Non-Lyapunov Homogeneous SISO Control Design

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***Abstract*— A simple method of single-input-single-output (SISO) homogeneous control design is proposed which does not employ Lyapunov technique and allows immediate easy construction of infinitely many homogeneous control forms for all relative and homogeneity degrees (HDs). In accordance with the HD the controllers feature finite-time or asymptotic stability, and fixed-time convergence to any vicinity of zero for positive HDs. Many known homogeneous sliding-mode controllers and finite-time stabilizers are obtained as particular cases. Computer simulation illustrates the theoretical results.**

I. Introduction

The classical output-regulation problem is to make the system output y track a real-time command signal $y_c(t)$, i.e. to make $\sigma = y - y_c$ vanish.

Consider an uncertain single-input single-output (SISO) system of a constant relative degree r. Its output dynamics is described by an integrator chain (Brunowsky system) $\sigma^{(r)} = h(t,x) + g(t,x)u$, $b \neq 0$, with uncertain functions h and g depending on the time t and the state x [16].

Denote $\vec{\sigma}_{r-1} = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$. A natural approach is to replace the unknown functions h, g with some *known* sets $H(\vec{\sigma}_{r-1}), G(\vec{\sigma}_{r-1})$, $h \in H$, $g \in G$. It reduces the problem to stabilizing the differential inclusion (DI) $\sigma^{(r)} \in H(\vec{\sigma}_{r-1}) + G(\vec{\sigma}_{r-1})u$. Any stabilizer $u = u(\vec{\sigma}_{r-1})$ is then equivalently effective for the whole class of uncertain systems featuring the same relative degree r and set functions H, G.

The problem can be solved in finite time (FT), or asymptotically, or only approximately due to the presence of noises and time delays. The control design method is often based on the homogeneity theory, whereas the homogeneity is provided by the appropriate choice of H, G and the feedback $u(\vec{\sigma}_{r-1})$. The homogeneity degree (HD) of the DI becomes the main classification parameter.

If $H(0) \neq \{0\}$ the control $u(\vec{\sigma}_{r-1})$ is to be discontinuous at $\vec{\sigma}_{r-1} = 0$. It corresponds to negative HDs and FT stabilization by means of sliding-mode (SM) control for H, G being constant segments. The case $r = 1$ corresponds to the classical (first order) SM [25], [10], whereas $r > 1$ corresponds to rth-order SMs (r-SMs) [4], [13], [7], [9], [14], [17], [18]. The well-known continuous controllers [15] FT stabilize power-integrator chains, and feature $H = \{0\}$ and $G = \{1\}$ in the considered case of pure integrator chains.

Practical stabilization is said to be in fixed time (FxT) [23], if the transient time is uniformly bounded for all initial conditions of $\vec{\sigma}_{r-1}$. Positive HDs are employed if the FxT convergence of system trajectories to a vicinity of zero is required [1], [8], [2], [24], [23], [24]. Classical linear controllers correspond to the HD 0, $H = \{0\}$ and $G = \{1\}$. Classical SM control of linear systems [25] also produces systems of the HD 0 for $H = I||\vec{\sigma}_{r-1}||$, where I and G are some segments.

Whereas most controllers are represented by some formulas (often recursive), the implicit-Lyapunov-function method [24] provides not a control formula, but a numeric procedure for calculating the control at each sampling step.

Each class of controllers usually has at least one paper devoted to it. Formula-based controllers often require complicated design of a Lyapunov function.

The alternative approach [17], [18] involves direct convergence proof. Each control formula $u = u(\vec{\sigma}_{r-1})$ has a number of parameters to be defined. Usually all parameters except the control magnitude gain can be fixed forever for each predefined $r = 1, 2, ...$. In practice $r > 4$ almost never appears, and the cases $r = 1, 2$ are very simple. Thus the number of needed parameters varies from 1 to 3.

Even if the Lyapunov analysis is available, the calculation of these parameters is cumbersome and often leads to excessively large numbers (thousands instead of tens). Therefore, the main practical method of determining the parameters is by numeric simulation.

The non-Lyapunov-based methods [9], [19], [22] are extended in this paper to allow easy construction of infinitely many new controller forms/templates for any relative and homogeneity degrees. In particular, many known homogeneous controllers of the form $u = u(\vec{\sigma}_{r-1})$ are obtained. The lacking parameters' values are to be found by simulation.

The method is presented for the perturbed integrator chains, but is easily extendible to the power-integrator chains as well. Simulation shows the method effectiveness.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b | \ a \in A, b \in B\}$. A function of a set is the set of function values on this set. The norm $||x||$ stays for the standard Euclidian norm of x, $B_\varepsilon = \{x \mid ||x|| \leq \varepsilon\}$; $||x||_h$ is a homogeneous norm; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$.

II. Coordinate homogeneity basics

Recall that solutions of the DI

$$\dot{x} \in F(x), \ F(x) \subset T_x\mathbb{R}^n, \qquad (1)$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all t. Here $T_x\mathbb{R}^n$ denotes the space of tangent vectors to $\mathbb{R}^n$ at the point x.

We call the DI (1) *Filippov DI*, if the vector-set field $F(x) \subset T_x\mathbb{R}^n$ is non-empty, compact and convex for any

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x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

Filippov DIs feature most standard properties (existence, extendability of solutions, etc.), but not the uniqueness of solutions [12]. DI (1) appears if an uncertain dynamic system $\dot{x} = \hat{f}(t,x)$ is considered, $\hat{f}(t,x) \in F(x)$. Also solutions of the differential equation (DE)

$$\dot{x} = f(x),\ f : \mathbb{R}^n \to T\mathbb{R}^n, \tag{2}$$

are defined as solutions of the Filippov DI corresponding to the discontinuous vector field $f(x)$ [12].

A. Homogeneity notions

Introduce the weights $m_1, ..., m_n > 0$ of the coordinates $x_1, ..., x_n$ in $\mathbb{R}^n$, $\deg x_i = m_i$, and the dilation [3]

$$d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n),$$

where $\kappa \geq 0$. Recall [3] that a function $g : \mathbb{R}^n \to \mathbb{R}^m$ is said to have the HD (weight) $q \in \mathbb{R}$, $\deg g = q$, if the identity $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^n$ and $\kappa > 0$.

We distinguish a vector function $f : \mathbb{R}^n \to \mathbb{R}^n$, $f : x \mapsto f(x) \in R^n$, and a vector field $f : \mathbb{R}^n \to T\mathbb{R}^n$, $f : x \mapsto f(x) \in T_x\mathbb{R}^n$. The vector field $f(x) \in T_x\mathbb{R}^n$ (vector-set field $F(x) \subset T_x\mathbb{R}^n$) does not in fact differ from the DE (2) (DI (1)).

Contrary to the case of a function, DI (1) (set field F) implicitly involves time in the derivative. Consider the combined time-coordinate transformation

$$(t, x) \mapsto (\kappa^{-q} t, d_\kappa x), \quad \kappa > 0, \tag{3}$$

where the number $-q \in \mathbb{R}$ might naturally be considered as the weight of t. The DI $\dot{x} \in F(x)$ and the vector-set field $F(x)$ are called homogeneous of the HD q, if the DI is invariant with respect to (3).

The respective homogeneity property can be rewritten as $\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^{-q} t)} \in F(d_\kappa x)$. Thus we come to the equivalent formal definition.

Definition 1: [18] A vector-set field $F(x) \subset T_x\mathbb{R}^n$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^n$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$.

The DE (2), $\dot{x}_i = f_i(x)$, is a particular case of DI, when the set $F(x)$ contains only one vector $f(x)$. Then the above definition is reduced to the standard definition $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg f_i$ [3]. Note that if f is discontinuous, (2) is equivalent to the corresponding homogeneous Filippov DI (1).

The identity $\kappa^{m_i} x_i = (\kappa^{1/\gamma})^{m_i \gamma} x_i$, $\gamma > 0$, shows that the weights $-q, m_1, ..., m_n$ are defined up to proportionality. In particular, the non-zero HD q of a vector-set field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, ..., m_n$. The sign of the HD determines many properties of DIs.

Any continuous positive-definite function of the HD 1 is called a homogeneous norm. We denote it $||x||_h$. A function which is a homogeneous norm for the weight vector $m \in \mathbb{R}^n$ is not a homogeneous norm under another equivalent weight distribution γm, $\gamma \neq 1$, $\gamma > 0$. The homogeneous norm

$$||x||_{h\varpi} = \left(\sum_i |x_i|^{\frac{\varpi}{m_i}} \right)^{\frac{1}{\varpi}},\ \varpi > 0. \tag{4}$$

is continuously differentiable for $x \neq 0$ if $\varpi > \max m_i$. Note that each two homogeneous norms $|| \cdot ||_{h_1}$, $|| \cdot ||_{h_2}$ are equivalent in the sense that there exist $\gamma_1 \geq \gamma_2 > 0$, such that $\gamma_2 ||x||_{h_1} \leq ||x||_{h_2} \leq \gamma_1 ||x||_{h_1}$.

A function $f(x)$ is called *quasi-continuous* [19] if it is continuous everywhere except possibly at $x = 0$.

B. Stability of homogeneous inclusions

Let the DI (1) be an asymptotically stable (AS) homogeneous Filippov DI. In particular, it implies that $m_i + q \geq 0$ for all $i = 1, ..., n$ [21]. In the following the DI has equilibrium at zero, and only the strong stability of zero is studied.

DI (1) is called *finite-time stable* (FTS) if it is AS, and each solution converges to zero in FT. DI (1) is called *fixed-time* (FxT) *attracted* [23], [24] to some vicinity Ω of zero, if each solution converges into Ω in FT, and the transient times posses a common finite upper bound.

Definition 2 ([18]): A set $D \subset \mathbb{R}^n$ is called *dilation retractable* if $\forall \kappa \in [0, 1]$ $d_\kappa D \subset D$. A homogeneous DI (1) is called *contractive*, if there exist two nonempty compact sets D_1, D_2 and $T > 0$ satisfying the following conditions. The set D_1 is dilation retractable, D_2 lies in the interior of D_1, and each solution which starts in D_1 at time $t = 0$ is in D_2 at $t = T$.

Note that any ball $B_R = \{x \in \mathbb{R}^n, ||x|| \leq R\}$ is dilation retractable. The following Theorem [20] summarizes stability features of DIs with arbitrary homogeneous degrees.

Theorem 1: A homogeneous Filippov DI (1) *is AS iff it is contractive. Moreover:*

1) *If the HD is negative, asymptotic stability is equivalent to FT stability.*
2) *If the HD is zero, asymptotic stability is exponential.*
3) *If the HD is positive, any open vicinity of zero attracts solutions in FxT. The convergence to zero is slower than exponential.*

The following theorem [20] establishes the method of local homogeneous approximations [1], [3].

Theorem 2: Let $\dot{x} \in \tilde{F}(x)$ be any other homogeneous Filippov DI with the same HD and dilation. Then for any homogeneous norm $|| \cdot ||_h$ there exists such $\delta > 0$ that the inclusion $\tilde{F}(x) \subset F(x) + B_\delta$ on $||x||_h = 1$ necessarily implies the asymptotic stability of the DI $\dot{x} \in \tilde{F}(x)$.

III. Homogeneous SISO control design

A. Problem statement

Consider a dynamic system of the form

$$\dot{x} = a(t, x) + b(t, x)u,\ \ \sigma = \sigma(t, x), \tag{5}$$

where $x \in \mathbb{R}^n$, $a : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $b : \mathbb{R}^{n+1} \to \mathbb{R}^{n \times m}$ and $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}$ are uncertain smooth functions, $u \in \mathbb{R}$ is the control. The output function σ can be considered as a tracking deviation. For simplicity we assume that any solution of (5) is forward complete, i.e. indefinitely extended in time, provided the control remains bounded along the solution trajectory.

The system is assumed to have the known relative degree r [16]. Respectively,

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{6}$$

holds, where $g \neq 0$ [16]. The functions $h(t,x)$ and $g(t,x)$ are unknown and smooth.

Denote $\vec{\sigma}_k = (\sigma, \dot{\sigma}, \ldots, \sigma^{(k)}) \in \mathbb{R}^{k+1}$. The task is to establish and keep $\sigma \equiv 0$ by means of a feedback control

$$u = u_*(\vec{\sigma}_{r-1}) \in U(\vec{\sigma}_{r-1}), \tag{7}$$

where u_* is a Lebesgue-measurable locally bounded function, and $U(\cdot)$ is the corresponding Filippov set-function [12]. Recall that $U(\vec{\sigma}_{r-1}) \subset T_{\vec{\sigma}_{r-1}}\mathbb{R}^r$ is nonempty, convex, compact and $U(\cdot)$ is upper semi-continuous.

Let $\deg \sigma^{(i)} = m_i > 0$, the system HD $q \in \mathbb{R}$ and the corresponding dilation be as follows:

$$\begin{array}{l} m_i = 1 + iq,\ m_r = 1 + rq \geq 0,\ i = 0, 1, ..., r, \\ d_\kappa \vec{\sigma}_{r-1} = (\kappa^{m_0}\sigma, \kappa^{m_1}\dot{\sigma}, ..., \kappa^{m_{r-1}}\sigma^{(r-1)}). \end{array} \tag{8}$$

The weight $m_0 = 1$ is taken fixed, since the weights are defined up to proportionality. Obviously $m_i > 0$ for $i < r$. The HD $q \in \mathbb{R}$ is the design parameter, $q \geq -1/r$.

Fix some homogeneous norm $||\cdot||_h$. Assume that

$$\begin{array}{l} |h(t,x)| \leq C||\vec{\sigma}_{r-1}||_h^{m_r},\ C \geq 0, \\ 0 < K_m \leq g(t,x) \leq K_M. \end{array} \tag{9}$$

Thus any solution of (5), (7), (9) satisfies the Filippov differential inclusion

$$\sigma^{(r)} \in [-C, C]\, ||\vec{\sigma}_{r-1}||_h^{m_r} + [K_m, K_M]U(\vec{\sigma}_{r-1}). \tag{10}$$

According to Section II-A the closed-loop inclusion (10) is homogeneous iff $\deg U = m_r$, i.e. if

$$\forall \kappa > 0\ u_*(\vec{\sigma}_{r-1}) \equiv \kappa^{-m_r} u_*(d_\kappa \vec{\sigma}_{r-1}). \tag{11}$$

The task is to construct homogeneous controllers (7), (11) which asymptotically stabilize (10) and the output σ of the system (5) as a result. *Analysis, tuning and comparison of the obtained controllers is out of the scope of this paper.*

B. Design method

Obviously not all systems satisfy conditions (9) for some q. It determines the applicability of the method to each concrete problem.

In particular $m_r = 0$ corresponds to $q = -1/r < 0$, $\deg u = 0$ and boundedness of h. The corresponding controllers are inevitably discontinuous at $\vec{\sigma}_{r-1} = 0$ for any $C > 0$, respectively DI (10) is stabilized in FT [19]. The motion of (5) on the set $\vec{\sigma}_{r-1} = 0$ is called rth-order SM (r-SM), and the controllers are called r-sliding homogeneous [18]. Conditions (9) hold at least locally for any smooth system (5), (6). Derivatives $\vec{\sigma}_{r-1}$ are optionally calculated in FT by homogeneous robust exact differentiator [17].

The case $m_r > 0$, i.e. $q > -1/r$, usually requires that the output dynamics be separated from the rest of the system. Otherwise conditions (9) become unrealistic. The method is applicable for stabilization of feedback-linearizable SISO systems [16] at their equilibria by continuous control. Note that neither the system nor the corresponding linearizing transformation are to be known exactly, since at least locally derivatives $\vec{\sigma}$ can be exactly calculated in real time [17].

The case $-1/r \leq q < 0$ is used to get the FT stability [1], [6], [15]. The choice $q = 0$ also includes the classical linear control theory methods and implies exponential convergence, $q > 0$ ensures asymptotic stability and the FxT convergence to any vicinity of the equilibrium [1], [23], [5], [24].

There are a number of homogeneous SM controllers solving the problem in the case $m_r = 0$, $q = -1/r < 0$, $\deg u = 0$. The recently established powerful method [7] exploits the knowledge of a concrete control Lyapunov function for the system $\sigma^{(r)} = u$ to generate a number of r-SM controllers. Synthesis of new control Lyapunov functions becomes the initial non-trivial step.

The alternative method [22], [9] starts from the knowledge of a homogeneous AS DE. It removes any control differentiability conditions and respectively yields more controllers. The corresponding AS DEs are constructed below for any r and q in a simple way.

1) ***Control construction****:* Consider an AS homogeneous DE

$$\sigma^{(r-1)} + \tilde{\varphi}(\vec{\sigma}_{(r-2)}) = 0, \tag{12}$$

where $\tilde{\varphi}$ is a continuous function, $\deg \tilde{\varphi} = m_{r-1}$. Let it be "algebraically" equivalent to the homogeneous DE

$$\varphi(\vec{\sigma}_{r-1}) = 0,\ \deg \varphi = k_s \geq 0, \tag{13}$$

which means that (13) and (12) define the same set in the space $\vec{\sigma}_{r-1} \in \mathbb{R}^r \backslash \{0\}$. Also assume that φ is quasi-continuous, and $\varphi(\vec{\sigma}_{r-1}) > 0 \Leftrightarrow \sigma^{(r-1)} + \tilde{\varphi}(\vec{\sigma}_{r-2}) > 0$.

Theorem 3: Let $N(\vec{\sigma}_{r-1})$ be any positive-definite continuous homogeneous function, $\deg N = 1$ *(i.e. N is a homogeneous norm). Consider the homogeneous controls*

$$u = -\alpha[N(\vec{\sigma}_{r-1})]^{m_r - \deg \varphi}\varphi(\vec{\sigma}_{r-1}), \tag{14}$$

$$u = -\alpha[N(\vec{\sigma}_{r-1})]^{m_r}\operatorname{sign}\varphi(\vec{\sigma}_{r-1}). \tag{15}$$

Then under conditions (8), (9) *DI* (10) *is homogeneous, and controllers* (14), (15) *asymptotically stabilize DI* (10) *for any sufficiently large $\alpha > 0$. The output σ of system* (5) *closed by these controls is asymptotically stabilized.*

Controller (14) *is continuous for $q > -1/r$. For $q = -1/r$ it is quasi-continuous (i.e. discontinuous only at $\vec{\sigma}_{r-1} = 0$).*

DI (10) *is FT stable if $q < 0$, and exponentially stable for $q = 0$. If $q > 0$ any open vicinity of $\vec{\sigma}_{r-1} = 0$ attracts solutions in FxT.*

In the case of controller (14), $q > -1/r$, $m_r - \deg \varphi < 0$, $u(0) = 0$ is assigned, which removes the discontinuity at 0. Multiplication of controllers by any locally bounded Lebesque-measurable function $k(t,x) \geq 1$ does not interfere with the convergence.

The proofs are provided in Section IV. Note that the peculiarity of the quasi-continuous SM controller (14) (the case $q = -1/r$, $m_r = 0$) is that it features significantly less chattering than other SM controllers, (15) in particular [19]. Another remark is that although controller (15) looks as a classical SM controller, in general it cannot establish the SM $\varphi(\vec{\sigma}_{r-1}) = 0$ due to the non-smoothness and infinite gradients of $\varphi(\vec{\sigma}_{r-1})$.

*2) **Recursion in** r:* An AS DE (12) is assumed known. Let $m_r > 0$, i.e. $q > -1/r$, otherwise no recursion is possible.

Choose two arbitrary homogeneous norms $||\cdot||_{h_1}, ||\cdot||_{h_2}$ and $m_\varphi > 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be any continuous strictly growing function, $\phi(0) = 0$. Then the following are some possible simple options for choosing φ:

$$\begin{array}{ll} 1. & \varphi = \lfloor \sigma^{(r-1)} + \tilde{\varphi} \rceil^{m_\varphi},\ m_\varphi > 0, \\ 2. & \varphi = \lfloor \sigma^{(r-1)} \rceil^{m_\varphi} + \lfloor \tilde{\varphi} \rceil^{m_\varphi}, \\ 3. & \varphi = \phi\left(\frac{\lfloor \sigma^{(r-1)} + \tilde{\varphi} \rceil^{m_\varphi}}{||\vec{\sigma}_{r-1}||_{h_1}^{m_\varphi m_{r-1}}} \right),\ \deg \varphi = 0. \end{array} \tag{16}$$

More options are available in (16), e.g. algebraic combinations of 1-3 are possible, etc. Obviously the number of such constructions is infinite at each recursion step for $r \geq 2$.

Now, taking $C = 0$, $K_m = K_M = 1$ and sufficiently large $\alpha > 0$, due to Theorem 3 obtain the new homogeneous AS DE of the order r

$$\sigma^{(r)} + \alpha ||\vec{\sigma}_{r-1}||_{h_2}^{m_r - \deg \varphi} \varphi(\vec{\sigma}_{r-1}) = 0 \tag{17}$$

from equation (12) of the order $r - 1$,

C. Demonstration of recursive homogeneous control design

Start with $r = 1$ and any $q \geq -1$. The weight $\deg \sigma = 1$ is fixed in advance. If $q = -1$ then

$$\dot{\sigma} \in [-C, C] + [K_m, K_M]u_1,\ u_1 = -\alpha \operatorname{sign} \sigma,\ \alpha > \tfrac{C}{K_m},$$

is the only option. One can also take different values of α for $\sigma > 0$ and $\sigma < 0$.

Let $q > -1$ then the only AS homogeneous DE is

$$\dot{\sigma} + \beta \lfloor \sigma \rceil^{1+q} = 0,\ \beta > 0. \tag{18}$$

It is FT stable for $-1 < q < 0$, exponentially stable for $q = 0$ and AS for $q > 0$. In the latter case convergence into any vicinity of 0 is in FxT. Also here one can take different values of β for $\sigma > 0$ and $\sigma < 0$.

Provided $q \geq -0.5$ the recursion (16), (17) is possible to $r = 2$. In particular, for $q = -0.5$ Theorem 3 produces

$$u_2 = -\alpha ||\vec{\sigma}_1||_h^{1+2q} \operatorname{sign}(\lfloor \dot{\sigma} \rceil^{\frac{1}{1+q}} + \beta^{\frac{1}{1+q}} \sigma),\ q \geq -\tfrac{1}{2},$$

from the FT stable DE (18). For this end one takes controller (15) and option 2 from (16). Replacing the multiplier $||\vec{\sigma}_1||_h^{1+2q}$ with a non-vanishing function get the non-singular terminal 2-SM control [11]. Note that (14) produces a quasi-continuous 2-SM controller.

Some other possible continuous controllers for $q > -0.5$ and $r = 2$ are of the form

$$\begin{array}{l} u_2 = -\alpha \left\lfloor \lfloor \dot{\sigma} \rceil^{\frac{1}{1+q}} + \beta_0 \sigma \right\rceil^{1+2q}, \\ u_2 = -\alpha (|\sigma| + |\dot{\sigma}|^{\frac{1}{1+q}})^q (\dot{\sigma} + \beta_0 \lfloor \sigma \rceil^{1+q}), \\ u_2 = -\alpha (\lfloor \dot{\sigma} \rceil^{\frac{1+2q}{1+q}} + \beta_0 \lfloor \sigma \rceil^{1+2q}), \\ u_2 = -\alpha \left\lfloor \lfloor \dot{\sigma} \rceil^{\frac{1+3q}{1+q}} + \beta_0 \lfloor \sigma \rceil^{1+3q} \right\rceil^{\frac{1+2q}{1+3q}} \text{ for } q > -\tfrac{1}{3}, \end{array} \tag{19}$$

where $\beta_0 > 0$ is any number, $\alpha > 0$ is sufficiently large. Using the third controller for $C = 0$, $K_m = K_M = 1$ get the new AS DE

$$\ddot{\sigma} + \tilde{\beta}_1 \lfloor \dot{\sigma} \rceil^{\frac{1+2q}{1+q}} + \tilde{\beta}_0 \lfloor \sigma \rceil^{1+2q} = 0, \tag{20}$$

where $\tilde{\beta}_1 = \alpha$ and $\tilde{\beta}_0 = \alpha \beta_0$. That is only one of many possibilities.

Taking the last controller of (19) obtain for $q > -1/3$ the AS DE $\ddot{\sigma} + u_2(\vec{\sigma}_1) = 0$. Then using the 2nd recursion option of (16) with $m_\varphi = (1 + 3q)/(1 + 2q)$ get

$$u_3 = -\alpha [\lfloor \ddot{\sigma} \rceil^{\frac{1+3q}{1+2q}} + \beta_1 \lfloor \dot{\sigma} \rceil^{\frac{1+3q}{1+q}} + \beta_0 \lfloor \sigma \rceil^{1+3q}], \tag{21}$$

$$\dddot{\sigma} + \tilde{\beta}_2 \lfloor \ddot{\sigma} \rceil^{\frac{1+3q}{1+2q}} + \tilde{\beta}_1 \lfloor \dot{\sigma} \rceil^{\frac{1+3q}{1+q}} + \tilde{\beta}_0 \lfloor \sigma \rceil^{1+3q} = 0. \tag{22}$$

Controller (21) generates AS DE (22) for $q > -1/3$, which in its turn produces many more controllers, etc.

Asymptotic stabilizers of the form (21) (DE (22)) are obtained for any r and $q > -1/r$. Note that in [6] they are developed for $q < 0$ close to 0. Similarly the controller

$$u = -\alpha \left\lfloor \lfloor \sigma^{(r-1)} \rceil^{\frac{\omega}{1+(r-1)q}} + \beta_{r-2} \lfloor \sigma^{(r-2)} \rceil^{\frac{\omega}{1+(r-2)q}} + ... + \beta_0 \lfloor \sigma \rceil^{\omega} \right\rceil^{\frac{1+rq}{\omega}} \tag{23}$$

is shown to be valid for any $\omega > 0$. In the case $q = -1/r$ it becomes a relay discontinuous controller, whereas the following is the quasi-continuous version for $q = -1/r$:

$$u = -\alpha \frac{\lfloor \sigma^{(r-1)} \rceil^{\frac{\omega}{1+(r-1)q}} + \beta_{r-2} \lfloor \sigma^{(r-2)} \rceil^{\frac{\omega}{1+(r-2)q}} + ... + \beta_0 \lfloor \sigma \rceil^{\omega}}{|\sigma^{(r-1)}|^{\frac{\omega}{1+(r-1)q}} + \beta_{r-2} |\sigma^{(r-2)}|^{\frac{\omega}{1+(r-2)q}} + ... + \beta_0 |\sigma|^{\omega}}. \tag{24}$$

Similar results are obtained in [7], [9] for $q < 0$. In particular, for $r = 5$, $q = -0.2$, $\omega = 0.2$ obtain the following quasi-continuous 5-SM controller (24) [9]:

$$u = -\alpha \frac{\sigma^{(4)} + 6 \lfloor \dddot{\sigma} \rceil^{\frac{1}{2}} + 5 \ddot{\sigma}^{\frac{1}{3}} + 3 \lfloor \dot{\sigma} \rceil^{\frac{1}{4}} + \sigma^{\frac{1}{5}}}{|\sigma^{(4)}| + 6 |\dddot{\sigma}|^{\frac{1}{2}} + 5 |\ddot{\sigma}|^{\frac{1}{3}} + 3 |\dot{\sigma}|^{\frac{1}{4}} + |\sigma|^{\frac{1}{5}}}.$$

The coefficients are found one by one by simulation according to the design recursion steps. Once one valid parametric set $\{\beta_j\}$ is found, the convergence rate is easily regulated in a standard way by changing β_j [9]. The accuracy of the controllers for any q in the presence of noises and discrete sampling is calculated in the recent paper [20].

IV. Proof of Theorem 3

Due to the lack of place the proofs are concise. The following lemmas are used in the proof of Theorem 3.

Lemma 1: In the notation of Section II let quasi-continuous functions $\phi_1(x), \phi_2(x)$ of the weight 0, $\deg \phi_1 = \deg \phi_2 = 0$, have the same zero-set, $\phi_1(x) = 0 \iff \phi_2(x) = 0$. Then for any $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\{x \in \mathbb{R}^n \backslash \{0\} |\, |\phi_1(x)| \leq \varepsilon\} \subset \{x \in \mathbb{R}^n \backslash \{0\} |\, |\phi_2(x)| \leq \delta\}.$$

Proof. Suppose that such $\varepsilon > 0$ does not exist for some $\delta > 0$. Choose some homogeneous norm $||\cdot||_h$, and consider the homogeneous sphere $S_{h1} = \{x \in \mathbb{R}^n \,|\, ||x||_h = 1\}$. Due to the homogeneity functions ϕ_1, ϕ_2 take all their values on S_{h1}.

Take some sequence $\varepsilon_j \to 0$, $\varepsilon_j > 0$. According to our assumption their exists a sequence $x_j \in S_{h1}$, such that $|\phi_2(x_j)| > \delta$ and $|\phi_1(x_j)| \le \varepsilon_j$. Since S_{h1} is compact, there exists a convergent subsequence $x_{j_l} \to x_* \in S_{h1}$. Obviously, $\phi_1(x_*) = 0$ due to the continuity of ϕ_1, and $\phi_2(x_*) \ge \delta$ due to the continuity of ϕ_2. It contradicts the lemma condition. □

Lemma 2: [9] Let $B \ge 0$, $|\theta| \le 1$, $0 \le \xi < 1$. Then the inequality $\frac{|A+B\theta|}{|A|+B} \le \xi$ implies that $|A + B\theta| \le \frac{2\xi}{1-\xi} B$.

We call a set homogeneous if it is invariant with respect to the dilation. The inequality $|\Psi(\vec{\sigma}_{r-1})| \le \delta$, $\Psi = \varphi/N^{k_s}$, $\deg \varphi = k_s$, describes a homogeneous region in the $\vec{\sigma}_{r-1}$ space and the Filippov DI of the order $r-1$. Fix some $\varpi > 1$ (see (4)).

Lemma 3: Under the conditions of Theorem 3 for any sufficiently small $\delta > 0$ the homogeneous region $|\Psi(\vec{\sigma}_{r-1})| \le \delta$ is a subset of another homogeneous region

$$|\sigma^{(r-1)} + \tilde{\varphi}(\vec{\sigma}_{r-2})| \le M \tfrac{2\varepsilon}{1-\varepsilon} ||\vec{\sigma}_{r-2}||_{h\varpi}^{k_s}, \tag{25}$$

where $\varepsilon = \varepsilon(\delta)$, $\varepsilon < 1$, $\lim_{\delta \to 0} \varepsilon(\delta) = 0$, $M > 0$, and (25) is a homogeneous AS DI in the space $\vec{\sigma}_{r-2}$.

Proof. Let $|\tilde{\varphi}(\vec{\sigma}_{r-2})|/||\vec{\sigma}_{r-2}||_{h\varpi}^{k_s} \le M$. Such number M exists, since $\tilde{\varphi}(\vec{\sigma}_{r-2})/||\vec{\sigma}_{r-2}||_{h\varpi}^{k_s}$ is a function of the weight 0 continuous on any sphere. Denote $N_1(\vec{\sigma}_{r-1}) = |\sigma^{(r-1)}| + M||\vec{\sigma}_{r-2}||_{h\varpi}^{k_s}$. According to Lemma 1 the DI $|\Psi(\vec{\sigma}_{r-1})| \le \delta$ implies $|\sigma^{(r-1)} + \tilde{\varphi}(\vec{\sigma}_{r-2})|/N_1(\vec{\sigma}_{r-1}) \le \varepsilon$. It in its turn implies (25) (Lemma 2). DI (25) is FTS for $\varepsilon = 0$, therefore it is also AS for any sufficiently small ε (Theorem 2). □

Fix such value of ε. We are still allowed to decrease δ while preserving ε and the asymptotic stability of (25).

Since the boundaries of (25) are not smooth, (25) is replaced with a smaller internal homogeneous region Ω_ε having smooth boundaries with locally bounded gradients.

Lemma 4: Under the conditions of Lemma 3 the homogeneous region (25) contains another homogeneous set Ω_ε,

$$\Omega_\varepsilon = \{\vec{\sigma}_{r-1} |\ \Phi_-(\vec{\sigma}_{r-2}) \le \sigma^{(r-1)} \le \Phi_+(\vec{\sigma}_{r-2})\},$$

where the homogeneous functions Φ_+, Φ_- are smooth everywhere except 0. The region Ω_ε is forward invariant for sufficiently large α and contains the set $|\Psi(\vec{\sigma}_{r-1})| \le \delta$ for any sufficiently small δ.

Proof. The proof is standard and follows [19], [22], [9]. The boundaries of the region (25) are continuous and are approximated over the sphere $||\vec{\sigma}_{r-2}|| = 1$ by smooth functions $\tilde{\Phi}_+$, $\tilde{\Phi}_-$ so that the needed inclusions are satisfied over the sphere. Then the functions $\tilde{\Phi}_+$, $\tilde{\Phi}_-$ are extended by homogeneity to Φ_+, Φ_- over the whole space $\vec{\sigma}_{r-1}$.

Outside of the region $|\Psi(\vec{\sigma}_{r-1})| \le \delta$ the control can be done large: control (15) satisfies $u = \pm\alpha$, whereas control (14) satisfies $|u| \ge \alpha\delta$.

Since boundaries of (25) are not smooth, (25) is replaced with a smaller internal homogeneous region Ω_ε having smooth boundaries with bounded gradients of the weight 0. Then the trajectories are shown to cross the boundaries of Ω_ε only into Ω_ε. Respectively, Ω_ε is an AS invariant attractor. The value of δ is decreased to keep the set $|\Psi(\vec{\sigma}_{r-1})| \le \delta$ inside Ω_ε, whereas ε remains untouched, and α is increased keeping a constant value $u_0 = \alpha\delta$. Hence, $|u| \ge u_0$ holds outside of Ω_ε. □

Proof of Theorem 3. Let $B_{hR} = \{||\vec{\sigma}_{r-1}||_h \le R\}$ be the homogeneous ball. Show the contractivity property of the closed-loop DI.

Note that the region (25) corresponds to the AS DI (Lemma 3). Moreover, the region Ω_ε is forward-time invariant: no trajectory can leave it. The standard proof [19], [22], [9] shows that for $q < 0$ the set Ω_ε is a FT attractor, which would finish the proof in that case.

Consider the functions $\pi_+ = \sigma^{(r-1)} - \Phi_+$, $\pi_- = \sigma^{(r-1)} - \Phi_-$. Due to the homogeneity $|\dot{\Phi}_+|, |\dot{\Phi}_-| \le k_N N^{m_r}$ for some $k_N > 0$. Consider the case $\pi_+ > 0$. The case $\pi_- < 0$ is similar. Calculations show [9] that

$$\dot{\pi}_+ \le -(\alpha\delta - k_N - \gamma_c C) N^{m_r} \tag{26}$$

holds outside of Ω_ε for some $\gamma_c > 0$. If π_+ vanishes, the trajectory enters Ω_ε and then converges to zero.

It is easy to see that for any large α the trajectories starting from the layer $||\vec{\sigma}_{r-1}||_h \in [0.5, 1]$ remain uniformly bounded and uniformly separated from zero during some time interval T also independent of α. Indeed, if the trajectory enters Ω_ε, the statement is trivial. Otherwise, for any large bounds and large enough α the uniform boundedness of σ_{r-1} follows from (26) and $\pi_+ > 0$.

Thus it follows from (26) that for sufficiently large α all trajectories starting in the layer $||\vec{\sigma}_{r-1}||_h \in [0.5, 1]$ enter Ω_ε in the time T. Taking all trajectories' segments starting in the ball B_{h1} over the time interval T and adding to it $\bar{\Omega}_\varepsilon = \Omega_\varepsilon \cap B_{hR}$ for sufficiently large R obtain a forward-invariant compact dilation-retractable set D. Fix such α.

Due to the homogeneity of the DI also $d_{1/2}D$ is an invariant set, $d_{1/2}D \subset D$. Moreover all trajectories starting in the layer $||\vec{\sigma}_{r-1}||_h \in [0.25, 0.5]$ enter $d_{1/2}\bar{\Omega}_\varepsilon$ in the time $2^q T$. Due to the asymptotic stability of (25) all solutions starting in $\bar{\Omega}_\varepsilon$ concentrate in $d_{1/2}\bar{\Omega}_\varepsilon$ in some time T_0 [20].

Take $T_1 = \max(T_0, 2^q T, T)$. Obviously all trajectories starting in D at time $t = 0$ gather in $d_{1/2}D$ at the time T_1. It proves the contractivity property. □

V. Simulation results

Consider an academic example of the HD q

$$\sigma^{(r)} = \cos(11t)|\sigma|^{1+qr} + (2 + \sin(5t))u, \tag{27}$$

which is understood in the Filippov sense [12]. The solutions obviously satisfy the DI

$$\begin{aligned} &\sigma^{(r)} \in [-1,1]||\vec{\sigma}_{r-1}||_{h\omega}^{1+qr} + [1,3]u, \\ &||\vec{\sigma}_{r-1}||_{h\omega} = \left(|\sigma|^\omega + ... + |\sigma^{(r-1)}|^{\frac{\omega}{1+(r-1)q}}\right)^{\frac{1}{\omega}}, \\ &\omega = \max[1, 1 + q(r-1)]. \end{aligned} \tag{28}$$

The system satisfies assumptions of the Section III-A for $q \ge -1/r$. Choose the controller (23) for $q > -1/r$. In the case $q = -1/r$ the quasi-continuous SM control (24) is chosen.

Let $r = 3$, $\beta_0 = 1$, $\beta_1 = 2$, $\alpha = 5$ and $\vec{\sigma}_2(0) = (100, 100, -100)$. The Euler integration step is 10^{-4}.

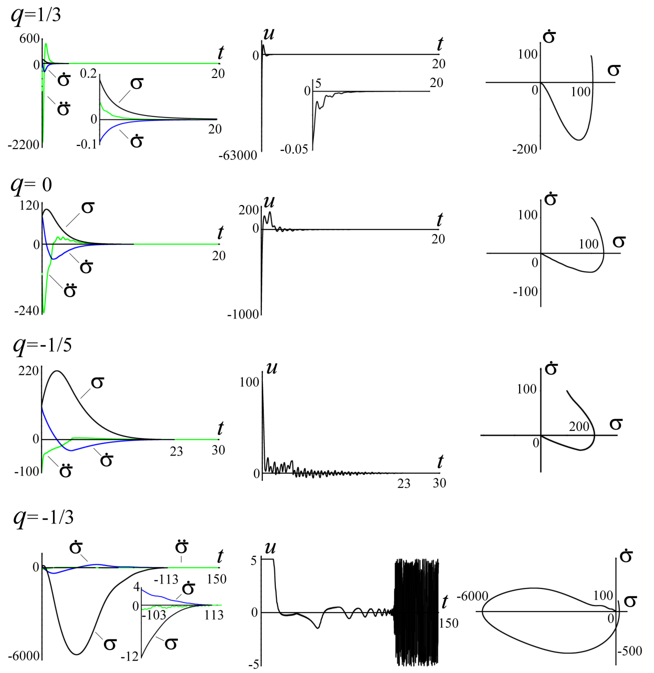

Fig. 1. Performance of (27), (23) for HDs $q = \frac{1}{3}, 0, -\frac{1}{5}$ and (27), (24) for $r = 3$ and $q = -\frac{1}{3}$. Projections of $\vec{\sigma}_2$ onto $\vec{\sigma}_1$ are shown on the right.

Performance of the resulting controllers

$$
\begin{aligned}
&u = -5\left\lfloor \ddot{\sigma} + 2\lfloor\dot{\sigma}\rceil^{\frac{5}{4}} + \sigma^{\frac{5}{3}}\right\rceil^{\frac{6}{5}} && \text{for } q = \tfrac{1}{3},\\
&u = -5(\ddot{\sigma} + 2\dot{\sigma} + \sigma) && \text{for } q = 0,\\
&u = -5\left\lfloor \ddot{\sigma}^{\frac{5}{3}} + 2\lfloor\dot{\sigma}\rceil^{\frac{5}{4}} + \sigma\right\rceil^{\frac{2}{5}} && \text{for } q = -\tfrac{1}{5},\\
&u = -5\frac{\ddot{\sigma}^3 + 2\lfloor\dot{\sigma}\rceil^2 + \sigma}{|\ddot{\sigma}|^3 + 2\dot{\sigma}^2 + |\sigma|} && \text{for } q = -\tfrac{1}{3}
\end{aligned}
$$

is shown in Fig. 1. Zooms of the graphs are provided for some HDs.

The asymptotic convergence for $q = 1/3$ is very slow near the origin, but very fast at large distances, $|\sigma| \leq 0.003$ is maintained for $t > 20$.

The HD $q = 0$ yields $\omega = 1$ and $||\vec{\sigma}_2||_{h\omega} = |\sigma| + |\dot{\sigma}| + |\ddot{\sigma}|$. The resulting linear control keeps $|\sigma| \leq 0.0001$ for $t \geq 20$ and features the exponential convergence in spite of time-variable "uncertainties".

The system is FT stable for any $q < 0$. The accuracy $|\sigma| \leq 10^{-17}$ is obtained for $q = -1/5$ and $t \geq 23$. In the case $q = -1/3$ the disturbances do not vanish at $\vec{\sigma}_2 = 0$, the 3-SM control is quasi-continuous and bounded, $|u| \leq 5$. The resulting FT convergence is very slow at large distances, $|\sigma| \leq 3 \cdot 10^{-11}$ is kept for $t \geq 113$.

VI. Conclusions

The proposed simple method creates infinite number of homogeneous asymptotic stabilizers for all combinations of relative and homogeneity degrees. The resulting controller forms/templates already do not need stability proof. At the same time each template contains a number of parameters to be assigned by simulation or, possibly, Lyapunov analysis.

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