On Fixed and Finite Time Stability in Sliding Mode Control

Arie Levant, *Member IEEE*

Abstract— **Any finite-time convergent homogeneous sliding-mode controller can be transformed into a fixed-time convergent one, featuring an upper bound of convergence time, which does not depend on initial conditions. Output feedback controller is optional. Continuity of the convergence time functions of homogeneous differential inclusions is studied. Feasibility of fixed-time-stable systems is considered.**

I. Introduction

CONTROL under heavy uncertainty conditions is one of the main subjects of the modern control theory, and the main idea to deal with such problems is to single out and keep properly chosen constraints successively killing the system dimensions. Sliding-mode (SM) or high-gain control are the corresponding methods [8,22].

Sliding mode is accurate and insensitive to disturbances [8,22]. While standard SMs are applicable to nullify sliding variables of the relative degree (RD) 1, higher order sliding modes (HOSMs) [3,4,6,10,14-16,19,23] are used to keep constraints of higher RD. One of the main reasons for their application is the possibility [2,11] to effectively attenuate the so-called chattering effect [8,12,17,22] caused by the high control-switching frequency.

Sometimes the uncertain system structure significantly changes at some discrete time instants, whereas the time period between the switches is never less than some positive constant known in advance. That situation is typical in control of switched and hybrid systems [18]. Since the system dynamics lacks uncertainty in the sliding mode, it is crucially important to provide for the convergence to SM in finite time, which does not depend on initial conditions.

There are a few options to solve the problem. Since in the practice the operational region is always bounded, traditional control methods can be applied. In the case when the operational region is unknown, a theoretical problem arises with unbounded operational region. It was recently found that HOSM controllers exist, that feature convergence time uniformly bounded in the whole state space. The corresponding convergence and stability are called fixed-time ones [20]. In particular fixed-time observers [1,7] were developed. Fixed-time HOSM controllers were proposed for controllable LTI systems with matched uncertainties [20]. Note that in the latter paper the matrix of control coefficients is assumed constant and known. No output-feedback results were proposed, for observation of the resulting extremely fast processes is very difficult.

It is proved in this paper that the minimal and maximal convergence-time functions of any finite-time-stable homogeneous Filippov differential inclusion are respectively upper- and lower-semicontinuous functions, taking its values on some inclusion solutions.

It is shown that any finite-time stable HOSM controller can be transformed into a fixed-time one by means of a discrete dynamic extension. Output-feedback version is also available. It is shown that for any compact set of initial conditions with sufficiently small sampling interval the system demonstrates the standard robustness features of homogeneous sliding modes.

Robustness issues of fixed-time stable systems are studied. It is shown that such systems always feature Euler solutions which tend to infinity faster than any exponent, provided they start sufficiently far away.

II. Problem statement and preliminaries

A. Problem statement

Following is the standard problem of HOSM control [14-16]. Consider a SISO dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is the control, a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, n can be also uncertain. The output σ is measured in real time. The task is to make σ vanish and to keep $\sigma \equiv 0$ afterwards. The convergence time of σ to zero is to be bounded by a predefined constant $T_{max} > 0$. In the following the standard approach is represented, which only provides for the finite-time convergence, with the convergence time strongly depending on initial conditions.

The relative degree r of system (1) is assumed to be constant and known. That means [13] that

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x)$, $g(t,x) \neq 0$ are some unknown smooth functions which can be expressed via Lie derivatives. It is supposed that

$$0 < K_{\mathrm{m}} \leq g(t,x) \leq K_{\mathrm{M}}, \ | h(t,x)| \leq C \tag{3}$$

for some K_{m}, K_{M}, $C > 0$. It is also assumed that trajectories of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control $u(t)$.

Usually in practice the operation region of any plant is inevitably bounded. Then the boundedness condition (3) turns out to be local and not restrictive. Nevertheless for the fixed-time convergence the condition is needed to be global. It will be further shown that it can be somewhat weakened.

A. Levant is with the School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978 Tel-Aviv, Israel (levant@post.tau.ac.il).

Obviously, (2), (3) imply the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \tag{4}$$

The problem is usually solved by a bounded control

$$u = \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}), \tag{5}$$

making all trajectories of (4), (5) converge in finite time to the origin $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ of the r-sliding phase space $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$. The closed-loop inclusion is minimally extended at the discontinuity points of (5) so that a compact convex non-empty and upper-semicontinuous inclusion is produced (Filippov inclusion [9,15]).

The function φ is assumed to be a locally-bounded Borel-measurable function. Thus, substituting any Lebesgue-measurable estimations of $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$ obtain a Lebesgue-measurable control. At the next step the lacking derivatives are real-time evaluated, producing an output-feedback controller.

It is easy to see that control (5) is to be discontinuous at the set $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ [15,16], which is called r-sliding set. The corresponding solutions $\sigma \equiv 0$ are said to exist in r-sliding (rth order sliding) mode [14].

B. Homogeneous sliding modes

A function f: $\mathbf{R}^n \rightarrow \mathbf{R}$ (respectively a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field f: $\mathbf{R}^n \rightarrow \mathbf{R}^n$) is called *homogeneous of the degree* $q \in \mathbf{R}$ *with the dilation*

$$d_\kappa: (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n),$$

and *the weights* $\deg x_i = m_i > 0$, if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds (respectively $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$, or $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$). The non-zero homogeneity degree q of a vector field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, ..., m_n$.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot x = f(x)$ (differential inclusion $\dot x \in F(x)$) with respect to the combined time-coordinate transformation

$$G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x),$$

where p, $p = -q$, might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as

$$\dot x \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x).$$

Suppose that feedback (5) imparts homogeneity properties to the closed-loop inclusion (4), (5). Due to the presence of the term $[-C, C]$, with $C \neq 0$ the right-hand side of (4) can only have the homogeneity degree 0. Thus, $\deg \sigma^{(r)} = \deg \sigma^{(r-1)} - p = \deg \sigma^{(r-1)} + q = 0$.

Scaling the system homogeneity degree q to -1, achieve that the homogeneity weights of $t, \sigma, \dot\sigma, ..., \sigma^{(r-1)}$ are $1, r, r-1, ..., 1$ respectively. This homogeneity is called the *r-sliding homogeneity* [15]. The inclusion (4), (5) is r-sliding homogeneous, if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: (t, \sigma, \dot\sigma, ..., \sigma^{(r-1)}) \mapsto (\kappa t, \kappa^r \sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \tag{6}$$

preserves the closed-loop inclusion (4), (5).

Transformation (6) transfers (4), (5) into

$$\sigma^{(r)} = \frac{d^r(\kappa^r\sigma)}{(d\kappa t)^r} \in [-C, C] + [K_m, K_M]\, \varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}). \tag{7}$$

Hence, the r-sliding homogeneity condition (7) requires

$$\varphi(\kappa^r\sigma, \kappa^{r-1}\dot\sigma, ..., \kappa\sigma^{(r-1)}) \equiv \varphi(\sigma, \dot\sigma, ..., \sigma^{(r-1)}). \tag{8}$$

Controller (5) is called r-sliding homogeneous if the identity (8) holds for any positive κ and any arguments. Also the corresponding r-sliding mode $\sigma \equiv 0$ is called homogeneous in that case. Thus, the relay control $u = -\alpha\, \mathrm{sign}\, \sigma$ is 1-sliding homogeneous, as well as the corresponding sliding mode.

Control (5) is assumed being r-sliding homogeneous. Being locally bounded, due to (8) it is also globally bounded.

A number of such sliding mode controllers is known [14-16]. Only the control amplitude α is to be adjusted in order to control any system (1), (3) of the corresponding relative degree. The following controllers are called quasi-continuous, for they produce control continuous everywhere except the r-sliding set $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ [16]. The controllers with $r \leq 4$ and valid parameters are listed below:

1. $u = -\alpha\, \mathrm{sign}\, \sigma$,
2. $u = -\alpha\, (\dot\sigma + |\sigma|^{1/2}\mathrm{sign}\, \sigma) / (|\dot\sigma| + |\sigma|^{1/2})$,
3. $u = -\alpha\, [\ddot\sigma + 2\, (|\dot\sigma| + |\sigma|^{2/3})^{-1/2}(\dot\sigma + |\sigma|^{2/3}\mathrm{sign}\, \sigma)] / [|\ddot\sigma| + 2\, (|\dot\sigma| + |\sigma|^{2/3})^{1/2}]$,
4. $M = \dddot\sigma + 3[\ddot\sigma + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{-1/3}(\dot\sigma + 0.5\, |\sigma|^{3/4}\mathrm{sign}\, \sigma)] [|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{-1/2}$,

   $N = |\dddot\sigma| + 3\, [|\ddot\sigma| + (|\dot\sigma| + 0.5|\sigma|^{3/4})^{2/3}]^{1/2}$, $u = -\alpha\, M/N$.

An r-sliding homogeneous norm is any continuous positive-definite function of the coordinates $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$ of the weight 1. In particular, denote $\vec\sigma = (\sigma, \dot\sigma, ..., \sigma^{(r-1)})$,

$$\|\vec\sigma\|_h = |\sigma|^{1/r} + |\dot\sigma|^{1/(r-1)} + ... + |\sigma^{(r-1)}|.$$

Note that, contrary to its name, it is not a norm. This homogeneous norm is used in the sequel.

C. Output-feedback homogeneous control

Any r-sliding homogeneous controller can be complemented by an $(r$-1)th order differentiator [14] producing an output-feedback controller and preserving the exactness, finite-time stability and the corresponding asymptotic accuracies [15]. Its application is possible due to the uniform boundedness of $\sigma^{(r)}$ provided by the boundedness of the feedback function φ in (5).

Let $\omega(t)$, $|\omega^{(k+1)}| \le L$, be the signal to be differentiated in real time, $L > 0$ is a known constant. Following is the kth-order robust finite-time-convergent exact homogeneous differentiator [14]:

$$\begin{aligned}
\dot{\varpi}_0 &= \upsilon_0,\ \upsilon_0 = -\lambda_{r-1} L^{1/(k+1)} |\varpi_0 - \omega(t)|^{1/(k+1)} \operatorname{sign}(\varpi_0 - \omega(t)) + \varpi_1,\\
\dot{\varpi}_1 &= \upsilon_1,\ \upsilon_1 = -\lambda_{r-2} L^{1/k} |\varpi_1 - \upsilon_0|^{(k-1)/k} \operatorname{sign}(\varpi_1 - \upsilon_0) + \varpi_2,\\
&\dots\\
\dot{\varpi}_{k-1} &= \upsilon_{k-1},\ \upsilon_{k-1} = -\lambda_1 L^{1/2} |\varpi_{k-1} - \upsilon_{k-2}|^{1/2} \operatorname{sign}(\varpi_{k-1} - \upsilon_{k-2}) + \varpi_k,\\
\dot{\varpi}_k &= -\lambda_0 L \operatorname{sign}(\varpi_k - \upsilon_{k-1}).
\end{aligned} \tag{9}$$

Here ϖ_i is the estimation of $\omega^{(i)}$, and parameters λ_i of differentiator (9) are chosen in advance for each k. An infinite sequence of parameters λ_i can be built, valid for all k [14]. In particular, one can choose $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 5$, $\lambda_5 = 8$ [16], which is enough for $1 \le k \le 5$.

In the absence of noises the equalities $\varpi_i = \omega^{(i)}$ are established in finite time. In the presence of a Lebesgue-measurable sampling noise with the magnitude γ the accuracy $|\varpi_i - \omega^{(i)}| = O(\gamma^{i/(k+1)})$ is obtained, and this asymptotics cannot be improved [14].

In the sequel differentiator (9) is symbolically denoted as $\varpi = D_k(\omega, \lambda, L)$, $\varpi_j = D_k^j(\omega, \lambda, L)$, $j = 0, 1, \dots, k$.

D. Convergence-time functions of homogeneous differential inclusions

It was erroneously stated in [15] that the maximal convergence time of a finite-time stable homogeneous differential inclusion is a continuous function of initial conditions. The following proposition corrects the statement.

Proposition 1. *Consider any finite-time stable homogeneous Filippov differential inclusion*

$$\dot{x} \in F(x),\ x \in \mathbf{R}^n, \tag{10}$$

of the negative homogeneity degree q. Then the maximal convergence time $T^*(x)$ is an upper semicontinuous function of initial conditions, whereas the minimal convergence time $T_*(x)$ is a lower semicontinuous function. Both functions are homogeneous of the weight -q.

Proof. It follows from [5] that the inclusion is uniformly asymptotically stable [15]. Thus it is uniformly finite-time stable, which means that the convergence times of the trajectories starting at a fixed point x are bounded from above and from below by the homogeneous norms with some fixed coefficients.

In particular, it means that irrespectively of a point x the supremum $T^*(x)$ and the infimum $T_*(x)$ of the convergence times satisfy the inequality

$$c_* \| x \|_h \le T_*(x) \le T^*(x) \le c^* \| x \|_h \tag{11}$$

for some fixed $0 < c_* \le c^*$.

Thus, having been taken over a sufficiently large closed convergence time segment, the trajectories constitute a compact set in the C-metric [9]. Recall also that a uniform limit of solutions is also a solution [9]. Prove that $T^*(x)$ is realized on some trajectory, i.e. is a maximum. Indeed there is a sequence of trajectories $x_k(t)$ starting at x, whose convergence times tend to $T^*(x)$, and which uniformly converge to some solution $x^*(t)$ of (10). Then due to (11) the convergence time of $x^*(t)$ cannot be larger than $T^*(x)$. On the other hand, due to the same inequality, if $x^*(t)$ converges to 0 in some time $T_0 < T^*(x)$, then also the upper limit of the convergence times of $x_k(t)$ cannot exceed T_0, which leads to contradiction.

Similarly also $T_*(x)$ is realized on some trajectory. Both functions are obviously homogeneous of the weight q.

Consider now the trajectories $x_k(t)$ of the maximal convergence time $T^*(x_k(0))$, which start at points $x_k(0)$, $x_k(0) \to x_0$. Due to the continuous dependence on initial conditions [9] get $x_k(T^*(x_0)) \to 0$. It follows now from (11) that $\limsup T^*(x_k(0)) \le T^*(x_0)$, which exactly corresponds to the upper semicontinuity.

Consider the trajectories $x_k(t)$ of the minimal convergence time $T_*(x_k(0))$ starting at points $x_k(0)$, $x_k(0) \to x_0$. Choose a subsequence $x_{k_s}(t)$ which uniformly converges to some solution $x_*(t)$, $x_*(0) = x_0$, and features $\lim T_*(x_{k_s}(0)) = \liminf T_*(x_k(0))$. Then, similarly to the above reasoning, the convergence time of $x_*(t)$ exactly equals the limit $\liminf T_*(x_k(0))$. This implies $T_*(x_0) \le \liminf T_*(x_k(0))$, which exactly corresponds to the lower semicontinuity. ■

It follows from Proposition 1 that the function $T^*(x)$ takes on its maximal value on any compact set of initial conditions. Similarly the function $T_*(x)$ takes on its minimum. Both functions are continuous at the origin. The author has used the continuity of the function $T^*(x)$ in a number of papers. The proved upper-semicontinuity of $T^*(x)$ provides for the validity of all results obtained there.

III. Control Design

Consider any r-sliding homogeneous finite-time stable controller (5). Let R_k be an arbitrary monotonously growing sequence, $R_k \to \infty$, $R_0 = 0$, $k = 0, 1, 2, \dots$. Let $\hat{T}$ be an upper estimation of the maximal time of convergence to zero from the homogeneous sphere $\| \vec{\sigma} \|_h = 1$. Note that such a number exists due to Proposition 1. Let $T(R)$ be a monotonously increasing positive-definite function $T(R)$, $R \ge 0$, $T(0) = 0$.

Define the function $n(s) = k \Leftrightarrow s \in [R_{k-1}, R_k)$. Obviously $n(s) \ge 1$ for any $s \ge 0$. Introduce an additional variable $\mu(t)$ that is left-hand continuous and features the dynamics

$$\mu(t+0) = \max\left\{1, \min\left[\mu(t), \frac{R_{n(\|\vec\sigma\|_h)}\hat{T}}{T(R_{n(\|\vec\sigma\|_h)}) - T(R_{n(\|\vec\sigma\|_h)-1})}\right]\right\}, \quad \text{with} \quad n\left(\|\vec\sigma\|_h\right) > 1; \qquad (12)$$

$$\mu = 1, \quad \text{with} \quad n\left(\|\vec\sigma\|_h\right) = 1.$$

Obviously $\mu(t)$ is piece-wise continuous. Define the control

$$u = \mu^r \varphi(\vec\sigma / \mu^r). \qquad (13)$$

Theorem 1. *Under the control* (12), (13) *the convergence time* T_{conv} *of any trajectory starting at the point* $\vec\sigma$ *satisfies the inequality*

$$T_{conv} \le T\left(R_{n(\|\vec\sigma\|_h)}\right). \qquad (14)$$

Note that the fixed-time convergence is obtained with $\lim_{R\to\infty} T(R) < \infty$. The proof easily follows from the following simple Lemma.

Lemma 1. *Let the maximal stabilization time of the trajectories of* (4), (5) *starting* $\|\vec\sigma\|_h \le R$ *be* $T_{\max}$, *then with* $\mu \ge 1$ *the maximal convergence time to zero of the system* (4), (13) *from the same region does not exceed* $T_{\max}/\mu$.

Proof. Note that with $\mu \ge 1$ all solutions of (4), (13) satisfy

$$\sigma^{(r)} \in \mu^r[-C, C] + \mu^r[K_m, K_M]\varphi(\sigma, \dot\sigma/\mu, ..., \sigma^{(r-1)}/\mu^{r-1}). \qquad (15)$$

Identity (8) is used here. Rewrite (15) in the form

$$\tfrac{d}{d\mu t} s_0 = s_1,\ \tfrac{d}{d\mu t} s_1 = s_2,\ \ldots,\ \tfrac{d}{d\mu t} s_{r-2} = s_{r-1},$$
$$\tfrac{d}{d\mu t} s_{r-1} \in [-C, C] + [K_m, K_M]\varphi(s_0, s_1, ..., s_{r-1}),$$

where $s_0 = \sigma$, $s_1 = \dot\sigma/\mu$, $\ldots, s_{r-1} = \sigma^{(r-1)}/\mu^{r-1}$. Thus, the convergence time to zero does not exceed $T_{\max}/\mu$, provided $\|\sigma, \dot\sigma/\mu, ..., \sigma^{(r-1)}/\mu^{r-1}\|_h \le R$ holds at the initial moment. The latter condition is automatically satisfied, provided $\|\vec\sigma\|_h \le R$. ■

Proof of the Theorem 1. It follows from Lemma 1 that with constant $\mu \ge 1$ the trajectories of (4), (13) starting from the homogeneous ball $\|\vec\sigma\|_h \le R_k$ stabilize at zero in time $R_k\hat{T}/\mu$. Thus, with $\mu = R_k\hat{T}/(T(R_k) - T(R_{k-1}))$ the trajectories enter the ball $\|\vec\sigma\|_h \le R_{k-1}$ in time $T(R_k) - T(R_{k-1})$. Therefore the total convergence time of the controller (12), (13) is estimated from above by the sum $T(R_k) - T(R_{k-1}) + T(R_{k-1}) - ... - T(R_1) + T(R_1) = T(R_k)$. ■

Include a differentiator of the form (9) to get an output feedback control. The differentiator needs to be initialized, so that it will provide for the exact derivative estimations. One way is to use a fixed-time convergent differentiator [1,7], and only after it has converged to apply the control (15). Another and maybe more practical way is to initialize it numerically, using finite differences as follows. A sampling time interval $\Delta t_* > 0$ is fixed, $u = 0$ is taken, and σ is sampled r times producing $\sigma(t_0)$, …, $\sigma(t_{r-1})$. Denote $\delta_j^0 = \sigma(t_j)$, $\delta_j^k = (\delta_j^{k-1} - \delta_{j-1}^{k-1})/\Delta t_*$, $k = 1, ..., r-1$. Then the differentiator is initialized by the values

$$z_0(t_{r-1}) = \delta_{r-1}^0, \ldots, z_{r-1}(t_{r-1}) = \delta_{r-1}^{r-1}, \qquad (16)$$

and converges during the additional time Δt_{**},

$$z = D_{r-1}(\sigma, \lambda, L) \text{ with } L \ge C + K_M \sup\varphi, \quad t \in [t_{r-1}, t_{r-1} + \Delta t_{**}) \qquad (17)$$
$$u = 0 \quad \text{with } t \in [t_0, t_{r-1} + \Delta t_{**}).$$

It is used here that any *r*-sliding homogeneous controller is bounded. Then denoting (12) symbolically as $\mu(t+0) = \mathrm{M}(\mu(t), \vec\sigma)$, define the output-feedback controller

$$\mu(t+0) = \mathrm{M}(\mu(t), z) \text{ with } t \ge t_{r-1} + \Delta t_{**}, \quad u = \mu^r\varphi(z/\mu^r), \quad z = D_{r-1}(\sigma, \lambda, \mu^r L). \qquad (18)$$

Theorem 2. *Let* $\delta, \gamma, \Delta t_{**}$ *be any fixed positive numbers. Assume that* σ *is measured with a Lebesgue-measurable noise which does not exceed* $\varepsilon > 0$ *in its absolute value. Let* $\Delta t_* = \gamma\varepsilon^{1/r}$, *then with any ball of initial values and sufficiently small* ε *the controller* (16)-(18) *provides for the accuracies* $|\sigma^{(i)}| \le \eta_i \max[\delta_t^{r-i}, \varepsilon^{(r-i)/r}]$ *to be established in the time* $T\left(R_{n(\|\vec\sigma(t_{r-1}+\Delta t_{**})\|_h)}\right) + \delta$. *Here the constants* η_i *only depend on the parameters of the differentiator* (9) *and the choice of the controller* (5).

The proof is straight-forward.

IV. Feasibility of Fixed-Time Stable Systems

One of the most important stages of control system design is the computer simulation. It is usually assumed that a system that cannot be reliably simulated in computer probably cannot be realized at all.

Consider a Filippov differential inclusion (10). Its Euler solution is a solution of the differential equation

$$\dot x = \xi_k,\ \xi_k \in F(x(t_k)),\ t \in [t_k, t_{k+1}).$$

It is defined by an initial condition $x(t_0)$, a strictly monotonously increasing sequence t_k and the sequence ξ_k. The solution is further called τ-solution, if $0 < t_{k+1} - t_k \le \tau$.

Proposition 2. *With* $t_k \to \infty$ *all τ-solutions of any r-sliding homogeneous finite-time stable inclusion* (4), (5) *in finite time converge into a small vicinity of the r-sliding mode* $\sigma^{(i)} = O(\tau^{r-i})$. *All τ-solutions of any asymptotically stable linear time-invariant system converge into a region, whose diameter is proportional to τ.*

Proposition 2 presents only two of many examples of systems, which can be reliably simulated by the Euler method. The proof of the first statement is similar to [15], while the proof of the second one is based on the successive scaling of linear time-invariant systems. The following Proposition is a trivial consequence of the uniform convergence of τ-solutions to the solutions of a differential inclusion over any closed time interval [9].

Proposition 3. *Let inclusion* (10) *be fixed-time stable (or only practically fixed-time stable, which means that all solutions in finite time converge into some bounded region). Then, for any compact set of initial conditions, with sufficiently small* $\tau > 0$, *all τ-solutions converge into an infinitesimal vicinity of the origin* (*converge into a bounded region*).

The following proposition demonstrates the difficulty of the realization of fixed-time stable systems. One cannot expect that with infinitesimally small sampling periods the behavior of the system be similar to its theoretical behavior. Indeed, one Euler step can be much incomparably larger than distance from the origin, even when the distance is large.

Proposition 4. *Let inclusion* (10) *be fixed-time stable or practically fixed-time stable. Then for any* $\tau, \gamma > 0$ *and any* $R > 0$ *there exist* $x_0, \dot{x}_0$, $\| x_0 \| \ge R$, $\dot{x}_0 \in F(x_0)$, such *that* $\| \dot{x}_0 \| \tau \ge \gamma \| x_0 \|$.

Let now $T_{enter}(R)$ be the supremum of the times needed for the solutions of (10) to enter the region $\| x \| \le R$. Obviously $T_{enter}(R)$ is a monotonously decreasing positive function. Therefore, $\lim_{R\to\infty} T_{enter}(R) \ge 0$. The equality $\lim_{R\to\infty} T_{enter}(R) = 0$ seems to hold for most known (practically) fixed-time stable systems. In particular, it holds for the systems homogeneous at the infinity with the positive homogeneity degree [1], systems with fixed-time stabilizing Lyapunov functions [20], and also for the controller (16)-(18).

Theorem 3. *Let inclusion* (10) *be fixed-time stable or practically fixed-time stable,* $\lim_{R\to\infty} T_{enter}(R) = 0$. *Then for any* $\tau > 0$ *there exists such* $R_\tau > 0$ *that for any initial condition* $x(t_0)$, $\| x(t_0) \| \ge R_\tau$, *there is a τ-solution, whose norm tends to infinity. Moreover, there are solutions that diverge faster than any predefined exponent.*

Proof. Define a sequence $R_k = 2^{n_k}$, $n_k \in \mathbb{N}$, $T_{enter}(R_k) \le \tau / (k+6)$, $k = 1, 2, \ldots$.

Let $R_\tau = 2R_1$, obviously $R_\tau \le R_2$. Any solution starting at some $R_2 \ge \| x(t_0) \| \ge R_\tau$ enters $\| x \| \le R_1$ in time which does not exceed $\tau / 7$. Choose an Euler solution with sufficiently small sampling/integration intervals, so that the resulting solution be close to an ideal solution of (10). This *τ*-solution enters $\| x \| \le R_1$ in time not exceeding $\tau / 7$. Thus, at some moment t_{k_1} the velocity value $\| \dot{x}(t_{k_1}) \| \ge 3 \| x(t_0) \| / \tau$ is to appear, while $R_1 \le \| x(t_{k_1}) \| \le \| x(t_0) \|$. At that moment, instead of continuing the *τ*-solution with small sampling intervals, apply the whole interval τ. As the result get that $\| x(t_{k_1+1}) \| \ge 2 \| x(t_0) \|$, which is achieved in the time less than 2τ. Continuing the process get $\| x(t_{k_2+1}) \| \ge 2^2 \| x(t_0) \|$ in time less than $(2+2)\tau$, etc.

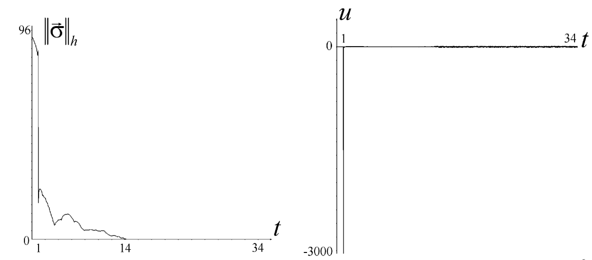

Figure 1. Fixed-time stabilization. The Euler integration step is 10^{-6}.

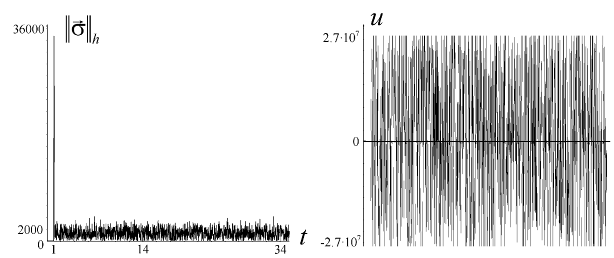

Figure 2. The same initial values. The Euler integration step is 10^{-5}.

At some moment the solution will exceed $2R_2$ in its norm. From that time on by similar construction $\| x \|$ will be multiplied at least by 3 during each 2τ interval, etc. ■

V. Simulation

Consider the academic example

$$\dddot{y} = \cos(5t + y - \dot{y} - \ddot{y}) + (2 + \sin(3t - y + \ddot{y}))u,$$
$$\sigma = y - \sin(0.5t) + \cos(t).$$

The relative degree of the system is 3. The task is to make σ vanish in fixed time. Controller (16)-(18) is constructed as in Section III. The convergence time function $T(R) = 40\left(1 - \dfrac{1}{0.083R + 1}\right)$, which corresponds to fixed-time convergence, and the sequence $R_k = 20k$, $k = 0, 1, 2, \ldots$, are chosen. The quasi-continuous controller N# 3 with $\alpha = 10$ from the list in Section IIB is taken as the basic controller φ.

Second-order differentiator (12) with the parameters λ_0 = 1.1, λ_1 = 1.5, λ_2 = 2 is taken. Its parameter $L = 30\mu^r$ switches together with μ in (12), $\hat{T} = 0.8$. Parameters $\Delta t_* = 0.001$, $\Delta t_{**} = 1$ are taken for its initialization.

Let the initial values be y = 60, $\dot{y}$ = -70, $\ddot{y}$ = 80. First demonstrate the fixed-time output-feedback convergence with sufficiently small sampling steps. The Euler step should be chosen so that the corresponding Euler step shift be small compared with the vector of the initial conditions. It is not trivial, since the control magnitude inevitably rapidly grows with the growth of the initial conditions' vector.

With the Euler step equal to 10^{-6} the fixed-time convergence is observed, in correspondence with Theorem 2. The graphs of the homogeneous norm of the solution and of the corresponding control are shown in Fig. 1. The convergence takes about 13 time units after the control is employed. The accuracy $|\sigma| \leq 4.6 \cdot 10^{-14}$, $|\dot{\sigma}| \leq 6.3 \cdot 10^{-7}$, $|\ddot{\sigma}| \leq 2.9 \cdot 10^{-4}$ is maintained. The control drops fast from very large values. The steady state control magnitude is about 10. Further decreasing the Euler integration step does not change the graphs, but improves the accuracy.

Next demonstrate the lack of the controller robustness with respect to Euler approximation. Let now the Euler step be equal to 10^{-5}. The system performance drastically changes. The convergence takes negligibly small time, the accuracies $|\sigma| \leq 8.9 \cdot 10^{-7}$, $|\dot{\sigma}| \leq 8.5 \cdot 10^{-2}$, $|\ddot{\sigma}| \leq 5.3 \cdot 10^{3}$ are obtained. In the steady state the control takes on unsustainable values of $\pm 2.7 \cdot 10^{7}$.

VI. Conclusions

Any homogeneous sliding mode controller can be transformed into a fixed-time convergent sliding-mode controller. Such controller provides for the transient time uniformly bounded by a constant independent of the initial conditions. Any convergence-time function given as a monotonous function of the homogeneous norm of the initial conditions can be approximated from below. The proposed controllers can be equipped with the differentiators producing fixed-time-convergent output-feedback controllers.

Computer realization of fixed-time stable systems was studied. It was shown that such realizations are not reliable. Indeed, it actually follows from the fact that, with any fixed sampling/integration period, sufficiently far from the origin any discretized system trajectory at each step performs enormous jumps, which are significantly larger than the distance from the point to the origin. Note that such phenomenon is impossible with linearly growing controls. It is proved that there are solutions that rapidly escape to infinity.

At the same time such systems are realizable for any bounded region of initial conditions, provided the sampling/integration step is taken small enough. Thus, in practice the fixed-time convergence is useful, when the region of initial conditions is bounded, but large. Sampling step should be carefully chosen. One cannot expect robust performance of a fixed-time stable system over an unbounded set of initial conditions.

Settling time functions of general finite-time stable homogeneous systems were studied. It was shown that the maximal and the minimal convergence-time functions are well defined and feature respectively upper and lower semicontinuity.

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