SISO Sliding Mode Control 2019: Robust Finite-Time-Exact Regulation and Observation.

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Abstract

Sliding Mode (SM) Control (SMC) is used to control systems under tough uncertainty conditions by properly choosing and exactly keeping a constraint involving system outputs and their derivatives. The constraint relative degree turns out to be the main approach parameter. Modern SMC establishes the constraint in finite time and uses high-order real-time robust and exact output differentiation. Closed-loop SMC systems are robust to the unaccounted-for dynamics of actuators and sensors, as well as to noises and discrete sampling.

Keywords: Sliding Mode, Relative Degree, Filtering, Differentiation, Robustness, Uncertainty

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Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$, $a \diamond B = \{a\} \diamond B$. A function of a set is the set of function values on this set. $\mathbb{R}_+ = [0, \infty)$; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$.

1. Introduction

Sliding mode control (SMC) systems, often also called variable-structure systems, have appeared as a theoretical and practical response to the challenge of control under heavy uncertainty conditions. SMC has already celebrated its half-century active-development milestone [19, 20, 14, 59, 58, 18, 56, 57, 28], though first controls using SMC technique appeared as early as in 1930s. This article presents some of the main SMC methods and is a modified and significantly extended version of [39].

Consider stabilizing a simple uncertain system

$$\ddot{x} = \hat{a}(t, x, \dot{x}) + \hat{b}(t, x, \dot{x})u,\ x, u \in \mathbb{R},\ |a| \leq 1,\ b \in [1, 2].$$

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The idea of the SMC approach is very intuitive. Any line in the phase plane $x, \dot{x}$ also has the meaning of a differential equation. Thus, keeping the trajectory on the line $\sigma = \dot{x} + x = 0$ asymptotically stabilizes the system. The corresponding control $u = -(2 + |\dot{x}|)\,\mathrm{sign}(\dot{x} + x)$ is the classical SMC (Fig. 1a, [20, 59]). The motion on the line $\sigma = 0$ is called SM. Since the relative degree of σ is 1 (i.e. the control appears already in $\dot{\sigma}$), it is called the 1st-order SM (1-SM) keeping $\sigma = \dot{x} + x = 0$.

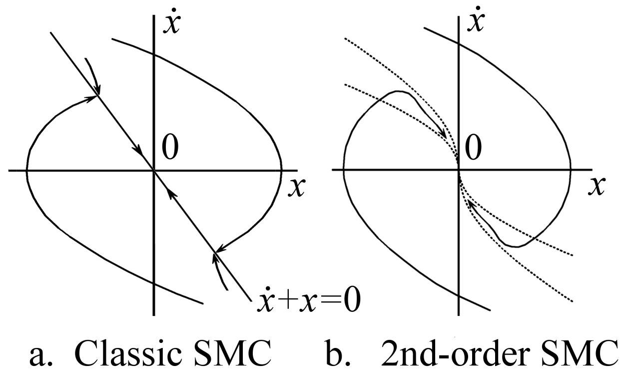

Figure 1: Trajectories of a. classic SMC, b. quasicontinuous second-order SMC.

Note that keeping $\dot{x} + \lfloor x \rceil^{1/2} = 0$ would provide for the finite-time (FT) stabilization. The corresponding control $u = -2\,\mathrm{sign}(\dot{x} + \lfloor x \rceil^{1/2})$ directly provides for the FT establishment of $\sigma = x = 0$ [49], [30]. Since the relative degree of $\sigma = x$ is 2 (i.e. for the first time the control appears in $\ddot{\sigma}$), the corresponding SM at the point $x = \dot{x} = 0$ is called the second-order SM (2-SM). Note that $\dot{x} + \lfloor x \rceil^{1/2}$ is not smooth and does not have a relative degree at the origin.

Another option is to apply the control $u = -2\frac{\lfloor \dot{x} \rceil^2 + x}{\dot{x}^2 + |x|}$. It also provides for $\sigma = x \equiv 0$ in FT, but it remains continuous till the very entrance into the 2-SM at the origin. Correspondingly it is called a quasi-continuous (QC) 2-SMC (Fig. 1b, [34, 56]).

Thus, the system uncertainty has been completely removed, but for the price of the control discontinuity. The corresponding solutions cannot be understood in the standard or the Caratheodory sense [21]. One also needs a differentiator to obtain a high-accuracy real-time estimation of $\dot{x}$.

Realization of SMC generates undesired system vibrations, called chattering [59, 22, 35]. The chattering effect is considered to be the main drawback of SMC systems [59, 5, 25, 57, 9, 22].

The traditional way to overcome the chattering effect is to introduce a switching regularization, making the control continuous. In particular, the relay function $\mathrm{sign}\,\sigma$ is often replaced by a "sigmoid" function, like $s/(|s| + \epsilon)$ or $\frac{2}{\pi}\arctan(\sigma/\epsilon)$, $0 < \epsilon << 1$ [57], [28]. Unfortunately, in that case the system remains sensitive to uncertainties for finite $1/\epsilon$, and hard chattering is generated by small high-frequency sampling noises for small ϵ [35].

The chattering is significantly diminished by inserting an integrator in the controller [30], [6], [56], provided σ and its derivatives are kept close to zero [35].

In the following text we provide the reader with the main SMC notions and tools for the simplest case of the single-input single-output (SISO) control.

2. Basic notions

Filippov definition. Consider a differential equation $\dot{x} = v(t,x)$, $x \in \mathbb{R}^{n_x}$, where v is a locally-essentially-bounded Lebesgue-measurable function. It is said to be understood in the Filippov sense [21], if it is replaced by the differential inclusion $\dot{x} \in K_F[v]$, where

$$K_F[v](t,x) = \bigcap_{\delta>0} \bigcap_{\mu_L N=0} \overline{\mathrm{co}}\, v(t, O_\delta(x) \backslash N). \tag{1}$$

Here μ_L is the Lebesgue measure, $O_\delta(x)$ is the δ-vicinity of x, and $\overline{co}M$ denotes the convex closure of M, (1) introduces the celebrated Filippov procedure.

Thus, a solution is defined as any locally absolutely-continuous function $x(t)$ which satisfies $\dot{x} \in K_F[v](t,x)$ almost everywhere.

In the most usual case, when v is continuous almost everywhere, the procedure results in taking the convex closure $K_F[v](t,x)$ of the set of all possible limit values of $v(t,y)$ at a given point (t,x), obtained when its continuity point (t,y) tends to (t,x). Values of v on sets of the measure 0 do not influence the solutions. Filippov differential equations posses all standard features of the solutions of ordinary differential equations, in particular existence and extension properties, but do not feature the solution uniqueness [21].

Relative degree. In the autonomous case the following definition is equivalent to the standard one based on Lie derivatives [27] provided one adds the fictitious equation $\dot{t} = 1$. Consider a smooth SISO system

$$\dot{x} = a(t,x) + b(t,x)u,\ \sigma = \sigma(t,x), \tag{2}$$

where $x \in \mathbb{R}^{n_x}$, a, b, and σ are smooth functions, $u, \sigma(t,x) \in \mathbb{R}$.

The *relative degree* of σ with respect to u at the point (t_0, x_0) is defined as the natural number r that satisfies two requirements: 1. it is the lowest total-derivative order of the output s which contains control,

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{3}$$

with the functional coefficient g, which locally differs from identical zero; 2. $g(t,x)$ does not vanish in some vicinity of the point (t_0, x_0).

It is easy to prove that the gradients of $t, \sigma, \dot{\sigma}, ..., \sigma^{(r-1)}$ are linearly independent, and, therefore, $r \leq n_x$. Note that the relative degree may not exist. The zero dynamics corresponds to the motion on the manifold $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$ described by equation (2) for $u = -h/g$.

The vector relative degree is defined along the same lines in the multi-input multi-otput (MIMO) case [27]. For the simplicity in the following we restrict ourselves to the SISO case.

It is important to mark that the calculation of relative degree is usually very simple, and is done orally. One only needs to track the shortest way of differentiation in which the control is to appear.

Real systems are often built in such a way that their mathematical models posses well-defined relative degrees and stable zero dynamics. Moreover, almost always $r = 2, 3, 4$, for the engineer needs a simple model. Correspondingly, significant parts of a real system are voluntarily removed to actuators or sensors, or are simply ignored as insignificant functional and singular perturbations.

Sliding mode. Any Filippov solution lying on the discontinuity surface/set of a differential equation is said to be in SM if the set of Filippov velocities contains at least two vectors. If a constraint $\sigma = 0$ is kept, the notation SM $\sigma \equiv 0$ is used, σ is called *the sliding variable*.

SM order. Suppose that the equality $\sigma = 0$ is kept on the SM solutions of a closed-loop system. Let σ be a scalar function. Then *the sliding order* k is defined as the lowest integer k, such that the kth-order total time derivative $\sigma^{(k)}$ is not a continuous function of the state variables and time [30, 32]. The corresponding motion $\sigma \equiv 0$ is called the kth-order SM, or k-sliding mode (k-SM). In the case of a vector sliding variable σ also the sliding order is a vector.

Connection to the relative degree. Consider system (2) with a scalar sliding variable σ and the relative degree r. Then, σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are continuous functions of t, x, i.e. the sliding order k is never less than r.

In the usual case of the control discontinuity obtain $k = r$. In that case the SM motion coincides with the system zero dynamics. The function $u_{eq} = -h/g$ found from the equation $\sigma^{(r)} = 0$ is traditionally called the *equivalent control* [59]. The classic SMs [18, 59] (Fig. 1a) correspond to $r = 1$ and the 1-SM $\sigma = 0$.

SMC is known to completely remove *matched disturbances*. Indeed, let (2) have the form $\dot{x} = a + b(u + \xi)$ where ξ is a disturbance. Then the SM motion (the zero dynamics) does not depend on ξ.

Chattering attenuation by HOSMs. High-Order SMs (HOSMs) were historically proposed to overcome the chattering-effect problem. Suppose the sliding order is r. In order to diminish the chattering one inserts l integrators in the feedback. Then the virtual discontinuous control $u^{(l)}$ is applied to establish the $(r + l)$-SM. Correspondingly, $u, \dot{u}, ..., u^{(l-1)}$ are formally included in the system state.

Note that the chattering reduction is not due to the continuity of the resulting actual control $u(t)$, but due to simultaneously keeping σ, $\dot{\sigma}$, ..., $\sigma^{(r+l-1)}$ at zero [35], while only σ, $\dot{\sigma}$, ..., $\sigma^{(r-1)}$ are the physical plant coordinates. Nothing theoretically prevents using any number of integrators, shifting the dangerous chattering deeper into a computer chip numerically producing the control.

HOSMs are also typically characterized by high accuracy in the presence of discrete sampling, small switching imperfections and noises [30, 33].

3. FT output regulation

Consider an uncertain smooth nonlinear SISO system of the form $\dot{x} = f(t, x, u)$, $x \in \mathbb{R}^{n_x}$, $\tilde{u} \in \mathbb{R}$, with a smooth output $\sigma(t, x) \in \mathbb{R}$. Let σ be the difference between some system output and a command signal available in real time. Thus, σ is the tracking error to be zeroed in FT and kept at zero afterwards.

The relative degree of σ is not defined for systems nonlinear in control. Moreover, in that case SM motions can be non-unique, and even generate non-Filippov solutions [59, 7]. Introduction of an integrator immediately resolves all these issues. Indeed, introducing the auxiliary control, $\dot{\tilde{u}} = u$, obtain the *affine-in-control* system of the form (2). See [40, 17] for the concrete SMC design details.

SMC problem. Consider now system (2) of the relative degree r, and assume that (3) holds with

$$|h(t,x)| \leq C,\ 0 < K_m \leq g(t,x) \leq K_M. \tag{4}$$

Such bounds are true at least for any compact operational region. The case $0 > -K_m \geq g(t,x) \geq -K_M$ is reduced to (4) by the control transformation $\tilde{u} = -u$.

Any solution of (2) is assumed infinitely extendable in time, provided σ, its derivatives $\dot{\sigma}, ..., \sigma^{(r-1)}$ and u remain bounded along the solution.

We search for a feedback control $u = u(\vec{\sigma})$, $\vec{\sigma} = (\sigma, \dot{\sigma},\ ...,\ \sigma^{(r-1)})$. Due to the uncertainty of the functions g, h in (3) one needs a discontinuous control u [33]. In other words, the stated problem is to establish the r-SM $\sigma = 0$.

The uncertain dynamics (3) can be replaced by the concrete differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u. \tag{5}$$

Most r-SM controllers are build as controllers for (5) making $\vec{\sigma}$ vanish in finite time. Though inevitably discontinuous at $\vec{\sigma} = 0$, the control $u = u(\vec{\sigma})$ can be continuous for any $\vec{\sigma} \neq 0$. Such control is called quasi-continuous (QC) and features significantly less chattering.

3.1. Homogeneous SMC

There are many known controllers solving the stated problem. Probably the simplest QC controller has the form [10, 16]

$$u = -\alpha\Psi_r(\vec{\sigma}) = -\alpha \frac{\lfloor \sigma^{(r-1)} \rceil^{\frac{\omega}{1}} + \beta_{r-2} \lfloor \sigma^{(r-2)} \rceil^{\frac{\omega}{2}} + ... + \beta_0 \lfloor \sigma \rceil^{\frac{\omega}{r}}}{|\sigma^{(r-1)}|^{\frac{\omega}{1}} + \beta_{r-2} |\sigma^{(r-2)}|^{\frac{\omega}{2}} + ... + \beta_0 |\sigma|^{\frac{\omega}{r}}},\ \ \omega > 0. \tag{6}$$

The theorem says that for any $\omega > 0$ there exist such $\beta_0, ..., \beta_{r-2} > 0$ that controller (6) stabilizes σ in FT for any sufficiently large $\alpha > 0$ only depending on K_m, K_M, C.

Functions $\Psi_r(\vec{\sigma})$ are invariant with respect to the transformation $\sigma^{(i)} \mapsto \kappa^{r-i}\sigma^{(i)}$, $\kappa > 0$, $i = 0, 1, ..., r-1$. Such controllers are called r-SM homogeneous [33, 56]. It is easy to see that $\Psi_r(\vec{\sigma})$ are continuous everywhere accept $\vec{\sigma} = 0$ and $|\Psi_r(\vec{\sigma})| \leq 1$, i.e. $|u| \leq \alpha$.

The following are valid QC controllers (6) for $r = 1, 2, 3, 4, 5$, $\omega = r$:

$$\begin{array}{ll} r=1. & u = -\alpha \operatorname{sign}\sigma, \\ r=2. & u = -\alpha \frac{\lfloor \dot\sigma \rceil^2 + \sigma}{\dot\sigma^2 + |\sigma|}, \\ r=3. & u = -\alpha \frac{\ddot\sigma^3 + 2\lfloor \dot\sigma \rceil^{\frac{3}{2}} + \sigma}{|\ddot\sigma|^3 + 2|\dot\sigma|^{\frac{3}{2}} + |\sigma|}, \\ r=4. & u = -\alpha \frac{\lfloor \dddot\sigma \rceil^4 + 2\lfloor \ddot\sigma \rceil^2 + 2\lfloor \dot\sigma \rceil^{\frac{4}{3}} + \sigma}{\dddot\sigma^4 + 2\ddot\sigma^2 + 2\dot\sigma^{\frac{4}{3}} + |\sigma|}, \\ r=5. & u = -\alpha \frac{\lfloor \sigma^{(4)} \rceil^5 + 6\lfloor \dddot\sigma \rceil^{\frac{5}{2}} + 5\lfloor \ddot\sigma \rceil^{\frac{5}{3}} + 3\lfloor \dot\sigma \rceil^{\frac{5}{4}} + \sigma}{|\sigma^{(4)}|^5 + 6|\dddot\sigma|^{\frac{5}{2}} + 5|\ddot\sigma|^{\frac{5}{3}} + 3|\dot\sigma|^{\frac{5}{4}} + |\sigma|}. \end{array} \tag{7}$$

Parameter α is usually found by simulation.

Note that in the case $g < 0$ **in** (3)**, and** $g(t,x) \in [-K_M, -K_m]$**, one has to take** $\alpha < 0$**.**

3.2. Differentiation and filtering

Let $\mathrm{Lip}_n(L)$ be the set of all functions $\mathbb{R}_+ \to \mathbb{R}$, whose nth derivative has the Lipschitz constant $L > 0$.

Let the input signal $f(t)$, $f(t) = f_0(t) + \eta(t)$, consist of a bounded Lebesgue-measurable noise $\eta(t)$ and an unknown basic signal $f_0(t)$, $f_0 \in \mathrm{Lip}_n(L)$. The noise η is bounded, $|\eta| \le \varepsilon_0$. The number $\varepsilon_0 \ge 0$ is unknown.

Differentiation problem [32]. The problem is to evaluate the derivatives $f_0^{(i)}(t)$, $i = 0, 1, ..., n$, in real time by some functions $z_i(t)$. The estimation is to be exact in the absence of noises after some FT transient, $z_i \equiv f_0^{(i)}$. The maximal steady-state errors are to continuously depend on ε_0.

Asymptotically optimal differentiation. It is proved that any differentiator exact on noise-free inputs $f_0, f_1 \in \mathrm{Lip}_n(L)$ has the worst-case steady-state accuracy $\sup|z_i - f_0^{(i)}| = 2^{\frac{i}{n+1}} K_{n,i} L^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}}$ for some f_0 and $\eta = f_1 - f_0$ [45]. Here $K_{n,i} \in [1, \pi/2]$ are the Kolmogorov constants [29, 45]. For example, $K_{1,1} = \sqrt{2}$.

Correspondingly, a differentiator is called *asymptotically optimal* [31, 32, 45, 47] if its steady-state accuracy satisfies

$$|z_i(t) - f_0^{(i)}(t)| \le \nu_i L^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}, \; i = 0, 1, ..., n, \tag{8}$$

for some constant coefficients ν_i independent of the basic input $f_0 \in \mathrm{Lip}_n(L)$, the Lebesgue-measurable noise η, $|\eta| \le \varepsilon_0$, and L, ε_0.

Introduce the number $n_f \ge 0$ which is further called *the differentiator filtering order*. The following differentiator [37], [47], [44] is called **the filtering differentiator** :

$$\begin{array}{l} \dot w_1 = -\tilde\lambda_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+n_f}{n+n_f+1}} + w_2, \\ \quad \dots \\ \dot w_{n_f-1} = -\tilde\lambda_{n+2} L^{\frac{n_f-1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+2}{n+n_f+1}} + w_{n_f}, \\ \dot w_{n_f} = -\tilde\lambda_{n+1} L^{\frac{n_f}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+1}{n+n_f+1}} + z_0 - f(t), \end{array} \tag{9}$$

$$
\begin{aligned}
\dot z_0 &= -\tilde\lambda_n L^{\frac{n_f+1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n}{n+n_f+1}} + z_1,\\
&\dots\\
\dot z_{n-1} &= -\tilde\lambda_1 L^{\frac{n+n_f}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{n+n_f+1}} + z_n,\\
\dot z_n &= -\tilde\lambda_0 L \,\mathrm{sign}(w_1),\ |f_0^{(n+1)}| \le L.
\end{aligned} \tag{10}
$$

In the case $n_f = 0$ the equations (9) disappear, and $w_1 = z_0 - f(t)$ is formally substituted in (10) yielding the well-known "standard" differentiator [32]. In the case $n = 0$ only the equation for z_0 remains in the lower part.

Parameters $\tilde\lambda_i$ are most easily calculated using the parameters $\lambda_0, ..., \lambda_n$ of the differentiator recursive form [32, 47]

$$
\begin{aligned}
\dot w_1 &= -\lambda_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+n_f}{n+n_f+1}} + w_2,\\
\dot w_2 &= -\lambda_{n+n_f-1} L^{\frac{1}{n+n_f}} \lfloor w_2 - \dot w_1 \rceil^{\frac{n+n_f-1}{n+n_f}} + w_3,\\
&\dots\\
\dot w_{n_f-1} &= -\lambda_{n+2} L^{\frac{1}{n+3}} \lfloor w_{n_f-1} - \dot w_{n_f-2} \rceil^{\frac{n+2}{n+3}} + w_{n_f},\\
\dot w_{n_f} &= -\lambda_{n+1} L^{\frac{1}{n+2}} \lfloor w_{n_f} - \dot w_{n_f-1} \rceil^{\frac{n+1}{n+2}} + z_0 - f(t),
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\dot z_0 &= -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) - \dot w_{n_f} \rceil^{\frac{n}{n+1}} + z_1,\\
\dot z_1 &= -\lambda_{n-1} L^{\frac{1}{n}} \lfloor z_1 - \dot z_0 \rceil^{\frac{n-1}{n}} + z_2,\\
&\dots\\
\dot z_{n-1} &= -\lambda_1 L^{\frac{1}{2}} \lfloor z_{n-1} - \dot z_{n-2} \rceil^{\frac{1}{2}} + z_n,\\
\dot z_n &= -\lambda_0 L \,\mathrm{sign}(z_n - \dot z_{n-1}),\ |f_0^{(n+1)}| \le L.
\end{aligned} \tag{12}
$$

In the case $n_f = 0$ one simply removes equations (11) and substitutes $\dot w_{n_f} = 0$ in the first equation of (12).

An infinite sequence of parameters $\vec\lambda = \{\lambda_0, \lambda_1, ...\}$ is proved to exist for any $\lambda_0 > 1$ [32], which is valid for any $n + n_f = 0, 1, ...$. In particular, $\vec\lambda = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32, ...\}$ suffice for $n + n_f \le 12$ (up to 7 [43, 45]).

Table 1: Parameters $\tilde\lambda_0, \tilde\lambda_1, ..., \tilde\lambda_{n+n_f}$ of differentiator (9), (10) for $n + n_f = 0, 1, ..., 12$

0	1.1												
1	1.1	1.5											
2	1.1	2.12	2										
3	1.1	3.06	4.16	3									
4	1.1	4.57	9.30	10.03	5								
5	1.1	6.75	20.26	32.24	23.72	7							
6	1.1	9.91	43.65	101.96	110.08	47.69	10						
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12					
8	1.1	19.66	171.73	795.63	1703.9	1464.2	608.99	120.79	14				
9	1.1	26.93	322.31	2045.8	6002.3	7066.2	4026.3	1094.1	173.72	17			
10	1.1	36.34	586.78	5025.4	19895	31601	24296	8908	1908.5	251.99	20		
11	1.1	48.86	1061.1	12220	65053	138954	143658	70830	20406	3623.1	386.7	26	
12	1.1	65.22	1890.6	29064	206531	588869	812652	534837	205679	48747	6944.8	623.30	32

Successively substituting the derivative $\dot w_1$ from the first equation into the equation for $\dot w_2$, then $\dot w_2$ into the equation for $\dot w_3$, etc., obtain that $\tilde\lambda_0 = \lambda_0$, $\tilde\lambda_n = \lambda_n$, and $\tilde\lambda_j = \lambda_j \tilde\lambda_{j+1}^{j/(j+1)}$, $j = n-1, n-2, \dots, 1$. The corresponding parameters $\tilde\lambda_i$ are listed in Table 1.

For example, the filtering differentiator of the order $n = 0$ and the filtering order $n_f = 2$ gets the form

$$\begin{aligned}
\dot{w}_1 &= -2\, L^{\frac{1}{3}} \lfloor w_1 \rceil^{\frac{2}{3}} + w_2,\\
\dot{w}_2 &= -2.12 L^{\frac{2}{3}} \lfloor w_1 \rceil^{\frac{1}{3}} + z_0 - f(t),\\
\dot{z}_0 &= -1.1 L \operatorname{sign} w_1,\ |\dot{f}_0| \le L,
\end{aligned} \tag{13}$$

where the parameters $\tilde{\lambda}_0 = 1.1, \tilde{\lambda}_1 = 2.12, \tilde{\lambda}_2 = 2$ are taken from the row $n + n_f = 2$ of Table 1. Its output z_0 estimates the component f_0 of the noisy signal $f = f_0 + \eta$ under the condition $|\dot{f}_0| \le L$.

The differentiator of the order $n = 1$ and the filtering order $n_f = 0$ (i.e. the "standard" differentiator [32]) has the equations

$$\begin{aligned}
\dot{z}_0 &= -1.5\, L^{\frac{1}{2}} \lfloor z_0 - f(t) \rceil^{\frac{1}{2}} + z_1,\\
\dot{z}_1 &= -1.1 L \operatorname{sign}(z_0 - f(t)),\ |\ddot{f}_0| \le L,
\end{aligned} \tag{14}$$

where the parameters $\tilde{\lambda}_0 = 1.1, \tilde{\lambda}_1 = 1.5$ are taken from the row $n + n_f = 1$ of Table 1. Its output z_0 estimates the component f_0 of the noisy signal f, z_1 estimates $\dot{f}_0$ under the condition $|\ddot{f}_0| \le L$.

The differentiator of the orders $n = n_f = 0$ has the simple equation

$$\dot{z}_0 = -1.1 L \operatorname{sign}(z_0 - f(t)),\ |\dot{f}_0| \le L.$$

The differentiator of the order 2 and the filtering order 0 is the standard differentiator

$$\begin{aligned}
\dot{z}_0 &= -2\, L^{\frac{1}{3}} \lfloor z_0 - f(t) \rceil^{\frac{2}{3}} + z_1,\\
\dot{z}_1 &= -2.12 L^{\frac{2}{3}} \lfloor z_0 - f(t) \rceil^{\frac{1}{3}} + z_2,\\
\dot{z}_2 &= -1.1 L \operatorname{sign}(z_0 - f(t)),\ |\dddot{f}_0| \le L.
\end{aligned} \tag{15}$$

Note the structure similarity of (13) and (15).

And here is the last example, differentiation order 2 and the filtering order 2, the coefficients are taken from row $2 + 2 = 4$ of the table:

$$\begin{aligned}
\dot{w}_1 &= -5\, L^{\frac{1}{5}} \lfloor w_1 \rceil^{\frac{4}{5}} + w_2,\\
\dot{w}_2 &= -10.03 L^{\frac{2}{5}} \lfloor w_1 \rceil^{\frac{3}{5}} + z_0 - f(t),\\
\dot{z}_0 &= -9.30\, L^{\frac{3}{5}} \lfloor w_1 \rceil^{\frac{2}{5}} + z_1,\\
\dot{z}_1 &= -4.57 L^{\frac{4}{5}} \lfloor w_1 \rceil^{\frac{1}{5}} + z_2,\\
\dot{z}_2 &= -1.1 L \operatorname{sign} w_1,\ |\dddot{f}_0| \le L.
\end{aligned} \tag{16}$$

Also see the discretization examples in (26), (29).

For brevity denote (9), (10) by

$$\dot{w} = \Omega_{n,n_f}(w, z_0 - f, L),\ \dot{z} = D_{n,n_f}(w_1, z, L), \tag{17}$$

with the tracking difference $z_0(t) - f(t)$ singled out as the separate argument.

Extend the above conditions on the input by letting the noise have the form $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, where each η_k, $k = 0, ..., n_f$, is a Lebesgue-measurable signal. For each k assume that there exists a uniformly bounded solution $\xi_k(t)$ of the equation $\xi^{(k)} = \eta_k$, $|\xi| \leq \varepsilon_k$.

Neither the expansion $\eta = \eta_0 + ... + \eta_{n_f}$ nor $\varepsilon_0, ..., \varepsilon_{n_f}$ are assumed to be known. The expansion is also not unique. Components $\eta_1, ..., \eta_{n_f}$ are possibly unbounded, but one can say that *they are bounded (small) in the average.*

Then [44] differentiator (17) in FT provides the accuracy

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L \rho^{n+1-i},\ i = 0, 1, ..., n, \\ &|w_1(t)| \leq \mu_{w1} L \rho^{n+n_f+1}, \end{aligned} \tag{18}$$

$$\rho = \max[(\tfrac{\varepsilon_0}{L})^{1/(n+1)}, ..., (\tfrac{\varepsilon_{n_f}}{L})^{1/(n+n_f+1)}] \tag{19}$$

for some $\mu_0, ..., \mu_n, \mu_{w1} > 0$ only depending on the parameters $\lambda_0, ..., \lambda_{n+n_f}$. Magnitudes of $w_2, ..., w_{n_f}$ depend on the concrete noises.

Taking $\eta_1 = ... = \eta_{n_f} = 0$, obtain that the filtering differentiator (9), (10) is asymptotically optimal. Moreover, it is proved that the differentiator is also applicable in the case when the multiple integrals of the noise components η_k are only small over *finite* time intervals not exceeding some $T_k > 0$ in their length [37, 38]. The error dynamics of the differentiator are homogeneous [37, 32, 33].

Let the input be sampled at the times $t_0, t_1, ..., \tau_j = t_{j+1} - t_j$, $\tau_j \leq \tau$, $\tau > 0$, $t_j \to \infty$. Also let the differentiator be applied as

$$\dot{w} = \Omega_{n,n_f}(w, z_0(t_k) - f(t_k), L),\ \dot{z} = D_{n,n_f}(w_1, z, L) \text{ for } t \in [t_k, t_{k+1}),$$

and once more let $\eta_1 + ... + \eta_{n_f} = 0$. Then the standard accuracy (18) is maintained, but for

$$\rho = \max[(\tfrac{\varepsilon_0}{L})^{1/(n+1)}, \tau]. \tag{20}$$

The case $\tau = 0$ formally corresponds here to continuous sampling.

The general case is more complicated, since, for example, a switching signal ± 1 with small integral, can be sampled as $+1$ with large integral. Additional theory and assumptions are employed [37, 38].

3.3. Homogeneous output feedback SMC

The stated SMC problem of the FT exact stabilization of σ is solved by the output feedback SMC

$$\begin{aligned} &\dot{w} = \Omega_{r-1,n_f}(w, z_0 - \sigma, L),\ \dot{z} = D_{r-1,n_f}(w_1, z, L), \\ &u = -\alpha \Psi(z),\ L \geq C + K_M \alpha \end{aligned} \tag{21}$$

for any filtering order $n_f \geq 0$. The proof is trivial, since the separation principle [4] is trivial in our case, and $\sigma \in \mathrm{Lip}_{r-1}(L)$.

Let the sliding variable be sampled in the same way and with the same noise $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$ as in Section 3.2. Then for any sufficiently large $\alpha > 0$ control (21) in FT provides for the accuracy

$$\begin{aligned} &|\sigma_0^{(i)}(t)| \leq \tilde{\mu}_i \rho^{r-i},\ i = 0, 1, ..., r-1, \\ &|w_1(t)| \leq \tilde{\mu}_{w1} L \rho^{r+n_f}, \end{aligned} \tag{22}$$

for the corresponding parameter ρ as in (19) or (20), and for some $\tilde{\mu}_i, \tilde{\mu}_{w1} > 0$ only depending on the parameters $\lambda_0, ..., \lambda_{r+n_f-1}$, L, α, C, K_m, K_M .

Note that the bound L can be very rough (often 50 times larger than required), and the values of C, K_m, K_M are not really needed, since the control parameter α is usually adjusted by simulation.

3.4. Discretization

In reality the system evolves in the continuous time whereas the sampling and the control input are performed and calculated at discrete times. The closed-loop system is necessarily a hybrid one, and the internal dynamics of the differentiator is replaced with some numeric integration of the corresponding differential equations.

Discretization of the output-feedback dynamic control (21) is performed by the simplest one-step Euler discretization with the control and its internal state kept constant over each sampling interval $[t_j, t_{j+1}]$ of the length $\tau_j = t_{j+1} - t_j$.

Denote $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$ for any $\phi(t)$. Then the discrete version of (21) gets the simplest form

$$\begin{array}{l} \delta_j w = \Omega_{r-1,n_f}(w(t_j), z_0(t_j) - \sigma(t_j), L)\tau_j, \\ \delta_j z = D_{r-1,n_f}(w_1(t_j), z(t_j), L)\tau_j, \ L \geq C + K_M \alpha, \\ u(t) = -\alpha \Psi(z(t_j)), \ t \in [t_j, t_{j+1}). \end{array} \tag{23}$$

Here and further the short form $\sigma(t_j)$ is used instead of the complete formula $\sigma(t_j, x(t_j))$.

The realization preserves the same accuracy (22), (20) with possibly changed coefficients $\tilde{\mu}_i, \tilde{\mu}_{wk}$, provided $\eta_1(t) + ... + \eta_{n_f}(t) \equiv 0$, i.e. only the bounded noise is present [41]. In the general case the formula is more complicated.

One can consider providing some time for the differentiator transient before applying the control. Note that the system dynamics (2) are independent of the system engineer, and, therefore, do not undergo discretization.

The stand alone application of the differentiator (17) can also employ the simplest Euler scheme as above, but in that case the accuracy becomes proportional to τ in the absence of noises for constant steps $\tau_j = \tau$, and is proportional to lower powers of τ for variable sampling intervals [48].

The proper discretization of (17) contains additional terms H_n with the powers of τ_j exceeding 1, and takes the form [44]

$$\begin{array}{l} \delta_j w = \Omega_{n,n_f}(w(t_j), z_0(t_j) - f(t_j), L)\tau_j, \\ \delta_j z = D_{n,n_f}(w_1(t_j), z(t_j), L)\tau_j + H_n(z(t_j), \tau_j), \\ H_n(z(t_j), \tau_j) = (H_{n,0}, ..., H_{n,n})^T, \ H_{n,n-1} = H_{n,n} = 0, \\ H_{n,i} = \frac{1}{2!} z_{i+2}(t_j)\tau_j^2 + ... + \frac{1}{(n-i)!} z_n(t_j)\tau_j^{n-i}, \ i = 0, 1, ..., n-2. \end{array} \tag{24}$$

Also see (26) for example. The additional Taylor-like terms H_n are only needed to restore accuracy (18) in the presence of very small noises [48]. Also here the formula (18) remains true when only the noise η_0 is present. The general formula is more complicated. Large α, L naturally require small sampling/integration intervals.

4. Examples

Numeric differentiation is difficult. Consider the simple input signal

$$f(t) = 0.8\cos t - \sin(0.2t) + \nu(t),$$

where $f_0(t) = 0.8\cos t - \sin(0.2t)$, ν is a noise. Obviously $|f_0^{(i)}| \leq 1$ for $i = 1, 2, \ldots$. Let the sampling step be constant, $\tau_j = \tau$. Consider the performance of the most popular differentiation methods.

The simplest method is based on the standard MatLab divided differences. Indeed, it has no transient and works quite well in the absence of noises. The estimation $\hat{f}_0^{(4)}$ of $f_0^{(4)}$ has the accuracy of about 0.01 for $\tau = 10^{-3}$ (Fig. 2a). Unfortunately, in spite of the absence of noises that estimation explodes already for $\tau = 10^{-4}$ due to the digital round-up errors (Fig. 2b). The error is already of the order of $6 \cdot 10^5$ for $\tau = 10^{-5}$.

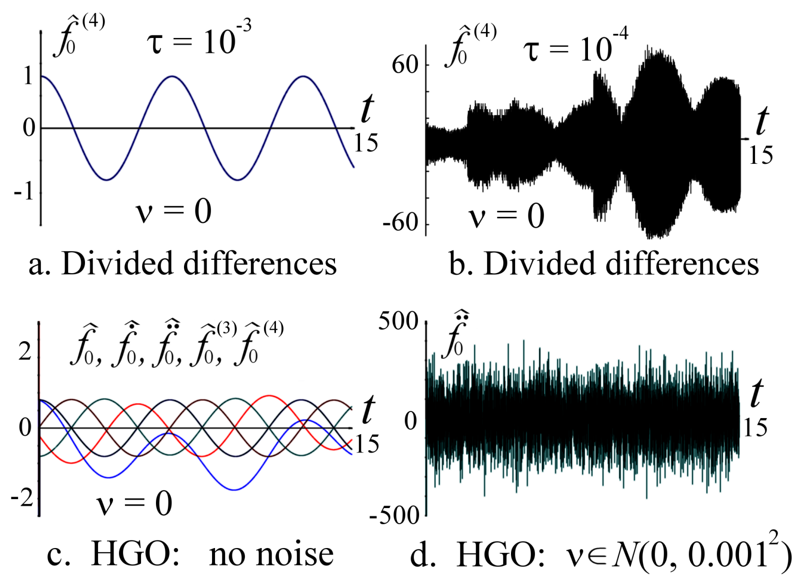

Figure 2: Difficulty of numeric differentiation. a: The divided-differences' estimation of $f_0^{(4)}$ in the absence of noises, $\nu = 0$, for $\tau = 10^{-3}$; b. the same estimation for $\tau = 10^{-4}$. Differentiation by the HGO with the multiple eigenvalue -1000 for $\tau = 10^{-6}$. The graphs are cut from above and from below to remove the high transient values (up to 10^{11}). c: in the absence of noises the accuracy is excellent; d: estimation $\hat{\ddot{f}}_0(t)$ of $\ddot{f}_0(t)$ in the presence of the Gaussian noise $\nu \in N(0, 0.001^2)$.

Another popular tool is the classical linear filter known as the high-gain observer (HGO) [4] with the characteristic polynomial $(p + 1000)^5$. Consider the sampling period $\tau = 10^{-6}$. In the absence of noises the HGO provides for very high accuracy (Fig. 2c). Its best accuracy is obtained for $\tau = 10^{-5}$, $\sup|z_i - f_0^{(i)}| \leq 2.2 \cdot 10^{-15},\ 8.5 \cdot 10^{-12},\ 1.2 \cdot 10^{-8},\ 9.8 \cdot 10^{-6},\ 4.3 \cdot 10^{-3}$ for

$i = 0, 1, 2, 3, 4$ respectively. It remains practically the same for smaller τ and coincides with the best accuracy obtained further by the filtering differentiator.

Unfortunately, in the presence of a small Gaussian noise with the distribution $N(0, 0.001^2)$ the accuracy of the HGO deteriorates to $\sup |z_i - f_0^{(i)}| \leq 3.3 \cdot 10^{-4},\ 0.58,\ 5.7{\cdot}10^2,\ 2.8{\cdot}10^5,\ 5.6{\cdot}10^7$ for $i = 0, 1, 2, 3, 4$ respectively for $\tau = 10^{-6}$ (Fig. 2d). Note [60] that reducing the eigenvalue one could get accuracies similar to those of the SM-based differentiators in the presence of noises not exceeding ± 0.002, but this requires the knowledge of the noise magnitude and deliberately sacrifices the differentiator accuracy in the absence of noises.

SM-based numeric differentiation. Consider the sampled signal

$$f(t) = f_0(t) + \eta(t),\ f_0(t) = 0.5 \sin t + 0.8 \cos(0.8t), \tag{25}$$

where $\eta(t)$ is the noise. Let the sampling interval be constant, $\tau_j = \tau$. The filtering differentiator (24) of the differentiation order $n = 5$ and the filtering order 2,

$$\begin{aligned}
&\delta_j w_1 = [-12L^{1/8} \lfloor w_1(t_j) \rceil^{7/8} + w_2(t_j)]\tau_j,\\
&\delta_j w_2 = [-84.14L^{2/8} \lfloor w_1(t_j) \rceil^{6/8} + z_0(t_j) - f(t_j)]\tau_j,\\
&\delta_j z_0 = [-281.37L^{3/8} \lfloor w_1(t_j) \rceil^{5/8} + z_1(t_j)]\tau_j,\\
&\qquad\qquad\qquad\qquad\qquad + z_2(t_j)\tfrac{\tau_j^2}{2} + z_3(t_j)\tfrac{\tau_j^3}{6} + z_4(t_j)\tfrac{\tau_j^4}{24} + z_5(t_j)\tfrac{\tau_j^5}{120},\\
&\delta_j z_1 = [-455.40L^{4/8} \lfloor w_1(t_j) \rceil^{4/8} + z_2(t_j)]\tau_j + z_3(t_j)\tfrac{\tau_j^2}{2} + z_4(t_j)\tfrac{\tau_j^3}{6} + z_5(t_j)\tfrac{\tau_j^4}{24},\\
&\delta_j z_2 = [-295.74L^{5/8} \lfloor w_1(t_j) \rceil^{3/8} + z_3(t_j)]\tau_j + z_4(t_j)\tfrac{\tau_j^2}{2} + z_5(t_j)\tfrac{\tau_j^3}{6},\\
&\delta_j z_3 = [-88.78L^{6/8} \lfloor w_1(t_j) \rceil^{2/8} + z_4(t_j)]\tau_j + z_5(t_j)\tfrac{\tau_j^2}{2},\\
&\delta_j z_4 = [-14.13L^{7/8} \lfloor w_1(t_j) \rceil^{1/8} + z_5(t_j)]\tau_j,\\
&\delta_j z_5 = [-1.1L\ \mathrm{sign}(w_1(t_j))]\tau_j,
\end{aligned} \tag{26}$$

is applied with $L = 1$, $\tau = 10^{-4}$ and zero initial conditions, $z(0) = 0$, $w(0) = 0$. Obviously $|f_0^{(6)}| \leq L$. The coefficients are taken from Table 1 from line $7 = 5{+}2$.

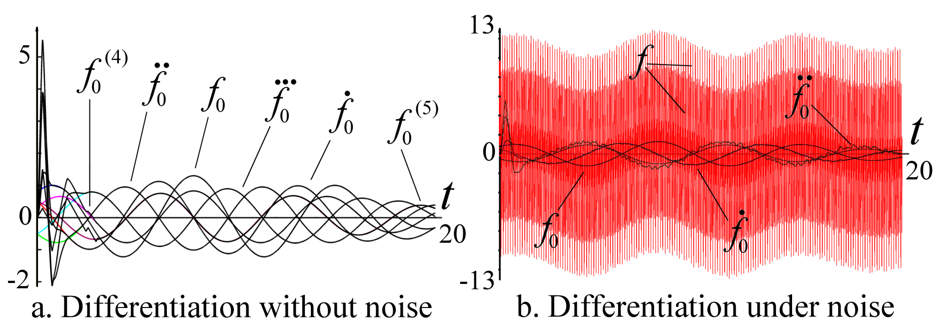

Figure 3: Performance of the filtering differentiator (26) with $n = 5$, $n_f = 2$, $L = 1$ for $\tau = 10^{-5}$ and the input (25). a: There is no noise. b: performance for noisy sampling (27); only estimations of f_0, $\dot f_0$, $\ddot f_0$ are shown.

Performance of the differentiator for $\eta = 0$ is demonstrated in Fig. 3a. Denote $|\Sigma|_{5,2} = (|w_1|, |w_2|, |z_0 - f_0|, ..., |z_5 - f_0^{(5)}|)$. Then the accuracy of the filtering differentiator for $t \in [10, 20]$ is provided by the component-wise inequality $|\Sigma|_{5,2} \le (3.0 \cdot 10^{-23}, 2.4 \cdot 10^{-19}, 1.3 \cdot 10^{-15}, 1.4 \cdot 10^{-12}, 1.2 \cdot 10^{-9}, 5.1 \cdot 10^{-7}, 1.1 \cdot 10^{-4}, 0.012)$. Note that this accuracy is practically the best possible because of the digital round-up errors [48].

Now introduce the noise

$$\begin{aligned} &\eta(t) = 3\cos(10000\,t) - 6\sin(20000\,t) - 4\cos(70000\,t) + \eta_G(t), \\ &\eta_G \in N(0, 0.1^2), \end{aligned} \tag{27}$$

where η_G is a random Gaussian signal with the standard deviation 0.1. The performance of the differentiator is demonstrated in Fig. 3b. The accuracy is provided by the component-wise inequality

$|\Sigma|_{5,2} \le (1.2 \cdot 10^{-5}, 1.8 \cdot 10^{-3}, 0.015, 0.14, 0.60, 1.4, 1.9, 1.1)$.

Car control. Consider a simple "bicycle" kinematic car model [55]

$$\dot{x} = V\cos(\varphi),\ \dot{y} = V\sin(\varphi),\ \dot{\varphi} = \tfrac{V}{\Delta}\tan\theta, \quad \dot{\theta} = u, \tag{28}$$

where x and y are the Cartesian coordinates of the middle point of the rear axle (Fig. 4a), $\Delta = 5m$ is the distance between the two axles, φ is the orientation angle, $V = 10m/s$ is the constant longitudinal velocity, θ is the steering angle (i.e. the actual input), and $u = \dot{\theta}$ is the control.

The goal is to move along some smooth trajectory $(x(t), y(t)) = (x(t), g(t))$, whereas $g(t), y(t)$ are sampled in real time. That is, the task is to make $\sigma = y(t) - g(t)$ vanish. The sliding variable σ is sampled with the time step τ and some noise $\eta(t)$. Let $g(t) = 10\sin(0.05x(t)) + 5$.

Obviously, $\dot{y}$ contains $\sin\varphi$, $\ddot{y}$ contains $\cos\varphi\tan\theta$ and $\dddot{y}$ contains $\frac{\cos\varphi}{\cos^2\theta}u$. Thus, the relative degree is $r = 3$ for $|\varphi| < \pi/2$, $|\theta| < \pi/2$.

Starting from $t = 0$ apply differentiator (24) of the differentiation order $r - 1 = 2$ and the filtering order $n_f = 2$ to the sampled noisy signal $\hat{\sigma}(t_j) = \sigma(t_j) + \eta(t_j)$. From $t = 1$ to $t = 40$ apply the standard 3-SM controller (21), (7)

$$\begin{aligned} &u = -\alpha \tfrac{z_2(t_j)^3 + 2\lfloor z_1(t_j)\rceil^{\frac{3}{2}} + z_0(t_j)}{|z_2(t_j)|^3 + 2|z_1(t_j)|^{\frac{3}{2}} + |z_0(t_j)|},\ t \in [t_j, t_{j+1}), \\ &\delta_j w_1 = [-5L^{1/5}\lfloor w_1(t_j)\rceil^{4/5} + w_2(t_j)]\tau_j, \\ &\delta_j w_2 = [-10.03L^{2/5}\lfloor w_1(t_j)\rceil^{3/5} + z_0(t_j) - \hat{\sigma}(t_j)]\tau_j, \\ &\delta_j z_0 = [-9.30L^{3/5}\lfloor w_1(t_j)\rceil^{2/5} + z_1(t_j)]\tau_j + \tfrac{\tau_j^2}{2} z_2(t_j), \\ &\delta_j z_1 = [-4.57L^{4/5}\lfloor w_1(t_j)\rceil^{1/5} + z_2(t_j)]\tau_j, \\ &\delta_j z_2 = [-1.1L\,\mathrm{sign}(w_1(t_j))]\tau_j. \end{aligned} \tag{29}$$

Due to the homogeneity of the applied output-feedback control (29) the additional term H_2 containing τ_j^2 can be omitted here while still preserving the accuracy (22), (20) [41].

Parameters

$$\alpha = 0.5,\ L = 50$$

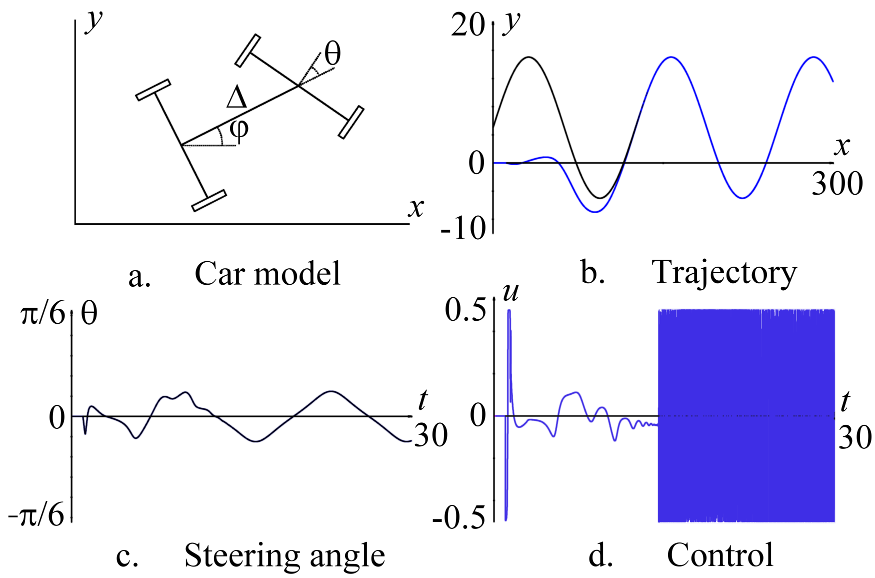

Figure 4: Performance of the 3-SM car control (21) for $r = 3$, $n_f = 2$, $L = 50$, $\alpha = 0.5$, for the integration/sampling step $\tau = 10^{-5}$: a. car model; b. the required and the resulting trajectories; c. steering angle; d. control.

are found by simulation. The integration of the closed-loop system is performed by the Euler method with the time step 10^{-5}.

First consider the case of the "exact" measurements, $\eta = 0$, $\tau_j = \tau = 10^{-5}$. The corresponding performance is shown in Fig. 4. The SM accuracy $|\sigma| \leq 1.1 \cdot 10^{-12} m$, $|\dot\sigma| \leq 1.3 \cdot 10^{-5} m/s$, $|\ddot\sigma| \leq 0.002 m/s^2$ is maintained.

Now introduce the sampling noise (Fig. 5c)

$$\eta(t) = 2\cos(10000\, t) + \eta_G(t),\ \eta_G \in N(0, 0.5^2). \tag{30}$$

The corresponding performance is shown in Fig. 5. The SM accuracy $|\sigma| \leq 0.041m$, $|\dot\sigma| \leq 0.67m/s$, $|\ddot\sigma| \leq 5.2m/s^2$ is maintained for the sampling step $\tau = 10^{-5}s$ (Fig. 5a). The accuracy deteriorates for the sampling step $\tau = 0.01s$ to $|\sigma| \leq 2.8m$, $|\dot\sigma| \leq 2.7m/s$, $|\ddot\sigma| \leq 6.8m/s^2$ (Fig. 5b,d). The performance is still quite acceptable.

4.1. Choice of the filtering order

In general, the higher the filtering order n_f the better accuracy asymptotics of the differentiator (9), (10) one can expect in the presence of noises. In particular, differentiators of higher filtering orders significantly better filter out high-frequency deterministic noises like $\cos(\omega t)$, since their n_f-order integrals decrease as ω^{-n_f}. On the other hand $n_f > 1$ has no advantage compared to $n_f = 1$ for stochastic noises with significant second moments.

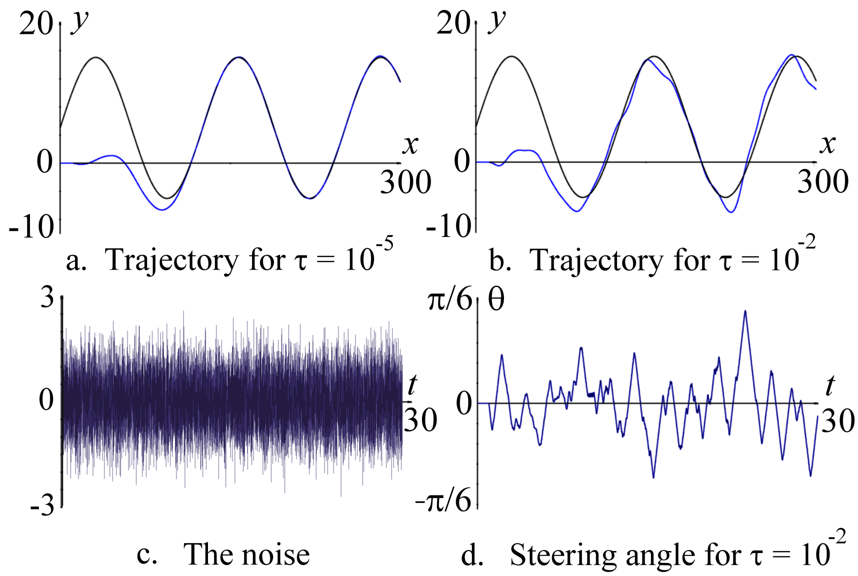

Figure 5: Performance of the 3-SM car control (21) for $r = 3$, $n_f = 2$, $L = 50$, $\alpha = 0.5$, for the noisy sampling with the step τ: a. trajectory for $\tau = 10^{-5}s$; b. trajectory for $\tau = 0.01s$; c. noise (30); d. steering angle for $\tau = 0.01s$.

Also note that the accuracy of the filter (9), (10) in the absence of noises slightly degrades for higher values of n_f due to the natural discrete delay. It results in a bit larger asymptotics coefficients.

5. Summary and Future Directions

Modern methods of SMC solve the SISO regulation problems exactly and in FT, provided the relative degree exists and is known, and the zero dynamics are practically stable. MIMO regulation problem is solvable provided some approximation of the high-frequency matrix is available [15, 46]. Further research is needed to remove this requirement.

Modern SMC methods are easily incorporated in other control approaches due to their accuracy, FT convergence and *smoothness* of the control. SM observers based on the filtering differentiators are capable of replacing expensive sensors in practical applications and signal processing.

SMC systems are proved to be robust with respect to noises, delays, singular and regular perturbations, and even with respect to relative-degree fluctuations [35, 23, 42]. The practical-relative-degree approach [36, 56] is expected to solve numerous problems when the relative degree does not exist or is too large, and even when a control process model is not available.

Lyapunov functions have been recently found for the main known SM controllers and observers [11, 12]. New discretization methods are actively developed. Bihomogeneous SMC allows fast and even fixed-time convergence of the controllers and observers [3, 13, 50, 52, 53, 54]. Different SMC discretization strategies are developed in order to minimize the chattering and improve the accuracy [1, 2].

Discrete SMs have not been considered in this chapter. The readers are referenced to [24, 8, 26] in that aspect. SMC of infinite-dimensional systems is considered by [51].

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