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CONSTRUCTION PRINCIPLES OF OUTPUT-FEEDBACK 2-SLIDING MODE DESIGN

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Abstract. Second order sliding modes are used to keep exactly a constraint of the second relative degree or just to avoid chattering, i.e. in the cases when the standard (first order) sliding mode application might be involved or impossible. A number of new 2-sliding controllers are designed by means of a proposed method based on some homogeneity reasoning. A recently developed robust exact differentiator being used as a standard part of the 2-sliding controllers, robust output feedback controllers with finite-time convergence are produced, capable to control any general uncertain single-input-single-output process with relative degree 2. Simulation results are presented.

1. Introduction

Control under heavy uncertainty conditions is one of the main problems of the modern control theory. The sliding-mode control approach [17, 18, 3] to the problem is based on keeping exactly a properly chosen constraint by means of high-frequency control switching. The approach exploits the main features of the sliding mode: its insensitivity to external and internal disturbances, ultimate accuracy and finite-time transient. Nevertheless, the standard-sliding-mode usage is bounded by some restrictions. The constraint being given by equality of an output variable σ to zero, the standard sliding mode may be implemented only if the relative degree [6] of σ is 1. In other words, control has to appear explicitly already in the first total time derivative $\dot\sigma$. Also, high frequency control switching leads to the so-called chattering effect which is exhibited by high frequency vibration of the controlled plant and can be dangerous in applications.

Consider a smooth dynamic system with smooth output function σ, and let the system be closed by some possibly-dynamical discontinuous feedback. Then, provided

- successive total time derivatives $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables and
- the set $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ is non-empty and consists locally of Filippov trajectories [4],

the motion on the set $\sigma = \dot\sigma = ... = \sigma^{(r-1)} = 0$ is called r-sliding mode (rth order sliding mode [2, 5, 7], Fig. 1). The additional condition of the Filippov velocity set containing more than 1 vector may be imposed in order to exclude some trivial cases.

The standard sliding mode is of the first order ($\dot\sigma$ is discontinuous). Higher order sliding modes (HOSM) remove the above-mentioned restrictions of the 1-sliding mode. Asymptotically stable HOSMs appear in many systems with traditional sliding-mode control and are deliberately introduced in systems with dynamical sliding modes [15, 3]. In particular 2-sliding modes are used to remove the chattering or to keep constraints of the second relative degree. While arbitrary-order sliding finite-time-convergent controllers are still theoretically studied [11], 2-sliding controllers are already successfully implemented for solution of real problems [10, 16]. Some homogeneity-based method is proposed in this paper for the construction of new finite-time convergent 2-sliding controllers featuring the highest accuracy of 2-sliding control [7]. A number of new controllers are presented.

Unfortunately, most of 2-sliding controllers explicitly use possibly-unavailable $\dot\sigma$ or sign $\dot\sigma$. The first difference of σ is usually used instead of $\dot\sigma$ in order to overcome the difficulty [2, 7]. Nevertheless, the resulting performance still critically depends on the sampling step which has to be chosen in accordance with the measurement-noise magnitude. Thus, the robustness of the controller is partially lost. This paper proposes to use a recently developed robust exact differentiator [8, 12] as a standard part of the 2-sliding controllers.

The resulting output-feedback controllers preserve the ultimate accuracy and finite-time convergence of the original controllers and do not require any information on the noises. Corresponding theorems and simulation results are presented.

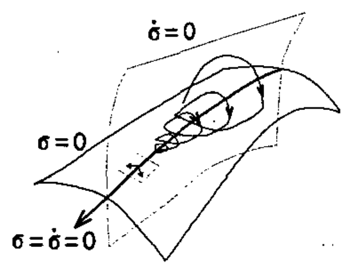

Fig. 1: 2-sliding mode

3. 2-sliding controllers

Consider a dynamic system of the form

$$\dot x = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is control, σ is a measured output, smooth functions a, b, σ and the dimension n are unknown.

The relative degree of the system is assumed to be 2. The task is to make the output σ vanish in finite time and to keep $\sigma \equiv 0$ by discontinuous feedback control. System trajectories are supposed to be infinitely extendible in time for any bounded input. The system is understood in the Filippov sense.

Calculating the second total time derivative $\ddot{\sigma}$ along the trajectories of (1) achieve that under these conditions

$$\ddot{\sigma} = h(t,x) + g(t,x)u, \qquad h = \ddot{\sigma}\big|_{u=0}, \; g = \tfrac{\partial}{\partial u}\ddot{\sigma} \neq 0 \qquad (2)$$

where the functions g, h are some unknown smooth functions [6]. Suppose that

$$0 < K_m \leq \tfrac{\partial}{\partial u}\ddot{\sigma} \leq K_M, \qquad |\ddot{\sigma}\big|_{u=0}| \leq C. \qquad (3)$$

for some K_m, K_M, $C > 0$. These conditions are satisfied at least locally for any smooth system (1). Assume here that (3) holds globally. Then (2), (3) imply the differential inclusion

$$\ddot{\sigma} \in [-C, C] + [K_m, K_M]u. \qquad (4)$$

Most 2-sliding controllers may be considered as controllers for (4) stirring σ, $\dot{\sigma}$ to 0 in (preferably) finite time. Inclusion (4) does not "remember" what was the original system (1). Thus, such controllers are obviously robust with respect to any perturbations preserving (3).

Hence, the problem is to find such a feedback

$$u = \varphi(\sigma, \dot{\sigma}), \qquad (5)$$

that all the trajectories of (4), (5) converge in finite time to the origin $\sigma = \dot{\sigma} = 0$ of the phase plane σ, $\dot{\sigma}$. Here φ is a locally bounded Borel-measurable function (actually all functions used in sliding-mode control satisfy this restriction). Differential inclusion (4), (5) is understood here in the Filippov sense [4], which means that the right-hand vector set is enlarged in a special way in order to satisfy certain convexity and semicontinuity conditions. Introduce a few simple auxiliary notions to be used further.

Inclusion (4), (5) and controller (5) itself are called *specially homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$G_\kappa: \quad (t, \sigma, \dot{\sigma}) \mapsto (\kappa t, \kappa^2\sigma, \kappa\dot{\sigma}) \qquad (6)$$

transfers its solutions into the solutions of the transformed inclusion. It is easily checked that (5) is specially homogeneous iff

$$\varphi(\kappa^2\sigma, \kappa\dot{\sigma}) \equiv \varphi(\sigma, \dot{\sigma}). \qquad (7)$$

Controller (5) is called *weakly stable* if all the trajectories starting from some centered at the origin disk on the plane σ, $\dot{\sigma}$ transfer together in some finite time into some disk of a smaller radius, the trajectories being uniformly bounded (i.e. being confined in a third disk).

Consider the case of noisy measurements, when

$$u = \varphi(\sigma + \eta_0(t), \; \dot{\sigma} + \eta_1(t)), \qquad (8)$$

where η_0, η_1 are some unknown bounded Lebesgue-measurable functions. Controller (5) is called further *weakly robust to input noises* if for some ε_0, $\varepsilon_1 > 0$ the restrictions $|\eta_0| \leq \varepsilon_0$, $|\eta_1| \leq \varepsilon_1$ cause controller (8) to be *uniformly* weakly stable (i.e. the three disks are the same for all possible noises $\eta_0(t)$, $\eta_1(t)$).

Most of known 2-sliding controllers [1, 2, 7] satisfy these properties. The homogeneity property (7) allows to prove their convergence in a very general way.

Theorem 1. *Let controller (5) be specially homogeneous and weakly stable. Then controller (5) provides for finite-time convergence of any trajectory of (1), (3) into the 2-sliding mode $\sigma = \dot{\sigma} = 0$, the convergence time being a locally bounded function of the initial conditions in the plane σ, $\dot{\sigma}$.*

Controller (5) requires availability of $\dot{\sigma}$. That information demand may be lowered if measurements are carried out at times t_i with constant step $\tau > 0$. Indeed, let

$$u = \varphi(\tau^2\sigma, \Delta\sigma_i), \qquad (9)$$

where $\sigma_i = \sigma(t_i, x(t_i))$, $\Delta\sigma_i = \sigma_i - \sigma_{i-1}$, $t \in [t_i, t_{i+1})$. Mark that identity (7) implies $\varphi(\tau^2\sigma, \Delta\sigma_i) \equiv \varphi(\sigma, \Delta\sigma_i/\tau)$.

Lemma 1. *Suppose that controller (5) is specially homogeneous and weakly robust to input noises, then with discrete measurements controller (9) provides in finite time for keeping the inequalities $|\sigma| < \gamma_0\tau^2$, $|\dot{\sigma}| < \gamma_1\tau$ with some positive constants γ_0, γ_1.*

Lemma 2. *Under the conditions of Lemma 1 let the magnitudes of noises η_0, η_1 be less than $\beta_0\delta$ and $\beta_1\delta^{1/2}$ respectively with some positive constants β_0 and β_1. Then for any $\delta>0$ controller (8) provides in finite time for keeping the inequalities $|\sigma| < \gamma_0\delta$, $|\dot{\sigma}| < \gamma_1\delta^{1/2}$ with some positive constants γ_0, γ_1.*

Remarks. The accuracy described in Lemma 1 is the best possible with discontinuous control and relative degree 2 [7]. Similar results can be formulated for any relative degree r and r-sliding controllers, only the homogeneity conditions (6), (7) are to be reformulated as in [11].

Many known 2-sliding controllers may be considered as particular cases of a generalized controller

$$u = -r_1 \,\mathrm{sign}(\mu_1\dot{\sigma} + \lambda_1|\sigma|^{1/2}\mathrm{sign}\,\sigma) - r_2 \,\mathrm{sign}(\mu_2\dot{\sigma} + \lambda_2|\sigma|^{1/2}\mathrm{sign}\,\sigma). \qquad (10)$$

It is easy to check that with appropriate r_1, r_2 (10) satisfies the conditions of Theorem 1 and the Lemmas. The following theorem is a straight-forward consequence of Theorem 1.

Theorem 2. *Let μ_i, λ_i, $i = 1,2$ be any non-negative numbers, such that $\mu_i^2 + \lambda_i^2 > 0$, $\mu_1^2 + \mu_2^2 > 0$, $\lambda_1^2 + \lambda_2^2 > 0$. Then there are such positive values of r_i that controller (10) provides for finite-time convergence of any trajectory of (1), (3) into the 2-sliding mode $\sigma = \dot{\sigma} = 0$.*

Since under the conditions of Theorem 3 controller (10) satisfies the conditions of the Lemmas, its discrete measurement version

$$u = - r_1 \operatorname{sign}(\mu_1 \Delta\sigma_i + \lambda_1 \tau |\sigma_i|^{1/2} \operatorname{sign} \sigma_i) - r_2 \operatorname{sign}(\mu_2 \Delta\sigma_i + \lambda_2 \tau |\sigma_i|^{1/2} \operatorname{sign} \sigma_i). \quad (11)$$

provides for the accuracy described in Lemma 1. Similarly, its noisy version has the accuracy provided by Lemma 2.

Controller (10) may be considered as a generalization of the so-called twisting controller [7, 2], when the switching takes place on parabolas $\mu\dot\sigma + \lambda|\sigma|^{1/2} \operatorname{sign} \sigma = 0$ instead of the coordinate axes. Consider some particular cases of controllers (10) and (11). With $\mu_1 = \lambda_2 = 0$, $r_1 > r_2 > 0$ achieve the twisting controller

$$u = - (r_1 \operatorname{sign} \sigma + r_2 \operatorname{sign} \dot\sigma),$$

the convergence condition being

$$(r_1 + r_2)K_m - C > (r_1 - r_2)K_M + C, \ (r_1 - r_2)\, K_m > C.$$

Its convergence on the plane σ, $\dot\sigma$ is shown in Fig. 2a. With $\mu_1 = \lambda_2 = 0$, $r_2 > r_1 > 0$ achieve from (11) the drift controller [7, 2] (it does not converge with continuous measurements).

A particular case of the controller with prescribed convergence law [7, 2] arises when $\mu_1 = \mu_2$, $\lambda_1 = \lambda_2$:

$$u = - \alpha \operatorname{sign}(\dot\sigma + \lambda|\sigma|^{1/2} \operatorname{sign} \sigma),$$

where $\alpha = r_1 + r_2$, $\lambda = \lambda_1/\mu_1$, and the convergence condition is

$$\alpha K_m - C > \lambda^2/2$$

(Fig 2b). This controller is very close to the terminal sliding mode controller [14]. An unexpected "twisting" finite-time convergence can appear here if the above condition is not satisfied, but still $\alpha K_m > C$ (Fig. 3b).

A new controller appears when $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$ and all components are nonzero, especially interesting are the cases when

$$r_1 = r_2, \ 4r_1K_m - 2C > \max(\lambda_1^2/\mu_1^2, \lambda_2^2/\mu_2^2).$$

In that case the trajectory is confined between two parabolas (Fig. 2c). The control vanishes in that region.

One may easily construct other 2-sliding controllers. List one more example:

$$u = \min\{R, \max[-R, - \alpha(\dot\sigma/|\sigma|^{1/2} + \lambda \operatorname{sign} \sigma)]\}, \qquad (12)$$

where $R, \alpha, \lambda > 0$, $RK_m - C > \lambda^2/2$, $\alpha\lambda > R$ (Fig. 2d). The control u is not defined here with $\sigma = 0$, but that is not significant, for the solutions of (1), (12) are understood in the Filippov sense [4] and are not influenced by the control values on a set of the measure 0. For example, u may be defined by continuity in time for any point except for the point $\sigma = \dot\sigma = 0$. An interesting feature of controller (12) is that in that case the discontinuity is concentrated at $\sigma = \dot\sigma = 0$, i.e. the control is a continuous function of time till the 2-sliding mode is attained in finite time. Thus in the presence of any measurement error the control signal turns out to be a continuous one!

Moreover, let $\phi(y)$ be any monotonously growing positive function of positive argument, then

$$u = \min\{R, \max[-R, -\alpha(\operatorname{sign}\dot\sigma\, \phi(|\dot\sigma|/|\sigma|^{1/2}) + \phi(\lambda) \operatorname{sign}\sigma]\}$$

provides for finite-time convergence to 2-sliding mode and the accuracies corresponding to Lemmas 1, 2 for any $\lambda > 0$ with R, α sufficiently large.

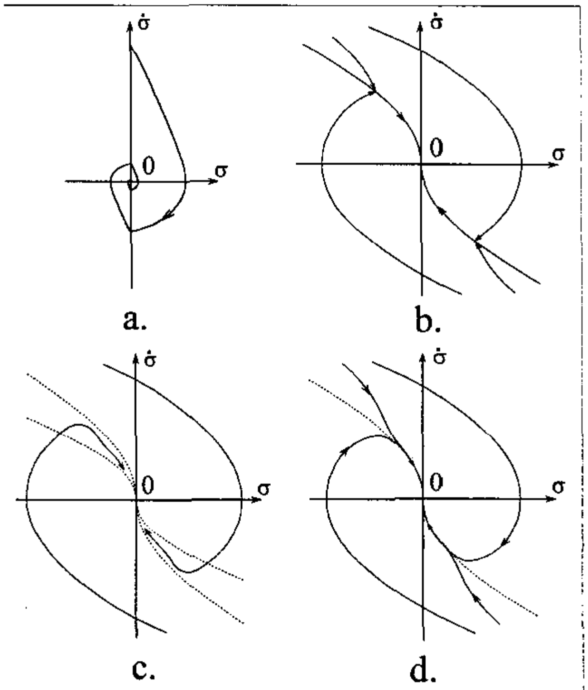

Fig. 2: Convergence of various 2-sliding controllers

Any listed controller $u = \varphi(\sigma, \dot\sigma)$ may also be used with relative degree 1 in order to remove the chattering and improve the sliding accuracy of the standard sliding mode. Indeed, suppose that the original relay control has the form

$$u = - \operatorname{sign} \sigma.$$

Then under certain natural conditions [7] it may be replaced by the chattering-free controller

$$\dot u = \begin{cases} -u, & |u| > 1 \\ \varphi(\sigma, \dot\sigma), & |u| \le 1. \end{cases}$$

3. Robust output-feedback 2-sliding controllers

The described 2-sliding controllers require direct measurement of $\dot\sigma$ which is not always possible. Therefore, first differences are used (Lemma 2). Unfortunately, that decision is also not perfect. Indeed, let the maximum possible error in the measurements of σ be $\delta > 0$. It is easy to check that the algorithm performance is preserved with δ sufficiently small. But it deteriorates when τ decreases or δ increases, for $\dot{\sigma}$ is bounded and the measurement error starts to dominate in the measured difference $\Delta\sigma$. Hence, the sampling time step τ is to be adjusted with respect to the often-unknown noise magnitude δ. Another solution is to make τ a function of measured σ [9], which is effective, but not convenient.

Assume that the function φ in feedback (5) be bounded, then another way is to calculate $\dot{\sigma}$ in real time by means of a robust exact differentiator [8, 12]. Its application is possible due to the boundedness of $\ddot{\sigma}$, as it follows from (4). The resulting controller has the form

$$\begin{aligned} &u = \varphi(z_0, z_1), \\ &\dot{z}_0 = -\lambda_0 |z_0 - \sigma|^{1/2}\operatorname{sign}(z_0 - \sigma) + z_1, \\ &\dot{z}_1 = -\lambda_1 \operatorname{sign}(z_0 - \sigma), \end{aligned} \tag{13}$$

where $\lambda_0 > 0$, $\lambda_1 > C + K_M U_M$ are the parameters, U_M is the correspondent maximal absolute value of the control found from (5). Adjustment of λ_0, λ_1 is described in detail in [8]. One of reasonable choices is to take

$$\lambda_0 = 2\,(C + K_M U_M)^{1/2}, \qquad \lambda_1 = 1.5\,(C + K_M U_M).$$

The outputs z_0, z_1 converge in finite time to σ and $\dot{\sigma}$ respectively in the absence of measurement noise. With the measurement-error magnitude δ the maximal deviations $|z_0 - \sigma|$ and $|z_1 - \dot{\sigma}|$ are proportional in the steady state to δ and $\delta^{1/2}$ respectively [8, 12].

Theorem 3. *Under the conditions of Lemma 1, with bounded function φ and properly chosen differentiator parameters* [8], *controller* (13) *provides in the absence of measurement noises for finite-time convergence of each trajectory to the 2-sliding mode $\sigma = \dot{\sigma} = 0$, otherwise convergence to a set defined by the inequalities $|\sigma| < \gamma_0\delta$, $|\dot{\sigma}| < \gamma_1\delta^{1/2}$ is assured for some positive constants γ_0, γ_1.*

Theorem 4. *Under the conditions of Theorem 3 the discrete-measurement version of controller* (13) *provides in the absence of measurement noises for the inequalities $|\sigma| < \gamma_0\tau^2$, $|\dot{\sigma}| < \gamma_1\tau$ for some $\gamma_0, \gamma_1 > 0$.*

Remarks. Also here similar results can be formulated for any relative degree r and r-sliding controllers, only the homogeneity definitions (6), (7) are to be reformulated as in [11, 12, 13].

4. Sketch of the proofs

Proof of Theorem 1. Let the trajectories of (4), (5) starting in a disk W_0 of the radius R finish in a disk W' of the radius $r < R$ in time T. Denote $\kappa = (r/R)^{1/2} < 1$. It is easy to see applying transformation (6) that $W_0 \supset G_\kappa W_0 \supset W'$. Thus, trajectories starting in W_0 come into $W_1 = G_\kappa W_0$ in time T. Denote $W_j = G_\kappa^j W_0$, $j \in \mathbf{Z}$ and achieve that trajectories starting in W_j in time $\kappa^j T$ finish in W_{j+1} and

$$\ldots \supset W_{-1} \supset W_0 \supset W_1 \supset \ldots, \quad \cup W_j = \mathbf{R}^2, \ \cap W_j = \{O\},$$

where O is the origin. Hence, any trajectory starting in W_j converges in finite time to the origin, the convergence time being estimated by the expression $\kappa^j T(1 + \kappa + \kappa^2 + \ldots) = \kappa^j T/(1 - \kappa)$. A similar procedure is performed for the larger disk containing the trajectories starting in the disk W_0 during the time T. As a result achieve a sequence of embedded sets retracting to the origin O. Thus, any trajectory starting at O has to belong to all of these sets and cannot leave O.

Proof of Lemma 1. Identity (7) implies that $\varphi(\tau^2\sigma, \Delta\sigma_i) \equiv \varphi(\sigma, \Delta\sigma_i/\tau)$, locally $|\Delta\sigma_i/\tau - \dot{\sigma}| \le \tau \sup|\ddot{\sigma}|$. Thus, the weak robustness conditions (Section 2) hold for any sufficiently small τ. It is easy to see that the transformation

$$\hat{G}_\kappa : (t, \sigma, \dot{\sigma}, \tau) \mapsto (\kappa t, \kappa^2\sigma, \kappa\dot{\sigma}, \kappa\tau)$$

preserves discrete sampling and transfers solutions of (4), (9) into solutions of the same inclusion but with different τ. It is proved like for Theorem 1 that with some small τ_0 all trajectories concentrate in some set $|\sigma| \le d_0$, $|\dot{\sigma}| \le d_1$ to stay there forever. Applying now $\hat{G}_\kappa$ with $\kappa = \tau/\tau_0$ achieve the needed asymptotics.

Proof of Lemma 2. A transformation $\hat{G}_\kappa$ is used,

$$\hat{G}_\kappa : (t, \sigma, \dot{\sigma}, \eta_0, \eta_1, \delta) \mapsto (\kappa t, \kappa^2\sigma, \kappa\dot{\sigma}, \kappa^2\eta_0, \kappa\eta_1, \kappa^2\delta),$$

transferring solutions of inclusion (4), (8) into solutions of the same inclusion but with δ changed to $\kappa^2\delta$.

Proof of Theorems 3, 4. Let $\xi_i = z_i - \sigma^{(i)}$, $i = 0,1$, then

$$\begin{aligned} &u = \varphi(\sigma + \xi_0, \dot{\sigma} + \xi_1), \\ &\dot{\xi}_0 \in -\lambda_0 |\xi_0 + [-\delta, \delta]|^{1/2}\operatorname{sign}(\xi_0 + [-\delta, \delta]) + \xi_1, \\ &\dot{\xi}_1 \in -\lambda_1 \operatorname{sign}(\xi_0 + [-\delta, \delta]), \end{aligned} \tag{14}$$

Consider now differential inclusion (4), (14) instead of (4), (13). With $\delta = 0$ variables ξ_0, ξ_1 vanish in finite time [8, 12], thus the first part of the Theorem 3 is a trivial consequence of Theorem 1. Let now the noise magnitude $\delta > 0$. It is easy to see that the transformation

$$\hat{G}_\kappa : (t, \sigma, \dot{\sigma}, \xi_0, \xi_1, \delta) \mapsto (\kappa t, \kappa^2\sigma, \kappa\dot{\sigma}, \kappa^2\xi_0, \kappa\xi_1, \kappa^2\delta)$$

transfers trajectories of (4), (14) into trajectories of (4), (14) but with changed δ. A homogeneity reasoning completes the proof.

5. Simulation results

Consider a variable-length pendulum with motions restricted to some vertical plane. A load of known mass m moves along the pendulum rod (Fig. 3a). Its distance from O equals $R(t)$ and is not measured. There is no friction. An engine transmits a torque u which is considered as control. The task is to track some function x_c given in real time by the angular coordinate x of the rod.

The system is described by the equation

$$\ddot{x} = -2\,\frac{\dot{R}}{R}\,\dot{x} - g\,\frac{1}{R}\sin x + \frac{1}{mR^2}\,u, \qquad (15)$$

where $g = 9.81$ is the gravitational constant, $m = 1$ was taken. Let $0 < R_m \le R \le R_M$, $\dot{R}$, $\ddot{R}$, $\dot{x}_c$ and $\ddot{x}_c$ be bounded, $\sigma = x - x_c$ be available. Following are the functions R and x_c considered in the simulation:

$$R = 1 + 0.25\sin 4t + 0.5\cos t,$$
$$x_c = 0.5\sin 0.5t + 0.5\cos t.$$

The relative degree of the system equals 2. The assumptions (3) are fulfilled only locally here. In order to simplify the demonstration no global free-of-chattering control [7] is constructed here. Thus, the controllers applied further are effective only for some bounded set of initial conditions. The controllers include a real-time differentiator and have the form

$$u = \varphi(z_0, z_1),$$
$$\dot{z}_0 = -6|z_0 - \sigma|^{1/2}\operatorname{sign}(z_0 - \sigma) + z_1, \qquad (16)$$
$$\dot{z}_1 = -35\operatorname{sign}(z_0 - \sigma), \qquad (17)$$

where z_0, z_1 are real-time estimations of σ, $\dot{\sigma}$ respectively. Differentiator (16), (17) is exact for input signals σ with second derivative not exceeding 30 in absolute value.

The initial conditions are $x(0) = \dot{x}(0) = 0$ were taken, $z_0(0) = x(0) - x_c(0) = -0.5$, $z_1(0) = 0$, the sampling step τ and the integration steps being the same, $\tau = 0.0001$. The Euler integration method was used, for it is the only method valid for sliding-mode simulation.

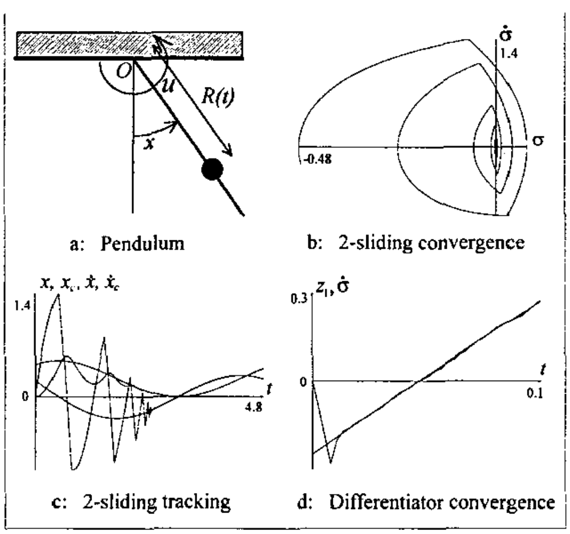

Fig. 3: Pendulum and performance of controller (16) - (18)

Consider a controller of form (10)

$$u = -6\operatorname{sign}(z_1 + 6|z_0|^{1/2}\operatorname{sign} z_0). \qquad (18)$$

The magnitude of the control is not sufficiently large here to establish a 1-sliding mode on the curve $\dot{\sigma} + 6|\sigma|\operatorname{sign}\sigma = 0$, nevertheless the 2-sliding mode $\sigma = \dot{\sigma} = 0$ is established here in finite time according to Theorem 1. The phase trajectories in the plane σ, $\dot{\sigma}$ and the first 0.1 seconds of the differentiator convergence are shown in Fig. 3b,c respectively, the corresponding accuracies being $|\sigma| = |x - x_c| \le 6.4\cdot 10^{-6}$, $|\dot{\sigma}| = |\dot{x} - \dot{x}_c| \le 6.4\cdot 10^{-3}$.

Consider a controller of form (12)

$$u = \min\{6, \max[-6, -5(z_1/|z_0|^{1/2} + 2\operatorname{sign} z_0)]\}. \qquad (19)$$

The trajectory and 2-sliding tracking performance in the absence of noise are shown in Figs. 4a,b, the corresponding accuracies being $|\sigma| = |x - x_c| \le 9.1\cdot 10^{-6}$, $|\dot{x} - \dot{x}_c| \le 2.0\cdot 10^{-3}$. After the sampling step τ was reduced from 10^{-4} to 10^{-5} the resulting accuracies changed to $|x - x_c| \le 9.6\cdot 10^{-8}$, $|\dot{x} - \dot{x}_c| \le 2.2\cdot 10^{-4}$ which corresponds to the statement of Theorem 4.

The tracking results and the differentiator performance in the presence of a noise with the magnitude 0.01 are demonstrated in Figs. 4c,d respectively, the corresponding accuracies being $|\sigma| = |x - x_c| \le 0.011$, $|\dot{x} - \dot{x}_c| \le 0.18$. The noise was a periodic nonsmooth function with nonzero average. The performance does not significantly change when the frequency of the noise varies from 10 to 10000.

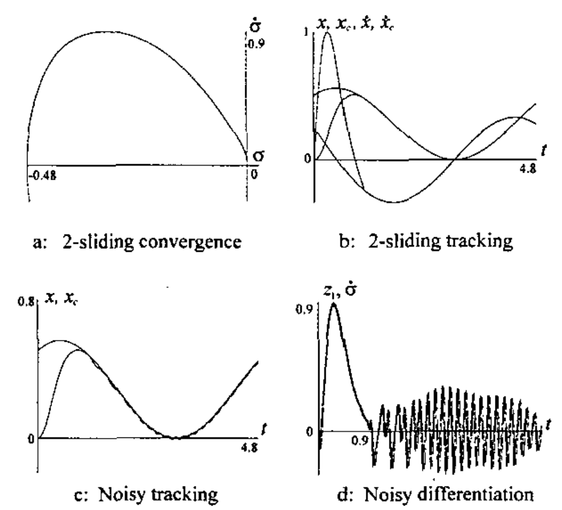

Fig. 4: Simulation of controller (16), (17), (19)

6. Conclusions

A method is proposed of second-order-sliding-mode controller design based on homogeneity reasoning. The resulting controllers feature finite-time convergence and the maximal possible for 2-sliding mode accuracy [7]. A number of new 2-sliding controllers were proposed using the new method which significantly increases the number of known 2-sliding controllers [2, 7].

The construction of a new 2-sliding controller is not difficult. One has only to find a specially-homogeneous weakly-stable controller, which is easy due to the simplicity of the plane geometry. The number of such 2-sliding controllers is obviously infinite.

Theorems 1, 3, 4 and Lemmas 1, 2 are almost literally extended to the case of arbitrary relative degree with corresponding definition of the homogeneity conditions. Unfortunately the very design of new higher-order sliding controllers is much more difficult due to the higher dimension of the problem.

A real-time robust exact differentiator having been used as a standard part of the 2-sliding controllers, the full single-input-single-output control is achieved based on the input measurements only. The only requirements are that the relative degree of the controlled uncertain process be 2 and boundedness restrictions (3) hold globally. Otherwise they are locally applicable for any smooth process of relative degree 2. No exact model of the process is needed.

The resulting robust output-feedback controllers preserve the ultimate accuracy of the original 2-sliding controllers with direct measurements of the input derivative. In the absence of noises the tracking accuracy proportional to τ^2 is provided, τ being a sampling period. That is the best possible accuracy with discontinuous second output derivative [7]. In the presence of a measurement noise the tracking accuracy is proportional to the unknown noise magnitude. That result does not depend on the unknown noise features.

The differentiator is to be used whenever the sampling step can be taken small. At the same time in the practically important case when the sampling step is sufficiently large compared with the noises and the output derivative, the differentiator is successfully replaced by the first finite difference (Lemma 1).

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