Homogeneous filtering and differentiation based on sliding modes[1]

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***Abstract*— The proposed nonlinear filtering differentiators combine the features of linear filters capable of rejecting large high-frequency signal components, with the exactness, finite-time convergence and optimal accuracy asymptotics of the sliding-mode-based observers and differentiators. New tracking filtering differentiators are proposed. Discretization of the filters is studied. Computer simulation illustrates the theoretical results.**

I. Introduction

Differentiation is the classic ill-posed problem, since possibly large derivatives of small noises require some voluntary distinction to be made between a noise to be ignored and the basic signal to be differentiated.

The classic filtering approach [1], [13], [30] distinguishes the basic signal and the noise by their frequencies. All high-frequency components are to be suppressed, and the approximate differentiation is provided for the components of a bounded frequency diapason. The resulting inaccuracy is a decent price for the robustness and filtering out even large high-frequency noises. Such observers are effectively used for the output-feedback stabilization of smooth nonlinear systems at their equilibria [1].

Sliding-mode (SM) control (SMC) [31], [10] features higher observation requirements due to the inevitable small high frequency system vibrations [15], [3]. An observer rejecting all high-frequency vibrations also wrongly estimates the state and prevents the very entrance into the final SM.

High-order SMs (HOSMs) were especially proposed to deal with high relative degrees of the outputs [4], [14], [5], [7], [9], [16], [19], [20]. Exact differentiators play essential part in the HOSMC theory [6], [8], [11], [19]. HOSM-based differentiators converge in finite time (FT) and exactly differentiate small high-frequency output components.

It is proved here that in the framework of the SMC one is able to simultaneously get the asymptotically-optimal exact robust differentiation and the filtering-out of *unbounded* noises of small average values.

Contrary to the recent papers [28], [24] this paper studies the discretization issues of filtering differentiation and for the first time introduces tracking filtering differentiators yielding smooth derivative estimations. Simulation shows the method effectiveness.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$. A function of a set is the set of function values on this set. The norm $||x||$ stays for the standard Euclidian norm of x, $B_\varepsilon = \{x \mid ||x|| \le \varepsilon\}$; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$.

II. Introduction to homogeneous differentiation

Let $\mathrm{Lip}_n(L)$ be the set of all scalar functions defined on $\mathbb{R}_+ = [0, \infty)$, and featuring the Lipschitz constant $L > 0$ of their nth derivative. Following [19] the differentiators are to be exact on functions from $\mathrm{Lip}_n(L)$.

Assumption 1: The input signal $f(t)$, $t \ge 0$ has the form $f(t) = f_0(t) + \eta(t)$, where f_0 is an unknown basic signal $f_0 \in \mathrm{Lip}_n(L)$, and $\eta(t)$ is a Lebesgue-measurable noise $\eta(t)$.

Assumption 2: The noise $\eta(t)$ is bounded, $|\eta| \le \varepsilon_0$. Whereas L, n are assumed known, $\varepsilon_0 \ge 0$ is unknown.

Differentiation problem [18], [19]: The task is to evaluate the derivatives $f_0(t), \dot f_0(t), ..., f_0^{(n)}(t)$ in real time, robustly with respect to small noises $\eta(t)$, and exactly in their absence.

Theorem 1 ([27]): For any $\varepsilon_* > 0$ there is $t_0 > 0$ (also $\forall t_0 > 0\ \exists \varepsilon_* > 0$) such that for any ε_0, $0 < \varepsilon_0 \le \varepsilon_*$, and any $f_0, f_1 \in \mathrm{Lip}_n(L)$ the inequality $\sup_{t \ge 0} |f_1(t) - f_0(t)| \le \varepsilon_0$ implies inequalities $\sup_{t \ge t_0} |f_1^{(i)}(t) - f_0^{(i)}(t)| \le K_{i,n} (2L)^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}$, $i = 0, 1, ..., n$, which turn into equalities on certain functions. Here $K_{i,n} \in [1, \pi/2]$ are the Kolmogorov constants [17].

Let $z_0(t), z_1(t), ..., z_n(t)$ be the real-time estimations of the derivatives $f_0(t), \dot f_0(t), ..., f_0^{(n)}(t)$ produced by a differentiator that is exact on any input from $\mathrm{Lip}_n(L)$ after a finite-time (FT) transient. Then, taking $f = f_1 = f_0 + \eta$, for $\eta = f_1 - f_0$, obtain that the best possible accuracy guarantied for *any* $f_0 \in \mathrm{Lip}_n(L)$ satisfies $\inf \sup |z_i - f_0^{(i)}| \ge K_{i,n} (2L)^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}$. In particular, $K_{1,1} = \sqrt{2}$.

A differentiator is called **asymptotically optimal** [27], if for some $\mu_i > 0$ under Assumptions 1, 2 it in FT provides the same accuracy $|z_i(t) - f_0^{(i)}(t)| \le \mu_i L^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}$, $i = 0, 1, ..., n$, for all inputs, noises and $\varepsilon_0 \ge 0$. Obviously, the inequalities $\mu_i \ge K_{i,n} \ge 1$ are always to hold.

The following is the asymptotically-optimal differentiator [19] in its so-called non-recursive form:

$$\begin{aligned}
\dot z_0 &= -\tilde\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1, \\
\dot z_1 &= -\tilde\lambda_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} + z_2, \\
&\dots \\
\dot z_{n-1} &= -\tilde\lambda_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} + z_n, \\
\dot z_n &= -\tilde\lambda_0 L \ \mathrm{sign}(z_0 - f(t)).
\end{aligned} \tag{1}$$

Here and further all differential equations are understood in the Filippov sense [12]. Differentiator (1) is called homogeneous, since its error dynamics satisfy a FT-stable homogeneous differential inclusion (DI).

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1. Accepted for presentation at IEEE CDC 2019

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Indeed, denote $\sigma_i = (z_i - f^{(i)})/L$. Now subtracting $f^{(i+1)}$ from the both sides of the equation for z_i, dividing by L and taking into account $f^{(n+1)} \in [-1,1]$ obtain the FT stable DI

$$\begin{array}{lcl}\dot\sigma_0 &=& -\tilde\lambda_n \lfloor\sigma_0\rceil^{\frac{n}{n+1}} + \sigma_1,\\ \dot\sigma_1 &=& -\tilde\lambda_{n-1} \lfloor\sigma_0\rceil^{\frac{n-1}{n+1}} + \sigma_2,\\ & & \ldots \\ \dot\sigma_{n-1} &=& -\tilde\lambda_1 \lfloor\sigma_0\rceil^{\frac{1}{n+1}} + \sigma_n,\\ \dot\sigma_n &\in& -\tilde\lambda_0 \operatorname{sign}(\sigma_0) + [-1,1],\end{array} \quad (2)$$

where $\operatorname{sign} 0 = [-1,1]$. The homogeneity of (2) is due to its invariance with respect to the transformation $\sigma_i \mapsto \kappa^{n+1-i}\sigma_i$, $t \mapsto \kappa t$ for $\kappa > 0$ (Appendix A).

Parameters $\tilde\lambda_i$ are most easily calculated using the parameters $\lambda_0, ..., \lambda_n$ of the differentiator recursive form [19]: $\tilde\lambda_0 = \lambda_0$, $\tilde\lambda_n = \lambda_n$, and $\tilde\lambda_j = \lambda_j \tilde\lambda_{j+1}^{j/(j+1)}$, $j = n-1, n-2, \ldots, 1$. An infinite sequence of parameters $\vec\lambda = \{\lambda_0, \lambda_1, ...\}$ can be built [19], providing coefficients $\tilde\lambda_i$ of (1) for all natural n. In particular, $\vec\lambda = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$ suffice for $n \le 7$ [26], [27]. The corresponding parameters $\tilde\lambda_i$ appear in Table I.

TABLE I

PARAMETERS $\tilde\lambda_0, \tilde\lambda_1, ..., \tilde\lambda_n$ OF DIFFERENTIATOR (1) FOR $n = 0, 1, ..., 7$

0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

In the presence of discrete measurements with the maximal sampling time interval $\tau > 0$ differentiator (1) in FT provides the accuracy

$$\begin{array}{l}|z_i(t) - f_0^{(i)}(t)| \le \mu_i L \rho^{n+1-i},\ i = 0, 1, ..., n,\\ \rho = \max[(\varepsilon_0/L)^{1/(n+1)}, \tau]\end{array} \quad (3)$$

for some $\mu_i > 0$ [19]. Here the case $\tau = 0$ formally corresponds to continuous sampling. The same accuracy asymptotics (with different constants μ_i) is maintained by properly discretized differentiator [29], [26].

III. FILTERING DIFFERENTIATORS

Introduce the number $n_f \ge 0$ which is further called *the filtering order*. The following **filtering differentiator** [24] is further called "standard":

$$\begin{array}{l}\dot w_1 = -\tilde\lambda_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor w_1\rceil^{\frac{n+n_f}{n+n_f+1}} + w_2,\\ \quad\ldots\\ \dot w_{n_f-1} = -\tilde\lambda_{n+2} L^{\frac{n_f-1}{n+n_f+1}} \lfloor w_1\rceil^{\frac{n+2}{n+n_f+1}} + w_{n_f},\\ \dot w_{n_f} = -\tilde\lambda_{n+1} L^{\frac{n_f}{n+n_f+1}} \lfloor w_1\rceil^{\frac{n+1}{n+n_f+1}} + z_0 - f(t),\\ \dot z_0 = -\tilde\lambda_n L^{\frac{n_f+1}{n+n_f+1}} \lfloor w_1\rceil^{\frac{n}{n+n_f+1}} + z_1,\\ \quad\ldots\\ \dot z_{n-1} = -\tilde\lambda_1 L^{\frac{n+n_f}{n+n_f+1}} \lfloor w_1\rceil^{\frac{1}{n+n_f+1}} + z_n,\\ \dot z_n = -\tilde\lambda_0 L \operatorname{sign}(w_1).\end{array} \quad (4)$$

Formally define that for $n_f = 0$ the first n_f equations disappear, and $w_1 = z_0 - f(t)$ is substituted for w_1, yielding the standard differentiator (1). Introduce the short notation for (4):

$$\dot w = \Omega_{n,n_f}(w, z_0 - f, L, \vec\lambda),\ \dot z = D_{n,n_f}(w_1, z, L, \vec\lambda), \quad (5)$$

where the parametric sequence $\vec\lambda$ is introduced in Section II. Consider the filtering properties of differentiator (4).

A (noise) function $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called a *signal of the (global) filtering order* $k \ge 0$, if ν is a locally integrable Lebesgue-measurable function, and there exists a globally bounded solution $\xi(t)$ of the equation $\xi^{(k)} = \nu$. Any number exceeding $\sup|\xi(t)|$ is called the *kth-order integral magnitude of* ν.

Assumption 3: Instead of Assumption 2 assume that the input signal also contains possibly-unbounded noise components, $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, where each η_k, $k = 0, ..., n_f$, is a signal of the filtering order k and the kth-order integral magnitude $\varepsilon_k \ge 0$.

The noise described in Assumption 2 is of the 0th filtering order and the 0th order integral magnitude ε_0. Thus, the standard differentiator (1) is of the filtering order $n_f = 0$, and is robust to the noises $\eta(t)$ of the filtering order 0 [19].

The following theorem shows that differentiators (4) of the filtering order $n_f \ge 0$ are robust with respect to *possibly unbounded* noises of the filtering orders not exceeding n_f.

Theorem 2: Under Assumptions 1, 3 differentiator (4) in FT provides for the accuracy

$$\begin{array}{l}|z_i(t) - f_0^{(i)}(t)| \le \mu_i L \rho^{n+1-i},\ i = 0, 1, ..., n,\\ |w_1(t)| \le \mu_{w1} L \rho^{n+n_f+1},\\ \rho = \max[(\frac{\varepsilon_0}{L})^{\frac{1}{n+1}}, (\frac{\varepsilon_1}{L})^{\frac{1}{n+2}}, ..., (\frac{\varepsilon_{n_f}}{L})^{\frac{1}{n+n_f+1}}],\end{array} \quad (6)$$

where $\{\mu_i\}$, μ_{w1} only depend on the choice of $\{\lambda_l\}$, $l = 0, ..., n + n_f$.

Thus, differentiator (4) is asymptotically optimal for any $n_f \ge 0$. Obviously, increasing the filtering order n_f preserves the accuracy asymptotics of the form (6) for noises of lower filtering orders. Here and further proofs and proof sketches appear in the Appendix.

The noise $\cos(\omega_* t)$ features any filtering order $k \ge 0$ with the integral magnitude $2/\omega_*^k$. It follows from Theorem 2 that the higher n_f the better is the accuracy, provided $\omega_* > 1$.

A (noise) function $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called a *signal of the local filtering order* $k \ge 0$ if ν is a locally integrable Lebesgue-measurable function, and there exist numbers $T, a_1, ..., a_k > 0$, such that for any t_1, t_2, $0 \le t_1 < t_2$, $t_2 - t_1 \le T$, there exists a solution $\xi(t)$, $t \in [t_1, t_2]$, of the equation $\xi^{(k)} = \nu$ which satisfies $|\xi^{(l)}| \le a_l$ for $l = 0, ..., k-1$. Numbers a_l are called the local *lth-order integral magnitudes of* ν.

Lemma 1: Any signal $\nu(t)$ of the local filtering order $k \ge 0$ from the above definition can be represented as $\nu = \eta_0 + \eta_1 + \eta_k$, where η_0, η_1, η_k are signals of the global filtering orders $0, 1, k$.

Fix any number $\rho_0 > 0$. Then, provided $\rho_* \le \rho_0$ holds for $\rho_* = \max[a_0^{1/k}, a_1^{1/(k-1)}, ..., a_{k-1}]$, the integral magnitudes of the signals η_0, η_1, η_k are calculated as $\gamma_0 \rho_*/T$, $\gamma_1 \rho_*$, $\gamma_k \rho_*^k$ respectively, where the constants $\gamma_0, \gamma_1, \gamma_k > 0$ only depend on k and ρ_0. In particular, in the important case $k = 1$ get $\rho_* = a_0$, $\nu = \eta_0 + \eta_1$, and independently of ρ_0 get $\gamma_0 = 1$, $\gamma_1 = 2$, i.e. $|\eta_0| \le a_0/T$, and the first-order integral magnitude of η_1 is $2a_0$.

Remark 1: Lemma 1 provides sufficient conditions for Assumption 3 and shows that the noise representation $\eta = \eta_0 + ... + \eta_{n_f}$ is not unique. Since the accuracy estimation (6) holds for any such noise representation, the realized accuracy inevitably corresponds to *the best possible one*.

Tracking filtering differentiators. Outputs $z_i(t)$ of differentiators (1) and (4) are not Lipschitzian due to the fractional powers on the right-hand sides of (1) and (4). Since in the reality the SM $z_0 - f(t) = 0$ is not kept ideally, also $\dot{z}_i \ne z_{i+1}$. One would like to ensure $z_i(\cdot) \in C^{n-i}$ and $\dot{z}_i \equiv z_{i+1}$, $i = 0, ..., n-1$, with z_n being only Lipschitzian.

The corresponding differentiators have been introduced for the identification of the practical relative degree in [21], [22], [27]. The following tracking filtering differentiator is their significantly *improved version* ready for application.

Let a homogeneous SMC $u = \alpha_n \psi_{n+1}(\sigma, \dot{\sigma}, ..., \sigma^n)$ in FT stabilize the DI $\sigma^{(n+1)} \in [-1, 1] + u$ for some $\alpha_n > 0$. Also assume $|\psi_{n+1}| \le 1$. Such controls are known as $(n+1)$th-order SM ($(n+1)$-SM) controls, [7], [9], [16], [20], [23]. The $(n+1)$-SM homogeneity means that $\psi_{n+1}(\sigma, \dot{\sigma}, ..., \sigma^n) \equiv \psi_{n+1}(\kappa^{n+1}\sigma, \kappa^n \dot{\sigma}, ..., \kappa \sigma^n)$ holds for any $\kappa > 0$ and $\sigma, \dot{\sigma}, ..., \sigma^n \in \mathbb{R}$.

Then the tracking filtering differentiator is defined as

$$\begin{array}{l} \dot{w} = \Omega_{n,n_f}(w, \zeta_0 - z_0 + f(t), L + \alpha_n L, \vec{\lambda}), \\ \dot{\zeta} = D_{n,n_f}(w_1, \zeta, L + \alpha_n L, \vec{\lambda}), \\ \dot{z}_0 = z_1, \ ..., \ \dot{z}_{n-1} = z_n, \\ \dot{z}_n = \alpha_n L \psi_{n+1}(\zeta/L). \end{array} \qquad (7)$$

Theorem 3: Under Assumptions 1, 3 differentiator (7) in FT provides for the accuracy asymptotics (6). Also $|\zeta_i| \le \mu_{\zeta i} L \rho^{n+1-i}$ are kept in the steady state for $i = 0, ..., n$. The constants μ_i, $\mu_{\zeta i}$, μ_{w1} only depend on the choice of $\{\lambda_l\}$, $l = 0, ..., n+n_f$, α_n and ψ_{n+1}.

IV. Discrete filtering differentiators

In modern practice a filter is a discrete dynamic system obtaining a discretely sampled input $f(t)$. Unlike Assumption 2, Assumption 3 is very sensitive to sampling, which is philosophically related to the Nyquist-Shannon sampling rate principle. Indeed, a sampled high-frequency periodic signal can become constant or slowly changing.

Let the sampling-times' sequence $t_0, t_1, ..., t_0 = 0$, feature bounded sampling steps $t_{j+1} - t_j = \tau_j \le \tau$. The upper bound $\tau > 0$ can be unknown. The admissible sequences t_j are assumed to exist for any $\tau > 0$.

Notation. Denote $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$ for any sampled vector signal $\phi(t_j)$.

A discretely sampled signal $\nu : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *signal of the (global) sampling filtering order* $k \ge 0$ and *the (global) kth order integral sampling magnitude* $a \ge 0$ if for each admissible sequence t_j there exists a discrete vector signal $\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1}$, $j = 0, 1, ...$, which satisfies the relations

$$\begin{array}{l} \delta_j \xi_i = \xi_{i+1}(t_j)\tau_j, \ i = 0, 1, ..., k-1, \\ \xi_k(t_j) = \nu(t_j), \ |\xi_0(t_j)| \le a. \end{array}$$

Assumption 4: The discretely sampled signals $\eta_l(t_j)$ are of the sampling filtering order l and integral magnitude ε_l.

Assumptions 1, 3, 4 are proved to hold in the steady-state SMs for the SMC $u(t)$, the equivalent control $u_{eq}(t)$, $f = u$, $f_0 = u_{eq}$, $\eta = u - u_{eq}$, and $n_f = 1$ [28].

In general one needs very small sampling steps to reveal the small average value of the noise. The following *alternative* assumption is natural in filtering theory and guaranties the stability of the average value with respect to sampling.

Assumption 5: Each noise η_l is absolutely continuous with $|\dot{\eta}_l| \le L_{\eta l}$, $L_{\eta l} > 0$, $l = 1, ..., n_f$.

Naturally, $L_{\eta l}$ can be unknown and large. Similarly to Remark 1, also here one does not need to check Assumptions 3,4,5 in order to use the differentiator.

The proposed homogeneous discretization of the standard filtering differentiator (5) has the form

$$\begin{array}{l} \delta_j w = \Omega_{n,n_f}(w(t_j), z_0(t_j) - f(t_j), L, \vec{\lambda})\tau_j, \\ \delta_j z = D_{n,n_f}(w_1(t_j), z(t_j), L, \vec{\lambda})\tau_j + T_n(z(t_j), \tau_j), \end{array} \qquad (8)$$

where the Taylor-like term $T_n \in \mathbb{R}^{n+1}$ is defined by

$$\begin{array}{l} T_{n,0} = \frac{1}{2!} z_2(t_j)\tau_j^2 + ... + \frac{1}{n!} z_n(t_j)\tau_j^n, \\ T_{n,1} = \frac{1}{2!} z_3(t_j)\tau_j^2 + ... + \frac{1}{(n-1)!} z_n(t_j)\tau_j^{n-1}, \\ ... \\ T_{n,n-2} = \frac{1}{2!} z_n(t_j)\tau_j^2, \\ T_{n,n-1} = 0, \ T_{n,n} = 0. \end{array} \qquad (9)$$

The proposed homogeneous discretization of the tracking differentiator (7) has the form

$$\begin{array}{l} \delta_j w = \Omega_{n,n_f}(w, \zeta_0 - z_0 + f, L + \alpha_n L, \vec{\lambda})|_{t=t_j} \cdot \tau_j, \\ \delta_j \zeta = D_{n,n_f}(w_1(t_j), \zeta(t_j), L + \alpha_n L, \vec{\lambda})\tau_j, \\ \delta_j z = \Psi_n(t_j)\tau_j + T_n(z(t_j), \tau_j), \\ \Psi_n(t_j) = (z_1, ..., z_n, \alpha_n L \psi_{n+1}(\zeta/L))^T|_{t=t_j}. \end{array} \qquad (10)$$

Theorem 4: Under Assumptions 1, 3, 4 discrete differentiators (8) and (10) provide the same accuracies as Theorems 2 and 3 respectively, but for

$$\rho = \max[\tau, \max_{0 \le l \le n_f} (\tfrac{\varepsilon_l}{L})^{\frac{1}{n+l+1}}] \qquad (11)$$

Formally define $L_{\eta 0} = 1$, then under Assumptions 1, 3, 5 the resulting accuracy (6) corresponds to

$$\rho = \max[\tau, \max_{0 \le l \le n_f} \max_{0 \le k \le l} (\tfrac{L_{\eta l}}{L} (\tfrac{\varepsilon_l}{L_{\eta l}})^{\frac{k+1}{l+1}})^{\frac{1}{n+k+1}}]. \qquad (12)$$

Assumptions 4 and 5 can be combined producing ρ calculated as the maximum of (11), (12). Similarly to Section III a signal of *the local sampling filtering order* k is defined. Also a lemma analogous to Lemma 1 holds.

Independent equally-distributed random sampling noises $\nu(t_j)$ of the zero mean value in practice feature the local sampling filtering order 1 [24]. Harmonic noises $\cos(\omega_* t)$ feature any global and local sampling filtering order.

Also here arbitrarily increasing the filtering order preserves the accuracy asymptotics for noises of lower filtering orders, though the actual asymptotics' coefficients can change. One also has to take into account the influence of the digital round-up errors for very small noises and sampling intervals [27], [29].

V. Simulation Results

Consider the noisy input signal

$$f(t) = f_0(t) + \eta(t),\ f_0(t) = \sin(0.5t) + 0.5\cos t, \quad (13)$$

$$\begin{aligned} &\eta(t) = \eta_1(t) + \eta_2(t) + \eta_3(t),\\ &\eta_1(t) = \cos(10000t + 1.237086),\\ &\eta_2(t) \in N(0, 0.5^2),\\ &\eta_3(t) = 0.001 \tfrac{d}{dt}\lfloor \cos(100t)\rceil^{\frac{1}{2}}\\ &\quad = -0.05\sin(100t)\lfloor\cos(100t)\rceil^{-\frac{1}{2}}, \end{aligned} \quad (14)$$

where η_1 is a high-frequency harmonic signal, η_2 is the random Gaussian signal of the standard deviation 0.5, and η_3 is an *unbounded* signal of the filtering order 1 and the integral magnitude 0.001. Obviously $|f_0^{(k)}| \leq 1$ for all k.

Two filtering differentiators (4) of the same order $n = 2$ are applied with parameters $L = 1$, $\tau_j = \tau = 10^{-6}$, using the discretization (8), (9) with $z(0) = 0$, $w(0) = 0$. The first differentiator has the filtering order $n_f = 3$, whereas the second has the filtering order $n_f = 5$. Their coefficients are taken from Table I from the lines $5 = 2 + 3$ and $7 = 2 + 5$ respectively.

The performance for the filtering order 3 is demonstrated in Fig. 1. The performance for the filtering order 5 does not differ neither visually nor even in the estimation accuracy. The resulting accuracy for $t \in [14, 20]$ is provided by the component-wise inequality

$$(|w_1|, |w_2|, |w_3|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq (5.1\cdot 10^{-8}, 7.0\cdot 10^{-6}, 1.1\cdot 10^{-3}, 6.6\cdot 10^{-3}, 0.060, 0.27).$$

The high filtering order $n_f = 5$ is beneficial for the high-frequency *extremely large* composite harmonic noise

$$\eta(t) = 1500\cos(10000t) + 3000\sin(20000t) + 2000\cos(70000t). \quad (15)$$

Apply the second-order differentiator (8), (9) with $n = 2$, $n_f = 5$, $L = 1$ and $\tau = 10^{-5}$. Its performance is demonstrated in Figs. 2a,b. The resulting accuracy for $t \in [35, 40]$ is provided by the component-wise inequality

$$\begin{aligned}(|w_1|, |w_2|, |w_3|, |w_4|, |w_5|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq\\ (2.7\cdot 10^{-17}, 1.7\cdot 10^{-13}, 1.9\cdot 10^{-9}, 2.0\cdot 10^{-5}, 0.32,\\ 9.0\cdot 10^{-5}, 3.3\cdot 10^{-3}, 0.063).\end{aligned}$$

Now apply the tracking differentiator (10) with the same parameters $n = 2$, $n_f = 5$, $L = 1$, and $\alpha_3 = 4$, $\psi_3(\zeta) = -\frac{\zeta_2^3 + \lfloor\zeta_1\rceil^{\frac{3}{2}} + \zeta_0}{|\zeta_2|^3 + |\zeta_1|^{\frac{3}{2}} + |\zeta_0|}$. Its performance is much worse than that of the standard filtering differentiator (8) due to the imposed outputs' smoothness task. Nevertheless, it still demonstrates remarkable filtering qualities (Fig 2c) and the accuracies

$$\begin{aligned}(|w_1|, |w_2|, |w_3|, |w_4|, |w_5|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq\\ (3.0\cdot 10^{-17}, 1.9\cdot 10^{-13}, 1.9\cdot 10^{-9}, 2.0\cdot 10^{-5}, 0.33,\\ 6.5\cdot 10^{-3}, 0.043, 0.50).\end{aligned}$$

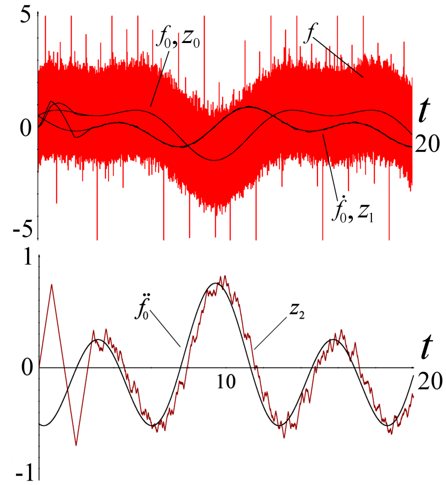

Fig. 1. Performance of differentiator (8), (9) with $n = 2$, $n_f = 3$, $L = 1$, $\tau = 10^{-6}$ for the input (13), (14). Estimations of f_0, $\dot{f}_0$ and $\ddot{f}_0$ are shown. The upper graph is cut from above and from below.

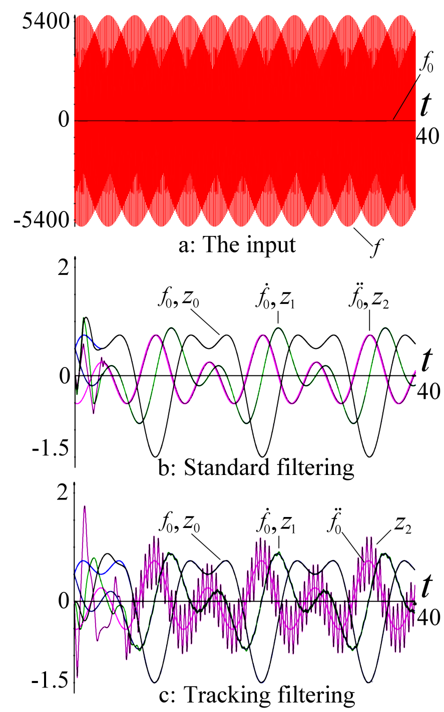

Fig. 2. Performance in the presence of extremely large noise (15) for $n = 2$, $n_f = 5$, $L = 1$, $\tau = 10^{-5}$. a: the input (13), (15); b: outputs of filtering differentiator (8), (9); c: outputs of tracking differentiator (10).

VI. Conclusions

Proposed homogeneous filtering differentiators are capable to filter out complicated noises of small average values. The filtering capabilities of the differentiators are determined by their filtering order. The higher the filtering order the higher the differentiation accuracy in the presence of noises. The accuracy asymptotics are calculated and the proposed homogeneous discretization preserves them.

The proposed nth-order filters/differentiators (5), (7) with fixed $L > 0$ and their discretizations feature the same optimal accuracy asymptotics as their predecessors [19] in the presence of bounded noises. In particular, in the absence of noises they are exact on the inputs $f_0 \in \mathrm{Lip}_n(L)$.

The noise is assumed representable as a sum of a finite number of noises of different filtering orders, whereas noises of the filtering order 0 are just bounded measurable noises of any nature. The calculated accuracy evaluation depends on that noise expansion. Since the expansion is not unique, due to Theorems 2, 3, 4 the actual accuracy corresponds to the *unknown* best possible expansion.

The proposed homogeneous tracking differentiators and their filtering modifications yield smooth derivative estimations $z_k \approx f_0^{(k)}$ satisfying relations $\dot{z}_k = z_{k+1}$ advantageous in some signal-processing applications. Their other features are the same as of the "standard" filtering differentiators.

Appendix

Recall that solutions of the differential inclusion (DI)

$$\dot{x} \in F(x),\ F(x) \subset T_x\mathbb{R}^{n_x}, \tag{16}$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all t. Here $T_x\mathbb{R}^{n_x}$ denotes the tangent space to $\mathbb{R}^{n_x}$ at $x \in \mathbb{R}^{n_x}$.

We call the DI (16) *Filippov DI*, if the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

Filippov DIs feature existence, extendability etc. of solutions, but not their uniqueness [12]. The Filippov definition [12] replaces a discontinuous vector field $f(x)$ with a Filippov DI.

A. Coordinate homogeneity basics

Introduce the weights $m_1, ..., m_{n_x} > 0$ of the coordinates $x_1, ..., x_{n_x}$ in $\mathbb{R}^{n_x}$, $\deg x_i = m_i$, and the dilation [2]

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}),$$

where $\kappa \geq 0$. Recall [2] that a function $g : \mathbb{R}^{n_x} \to \mathbb{R}^m$ is said to have the homogeneity degree (HD) (weight) $q \in \mathbb{R}$, $\deg g = q$, if the identity $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^{n_x}$ and $\kappa > 0$.

Consider the combined time-coordinate transformation

$$(t, x) \mapsto (\kappa^{-q} t, d_\kappa x), \quad \kappa > 0, \tag{17}$$

where the number $-q \in \mathbb{R}$ might naturally be considered as the weight of t. The DI $\dot{x} \in F(x)$ and the vector-set field $F(x)$ are called homogeneous of the HD q, if the DI is invariant with respect to (17), i.e. $\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^{-q} t)} \in F(d_\kappa x)$. The following is the formal definition.

Definition 1: [20] A vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^{n_x}$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$.

A system of differential equations (DEs) $\dot{x}_i = f_i(x)$, $i = 1, ..., n_x$, is a particular case of DI, when the set $F(x)$ contains only one vector $f(x)$ and is reduced to $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg f_i$ [2]. Note that if f is discontinuous, the DE is equivalent to the corresponding homogeneous Filippov DI (16).

Note that the weights $-q$, $m_1, ..., m_{n_x}$ are defined up to proportionality. The sign of the HD determines many properties of DIs.

Any continuous positive-definite function of the HD 1 is called a homogeneous norm. We denote it $||x||_h$.

It is proved in [20], [25] that if the HD of the DI (16) is negative then it is asymptotically stable iff it is FT stable. Moreover, in the presence of a maximal delay $\tau \geq 0$ and noises of the magnitudes $\varepsilon_i \geq 0$, $i = 1, 2, ..., n_x$, all extendable-in-time solutions of the disturbed DI

$$\dot{x} \in F(x(t - \tau[0, 1]) + [-\varepsilon_1, \varepsilon_1] \times ... \times [-\varepsilon_{n_x}, \varepsilon_{n_x}])$$

starting from some time satisfy the inequalities $|x_i| \leq \mu_i \rho^{m_i}$ for some $\mu_i > 0$ and $\rho = \max[\varepsilon_1^{1/m_1}, ..., \varepsilon_n^{1/m_{n_x}}, \tau]$.

In particular, the differentiator accuracy (3) is the result of the homogeneity of dynamics (2) with the HD -1.

B. The proof sketches of the main results

Proof of Theorem 2. According to the filtering-order definition introduce the functions $\xi_k(t)$, $|\xi_k| \leq \delta_k$, $\xi_k^{(k)}(t) = \nu_k(t)$. Let

$$\begin{array}{l} \omega_1 = w_1 + \xi_{n_f}, \omega_2 = w_2 + \dot{\xi}_{n_f} + \xi_{n_f - 1}, ..., \\ \omega_{n_f} = w_{n_f} + \xi_k^{(n_f - 1)} + ... + \dot{\xi}_2 + \xi_1; \\ \sigma_i = z_i - f_0^i,\ i = 0, ..., n. \end{array} \tag{18}$$

Then $f = f_0 + \eta + \dot{\xi}_1 + ... + \xi_{n_f}^{(n_f)}$, and one can rewrite (4) in the form

$$\begin{array}{l} \dot{\omega}_1 = -\tilde{\lambda}_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{n+n_f}{n+n_f+1}} + \omega_2 - \xi_{n_f - 1}, \\ \dot{\omega}_2 = -\tilde{\lambda}_{n+n_f-1} L^{\frac{2}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{n+n_f-1}{n+n_f+1}} + \omega_3 - \xi_{n_f - 2}, \\ \quad \dots \\ \dot{\omega}_{n_f - 1} = -\tilde{\lambda}_{n+2} L^{\frac{n_f - 1}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{n+2}{n+n_f+1}} + \omega_{n_f} - \xi_2, \\ \dot{\omega}_{n_f} = -\tilde{\lambda}_{n+1} L^{\frac{n_f}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{n+1}{n+n_f+1}} + \sigma_0 + \eta, \\ \dot{\sigma}_0 = -\tilde{\lambda}_n L^{\frac{n_f+1}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{n}{n+n_f+1}} + z_1, \\ \quad \dots \\ \dot{\sigma}_{n-1} = -\tilde{\lambda}_1 L^{\frac{n+n_f}{n+n_f+1}} \lfloor \omega_1 - \xi_{n_f} \rceil^{\frac{1}{n+n_f+1}} + \sigma_n, \\ \dot{\sigma}_n \in -\tilde{\lambda}_0 L\ \mathrm{sign}(\omega_1 - \xi_{n_f}) + [-L, L], \end{array} \tag{19}$$

which is a perturbation of the FT stable homogeneous error dynamics (2) of the standard $(n + n_f)$th-order differentiator (1) obtained by substituting $n + n_f$ for n. Obviously,

$\deg \omega_k = n+n_f+2-k$, $\deg z_i = n+1-i$, $\deg t = -q = 1$, the HD is -1.

It follows from [25] that $\sup|\sigma_i| = O(\rho^{n+1-i})$, $\sup|\omega_k| = O(\rho^{n+n_f+2-k})$. Now the accuracy of z_i is directly obtained from these relations. Taking into account that $\sup|\dot{\omega}_k| = O(\rho^{n+n_f+1-k-1})$ obtain the accuracy of w_k from (18). □

Proof of Theorem 3. First consider the case without noises. Denote $\sigma = (z_0 - f_0)/L$, $\tilde{\zeta} = \zeta/L$, $\tilde{w} = w/L$ and rewrite (7) in the form

$$\begin{aligned} &\dot{\tilde{w}} = \Omega_{n,n_f}(\tilde{w}, \tilde{\zeta}_0 - \sigma, 1+\alpha_n, \vec{\lambda}), \\ &\dot{\tilde{\zeta}} = D_{n,n_f}(\tilde{w}_1, \tilde{\zeta}, 1+\alpha_n, \vec{\lambda}), \\ &\sigma^{(n+1)} \in \alpha_n \psi_{n+1}(\tilde{\zeta}) + [-1,1]. \end{aligned} \tag{20}$$

It follows from $|\sigma^{(n+1)}| \le 1+\alpha_n$ and Theorem 2 that in FT obtain $\tilde{\zeta} \equiv \vec{\sigma}$, where $\vec{\sigma} = (\sigma, ..., \sigma^{(n)})^T$. Now due to the choice of $\psi_{n+1}(\cdot)$ in FT get $\vec{\sigma} = 0$. The observation that (20) is homogeneous with the weights $\deg \tilde{\zeta}_i = \deg \sigma^{(i)} = n+1-i$, $i = 0,1,...,n$, $\deg \tilde{w}_l = n+n_f+2-l$, $l = 1,...,n_f$, and the HD -1 implies the required accuracy due to the above results from Appendix A for $\eta_1 = ... = \eta_{n_f} = 0$. In the general case the proof is similar to Theorem 2. □

Proof of Lemma 1. Divide the time axis in the segments of the length T. There is the corresponding solution $\xi_j : [jT,(j+1)T] \to \mathbb{R}$, $\xi_j^{(k)} = \nu$, over each time interval.

First consider the case $k = 1$ ($k = 0$ is trivial). Then $\rho_* = a_0$, $|\xi_j| \le a_0$, and taking $\xi(t) = \int_{jT}^{t} u(s)ds$, $u(t) = (\xi_{j+1}((j+1)T) - \xi_j((j+1)T))/T$ for $t \in [jT,(j+1)T]$ proves the Lemma.

In the general case ξ is modified by a bounded shift smoothly vanishing at the jth-interval end so that at the right end the needed initial values of ξ_{j+1} are obtained. □

Proof of Theorem 4. Consider the case of differentiator (8). The case of the tracking differentiator is similar. The proof is straightforward in the case of Assumption 4.

In the case of Assumption 5 each noise component η_l is mapped to the bounded integral function ξ_l, $|\xi_l| \le \varepsilon_l$, $\xi_l^{(l)} = \eta_l$, $|\xi_l^{(l+1)}| \le L_{\eta l}$, $l = 1,...,n_f$. Due to Theorem 1 starting from some moment inequalities $|\xi_l^{(k)}| \le \frac{\pi}{2} L_{\eta l}^{\frac{k}{l+1}} \varepsilon_l^{\frac{l+1-k}{l+1}}$ hold for $k = 0,...,l+1$.

Similarly to (18) define

$$\begin{aligned} &\omega_1(t_j) = w_1(t_j) + \xi_{n_f}(t_j), \\ &\qquad\qquad ..., \\ &\omega_{n_f}(t_j) = w_{n_f}(t_j) + \xi_{n_f}^{(n_f-1)}(t_j) + ... + \dot{\xi}_2(t_j) + \xi_1(t_j), \\ &\sigma_i(t_j) = z_i(t_j) - f_0^{(i)}(t_j),\ i = 0,1,...,n. \end{aligned}$$

Now using Assumption 5 obtain the disturbed discrete error system similar to (19). The rest of the proof is similar to that from [29]. □

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