y)$.

Assume that the chattering of the vector coordinate x of the first subsystem is considered dangerous, while the chattering of the second subsystem is not important for some practical reason. In particular, this can be the case when the vector coordinate y of the second subsystem corresponds to some internal computer variables. In the following, the first subsystem is called **main** and may contain the models of any chattering-sensitive devices including actuators and sensors; the second subsystem is called **auxiliary**.

It is said that there is **infinitesimal** (L_p-)chattering in a closed-loop control system depending on a small vector chattering parameter if any local chattering family of *the main-subsystem* trajectories features infinitesimal chattering. The chattering is called **unbounded** if there exists a local chattering family of the main subsystem with unbounded chattering. The chattering is called **bounded** if it is not unbounded and there exists a local chattering family of the main subsystem with bounded chattering.

The least possible chattering in this classification is the infinitesimal one. In other words, infinitesimal chattering is present in any control system, as a result of infinitesimal disturbances of a different nature.

The prefix L_p- is omitted in the cases when the corresponding statement on chattering does not depend on $p \geq 1$. This is true everywhere in the sequel.

C. Chattering in Mechanical Systems

One of the most devastating effects caused by chattering is extensive heat emission, which can be estimated by energy oscillation. It is shown here that infinitesimal chattering excludes dangerous heat emission in mechanical systems.

Consider some closed-loop system. Let the main subsystem be a mechanical system described by the Lagrange equation

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}} - \frac{\partial K}{\partial q} = Q(t, q, \dot{q}, w, u_\varepsilon(t, q, \dot{q}, w)). \qquad (1)$$

Here $q \in \mathbf{R}^n$ is the vector of generalized coordinates, the smooth function $K(t, q, \dot{q})$ is the kinetic energy, and the smooth function Q is the vector of generalized forces produced by the auxiliary subsystem

$$\dot{w} = W(t, q, \dot{q}, w, u_\varepsilon(t, q, \dot{q}, w)). \qquad (2)$$

Assume that the forces Q and the initial values of q, at the time t_1 are bounded. Let the control u_ε provide for the uniform convergence of the main-subsystem trajectories $q_\varepsilon(t)$ to some smooth limit trajectory $q_*(t)$ with $\varepsilon \to 0$, $t \in [t_1, t_2]$.

The kinetic energy has the form

$$K = \dot{q}^t M_0(t, q)\dot{q} + M_1(t, q)\dot{q} + M_2(t, q) \qquad (3)$$

where M_0, M_1, M_2 are some smooth matrix functions of the corresponding dimensions, and M_0 is symmetrical and positive definite.

It is easy to see that the uniform convergence of $q_\varepsilon(t)$ to $q_*(t)$ also implies the uniform convergence of $\dot{q}_\varepsilon(t)$ to $\dot{q}_*(t)$ with $\varepsilon \to 0$. Indeed, (1), (3) imply the boundedness of $(q_\varepsilon(t), \dot{q}_\varepsilon(t))$ which, in turn, imply the boundedness of $\ddot{q}_\varepsilon(t)$. The uniform convergence of $\dot{q}_\varepsilon(t)$ to $\dot{q}_*(t)$ now follows from the boundedness of $\ddot{q}_\varepsilon(t)$ (for example, see Lemma 1 in [20]).

The energy of the system takes the form

$$E = \dot{q}^t M_0(t, q)\dot{q} + M_1(t, q)\dot{q} + p(t, q) \qquad (4)$$

where the first two terms come from (3). Any small decrease in energy corresponds to possible heat emission, which does not exceed the absolute value of the energy decrease. Introducing the function

$$\mathrm{Pos}(x) = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}$$

obtain that the integral heat emission H_ε is evaluated as

$$\begin{aligned} H_\varepsilon &\le \int_{t_1}^{t_2} \mathrm{Pos}(-\dot{E}_\varepsilon)dt \\ &\le \int_{t_1}^{t_2} \mathrm{Pos}(-\Delta\dot{E})dt + \int_{t_1}^{t_2} \mathrm{Pos}(-\dot{E}_*)dt \end{aligned} \qquad (5)$$

where $\Delta E = E_\varepsilon(t) - E_*(t)$ is the energy difference taken with respect to the energy $E_*(t)$, calculated along the limit trajectory $(q_*(t), \dot{q}(t))$. The last term characterizes the maximal heat emission along the limit trajectory, and is supposed to be practically acceptable. Using (4) and $E_\varepsilon(t) = \Delta E + E_*(t)$ obtain

$$\dot{E}_\varepsilon = (2\dot{q}^t M_0(t, q_*)\Delta\dot{q} + M_1(t, q_*))\Delta\ddot{q} + \dot{E}_*(t) + o(1). \qquad (6)$$

Now suppose that the chattering is infinitesimal. This means that $\int \Delta\ddot{q}dt \to 0$. It follows from (5), (6) that also $\int \mathrm{Pos}(-\Delta\dot{E})dt$ is infinitesimally small. Thus, also the additional heat emission is negligible.

Note that the above calculation does not take into account any possible heat emission in the devices yielding the generalized forces Q, i.e., the chattering of the auxiliary subsystem (2) (the actuator model). Including the actuator dynamics into the main subsystem might require smoothing the control.

III. Examples

Consider a smooth dynamic system

$$\dot{x} = a(t, x) + b(t, x)u \qquad (7)$$

$x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$. Let $\sigma(t, x) = 0$, $\sigma \in \mathbf{R}^m$, be a vector constraint to be kept in the standard sliding mode. Its physical meaning can be a tracking deviation, for example. Let the vector relative degree of σ be $(1, 1, \ldots, 1)$, which means that

$$\dot{\sigma} = \Theta_1(t, x) + \Theta_2(t, x)u \qquad (8)$$

with some smooth Θ_1, Θ_2 and $\det \Theta_2 \ne 0$. Taking

$$u = -K\Theta_2^{-1}\frac{\sigma}{\|\sigma\|}, \quad K > \sup\|\Theta_1\| \qquad (9)$$

obtain the local first-order sliding mode $\sigma \equiv 0$. Consider any regularization parameter ε having the physical sense of switching imperfections, such as switching delays, small measurement errors, hysteresis etc., which vanish when $\varepsilon = 0$.

Proposition 4: Variable Structure System (VSS) (7)–(9) features bounded chattering.

Proof: Fix any initial conditions. Then with $\varepsilon \to 0$ the trajectories uniformly converge to the Filippov solution [13], consisting of a transient part and a sliding-mode motion with $\sigma = 0$. Taking σ as a part of new coordinates according to Proposition 2, obtain that during the approximate sliding motion the difference $\dot{\sigma} - 0$ between the family trajectory and the ideal sliding-mode motion satisfies the inequality $K + \sup\|\Theta_1\| \ge \|\dot{\sigma}\| \ge K - \sup\|\Theta_1\| > 0$. Thus, due to Proposition 3, the chattering is bounded in any L_p-sense. ∎

Similarly, the optimal control problems featuring sliding-mode trajectories [16] reveal bounded chattering.

Proposition 5: Let system (7) be closed by some continuous feedback $u = U(t, x)$, and ε be the maximal magnitude of the measurement noise and control delays. Then only infinitesimal chattering is present in the system.

The **proof** immediately follows from the continuous dependence of the solutions on the right-hand side of differential equations (the uniqueness of the solutions is not required). ∎

Now let (7) be a Single-Input Single-Output (SISO) system, $u \in \mathbf{R}$, $\sigma \in \mathbf{R}$, and let the relative degree be r, which means that the system can be rewritten in the form [18]

$$\sigma^{(r)} = h(t, \theta, \Sigma) + g(t, \theta, \Sigma)u, K_M \ge g \ge K_m > 0 \qquad (10)$$

$$\dot{\theta} = \Theta(t, \theta, \Sigma), \quad \zeta \in \mathbf{R}^{n-r} \qquad (11)$$

where $\Sigma = \left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right)$, and, without any loss of generality, the function g is assumed positive. Suppose that h be uniformly bounded in any bounded region of the space ζ, Σ and

(11) features the Bounded-Input-Bounded-State (BIBS) property with Σ considered as the input.

In the case when the functions g and h are uncertain, a high-gain feedback is applied

$$u = -ks, s = \sigma^{(r-1)} + \beta_1\sigma^{(r-2)} + \cdots + \beta_{r-1}\sigma \quad (12)$$

where $\lambda^{r-1} + \beta_1\lambda^{r-2} + \cdots + \beta_{r-1}\lambda$ is a Hurwitz polynomial. It can be shown that, provided k is sufficiently large, such feedback provides for the semiglobal convergence into a set $\|\Sigma\| \leq d$, $d = O(1/k)$.

According to Proposition 4 system (10)–(12) features infinitesimal chattering with any fixed k and small noises. In order to improve the performance, one needs to increase k. It is easy to show that with the chattering parameter $\mu = 1/k \to 0$, a system with infinitesimal chattering is obtained in the absence of noise. Now introduce some infinitesimal noise of the magnitude $\varepsilon \to 0$ in the measurements of the function s.

Theorem 1: Let possible noises be any smooth functions of time of the magnitude ε. Then the chattering in system(10)–(12) is unbounded with the chattering parameters $\mu = 1/k \to 0$ and $\varepsilon \to 0$.

Note that this result applies also to the estimation of the chattering of multi-input multi-output (MIMO) systems. Indeed, it is sufficient to fix all feedback components except one in order to prove the possibility of unbounded chattering.

Proof: The relative degree of the auxiliary output s is 1. Hence, in some new coordinates the system has the form

$$\begin{aligned} \dot{s} &= \rho(t,\xi,s) - k\tilde{g}(t,\xi,s)(s+\eta), \\ \dot{\xi} &= \Xi(t,\xi,s) \end{aligned} \quad (13)$$

where ρ, Ξ are smooth bounded functions, and η is the measurement noise, $|\eta| \leq \varepsilon$, $KM \geq \tilde{g} \geq K_m > 0$. Since the consideration is local, all functions and their partial derivatives are bounded.

It is easy to see that the relation $s = O(\mu) + O(\varepsilon)$ is established in the time $O(1/k) = O(\mu)$. Thus, the trajectories converge to a discontinuous vector-function, if $s \neq 0$ at the initial moment. Therefore, only chattering families with initial conditions on $s = 0$ make sense. In such a case the limit trajectory satisfies the zero-dynamics [18] equations

$$s = 0, \quad \dot{\xi} = \Xi(t,\xi,0).$$

Introduce a new time $\tau = kt = t/\mu$, and denote $(\cdot)' = d/d\tau(\cdot) = \mu d/dt(\cdot)$. Then (13) takes the form

$$s' = \mu\rho(t,\xi,s) - \tilde{g}(t,\xi,s)(s+\eta).$$

Hence, $s' = O(\mu) + O(\varepsilon)$, and also the full derivatives ρ' and $\tilde{g}'$ are of the same order.

Differentiate once more, and obtain that

$$s'' = o(\mu) + o(\varepsilon) - \tilde{g}(t,\xi,s)(s' + \eta')$$

or, equivalently

$$(s' + \eta')' = o(\mu) + o(\varepsilon) + \eta'' - \tilde{g}(t,\xi,s)(s' + \eta').$$

This means that in finite τ-time s' starts to track $-\eta'$ with the accuracy $(o(\mu) + o(\varepsilon)) + \sup|\eta''|/K_m$. Let $\eta = \varepsilon \sin \nu\tau$, $1 > \nu > 0$. Then $\eta' = \varepsilon\nu \cos \nu\tau$. Take $\nu = 0.1\ \mathrm{K}_m^{1/2}$, then with ε small enough, $|s'| \geq 0.03\varepsilon$ is kept whenever $|\cos\ \nu\tau| \geq 0.5$. This means that $|s'| \geq 0.03\varepsilon$ is kept during the constant part $(4\pi/3)/(2\pi) = 2/3$ of each τ-period of $\cos \nu\tau$. The τ-period equals $2\pi/\nu = 20\pi K_m^{-1/2}$

Now return to the original time t, and take $\varepsilon = \mu^{1/2} = k^{-1/2}$. The t-period of the chosen noise equals $20\pi K_m^{-1/2}\mu \to 0$. Thus, with sufficiently large k, the inequality $|\dot{s}| \geq 0.03k\varepsilon = 0.03k^{1/2}$ is kept during $1/2 < 2/3$ of the t-length T of the considered trajectories. Therefore, the L_1-chattering of the coordinate s satisfies the relations

$$L_1 - \mathrm{chat}(s) \geq 0.5T0.03K^{1/2} = 0.015Tk^{1/2} \to \infty$$

which proves the Theorem according to Proposition 2. ■

Note that the introduction of control saturation in (12) removes the unbounded chattering. Indeed, as follows from (10), $\sigma(r)$ becomes uniformly bounded, implying the boundedness of the chattering functions. Such control can be considered as a regularization of the relay control $u = -\alpha$ sign s and corresponds to the standard method [32] of the sliding-mode chattering attenuation. Unfortunately, the bounded chattering of the coordinate s is still inevitable. Indeed, it is sufficient to choose the measurement noise $\eta = 2\lambda k^{-1}$ sign $(\sin \mu kt)$, where λ is the control saturation threshold and μ is a sufficiently small constant. As a result the control will remain at the saturation value during a certain part of the time, causing bounded chattering of the coordinate $\sigma^{(r-1)}$. Thus, this chattering attenuation method is not effective, if one uses large k to prevent significant loss of the sliding mode accuracy.

It can also be shown similarly that the fast coordinates of singularly perturbed systems demonstrate unbounded chattering. This does not necessarily mean the unbounded chattering of the whole system, if these coordinates are internal coordinates of some auxiliary unit which can be excluded from the main subsystem. In particular, the chattering of the internal sensor variables can often be ignored; in the case of computer-based control some coordinates might lack physical sense. Note that the main-subsystem chattering still accounts for the influence of the excluded unit.

IV. Chattering of High-Order Sliding Modes

Consider a smooth dynamic system with a smooth output function σ. Let the system be closed by some possibly dynamic discontinuous feedback and be understood in the Filippov sense [13]. Then, provided that successive total time derivatives σ, $\dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is said to be in r-sliding (rth-order sliding) mode [19], [22]. The standard sliding mode, used in the most variable structure systems, is of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous).

Consider once more the uncertain SISO dynamic system (2) with the output $\sigma = \sigma(t,x)$, which was considered in Theorem 1. Let a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ be unknown smooth functions. Assume that σ and possibly a number of its derivatives are measured in real time; n can also be uncertain. The relative degree r of the system is assumed to be constant and known, and, as previously, the task is to provide for $\sigma = 0$. It is also supposed that the functions h, g in (10) are bounded in the operational region, so that $0 < K_m \leq g \leq K_M$ and $|h| \leq C$.

Some bounded r-sliding control $u = U(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)})$ is applied, providing for the finite-time stability of the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \tag{14}$$

It is assumed to be r-sliding homogeneous [23], which means that the identity

$$u = U(\sigma, \dot{\sigma}, \dots, \sigma^{(r-1)}) \equiv U(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, \dots, \kappa\sigma^{(r-1)}) \tag{15}$$

is kept for any $\kappa > 0$. Most known high-order sliding-mode controllers [2]–[4], [11], [12], [19], [22]–[31] satisfy this assumption. It can be shown that the function U is inevitably discontinuous at least at $\Sigma = 0$.

All noises considered further in this paper are bounded Lebesgue-measurable functions of time of any nature. *No additional requirements are needed.*

Suppose that $\sigma^{(i)}, i = 0, 1, \dots, r-1$, is measured with noises of the magnitudes $\gamma_i \varepsilon^{r-i}$, and variable delays not exceeding $\tilde{\gamma}_i \varepsilon$, where $\tilde{\gamma}_i$, γ_i are some positive constants. It is proved [23] that the accuracy $|\sigma| < a_0 \varepsilon^r, |\dot{\sigma}| < a_1 \varepsilon^{r-1}, \dots, |\sigma^{(r-1)}| < a_{r-1}\varepsilon$ is established in finite time with some positive constants $a_0, a_1, \dots, a_{r-1}$ independent of ε. The result does not change when only σ is measured and all its derivatives are estimated by means of an $(r-1)$th order robust differentiator [22] based on 2-sliding modes.

Note that with $\varepsilon = 0$ the exact r-sliding mode $\sigma \equiv 0$ is established. The above connection between the measurement noise magnitudes and delays is not restrictive, since in reality there are concrete noises and delays, which can be considered as samples of a virtual family indexed by ε in a non-unique way. Moreover, actual noise magnitudes can be lower, preserving the same upper estimations and the worst-case asymptotics.

Following from the above result, there is no unbounded chattering in the system (2), (15). Indeed, after the coordinates are chosen as in (10), (11), it is obvious that the only coordinate which can reveal bounded or unbounded chattering is $\sigma^{(r-1)}$. Its chattering function is bounded due to the boundedness of $\sigma^{(r)}$. Thus, unbounded chattering is impossible. Actually, the author can prove that there is bounded chattering in most cases of the well-known 2-sliding controllers [3]–[6], [11], [12], [25], [29], or in the case of nested sliding-mode controllers [22] ($r = 1, 2, \dots,$). The situation is more complicated in the case of quasi-continuous controllers [24], which are continuous everywhere except $\Sigma = 0$. Simulation reveals bounded chattering also in that case.

The chattering attenuation procedure [3]–[6], [11], [12], [19]–[31] is based on treating the derivative $u^{(l)}$ as a new control. As a result, the relative degree is artificially increased to $r + l$, and $u^{(i)}, i = 0, \dots, l-1$, are included in the set of coordinates. Global ($r = l = 1$) [25] or semiglobal [27] convergence is ensured for the $(r+l)$-sliding mode. The above result provides for the accuracies $\sigma = O(\varepsilon^{r+l}), \dot{\sigma} = O(\varepsilon^{r+l-1}), \dots, \sigma^{(r)} = O(\varepsilon^l)$ with time delays of the order of ε and the measurement errors of $\sigma^{(i)}$ being $O(\varepsilon^{r+l-i})$. Thus, only infinitesimal chattering takes place in that case. Moreover, chattering functions of the plant trajectories are of the order $O(\varepsilon^l)$. These results are trivially extended to the MIMO case with a vector relative degree and a vector sliding order.

Consider now the influence of unaccounted-for fast stable actuators and sensors, when the $(r+l)$-sliding mode is the limit operational mode. The following consideration can also be extended to the MIMO case, provided the relative degree is well-defined and the control matrix coefficient g is known. Let time belong to a segment Ω_t, and $x, u, \dot{u}, \dots, u^{(l-1)}$ belong to some compact regions Ω_x, Ω_u. Suppose that the initial conditions belong to some smaller compact set $\Omega_{t0} \times \Omega_{x0} \times \Omega_{u0}$. This results in

$$\begin{aligned} \dot{x} &= a(t,x) + b(t,x)u, \quad \sigma^{(r)} = h(t,x) + g(t,x)u. \\ \sigma^{(r+1)} &= h_1(t,x,u,\dot{u},\dots,u^{(l-1)}) + g(t,x)u^{(l)}. \end{aligned} \tag{16}$$

The sliding-mode equalities $\sigma = \dot{\sigma} = \cdots = \sigma^{(r+l-1)} = 0$ define the values of $u, \dot{u}, \dots, u^{(l-1)}$ in a unique way. Thus, with sufficiently small Ω_t, Ω_x, Ω_u some inclusion of the form

$$\sigma^{(r+1)} \in [-C_1, C_1] + [K_m, K_M]u^{(l)}, \quad K_M \geq K_M > 0 \tag{17}$$

is valid in the considered vicinity of the $(r+l)$-sliding mode $\sigma = \dot{\sigma} = \cdots = \sigma^{(r+l-1)} = 0$. Introduce the functions $\sigma_0(t,x) = \sigma$, $\sigma_1(t,x) = \dot{\sigma}, \dots, \sigma_{r-1}(t,x) = \sigma^{(r-1)}$, $\sigma_r(t,x,u) = \sigma^{(r)}, \dots, \sigma_{r+l-1}(t,x,u,\dot{u},\dots,u^{(l-1)}) = \sigma^{(r+l-1)}$. They can be expressed by means of Lie derivatives, and can be supplemented up to the local coordinates [18].

Let now the output ν of an actuator be substituted for u

$$\dot{x} = a(t,x) + b(t,x)\nu, \quad \sigma^{(r)} = h(t,x) + g(t,x)\nu. \tag{18}$$

The mathematical model of the actuator is described by the equations

$$\mu \dot{z} = f(z,u), \quad \nu = \nu(z) + \eta_1(t) \tag{19}$$

where $z \in \mathbf{R}^m, u \in \mathbf{R}$ is the control and the input of the actuator, ν is a continuous output function, the time constant $\mu > 0$ is a small parameter, and $\eta_1(t)$ is some small deterministic Lebesgue-measurable noise of magnitude ε_1. Both ε_1 and μ are infinitesimal chattering family parameters. Note that the functions $\sigma_r, \dots, \sigma_{r+l-1}$ are no longer equal to the derivatives $\sigma^{(r)}, \dots, \sigma^{(r+l-1)}$, for $\sigma^{(r)}$ now depends on the noisy actuator output ν.

As previously, the new control $u^{(l)}$ is determined by the $(r+l)$-sliding homogeneous feedback

$$u^{(l)} = U(s_0, s_1, \dots, s_{r+l-1}) \tag{20}$$

$$U(s_0, s_1, \dots, s_{r+l-1}) \equiv U\left(\kappa^{r+1}s_0, \kappa^{r+l-1}s_1, \dots, \kappa s_{r+l-1}\right) \tag{21}$$

where U is a function continuous almost everywhere, bounded by some constant $u_M, u_M > 0$, in its absolute value, and s_i are some real-time estimations of $\sigma^{(i)}$ to be defined below. Being applied to (16) with

$$\nu = u, \quad s_i = \sigma^{(i)}, i = 1, \dots, r+l-1 \tag{22}$$

it provides for the finite-time convergence to the $(r+l)$-sliding mode $\sigma \equiv 0$. Let the constraint function σ have the sense of a tracking error, i.e.

$$\sigma(t,x) = \sigma_x(x) - \sigma_c(t)$$

where $\sigma_c(t)$ is a smooth function available in real time with an a-priori bounded $(r+l)$th derivative, and $\sigma_x(x)$ is a smooth function.

Now let an estimate of σ be obtained as the noisy output $s + \eta_2(t)$ of a sensor with the model

$$\lambda\dot{\varsigma} = \phi(\zeta, x), \quad s = S(\zeta, x) - \sigma_c(t). \tag{23}$$

Here $0 < \lambda \ll 1$, S, ϕ are continuous functions, and η_2 is a noise whose magnitude ε_2 also tends to zero with $\lambda \to 0$. The estimates s_i, $i = 1, \ldots, r+l-1$ are obtained as the outputs of the $(r+l-1)$th-order differentiator [22], which is denoted by

$$s(t) + \eta_2(t) \Longrightarrow D_{r+l-1} \Longrightarrow (s_0, s_1, \ldots, s_{r+l-1}). \tag{24}$$

Assumption 1: The actuator with $\mu = 1$ and the sensor with $\lambda = 1$ feature the BIBS property. Initial values of z, ζ belong to some compact sets Ω_{z0}, $\Omega_{\zeta 0}$.

Since $|u| \leq u_M$, this provides for z, ζ belonging to some compact regions $\Omega_z \supset \Omega_{z0}$, $\Omega_\zeta \supset \Omega_{\zeta 0}$ independent of μ, λ. Indeed, μ (respectively, λ) can be excluded by the time transformation $\tau = t/\mu$(respectively $\tau = t/\lambda$). This assumption also causes the actuator output ν to be bounded in its absolute value by some constant $\nu_M > u_M > 0$.

Assumption 2: The actuator and the sensor are assumed exact in the following sense. With $\mu = 1 (\lambda = 1)$ and any constant value of u (respectively x), the output v uniformly tends to u (respectively, s uniformly tends to σ). That means that for any t_0, $\delta > 0$ there exists $T > 0$ such that with any u, $u = \text{const}$, $|u| \leq u_M$, $z(0) \in \Omega_z$ (respectively constant $x \in \Omega_x$ and $\zeta(0) \in \Omega_\zeta$), the inequality $|v-u| \leq \delta$ (respectively $|s-\sigma| \leq \delta$) is kept, starting from the moment T.

Note that any linear actuator with the transfer function $P(\mu w)/Q(\mu w)$ satisfies Assumptions 1, 2, provided $\deg Q - \deg P > 0$, Q is a Hurwitz polynomial, $P(0)/Q(0) = 1$. Similar conditions can also be formulated for a linear sensor, s and σ.

Theorem 2: Let $l > 0$ and Assumptions 1, 2 hold. Then with any $\delta > 0$ the sliding accuracy $|\sigma_0| < \delta$, $|\sigma_1| < \delta, \ldots$, $|\sigma_{r+l-1}| < \delta$ is established with sufficiently small λ, μ, ε_1, ε_2. The differentiator errors are also arbitrarily small, $|s_i - \sigma_i| < \delta$.

Theorem 2 is true also in the case $l = 0$. It corresponds to the discontinuous operator input, and is considered in another paper [28]. The results presented are valid for general finite-time stable controllers (2), (21). Therefore, in order to avoid the consideration of specific convergence dynamics, only chattering families, approximating the $(r+l)$-sliding mode from the very beginning, are considered. Such families are further called *sliding chattering families*. Thus, the chattering family features initial conditions satisfying $\sigma_0 = s_0 = \cdots = \sigma_{r+l-1} = s_{r+l-1} = 0$. The additional coordinates $u, \dot{u}, \ldots, u^{(l-1)}$ (and respectively $\sigma_r, \ldots, \sigma_{r+l-1}$) are excluded from the main-subsystem coordinates. Note that most probably $u^{(l-1)}$ reveals bounded chattering.

Theorem 3: Under Assumptions 1 and 2 with $l > 0$ any sliding chattering family of the system(18)–(21), (23), (24) reveals only infinitesimal chattering when λ, μ, ε_1, $\varepsilon_2 \to 0$.

Proof of Theorem 2: Assumptions 1 and 2 imply the following important features of the system.

Lemma 1: Under Assumptions 1 and 2 let the input $u(t) + \omega(t)$ of the actuator(19) be a Lipschitz function of time $u(t)$ with some fixed Lipschitz constant, being corrupted by a noise $\omega(t)$ with the magnitude ϖ. Then for any δ, $\varepsilon > 0$ with sufficiently small μ, ϖ the inequality $|v-u| \leq \varepsilon$ is established in time δ and is kept afterwards.

Lemma 2: Under Assumptions 1 and 2 let the input $x(t) + \omega(t)$ of the sensor (23) be a Lipschitz vector function of time $x(t)$ with some fixed Lipschitz constant, being corrupted by a vector noise $\omega(t)$ with the magnitude ϖ. Then for any δ, $\varepsilon > 0$ with sufficiently small λ, ϖ the inequality $|s-\sigma| \leq \varepsilon$ is established in time δ and is kept afterwards.

Proofs of Lemmas 1 and 2 are very similar, therefore, only the first one is presented.

Proof of Lemma 1: Let the Lipschitz constant of $u(t)$ be $L > 0$. Consider the time transformation $t = \mu\tau$. Then (19) takes the form

$$\dot{z} = f(z, u_1(\tau) + \omega_1(\tau)),$$
$$\nu = \nu(z), \quad u_1(\tau) = u(\mu, \tau), \quad \omega_1(\tau) = \eta_1(\mu, \tau).$$

The function $u_1(\tau)$ is also Lipschitzian, but with the Lipschitz constant μL. Fix some initial value t_0 of time t corresponding to $\tau_0 = t_0/\mu$. Let $T > 0$ be the τ-time needed to establish the inequality $|\nu - u_1| \leq \varepsilon/4$ with any constant $u_1 = u_0$, $|u_0| \leq u_M$. Take $u_0 = u_1(t_0/\mu) = u(t0)$. With sufficiently small μ and ϖ the change of $u_1 + \eta_1$ is arbitrarily small during the τ-time $2T$. Thus, since the functions f and ν are uniformly continuous in their arguments, and due to the continuous dependence of the solution $z(t)$ on the right-hand side, the inequality $|\nu - u_1| \leq \varepsilon/2$ is established in the τ-time T, and $|\nu - u_1| \leq \varepsilon$ is kept during the next τ-interval of the same length T. Applying the same reasoning from the moment $\tau = t_0/\mu + T$ and taking the new value $u_0 = u_1(t_0/\mu + T) = u(t_0 + \mu_T)$, obtain that $|v - u_1| \leq \varepsilon$ holds also during the third τ-interval of the length T. Continuing this reasoning, obtain that $|\nu - u_1| \leq \varepsilon$ is kept forever. Returning to the original time $t = \mu\tau$ obtain the statement of the Lemma. ■

It follows from (20) and the boundedness of U that the conditions of Lemma 2 are automatically satisfied, and the sensor produces an estimate of σ with infinitesimal errors. Therefore, mathematically the robust finite-time convergent exact differentiator [22] can be considered as an additional virtual sensor producing noisy estimates of the derivatives of σ. The differentiator transient can be made arbitrarily short by the choice of the initial conditions, and/or differentiator parameters.

Lemma 3: The differentiator produces estimates of $\sigma_0, \sigma_1, \ldots, \sigma_{r+l-1}$. For any $\varepsilon > 0$ with sufficiently small λ, μ, ε_1, ε_2, the inequalities $|s_i - \sigma_i| \leq \varepsilon$ are established and kept afterwards.

Proof: According to Lemmas 1 and 2 the differences $v - u$ and $s - \sigma$ can be made arbitrarily small. Let $|\nu - u| \leq \theta$, $|s - \sigma| \leq$

θ. Thus, differentiating the functions σ_i and substituting v for u, when u appears as a result of differentiating x with respect to (15), obtain that

$$\begin{aligned}&\dot\sigma_0=\sigma_1,\ldots,\dot\sigma_{r-2}=\sigma_{r-1},\\&\dot\sigma_{r-1}=\sigma_r+O(\theta),\\&\qquad\ldots\\&\dot\sigma_{r+l-2}=\sigma_{r+l-1}+O(\theta),\\&\dot\sigma_{r+l-1}\in[-C_1,C_1]+[K_m,K_M]U(s_0,s_1,\ldots,s_{r+l-1})+O(\theta)\end{aligned}\tag{25}$$

Here $O(\theta)$ denotes any function φ with $|\varphi/\theta|$ uniformly bounded for all trajectories. Recall that the differentiator has the form

$$\begin{aligned}\dot z_0&=-\tilde\lambda_{r+l-1}|z_0-s|^{(r+l-1)/(r+l)}\mathrm{sign}(z_0-s)+z_1,\\\dot z_1&=-\tilde\lambda_{r+l-2}|z_0-s|^{(r+l-2)/(r+l)}\mathrm{sign}(z_0-s)+z_2,\\&\ldots\\\dot z_{r+l-2}&=-\tilde\lambda_1|z_0-s|^{1/(r+l)}\mathrm{sign}(z_0-s)+z_{r+l-1},\\\dot z_{r+l-1}&=-\tilde\lambda_0\mathrm{sign}(z_0-s).\end{aligned}$$

Introducing notation $\omega_i=z_i-\sigma_i$, and assuming that $|\dot\sigma_{r+l-1}|<L$, obtain

$$\begin{aligned}&\dot\omega_0\in-\tilde\lambda_{r+l-1}\left|\omega_0+\tilde c\theta[-1,1]\right|^{(r+l-1)/(r+l)}\\&\quad\times\mathrm{sign}(\omega_0+\tilde c\theta[-1,1])+\omega_1+\tilde c\theta[-1,1],\\&\dot\omega_1\in-\tilde\lambda_{r+l-2}\left|\omega_0+\tilde c\theta[-1,1]\right|^{(r+l-2)/(r+l)}\\&\quad\times\mathrm{sign}(\omega_0+\tilde c\theta[-1,1])+\omega_2+\tilde c\theta[-1,1],\\&\qquad\ldots\\&\dot\omega_{r+l-2}\in-\tilde\lambda_1\left|\omega_0+\tilde c\theta[-1,1]\right|^{1/(r+l)}\\&\quad\times\mathrm{sign}(\omega_0+\tilde c\theta[-1,1])+\omega_{r+l-1}+\tilde c\theta[-1,1],\\&\dot\omega_{r+l-1}\in-\tilde\lambda_0\mathrm{sign}(\omega_0+\tilde c\theta[-1,1])+[-L,L]\end{aligned}$$

where $\tilde c>0$ is an appropriate constant. This is a disturbed finite-time stable homogeneous differential inclusion with a negative homogeneity degree [23]. Therefore, it has an invariant region around the origin, attracting trajectories in finite time [23]. That attracting region retracts to zero with $\theta\to 0$. ■

Now let the accuracy of the differentiator (Lemma 3) and the actuator (Lemma 1) be of the order of θ_1. Then (25) can be rewritten as the differential inclusion

$$\begin{aligned}&\dot\sigma_0=\sigma_1,\ldots,\dot\sigma_{r-2}=\sigma_{r-1},\\&\dot\sigma_{r-1}\in\sigma_r+\tilde c_1\theta_1[-1,1],\\&\qquad\ldots\\&\tilde\sigma_{r+l-2}\in\sigma_{r+l-1}+\tilde c_1\theta_1[-1,1],\\&\tilde\sigma_{r+l-1}\in[-C_1,C_1]+\\&[K_m,K_M]U(\sigma_0+\tilde c_1\theta_1[-1,1],\ldots,\sigma_{r+l-1}+\tilde c_1\theta_1[-1,1])\\&\qquad\qquad+\tilde c_1\theta_1[-1,1].\end{aligned}\tag{26}$$

Here $\tilde c_1>0$ is another appropriate constant. Thus, (26) can be considered as a disturbed finite-time stable homogeneous inclusion with a negative homogeneity degree. Hence, it has an invariant region attracting trajectories in finite time and retracting to zero with $\theta_1\to 0$. ■

The proof of Theorem 3 is a simple consequence of Theorem 2 and (26) with $\theta_1\to 0$.

V. SIMULATION

A. High Order Sliding Mode Control

An already traditional example of the kinematic car model

$$\dot x=\nu\cos\varphi,\quad \dot y=\nu\sin\varphi,\quad \dot\varphi=\frac{v}{l}\tan\theta,\dot\theta=u_{act}$$

is chosen. Here x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, ν is the longitudinal velocity, l is the length between the two axles, θ is the steering angle, and u_{act} is the actuator output. The task is to steer the car from a given initial position to the trajectory $y=g(x)$ with $g(x)$ and y measured in real time. Define $\sigma=y-g(x)$. Let $\nu=const=10$ m/s, $l=5$ m

$$g(x)=10\sin(0.05x)+5,x=y=\varphi=\theta=0\ \mathrm{at}\ t=0.$$

The relative degree of the system is 3. The plant coordinates x, y, φ, θ are to be analyzed for chattering. The actuator coordinates are assumed excluded from the main-subsystem coordinates. As follows from the previous section, the chattering of σ, $\dot\sigma$ and $\ddot\sigma$, which is small with small $|\dot\sigma|$, $|\ddot\sigma|$ and $|\ddot\sigma^{\cdot}|$, is to be considered. The relative degree is artificially increased to 4, treating $\dot u$ as a new (discontinuous) control. Following the main idea of the high-order sliding mode control [22]–[25], the controller is known in advance; only one gain is adjusted here. The 4-sliding homogeneous quasi-continuous controller [24] was applied, as shown in the equation at the bottom of the page. The estimates s_i of the derivatives $\sigma^{(i)}$ are obtained as the result of the real-time differentiation of the sensor output s by the third-order differentiator

$$\begin{aligned}&\dot s_0=\xi_0,\quad \xi_0=-15.3|s_0-s|^{3/4}\mathrm{sign}(s_0-s)+s_1,\\&\dot s_1=\xi_1,\quad \xi_1=-17.8|s_1-\xi_o|^{2/3}\mathrm{sign}(s_1-\xi_0)+s_2,\\&\dot s_2=\xi_2,\quad \xi_2=-39.7|s_2-\xi_1|^{1/2}\mathrm{sign}(s_2-\xi_1)+s_3,.\\&\dot s_3=-770\mathrm{sign}(s_3-\xi_2).\end{aligned}$$

$$\begin{aligned}&\dot u=0\ \text{with}\ t\le 1,\\&\dot u=-5\frac{\left\{s_3+3[s_2+(|s_1|+0.5|s_0|^{3/4})^{-1/3}(s_1+0.5|s_0|^{3/4}\mathrm{sign}s_0)]\,[|s_2|+(|s_1|+0.5|s_0|^{3/4})^{2/3}]^{-1/2}\right\}}{\left\{|s_3|+3[|s_2|+(|s_1|+0.5|s_0|^{3/4})^{2/3}]^{1/2}\right\}}\ \text{with}\ t>1.\end{aligned}$$

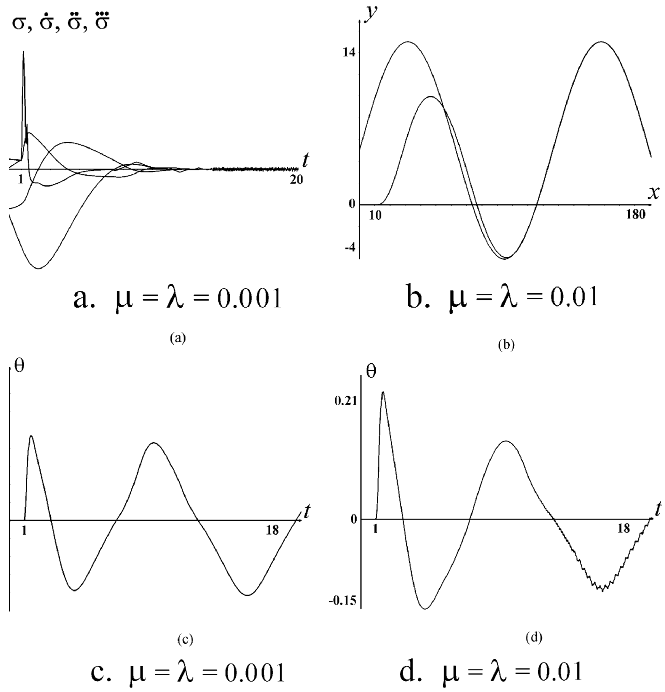

Fig. 1. 4-sliding car control in the presence of actuator and sensor.

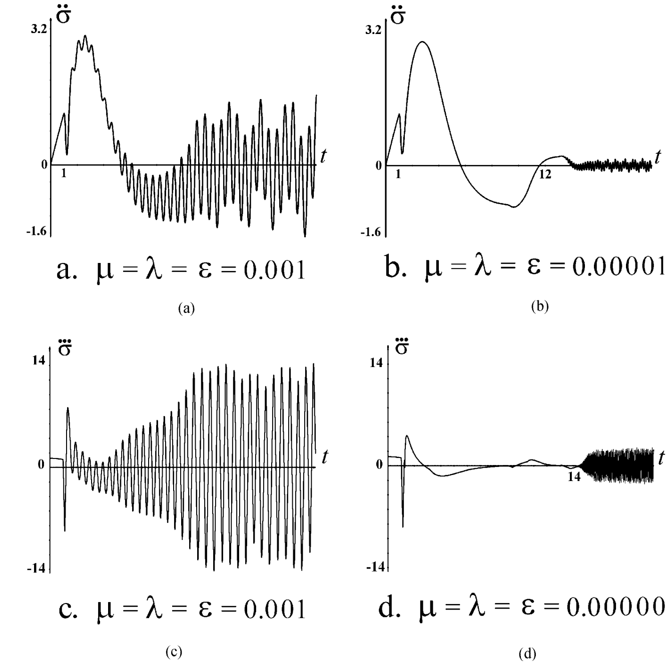

Fig. 2. Infinitesimal chattering in the presence of actuator, sensor and noise.

In its turn s is produced by means of a sensor with the linear model

$$s = \hat{y} - g(x), \quad \lambda^3 \dddot{\hat{y}} + 2\lambda^2 \ddot{\hat{y}} + 2\lambda \dot{\hat{y}} + \hat{y} = \lambda \dot{y} + y.$$

The initial values $\hat{y} = -10$, $\dot{\hat{y}} = 20$, $\ddot{\hat{y}} = -8$ were taken. The control u, which is defined above by its derivative, has the initial value 0, and enters the plant through an actuator with the nonlinear model

$$\mu \dot{z}_1 = z_2, \quad \mu \dot{z} = -(z_1 - u)^3 + (z_1 - u) + z_2, \quad u_{act} = z_1.$$

The actuator coordinates z_i have zero initial conditions.

The simulation results are demonstrated in Fig. 1. The 4-sliding accuracies $|\sigma| \leq 0.0045$, $|\dot{\sigma}| \leq 0.0026$, $|\ddot{\sigma}| \leq 0.0024$, $|\ddot{\sigma}^{\cdot}| \leq 0.34$ were obtained with the sensor and actuator time constants $\lambda = \mu = 0.001$. The accuracies changed to $|\sigma| \leq 0.045$, $|\dot{\sigma}| \leq 0.030$, $|\ddot{\sigma}| \leq 0.13$, $|\ddot{\sigma}^{\cdot}| \leq 3.1$ with the sensor and actuator time constants $\lambda = \mu = 0.01$. Hence, the simulation shows that the lower λ, μ, the less the chattering.

Now introduce a noise at the output of the sensor of the magnitude ε. The system reveals high sensitivity to such noises due to the presence of the third-order differentiator, i.e., the corresponding coefficients are large. Note that in practice additional sensors would probably be introduced. Alternatively, a simpler and more robust 3-sliding controller can be used here with the second-order differentiator [22].

The accuracies $|\sigma| \leq 0.076$, $|\dot{\sigma}| \leq 0.29$, $|\ddot{\sigma}| \leq 1.7$, $|\ddot{\sigma}^{\cdot}| \leq 14$ were obtained with $\lambda = \mu = \varepsilon = 10^{-3}$. The accuracies $|\sigma| \leq 0.0019$, $|\dot{\sigma}| \leq 0.0085$, $|\ddot{\sigma}| \leq 0.16$, $|\ddot{\sigma}^{\cdot}| \leq 4.5$ and $|\sigma| \leq 0.0013$, $|\dot{\sigma}| \leq 0.0039$, $|\ddot{\sigma}| \leq 0.053$, $|\ddot{\sigma}^{\cdot}| \leq 2.5$ were obtained with $\lambda = \mu = \varepsilon = 10 - 5$ and $\lambda = \mu = \varepsilon = 10^{-6}$ respectively. These asymptotical accuracies correspond to the accuracies $\sigma^{(i)} = O(\varepsilon^{(4-i)/4})$, which would be obtained *in the absence of sensors and actuators* [22]–[24]. The tracking remains practically ideal.

The chattering is infinitesimal here, since $\dot{\sigma}, \ddot{\sigma}, \ddot{\sigma}^{\cdot}$ vanish with λ, μ, $\varepsilon \to 0$ (Fig. 2). In particular, the integral of $|\ddot{\sigma}^{\cdot}|$ characterizes the chattering of the plant coordinate θ, chattering of x, y, φ corresponds to the integrals of $|\dot{\sigma}|$ and $|\ddot{\sigma}|$. The results are almost independent of the features of the noise.

B. Aircraft Pitch Control

The chattering of a mechanical actuator is demonstrated here. A practical aircraft control problem [21] is to get the pitch angle θ of a flying platform to track some signal θ_c given in real time. The actual nonlinear dynamic system is given by its linear 5-dimensional approximations, calculated for 42 equilibrium points within the Altitude—Mach flight envelope and containing significant uncertainties. The relative degree is 2. Details are presented in [21]. The actuator (stepper motor servo) output ν is to follow the input u. The output ν changes its value 512 times per second with a step of $\pm 0.2°$, or remains the same. It gets the input 64 times per second and stops to react for 1/32 s each time, when sign $(u - v)$ changes. The actuator output has the physical meaning of the horizontal stabilizer angle, and its significant chattering is not acceptable.

Following are unpublished simulation results (1994) revealing the chattering features of a linear dynamic control based on the H_∞ approach and a 3-sliding-mode control practically applied afterwards in the operational system (1997). In order to produce a Lipschitzian control, the 3-sliding-mode controller was constructed according to the above chattering attenuation procedure. The comparison of the performances is shown in Fig. 3. The control switches from the linear control to the 3-sliding-mode control at $t = 31.5$. The chattering is caused by the inevitably relatively large linear-control gain.

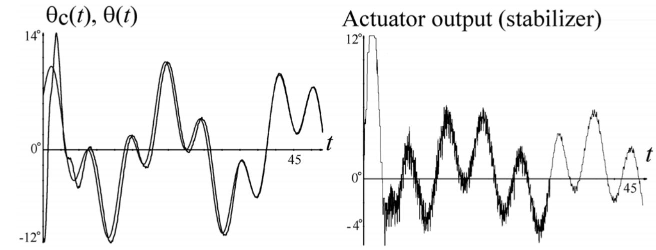

Fig. 3. Chattering of the aircraft horizontal stabilizer: a switch from a linear control to a 3-sliding one.

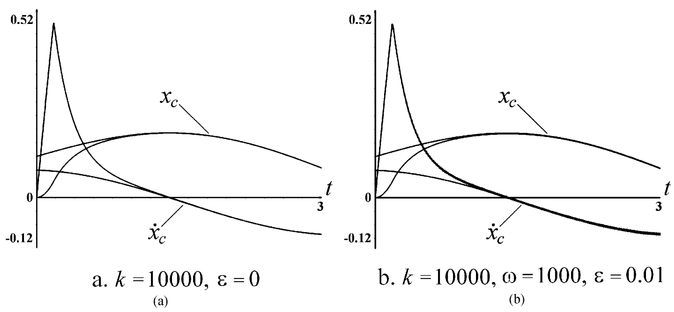

Fig. 4. Tracking performance of the high-gain pendulum controller.

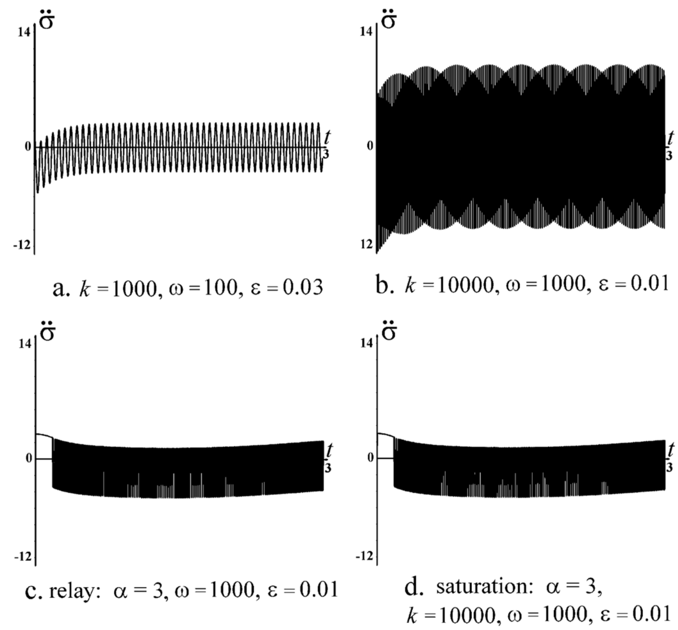

Fig. 5. Unbounded chattering of the high-gain controller, and bounded chattering of the saturation-based and relay controllers (the chattering is estimated by the integral of $|\ddot{\sigma}|$).

C. High-Gain and Sliding-Mode Controls

Now consider a simple rigid-pendulum control problem

$$\ddot{x} = -2\dot{x} - 9.81\sin x + u, \quad \sigma = x - x_c,$$
$$x_c = 0.5\sin 0.5t + 0.5\cos t$$

where x is the angle deviation from the vertical position, the length and the mass of the pendulum equal 1. The control u is the torque applied to the pendulum. The task is to track the "unknown-in-advance" signal x_c. The high-gain controller

$$u = k(5\sigma + \dot{\sigma} + \eta), \quad \eta = \varepsilon\cos(\omega t + 1.23)\dot{\sigma}$$

is applied, where η is the considered "measurement noise". The initial values of system coordinates x and $\dot{x}$ are zeroed.

The high-gain controller demonstrates ideal performance in the absence of noises ($\varepsilon = 0$) with $k = 10000$. In that case there is no chattering in the system (Fig. 4(a)). The tracking precision deteriorates only slightly in the presence of the noise with $\varepsilon = 0.01$, $\omega = 1000$ (Fig. 4(b)).

The chattering in the system is characterized by the integrals of the absolute values of $|\dot{\sigma}|$ and $|\ddot{\sigma}|$. The chattering of σ is infinitesimal, for $|\dot{\sigma}|$ is infinitesimal, as follows from Fig. 4. The graphs of $\ddot{\sigma}$ are demonstrated in Fig. 5(a) and (b). It is seen from the comparison with $k = 1000$, $\omega = 100$, $\varepsilon = 0.03$ [Fig. 5(a)] that the chattering grows significantly with the growth of k and ω and the reduction of ε. This corresponds to the unbounded chattering of $\dot{\sigma}$ and of the whole system (Theorem 1).

Now consider the controller with saturation

$$u = \max(-\alpha, k\min(\alpha, k(5\sigma + \dot{\sigma} + \eta)))$$

and the relay controller

$$u = -\alpha\,\mathrm{sign}(5\sigma + \dot{\sigma} + \eta)$$

with $\alpha = 3$. The noise parameters and k remain the same. The tracking graphs do not differ significantly from Fig. 4(b) and are not shown. It is seen from the graphs in Fig. 5(c) and (d) that both controllers feature bounded chattering and practically identical performance.

VI. Conclusion

Formal definitions of the chattering in mathematical control systems were proposed. The chattering can be classified as infinitesimal, bounded and unbounded. The infinitesimal chattering cannot be avoided. Smooth systems feature such chattering with continuous control.

Standard (first-order) sliding-mode systems feature bounded (i.e., finite-energy) chattering. Their chattering does not depend on the noise features, and can only be reduced by diminishing the discontinuous control component. Also high-order sliding modes feature bounded chattering, when directly applied to control systems with high relative degree.

Systems with high gains feature unbounded (i.e., unbounded energy) chattering in the presence of small high-frequency sampling noises. Thus, in certain cases, a standard sliding mode can be preferable to a large-gain linear control, if the discontinuous control component is relatively small. Saturation improves the performance of the high-gain controllers, but does not remove the bounded chattering.

High-order sliding mode excludes both dangerous kinds of chattering, if the well-known chattering attenuation procedure [3], [19] is used, when the control derivative is treated as a new (discontinuous) control. The presence of small measurement errors, delays and unaccounted-for fast stable actuators and sensors only generates infinitesimal chattering.

The practical approach [21] is justified when fast stable actuators and sensors are ignored at the preliminary control design stage, and are taken into account only during the simulation-based parameter adjustment of high-order sliding-mode controllers. It is proved that the faster the actuators and sensors the less is their influence on the system. At the same time the sliding controller parameters together with the plant determine whether the actuators and sensors can be considered really fast. Indeed, faster controllers require faster actuator and sensor responses.

An obvious drawback of the proposed approach is that the worst-case noises and disturbances are considered. Thus, even unbounded chattering may vanish with appropriate noises and disturbances. At the same time such an approach leads to the simplest chattering classification. Infinitesimal mathematical chattering presumably corresponds to negligible chattering in real systems.

The author did not consider theoretically discretization issues and small delays, which can also produce dangerous chattering in systems with large gains (Fig. 3). It follows from Section IV that such imperfections do not produce additional chattering in the systems with homogeneous discontinuous control.

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