Filtering Differentiators and Observers
VSS 2018, Graz, Austria, July 9-11, 2018

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***Abstract*— New finite-time-exact robust (FTER) differentiators, filters and observers based on sliding modes are produced which are capable of filtering out unbounded sampling noises and of the complete FTER output-dynamics observation in the case of known relative degrees and high-frequency gains.**

I. Introduction

Sliding modes (SMs) keep system outputs (sliding variables) at 0 by high-frequency control switching, and are used to suppress system uncertainties [27], [10]. High-order sliding modes (HOSMs) [4], [13] are effective for all relative degrees of sliding variables.

The HOSM theory uses the homogeneity theory [5], [14]. One of the main applications of homogeneous SMs is the finite-time (FT) exact and robust (FTER) differentiation [13]. Such differentiators have found extensive theoretical and practical applications [1], [3], [4], [6], [8], [9], [21], [24].

A general SM-observation lemma is formulated in this paper, which provides numerous modifications of differentiators [13]. New FTER differentiators filter out certain *unbounded* noises. Effectively rejectable noises are studied. These results extend the ideas [20], where such simplest FTER differentiator extracts equivalent control in SM.

High-order differentiation of signals corrupted by large Gaussian noises is demonstrated. A FTER observer of $y, \dot y, ..., y^{(r-1)}, h$ is proposed for the system $y^{(r)} = h(t) + u$, $|\dot h| \le L$, provided r, L are known, y, u are measured. It allows the output-feedback application of the generalized-super-twisting ("continuous twisting" [26]) control.

Application of the new differentiator in the 3rd-order SM (3-SM) output-feedback car control [15] is shown to significantly improve the system accuracy and robustness.

II. Preliminaries

Notation. For any $w, \gamma \in \mathbb{R}$ denote $\lfloor w \rceil^{\gamma} = |w|^{\gamma} \operatorname{sign} w$ if $\gamma > 0$ or $w \ne 0$; let $\lfloor w \rceil^{0} = \operatorname{sign} w$.

Standard differentiator. Let $f(t)$, $f: \mathbb{R} \to \mathbb{R}$, be n times continuously differentiable, $n \ge 0$, $f^{(n)}$ being Lipschitz with $|f^{(n+1)}| \le L$. The differentiator [13] is described by the Filippov differential equations (DEs) [11]

$$
\begin{aligned}
\dot z_0 &= -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1, \\
\dot z_1 &= -\lambda_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} + z_2, \\
&\dots \\
\dot z_{n-1} &= -\lambda_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} + z_n, \\
\dot z_n &= -\lambda_0 L \operatorname{sign}(z_0 - f(t)).
\end{aligned} \tag{1}
$$

Solutions z_i in FT exactly estimate $f^{(i)}$, $i = 0, ..., n$, for proper parameters $\lambda_i > 0$. A Lyapunov function for the error dynamics of (1) has been recently found [7], but the calculation of λ_i leads to redundantly large values for $n \ge 2$.

It is proved [13] that there is an infinite positive sequence $\tilde\lambda_0, \tilde\lambda_1, ...$, such that for any n the formulas $\lambda_0 = \tilde\lambda_0$, $\lambda_n = \tilde\lambda_n$, and $\lambda_j = \tilde\lambda_j {\lambda_{j+1}}^{j/(j+1)}$, $j = n-1, n-2, \dots, 1$, define a valid parametric set for the FT convergence of (1). Such sequence is built recursively, starting with any $\tilde\lambda_0 > 1$. For each n the value $\tilde\lambda_{n+1}$ is simply taken sufficiently large.

The sequence has been experimentally found at least up to $n = 7$, $\{\tilde\lambda_i\} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$ [14]. The corresponding parameters are listed in Table I. In the sequel some proper parametric set $\{\lambda_i\}$ is assumed fixed.

TABLE I

Parameters $\lambda_0, \lambda_1, ..., \lambda_n$ for $n = 0, 1, ..., 7$

0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

Alternative parametric sets λ_i, $i = 0, ..., n$, are listed in [23] for $n \le 10$. Higher order parameters is difficult to find due to digital accuracy restrictions.

Weighted homogeneity. A solution of a differential inclusion (DI) $\dot x \in F(x)$, $x \in \mathbb{R}^{n_x}$, is any locally absolutely continuous function $x(t)$, satisfying the DI for almost all t. A DI $\dot x \in F(x)$ is called *Filippov DI*, if $F(x)$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function [11]. The latter means that the maximal distance from the points of $F(x)$ to the set $F(y)$ tends to zero, as $x \to y$.

It is well-known that such DIs have most standard features, including existence and extendability of solutions, except the uniqueness of solutions [11].

Introduce the weights $m_1, m_2, \dots, m_{n_x} > 0$ of the coordinates $x_1, x_2, \dots, x_{n_x}$ in $\mathbb{R}^{n_x}$. Define the dilation

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}),$$

where $\kappa > 0$. Recall [2] that a function $f: \mathbb{R}^{n_x} \to \mathbb{R}$ is said to have the homogeneity degree (weight) $q \in \mathbb{R}$, $\deg f = q$, if the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds for any x and $\kappa > 0$.

The homogeneity of a vector-set field $F(x)$ is defined as the invariance of the DI $\dot x \in F(x)$ with respect to the

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transformation $(t,x) \mapsto (\kappa^{-q}t, d_\kappa x)$, $\kappa > 0$, where $-q$ might be understood as the weight of t.

Hence, a vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^{n_x}$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $\kappa^q d_\kappa F(x) = F(d_\kappa x)$ holds for any x and $\kappa > 0$ [14]. Any positive-definite continuous function $||x||_h$ of the homogeneity degree 1 is called a *homogeneous norm* of x.

III. Observation lemma

Fix any $L > 0$ and consider the disturbed system

$$
\begin{aligned}
\dot{\omega}_0 &= -\lambda_n L^{\frac{1}{n+1}} \lfloor \omega_0 + \zeta_0(t) + \eta_0(t) \rceil^{\frac{n}{n+1}} + \omega_1 \\
&\qquad\qquad + \zeta_1(t) + \eta_1(t), \\
\dot{\omega}_1 &= -\lambda_{n-1} L^{\frac{2}{n+1}} \lfloor \omega_0 + \zeta_0(t) + \eta_0(t) \rceil^{\frac{n-1}{n+1}} + \omega_2 \\
&\qquad\qquad + \zeta_2(t) + \eta_1(t), \\
&\dots \\
\dot{\omega}_{n-1} &= -\lambda_1 L^{\frac{n}{n+1}} \lfloor \omega_0 + \zeta_0(t) + \eta_0(t) \rceil^{\frac{1}{n+1}} + \omega_n \\
&\qquad\qquad + \zeta_n(t) + \eta_n(t), \\
\dot{\omega}_n &= -\lambda_0 L \,\mathrm{sign}(\omega_0 + \zeta_0(t) + \eta_0(t)) + \zeta_{n+1}(t).
\end{aligned} \tag{2}
$$

of Filippov DEs [11]. The parameters $\lambda_i > 0$, $i = 0, ..., n$, are chosen as above, $\eta(t) \in \mathbb{R}^{n+1}$ represents bounded Lebesgue-measurable noises, $|\eta_i| \le \varepsilon_i$. The disturbance $\zeta(t) \in \mathbb{R}^{n+2}$ is a locally-bounded Lebesgue-measurable function to be further specified.

Let $f : \mathbb{R} \to \mathbb{R}$ be d times continuously differentiable, $0 \le d \le n$, $f^{(d)}$ being a locally absolutely continuous function with a locally bounded derivative. We write $\zeta = Z(f, i_0, i_1)$, $0 \le i_0 < i_1 = i_0 + d + 1 \le n+1$, $i_1 - i_0 = d+1$, if $\zeta_i \equiv 0$ for any $i \ne i_0, i_1$, and $\zeta_{i_0} = -f(t)$, $\zeta_{i_1} = f^{(d+1)}(t)$. That means that $Z(f, i_0, i_1) = (0, ..., 0, -f, 0, ..., 0, f^{(i_1 - i_0)}, 0, ..., 0)^T$.

We also allow $i_1 = i_0$ in which case $\zeta_{i_0} = f$ is taken, all other components ζ_i of $Z(f, i_0, i_0)$ are zero, and f is a locally bounded Lebesgue-measurable function.

Denote (2) by $\dot{\omega} = \Omega_n(\omega, \zeta, \eta)$. Introduce a delay. Then solutions of the delayed system satisfy

$$
\dot{\omega}(t) \in \Omega_n(\omega(t - [0,\tau]), \zeta(t - [0,\tau]), \eta(t - [0,\tau])),\ \tau \ge 0. \tag{3}
$$

Note that in the systems based on sampling of the noisy signals $\zeta(t) + \eta(t)$ the velocity $\dot{\omega}(t)$ only depends on the last sampled values, and the current state and the signals. If the first sampling has been at $t = 0$ the system behavior for $t \ge 0$ never depends on the system history before $t = 0$.

Assumption 1: The right-hand side of the delayed system (3) does not depend on the values of $z(t), \zeta(t), \eta(t)$ for $t < 0$.

Lemma 1: Let the parameters λ_i, $i = 0, 1, ..., n$, be chosen as described above, and Assumption 1 hold. Consider the delayed system (3) with the composite disturbance $\zeta = Z(f_1, i_{10}, i_{11}) + ... + Z(f_k, i_{k0}, i_{k1}) + Z(g, n+1, n+1)$, where $i_{j0} < i_{j1}$ for $j = 1, ..., k$, $|\eta_j| \le \varepsilon_j$ and $|g(t)| \le L$. Then for any initial conditions and any extendable in time solution of (3) the functions $\sigma_i = \omega_i - \sum_{i,j: i_{j0} \le i < i_{j1}} f_j^{(i - i_{j0})}(t)$ in finite time stabilize in the region

$$
|\sigma_i| \le \mu_i L \rho^{n+1-i},\ \rho = \max\{\tau, (\tfrac{\varepsilon_j}{L})^{\frac{1}{n+1-j}},\ j = 0, 1, ..., n\}
$$

for some $\mu_i > 0$ only dependent on λ_i, $i = 0, ..., n$. In particular σ_i in FT converge to identical zero for $\varepsilon = \tau = 0$.

Proof: Subtract $\sum_{i,j: i_{j0} \le i < i_{j1}} f_j^{i - i_{j0}}(t)$ from the both sides of the equation for $\dot{\omega}_i$ of (2) and divide it by L. Taking $\tilde{\sigma}_i = \sigma_i / L$ obtain

$$
\begin{aligned}
\dot{\tilde{\sigma}}_0 &= -\lambda_n \left\lfloor \tilde{\sigma}_0 + \tfrac{\eta_0(t)}{L} \right\rceil^{\frac{n}{n+1}} + \tilde{\sigma}_1 + \tfrac{\eta_1(t)}{L}, \\
\dot{\tilde{\sigma}}_1 &= -\lambda_{n-1} \left\lfloor \tilde{\sigma}_0 + \tfrac{\eta_0(t)}{L} \right\rceil^{\frac{n-1}{n+1}} + \tilde{\sigma}_2 + \tfrac{\eta_2(t)}{L}, \\
&\dots \\
\dot{\tilde{\sigma}}_{n-1} &= -\lambda_1 \left\lfloor \tilde{\sigma}_0 + \tfrac{\eta_0(t)}{L} \right\rceil^{\frac{1}{n+1}} + \tilde{\sigma}_n + \tfrac{\eta_n(t)}{L}, \\
\dot{\tilde{\sigma}}_n &= -\lambda_0 \,\mathrm{sign}(\tilde{\sigma}_0 + \tfrac{\eta_0(t)}{L}) + \tfrac{g(t)}{L}.
\end{aligned} \tag{4}
$$

Taking into account the bounds of η and g obtain that $\tilde{\sigma}$ satisfies the DI

$$
\begin{aligned}
\dot{\tilde{\sigma}}_0 &\in -\lambda_n \lfloor \tilde{\sigma}_0 + [-\tfrac{\varepsilon_0}{L}, \tfrac{\varepsilon_0}{L}] \rceil^{\frac{n}{n+1}} + \tilde{\sigma}_1 + [-\tfrac{\varepsilon_1}{L}, \tfrac{\varepsilon_1}{L}], \\
\dot{\tilde{\sigma}}_1 &\in -\lambda_{n-1} \lfloor \tilde{\sigma}_0 + [-\tfrac{\varepsilon_0}{L}, \tfrac{\varepsilon_0}{L}] \rceil^{\frac{n-1}{n+1}} + \tilde{\sigma}_2 + [-\tfrac{\varepsilon_2}{L}, \tfrac{\varepsilon_2}{L}], \\
&\dots \\
\dot{\tilde{\sigma}}_{n-1} &\in -\lambda_1 \lfloor \tilde{\sigma}_0 + [-\tfrac{\varepsilon_0}{L}, \tfrac{\varepsilon_0}{L}] \rceil^{\frac{1}{n+1}} + \tilde{\sigma}_n + [-\tfrac{\varepsilon_n}{L}, \tfrac{\varepsilon_n}{L}], \\
\dot{\tilde{\sigma}}_n &\in -\lambda_0 \,\mathrm{sign}(\tilde{\sigma}_0 + [-\tfrac{\varepsilon_0}{L}, \tfrac{\varepsilon_0}{L}]) + [-1, 1].
\end{aligned} \tag{5}
$$

The last inclusion of (5) is understood for $\tilde{\sigma}_0 \in [-\varepsilon_0/L, \varepsilon_0/L]$ as $\dot{\tilde{\sigma}}_n \in [-1 - \lambda_0, 1 + \lambda_0]$, which provides for the upper semicontinuity of DI (5).

In the case $\varepsilon = 0 \in \mathbb{R}^{n+1}$, DI (5) describes the error dynamics of (1) for $\tilde{\sigma}_i = (z_i - f^{(i)})/L$. It is homogeneous of the degree -1 with $\deg \tilde{\sigma}_i = n + 1 - i$, and is FT stable (FTS) for the chosen parameters [13].

Denote system (5) by $\dot{\tilde{\sigma}} \in \Sigma(\tilde{\sigma}, \varepsilon/L)$. Consider the system with variable delay $\dot{\tilde{\sigma}}(t) \in \Sigma(\tilde{\sigma}(t - [0,\tau]), \varepsilon/L)$, $\tau \ge 0$. Some non-resrictive conditions on initial values are required here [18], [17]. These conditions trivially hold due to Assumption 1.

According to Theorem 1 of [18] all extendable in time solutions of the delayed system in FT converge into the region $||\tilde{\sigma}||_h \le \mu \max[||\varepsilon/L||_h, \tau]$ for some $\mu > 0$ (this formula is only valid for the homogeneity degree -1). ■

Note that piece-wise linear-in-time solutions generated by the Euler integration of (2) with variable steps not exceeding τ always satisfy conditions of the Lemma.

Example. Let $f(t) = f_0(t) - \eta_0(t)$, $\varepsilon_1 = ... = \varepsilon_n = 0$, $g = -f_0^{(n+1)}(t)$, $\zeta = Z(f_0, 0, n+1) + Z(g, n+1, n+1)$. Then after changing the notation, $z_i = \omega_i$, obtain the standard differentiator (1). Correspondingly $\sigma_i = z_i - f_0^{(i)}$, and Lemma 1 implies the convergence and the standard accuracy [13] of the differentiator (1).

IV. New differentiators

Assumption 2: Let the input signal $f(t) = f_0(t) + \nu(t)$, $t \ge 0$, consist of an unknown **locally** (essentially) bounded Lebesgue-measurable noise ν with unknown features, and an unknown basic signal f_0, whose $(n_d + 1)$th derivative has a known Lipschitz constant $L > 0$, $n_d \ge 0$.

Introduce the number $n_f \ge 0$ which is further called *the filtering order*. Correspondingly, n_d is further called *the differentiation order*. Let $n = n_d + n_f$. The new differentiator is formally defined as the standard one (1) for $n_f = 0$, otherwise for $n_f \geq 1$ it gets the new form

$$\begin{aligned}
&\dot{w}_1 = -\lambda_n L^{\frac{1}{n+1}} \lfloor w_1 \rceil^{\frac{n}{n+1}} + w_2,\ n = n_f + n_d,\\
&\quad \dots\\
&\dot{w}_{n_f-1} = -\lambda_{n_d+2} L^{\frac{n_f-1}{n+1}} \lfloor w_1 \rceil^{\frac{n_d+2}{n+1}} + w_{n_f},\\
&\dot{w}_{n_f} = -\lambda_{n_d+1} L^{\frac{n_f}{n+1}} \lfloor w_1 \rceil^{\frac{n_d+1}{n+1}} + z_0 - f(t),\\
&\dot{z}_0 = -\lambda_{n_d} L^{\frac{n_f+1}{n+1}} \lfloor w_1 \rceil^{\frac{n_d}{n+1}} + z_1,\\
&\quad \dots\\
&\dot{z}_{n_d-1} = -\lambda_1 L^{\frac{n}{n+1}} \lfloor w_1 \rceil^{\frac{1}{n+1}} + z_{n_d},\\
&\dot{z}_{n_d} = -\lambda_0 L\ \mathrm{sign}(w_1).
\end{aligned} \tag{6}$$

Note that taking $\zeta = Z(f_0, n_f, n+1) + Z(-f_0^{n_d+1}, n+1, n+1)$, $\eta_{n_f} = \nu$, $\eta_i = 0$ for any $i \neq n_f$, and $w_k = \omega_{k-1}$, $k = 1, ..., n_f$, $z_k = \omega_{k+n_f}$, $k = 0, 1, ..., n_d$, obtain the general observer form (2). Correspondingly, Lemma 1 implies that, provided $|\nu(t)| \leq \delta$, the inequalities

$$\begin{aligned}
&|w_1| \leq \mu_{w1} L\rho^{n_f+n_d+1},\ \dots,\ |w_{n_f}| \leq \mu_{wn_f} L\rho^{n_d+2};\\
&|z_0 - f_0(t)| \leq \mu_0 L\rho^{n_d+1},\\
&\quad \dots,\\
&|z_{n_d} - f_0^{(n_d)}(t)| \leq \mu_{n_d} L\rho
\end{aligned} \tag{7}$$

are kept after a FT transient for $\rho = (\delta/L)^{1/(n_d+1)}$. It means that (6) describes *an alternative asymptotically optimal [19] n_dth-order differentiator*. Remarkably this differentiator has new significant filtering properties.

A (noise) function $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called a *signal of the filtering order* $k \geq 0$ if ν is a locally (essentially) bounded Lebesgue-measurable function, and there exists a uniformly bounded solution $\xi(t)$ of the equation $\xi^{(k)} = \nu$. Any number exceeding $\sup |\xi(t)|$ is called the *kth-order integral magnitude of* ν.

Remark 1: The filtering order of a signal is vulnerable to discrete sampling or *variable* delay. Indeed, any alternating signal ± 1 with infinitesimally small integral can be sampled as a constant signal $+1$.

Assumption 3: Considered variable delays preserve the filtering order and the integral magnitude of the considered noises.

The above assumption is definitely restrictive. It requires that the sampling reliably represent the input signal. The subject is directly treated in the discretization section VI.

In particular the standard differentiator is robust with respect to the noises $\nu(t)$ of the filtering order 0 [13]. The following theorem shows that differentiators (6) of the filtering order n_f are robust with respect to the *possibly unbounded* noises of the filtering order not exceeding n_f.

Theorem 1: Under Assumptions 1-3 let $\nu = \nu_0 + ... + \nu_{n_f}$, ν_k being a noise of the filtering order k and the integral magnitudes δ_k, $k = 0, 1, ..., n_f$. Let also the input be sampled with the sampling step not exceeding $\tau \geq 0$. Then all solutions of (6) in FT provide for the accuracy (7), where $\{\mu_i\}$, $\{\mu_{wi}\}$ only depend on $\{\lambda_i\}$ and

$$\rho = \max[(\tfrac{\delta_0}{L})^{\frac{1}{n_d+1}}, (\tfrac{\delta_1}{L})^{\frac{1}{n_d+2}}, ..., (\tfrac{\delta_{n_f}}{L})^{\frac{1}{n_d+n_f+1}}, \tau]. \tag{8}$$

Assumptions 1, 3 are trivial for $\tau = 0$ which corresponds to the continuous (not discrete) sampling. Assumption 1 is not restricting also for $\tau > 0$. Assumption 3 always holds for constant delays.

The proposed differentiator with $n_d > 0$ is never worse than the standard differentiator (i.e. one with $n_d = 0$). Assumption 3 can be difficult to check, but the differentiator will "itself detect" whether it holds.

Proof: Let $\xi_k^{(k)} = \nu_k$, $|\xi_k| \leq \delta_k$, $k = 0, 1, ..., n_f$. Consider the general observer (2) of the order n and the disturbance $\zeta = Z(-\xi_1, n_f - 1, n_f) + ... + Z(-\xi_{n_f}, 0, n_f) + Z(f_0, n_f, n+1)$ and the noise $\eta_i = \xi_{n_f - i}$, $i = 0, ..., n_f - 1$, $\eta_i = 0$, $i \geq n_f$, where $\xi_0 = \nu_0$. The theorem now follows from Lemma 1. ■

It is not always easy to check the filtering order of the noise. The following simple Lemma helps in that case. Introduce the functional

$$I_j(\nu, t_0, t_1) = \underbrace{\int_{t_0}^{t_1} ... \int_{t_0}^{s_3} \int_{t_0}^{s_2}}_{j} \nu(s_1) ds_1 ds_2 ... ds_j \tag{9}$$

mapping any function ν integrable on $[t_0, t_1]$ into $\mathbb{R}$, $j = 1, 2, ...$. Obviously $\xi = I_j(\nu, t_0, t)$ satisfies $\xi^{(j)} = \nu$, $\xi(t_0) = \xi'(t_0) = ... = \xi^{(j-1)}(t_0) = 0$.

Lemma 2: Let the noise $\nu(t)$ satisfy Assumption 2 and the inequalities

$$|I_j(\nu, t_0, t_1)| \leq \delta_j,\ j = 1, 2, ..., k, \tag{10}$$

for some $T, \delta_1, ..., \delta_k > 0$ and for any t_0, t_1, $0 \leq t_1 - t_0 \leq T$, $t_0 \geq 0$. Then it can be represented as $\nu = \nu_0 + \nu_k$, where ν_0, ν_k are some noises of the filtering orders $0, k$ and the integral magnitudes $\tilde{\delta}_l = \sum_{j=0}^{k} \alpha_{l,j} \delta_j$ for some coefficients $\alpha_{l,j} > 0$, $l = 0, k$. One can choose $\nu_0 \in C^\infty$.

Call (10) the *jth-order integral boundedness condition*. In particular, if the discontinuity of ν_0 is allowed, for $k = 1$ get $\tilde{\delta}_0 = \delta_1/T$, $\tilde{\delta}_1 = 2\delta_1$. Indeed, it is enough to take

$$\begin{aligned}
&\nu_0(t) = \tfrac{1}{T} \textstyle\int_{mT}^{(m+1)T} \nu(s) ds,\ \nu_1 = \nu - \nu_0\\
&\text{for } t \in [mT, (m+1)T),\ m = 0, 1, ...
\end{aligned}$$

Obviously, the uniform boundedness or the 1st-filtering-order condition trivially imply the jth-integral-boundedness-order condition, and that in its turn implies the j_1th-integral-boundedness-order condition for any $j_1 > j$. Thus according to the lemma any signal of the 1st filtering order or satisfying the 1st-order integral-boundedness condition can be represented as the sum of a signal of any predefined filtering order and a bounded smooth signal. Hence, differentiators of any filtering order are applicable in these cases.

The concrete choice of applied differentiator should be the result of some accuracy optimization. In no case it is possible to improve the optimal differentiator accuracy with respect to general *bounded* noises [19].

Proof: Introduce the function

$$\Phi(s) = \begin{cases} 0 & \text{for } s \leq 0,\\ e^{-\frac{1}{s^2} - \frac{1}{(s-1)^2}} & \text{for } 0 < s < 1,\\ 0 & \text{for } s \geq 1. \end{cases} \tag{11}$$

Obviously $\Phi \in C^\infty$, $\Phi^{(l)}(0) = \Phi^{(l)}(1) = 0$ for $l = 0, 1, 2, ...,$ $0 < \Phi(s) < 1$ for $s \in (0,1)$. Note that $I_1(\Phi(\frac{\cdot}{T}), 0, T) = T\, I_1(\Phi(\cdot), 0, 1)$.

Build functions ν_0, ν_k such that $I_j(\nu_k, t_0, t_1) \le \tilde{\delta}_j$ for any t_0, t_1, $0 \le t_1 - t_0 \le T$, $j = 0, ..., k$, and $\nu_k(mT) = 0$ for any $m \ge 0$, $m \in \mathbb{Z}$.

First let $\nu_{0,0} = 0, \nu_{k,0} = \nu$. Now for any $j = 1, 2, ..., k$ and any $t \in [0,T]$ define

$$\begin{aligned}&\nu_{0,j}(t) = \nu_{0,j-1}(t) + T^{-(j+1)}\Phi^{(j+1)}(\tfrac{t}{T})\tfrac{I_{j+1}(\nu_{1,j-1},0,T)}{T\, I_1(\Phi,0,1)},\\ &\nu_{k,j}(t) = \nu(t) - \nu_{0,j}(t).\end{aligned}$$

Let $\nu_0 = \nu_{0,k}, \nu_k = \nu_{k,k} = \nu - \nu_{0,k}$. Obviously ν_0 is smooth and bounded, $I_j(\nu_k, 0, T) = 0$, $j = 1, 2, ..., k$. Define the solution $\xi(t)$ of the equation $\xi^{(k)} = \nu_k$ on $[0,T]$ as $\xi(t) = I_k(\nu_k, 0, t)$. Now by the same procedure extend the functions ν_0, ν_k, ξ to the segment $[T, 2T]$, etc. ■

V. Control applications

Equivalent control extraction. It follows from Lemma 2 and Theorem 1 that differentiator (6) can extract the equivalent contol $u_{eq}(t)$ from the chattering SM control $u(t)$, provided $n_f > 0$, $n_d \ge 0$, $|u_{eq}^{(n_d+1)}| \le L$. Indeed, $u(t)$ can be considered as u_{eq} corrupted by a noise of the filtering order 1 [20]. One simply differentiates $u(t)$ [20].

FT SM observation. Consider a SISO system $\dot{x} = a(t,x) + b(t,x)u$ of the relative degree r with the output $y(t,x)$, $x \in \mathbb{R}^{n_x}$, $y : \mathbb{R}^{n_x+1} \to \mathbb{R}$ [12]. The task is to make y vanish and keep it at zero.

Assuming the function $\frac{\partial}{\partial u} y^{(r)}$ is perfectly known, after some appropriate redefinition of the control the output dynamics can be rewritten in the form $y^{(r)} = h(t,x) + u$. That is the widespread case in robotics, flight control etc.

Assume that the function $h(t, x(t))$ is uncertain, but possesses a known Lipschitz constant L. Then there are three main SMC strategies to solve the problem for a fixed r by continuous control.

1. The term h is estimated and canceled by the control $u = -h + u_1$. Then y is stabilized by any appropriate control $u_1(y, \dot{y}, ..., y^{(r-1)})$. It is done in [25] for $r = 1$.

2. The output y is stabilized by means of a generalized super-twisting $(r+1)$th-order SM ($(r+1)$-SM) controller $u(y, \dot{y}, ..., y^{(r-1)})$. Such 3-SM controller, only needing availability of $y, \dot{y}$, is constructed in [26] for $r = 2$ (the so-called "continuous twisting controller").

3. Differentiation produces $y^{(r+1)} = \dot{h} + \dot{u}$, $|\dot{h}| \le L$. Now y is stabilized by any standard $(r+1)$-SM control $\dot{u}(y, \dot{y}, ..., y^{(r)})$ [13]. The arguments of the control are produced by the rth-order differentiator (1) or (6). That strategy is not considered here. Note that it is also applicable if the control coefficient is uncertain, but Lipschitzian [16].

The difficulty of the first two strategies is that $\dot{y}, ..., y^{(r-1)}, h$ are in general not available. Indeed, if $u(t)$ is not Lipschitzian, the simple differentiation of y by the rth-order differentiator (6) is not possible.

Since only the observation problem is considered, replace the arguments $(t, x(t))$ with t only, for brevity omitting $x(t)$. Then the output dynamics along the current trajectory is

$$y^{(r)} = h(t) + u(t). \qquad (12)$$

Observation problem. Let y satisfy (12) and be available by its noisy approximation $\hat{y}(t) = y(t) + \nu(t)$. The problem is to estimate $y, \dot{y}, ..., y^{(r-1)}, h$ using the knowledge of $u(t)$ and of the constant $L > 0$, $|\dot{h}| \le L$.

Fix any filtering order $n_f \ge 0$, the differentiation order r and apply the observer

$$\begin{aligned}
\dot{w}_1 &= -\lambda_{r+n_f} L^{\frac{1}{r+n_f+1}} \lfloor w_1 \rceil^{\frac{r+n_f}{r+n_f+1}} + w_2,\\
&\dots\\
\dot{w}_{n_f-1} &= -\lambda_{r+2} L^{\frac{n_f-1}{r+n_f+1}} \lfloor w_1 \rceil^{\frac{r+2}{r+n_f+1}} + w_{n_f},\\
\dot{w}_{n_f} &= -\lambda_{r+1} L^{\frac{n_f}{r+n_f+1}} \lfloor w_1 \rceil^{\frac{r+1}{r+n_f+1}} + z_0 - \hat{y}(t),\\
\dot{z}_0 &= -\lambda_r L^{\frac{n_f+1}{r+n_f+1}} \lfloor w_1 \rceil^{\frac{r}{r+n_f+1}} + z_1,\\
&\dots\\
\dot{z}_{r-1} &= -\lambda_1 L^{\frac{r+n_f}{r+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{r+n_f+1}} + z_r + u(t),\\
\dot{z}_r &= -\lambda_0 L \,\mathrm{sign}(w_1).
\end{aligned} \qquad (13)$$

As previously, in the case $n_f = 0$ the filtering variables w are excluded, and, correspondingly, $u(t)$ is added in the last-but-one equation of the standard differentiator (1) for $n_d = r$, $f(t) = \hat{y}(t)$.

Theorem 2: Let the noise $\nu(t)$ satisfy conditions of Theorem 1, the sampling step not exceed $\tau \ge 0$, $u(t)$ be *locally bounded and Lebesgue-measurable*. Then observer (13) in FT establishes the accuracy

$$\begin{aligned}
&|w_1| \le \mu_{w1} L \rho^{n_f+r+1},\ ...,\ |w_{n_f}| \le \mu_{wn_f} L \rho^{r+2};\\
&|z_0 - y(t)| \le \mu_0 L \rho^{r+1},\\
&\quad ...,\\
&|z_{r-1} - y^{(r-1)}(t)| \le \mu_{r-1} L \rho^2,\\
&|z_r - h(t)| \le \mu_r L \rho,\\
&\rho = \max[(\tfrac{\delta_0}{L})^{\frac{1}{r+1}}, (\tfrac{\delta_1}{L})^{\frac{1}{r+2}}, ..., (\tfrac{\delta_{n_f}}{L})^{\frac{1}{r+n_f+1}}, \tau],
\end{aligned} \qquad (14)$$

where the coefficients $\{\mu_{wi}\}, \{\mu_j\}$ are determined by $\{\lambda_i\}$.

Proof: Consider the general (n_f+r)th-order observer (2) with the disturbances $\zeta = Z(-\xi_1, n_f - 1, n_f) + ... + Z(-\xi_{n_f}, 0, n_f) + Z(y, n_f, n_f + r) + Z(h, n_f + r, n_f + r + 1) + Z(\dot{h}, n_f + r + 1, n_f + r + 1)$, and the generalized noise $\nu_0 = \xi_{n_f}, ..., \nu_{n_f} = \xi_0 = \nu_0$, $\nu_i = 0$ for $i > n_f$. Taking into account $y^{(r)} = h + u$ and the correspondence between ω and (w, z) obtain observer (13). The theorem now directly follows from Lemma 1. ■

VI. Discretization

Discrete sampling and computer realization require numeric integration between the sampling instants. A sampled signal can be considered as the same input signal taken with variable delay.

Let the sampling take place at the times $t_0, t_1, ..., t_0 = 0$, $t_{k+1} - t_k = \tau_k \le \tau$. A signal $\nu(t)$ is said to be of the *discrete filtering order* 1 if $|\sum_{k=0}^{N} \nu(t_k)\tau_k| \le \delta$ holds for some δ and any N.

Correspondingly the 1st-order discrete integral boundedness condition is that for some $\delta_1, T > 0$ and any $t_*, \Delta t \geq 0$, $\Delta \leq T$ implies $|\sum_{t_k \in [t_*, t_*+\Delta t]} \nu(t_k)\tau_k| \leq \delta_1$.

Though the same notions can be defined for any order, we skip it due to the lack of space. The discrete analogue of Lemma 2 is also true.

Denote the (n_f, n_d)-differentiator (6) by $(\dot{w}, \dot{z}) = D_{n_f,n_d}(w, z, L)$. Then the discrete differentiator has the form

$$\begin{aligned}(w(t_{j+1}), z(t_{j+1}))^T &= (w(t_j), z(t_j)) + \\ &D_{n_f,n_d}(w(t_j), z(t_j), L)\tau_j + T_{n_f,n_d}(z(t_j), \tau_j),\end{aligned}$$

$$\begin{pmatrix} T_0 \\ \ldots \\ T_{n_f-1} \\ T_{n_f} \\ \ldots \\ T_{n_f+i} \\ \ldots \\ T_{n_f+n_d-2} \\ T_{n_f+n_d-1} \\ T_{n_f+n_d} \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ \frac{1}{2!} z_2(t_j)\tau_j^2 + \ldots + \frac{1}{n_d!} z_{n_d}(t_j)\tau_j^{n_d} \\ \ldots \\ \sum_{s=i+2}^{n_d} \frac{1}{(s-i)!} z_s(t_j)\tau_j^{s-i} \\ \ldots \\ \frac{1}{2!} z_{n_d}(t_j)\tau_j^2 \\ 0 \\ 0 \end{pmatrix}. \tag{15}$$

Here $T_{n_f,n_d} \in \mathbb{R}^{n_f+n_d+1}$. In particular $T_{n_f,0}(w, z, \tau) = 0 \in \mathbb{R}^{n_f+1}$, $T_{n_f,1}(w, z, \tau) = 0 \in \mathbb{R}^{n_f+2}$.

Theorem 3: Let the noise $\nu = \nu_0 + \nu_1$ consist of the bounded noise ν_0 and of the noise ν_1 of the discrete filtering order 1. Then under Assumption 2 the discrete differentiator (15) provides for the same accuracy asymptotics (7), (8), as its continuous-time analogue (6).
The proof is similar to [20] and is omitted due to the lack of space.

A. Stochastic noise: qualitative analysis

Stochastic noises provide for the important class of discrete-sampling noises representable as the sum of a bounded noise of a small magnitude, and a noise of the filtering order 1 of a small integral magnitude.

The following reasoning is *qualitative*, and with some changes is extended to independent random sampling noises with bounded mean values, identical centralized distributions and a finite quadratic deviation.

Let each noise $\nu(t_k)$ be a Gaussian noise of the distribution $N(0, \sigma_\nu^2)$, and let $\tau_k = t_{k+1} - t_k = \tau$, $k = 0, 1, \ldots$. Denote

$$\Sigma(t_*, t_* + \Delta t) = \sum_{t_k \in [t_*, t_*+\Delta t]} \nu(t_k)\tau;\ N = \text{integer.part}(\tfrac{\Delta t}{\tau}).$$

Then $\Sigma(t_*, t_* + \Delta t)$ is normally distributed with the mean 0 and $\sigma_\Sigma \approx (\tau^2 N \sigma_\nu^2)^{1/2} \approx \sigma_\nu\sqrt{\tau\Delta t}$. Thus $|\Sigma(t_*, t_* + \Delta t)|$ with the probability 0.997 does not exceed $3\sigma_\nu\sqrt{\tau\Delta t}$ and, therefore, $\nu(t_k)$ practically satisfies the first-order integral-boundedness condition for any fixed $T > 0$, $\Delta t \leq T$.

Fix some $T > 0$. Due to Lemma 2 get $\nu = \nu_0 + \nu_1$, where $|\nu_0| \leq \delta_0 = 3\sigma_\nu(\tau/T)^{1/2}$, the integral magnitude of ν_1 is $\delta_1 = 6\tau\sigma_\nu(\tau T)^{1/2}$. Take $T = 1$. Now the accuracy of the discrete differentiator (15) for small τ is provided by (7) for $\rho = 3\frac{\sigma_\nu}{L}\tau^{\frac{1}{2(n_d+2)}}$. I.e. $z_i - f_0^{(i)} = O(L^{\frac{i+1}{n_d+2}}\sigma_\nu^{\frac{n_d+1-i}{n_d+2}}\tau^{\frac{n_d+1-i}{2(n_d+2)}})$, $i = 0, 1, \ldots, n_d$.

VII. Simulation results

Differentiation. Consider the input function, and the differentiation and filtering orders

$$f_0(t) = \sin t - \cos 0.5t, L = 2, n_d = 3, n_f = 2. \tag{16}$$

Differentiator (15) has been applied with the constant sampling step τ and the Gaussian sampling noise of the mean 0 and the standard deviation 5. According to Subsection VI-A the accuracies $z_0 - f_0 = O(\tau^{4/10})$, $z_1 - \dot{f}_0 = O(\tau^{3/10})$, $z_2 - \ddot{f}_0 = O(\tau^{2/10})$, $z_3 - \dddot{f}_0 = O(\tau^{1/10})$ are expected. Results of the differentiation for the sampling steps $\tau = 10^{-4}$ and $\tau = 10^{-7}$ are demonstrated in Fig. 1.

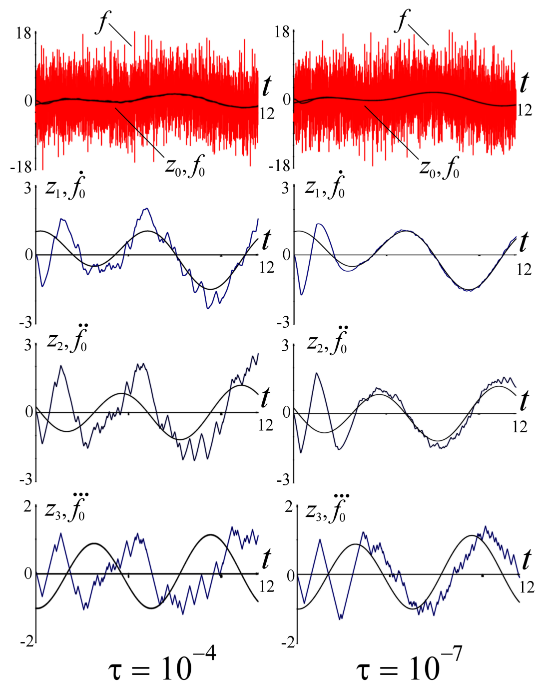

Fig. 1. Differentiation of the signal $\sin t - \cos 0.5t$ corrupted by the Gaussian noise with the standard deviation 5, $n_d = 3$, $n_f = 2$.

Car control. Consider the kinematic model of vehicle motion [22]

$$\begin{aligned} &\dot{x} = V\cos(\varphi), \quad \dot{y} = V\sin(\varphi) \\ &\dot{\varphi} = \tfrac{V}{\Delta}\tan\theta, \quad\ \ \dot{\theta} = u, \end{aligned} \tag{17}$$

where x and y are the Cartesian coordinates of the middle point of the rear axle (Fig. 2a), Δ is the distance between the two axles, φ is the orientation angle, V is the constant longitudinal velocity, θ is the steering angle (i.e. the actual input), and $u = \dot{\theta}$ is the control input.

The goal is to track some smooth trajectory $y = g(x)$, whereas $g(x(t)), y(t)$ are available in real time. That is, the task is to make $s(x, y) = y - g(x)$ as small as possible. The function s is measured with the sampling step $\tau = 0.001s$ and the Gaussian noise of the zero mean and the standard deviation 0.75 meters. Such deviation corresponds to the error magnitude of about $1.5m$, i.e. to the GPS accuracy.

The relative degree is 3. Starting from $t = 1$ and up to $t = 30$ apply the standard 3-SM controller [15]

$$u = -2\frac{z_2 + 2(|z_1| + |z_0|^{\frac{2}{3}})^{-\frac{1}{2}}(z_1 + z_0^{\frac{2}{3}} \operatorname{sign} z_0)}{|z_2| + 2(|z_1| + |z_0|^{\frac{2}{3}})^{\frac{1}{2}}}. \quad (18)$$

Controller inputs z_i are the outputs of the differentiator (15) with $n_d = 2$, $L = 100$ and the input s.

Two cases are considered: the filtering order n_f equals 0 (standard differentiator [13], Fig. 2b) and $n_f = 1$ (filtering differentiator, Fig. 2c). It is clearly seen from the graphs that the control is lost when the standard differentiator is used.

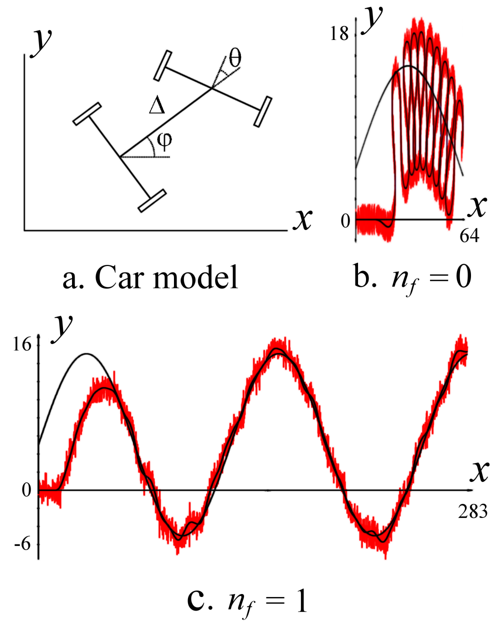

Fig. 2. a: The car model. Car trajectories for the sampling step 10^{-3}. The y coordinate corrupted by the Gaussian noise of the standard deviation $0.75m$ is shown in red. b: The loss of control when the standard differentiator is used. c: The performance of the new differentiator in the feedback.

Note that noisy measurements of $s(x, y) = y - g(x)$ only provide the current sampled width of the containing-the-car layer around the uncertain line $s = 0$. Since it cannot be graphically shown, instead the noise is added in red to the actual car trajectory in the vertical direction (Figs. 2b, c).

VIII. Conclusions

New differentiators feature considerable filtering abilities with respect to *unbounded* noises while preserving the asymptotically optimal accuracy, exactness, homogeneity and FT stability of the standard differentiator [13].

Lemma 1 provides for numerous options of SM observation design. In particular, observer (13) provides for the FT observation of $\dot{y}, ..., y^{(r-1)}, h$ for the system $y^{(r)} = h(t) + u$, $|\dot{h}| \leq L$, provided r, L are known.

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