Black-Box Control
in Theory and Applications
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SISO control problems

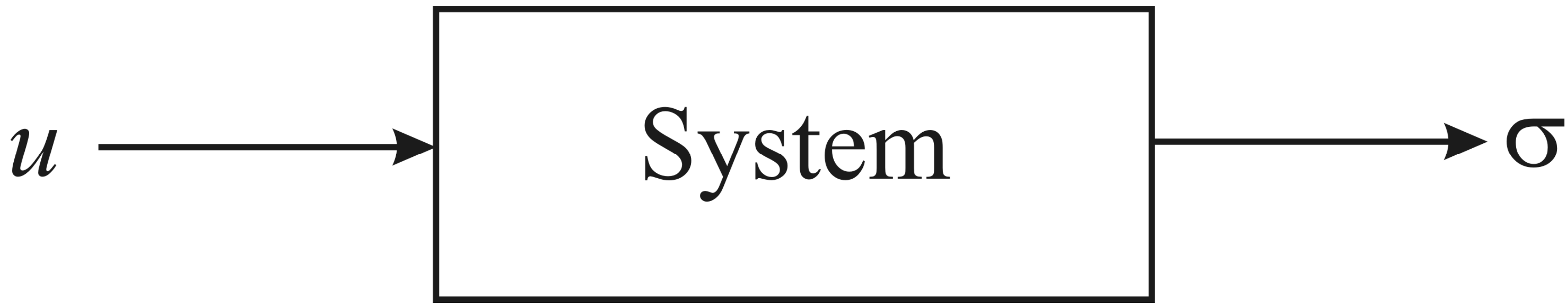
Contr. problems which maybe can be addressed

Finances: Macro-economic control by state bank, Taxes control, etc

Contr. problems which are addressed

Air condition, auto-pilots, keeping bicycle balance, targeting, tracking, orientation, hormonal levels in blood, etc.

The author mostly presents here results obtained with his participation, but he is completely aware of significant results by other researchers.



Tracking deviation: $\sigma = y - y_c(t)$

The goal: $\sigma = 0$

Any solution of the problem should be feasible and robust. We need some PSEUDO-MODEL

"Black Box" Models

1. Sliding-Mode Control (here):

$$\frac{d^r}{dt^r}\sigma = h(t) + g(t)u \; ,$$

$$r \in \mathbb{N},\, h \in [-C, C], \quad g \in [K_m, K_M]$$

2. Model-free control (Fliess, Join, Lafont, et al)

"Ultra-local model"

$$\frac{d^r}{dt^r}\sigma = F + Ku \; , \quad r = 1, 2 \; , \, F, K = const$$

PID (proportional, integral, derivative) control

In order to control a Black Box

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

one should at least identify *r*.

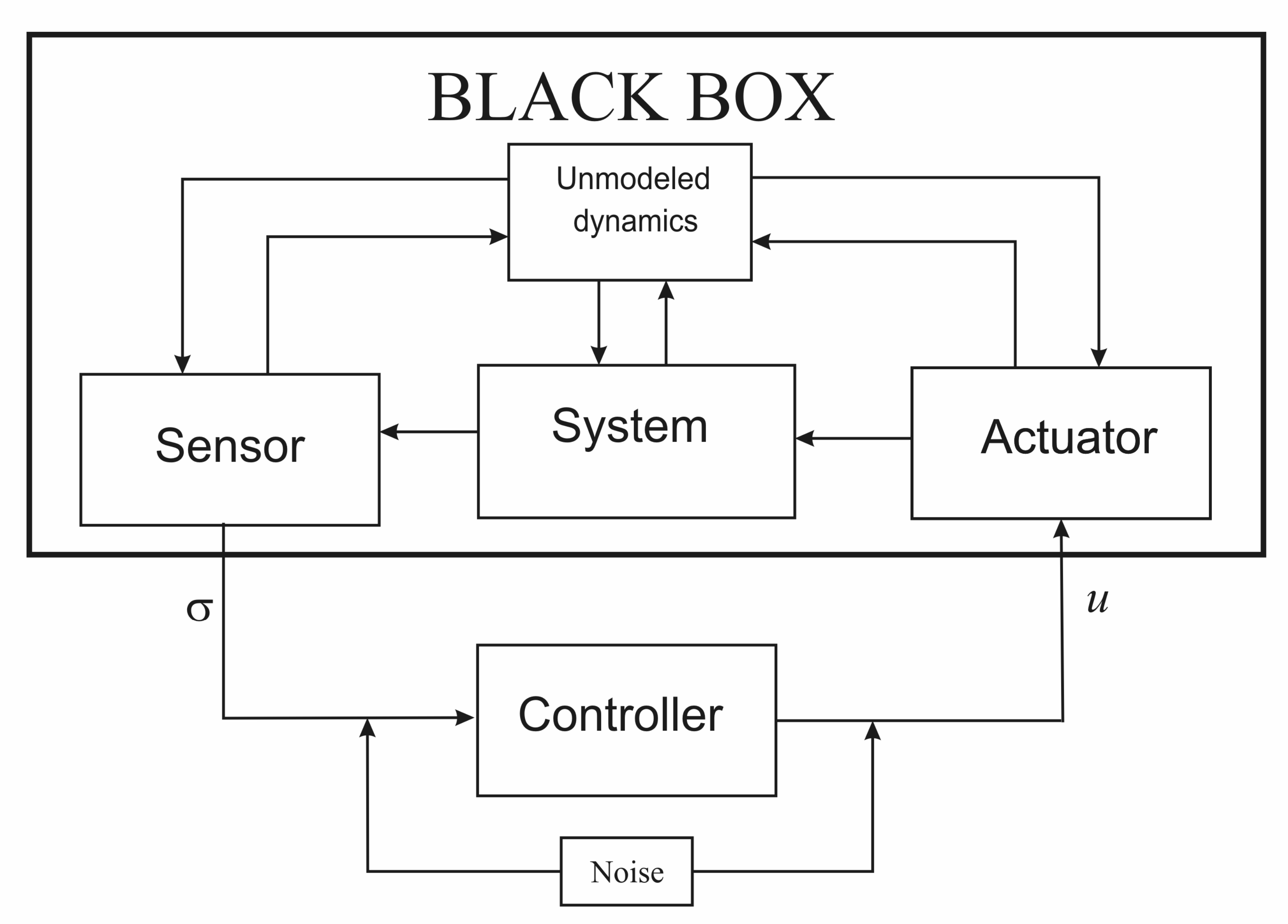
r is called the Practical Relative Degree (PRD)

In the framework by Fliess $r = 1, 2$

We also want some nice features:

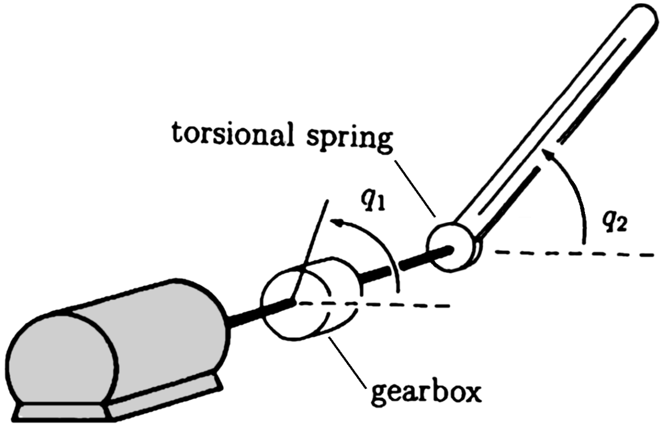
Lipschitzian (even smooth) bounded control

General Control Problem as Black-Box control



Any relative degree is possible

(example by Isidori)



$$J_1\ddot{q}_1+F_1\dot{q}_1-\frac{K}{N}(q_2-\frac{q_1}{N}) = u,$$

$$J_2\ddot{q}_2+F_2\dot{q}_2+K(q_2-\frac{q_1}{N})+mgl\cos q_2 = 0$$

The output is q_2,

$$q_2^{(4)} = \ldots + \frac{K}{NJ_1J_2}u, \quad q_2^{(5)} = \ldots + \frac{K}{NJ_1J_2}\dot{u}$$

The input: u. The relative degree $r = 4$

The input: $\dot{u} = v$. The relative degree $r = 4+1=5$

Any relative degree can be got in such a way.

Inevitable BAD subproblem

$$\dot{z}_0 = z_1,\ \dot{z}_1 = z_2,\ \ldots,\ \dot{z}_{r-2} = z_{r-1},$$

$$\dot{z}_{r-1} = u,\ \ \text{output: }\ y = z_0$$

The goal: $\sigma = y(t) - f(t) = 0$

$$\sigma^{(r)} = f^{(r)}(t) + u$$

If $\sigma \equiv 0$ then

$$z_i = f^{(i)}(t)\ ,\ i = 0,1,\ldots,r-1$$

Exact differentiation is included!

Changing the relative degree

Black-Box Control problem: $\sigma \to 0$

$$\sigma^{(r)} = h(t) + g(t)u$$

$$\sigma^{(r+1)} = \dot{h}(t) + \dot{g}(t)u + g(t)\dot{u}$$

$$v = \dot{u}, \quad \sigma^{(r+1)} \in [-C_1, C_1] + [K_m, K_M]v$$

Remark: u is to be kept bounded …

Systems non-affine in control

$$\dot{x} = f(t,x,u),\ \ x \in \mathbf{R}^n,$$

Output: $\sigma(t,x)$ (tracking error), input: $u \in \mathbf{R}^l$

The goal: $\sigma \equiv 0$

Nonlinearity in control and its discontinuity $\Rightarrow$

$v = \dot{u}$ is taken as a new control,

$$\begin{pmatrix} \dot{x} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} f(t,x,u) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} v$$

The new system is affine in control,
$u(t)$ is differentiable.

The main method

Black-Box Control problem: $\sigma \to 0$

$$\sigma^{(r)} = h(t) + g(t)u$$

Solution: $u = \alpha U_r(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$ or $\dot{u} = \alpha_1 U_{r+1}(\sigma, \dot{\sigma}, \ldots, \sigma^{(r)})$

U_r, U_{r+1} are discontinuous but bounded

Relative Degree (RD)

$$\dot{x} = a(t,x) + b(t,x)u,\ x \in \mathbf{R}^n,\ \sigma,\ u\ \in \mathbf{R}$$

Informally: RD is the number r of the first total derivative where the control explicitly appears with a non-zero coefficient.

$$\sigma^{(r)} = h(t,x) + g(t,x)u,\ g \neq 0$$

Newton law: $\ddot{x} = \frac{1}{m}F$, RD=2

In my practice the relative degrees

$r = 2, 3, 4, 5$

mechanical systems, Newton law, integrators

But the solution is valid for any r.

Sliding mode (SM)

(not a math. definition)

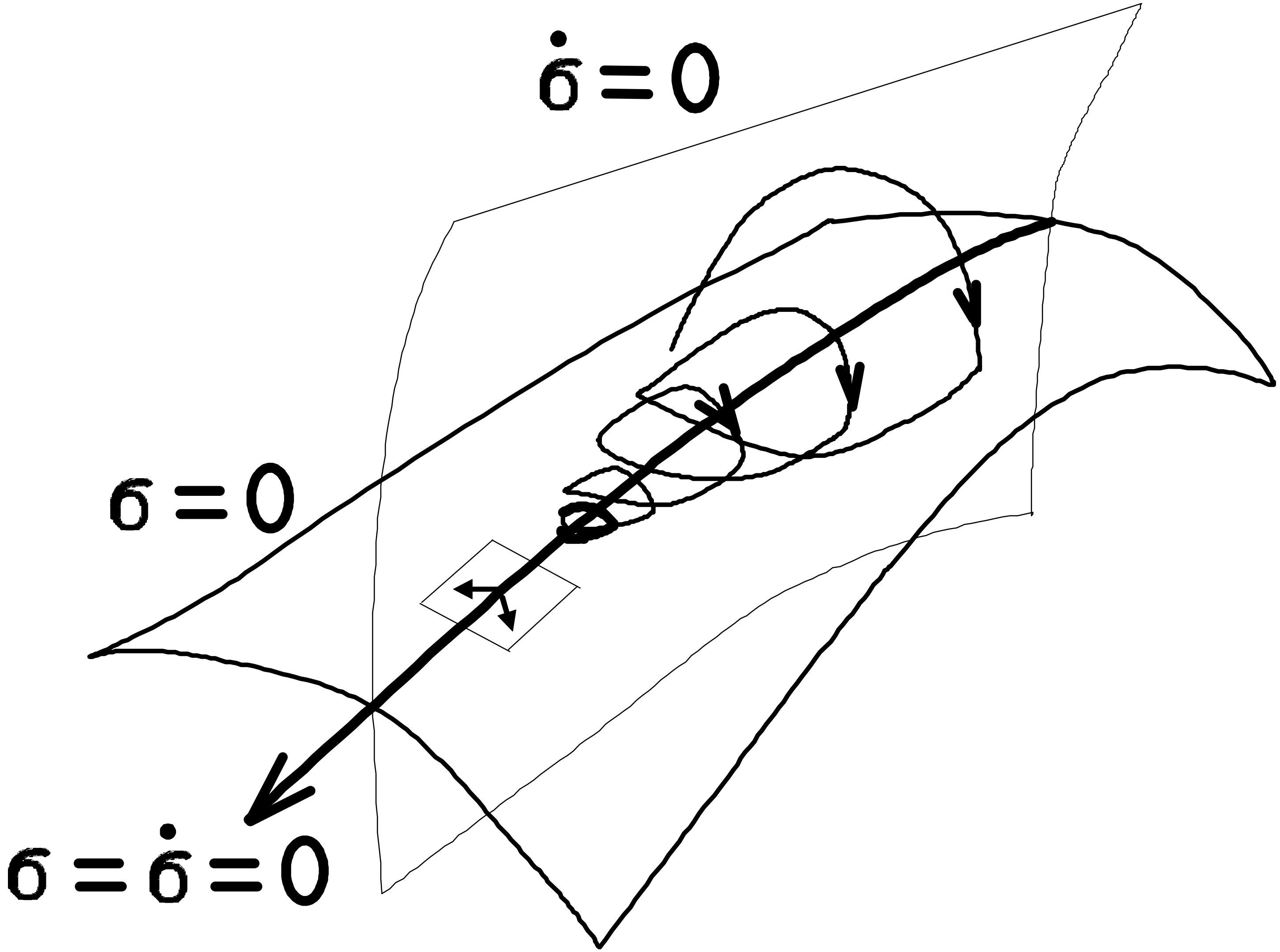
Any system motion mode existing due to high-frequency, theoretically infinite-frequency control switching is called SM.

*r*th-order sliding mode (*r*-SM)

(not a math. definition)

r-SM is a SM keeping $\sigma \equiv 0$ for RD = *r* by means of high-frequency switching of *u*.

Example: 2-SM phase portrait



Some abbreviations till now

SM - sliding mode,

r-SM – *r*th order SM

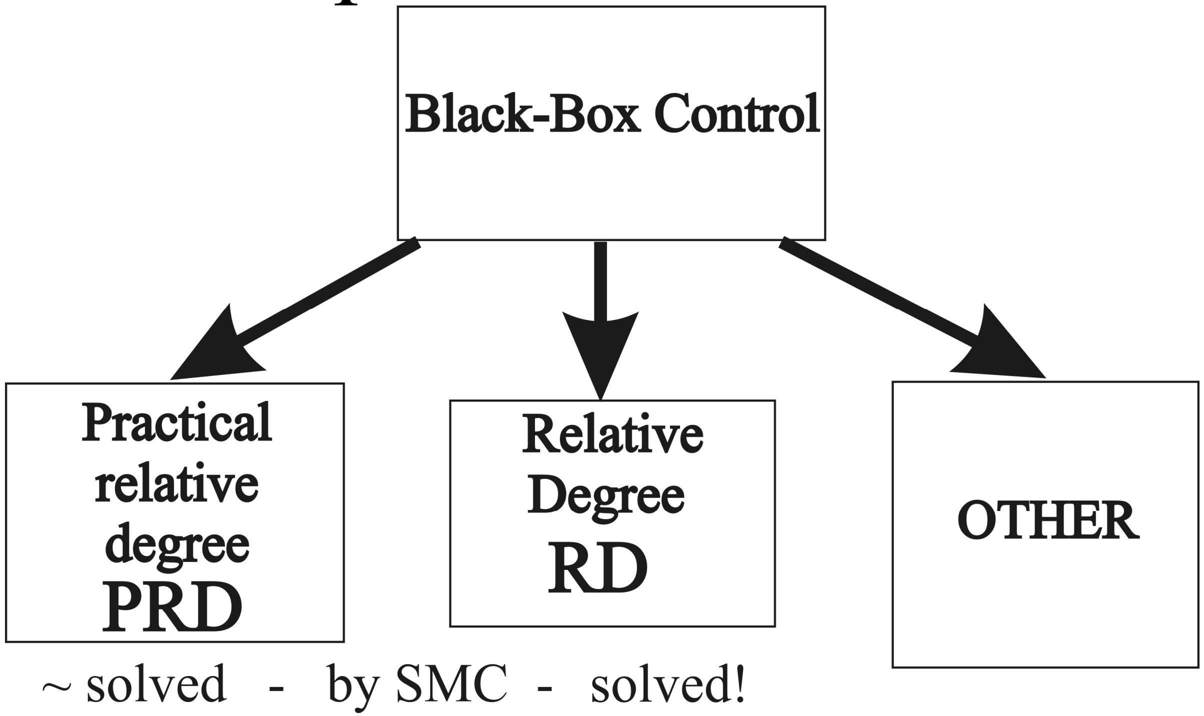
SMC – sliding mode control

RD – relative degree

PRD – practical relative degree

Preliminary conclusions

SMC theoretically "almost" solves the classical Black-Box control problem.

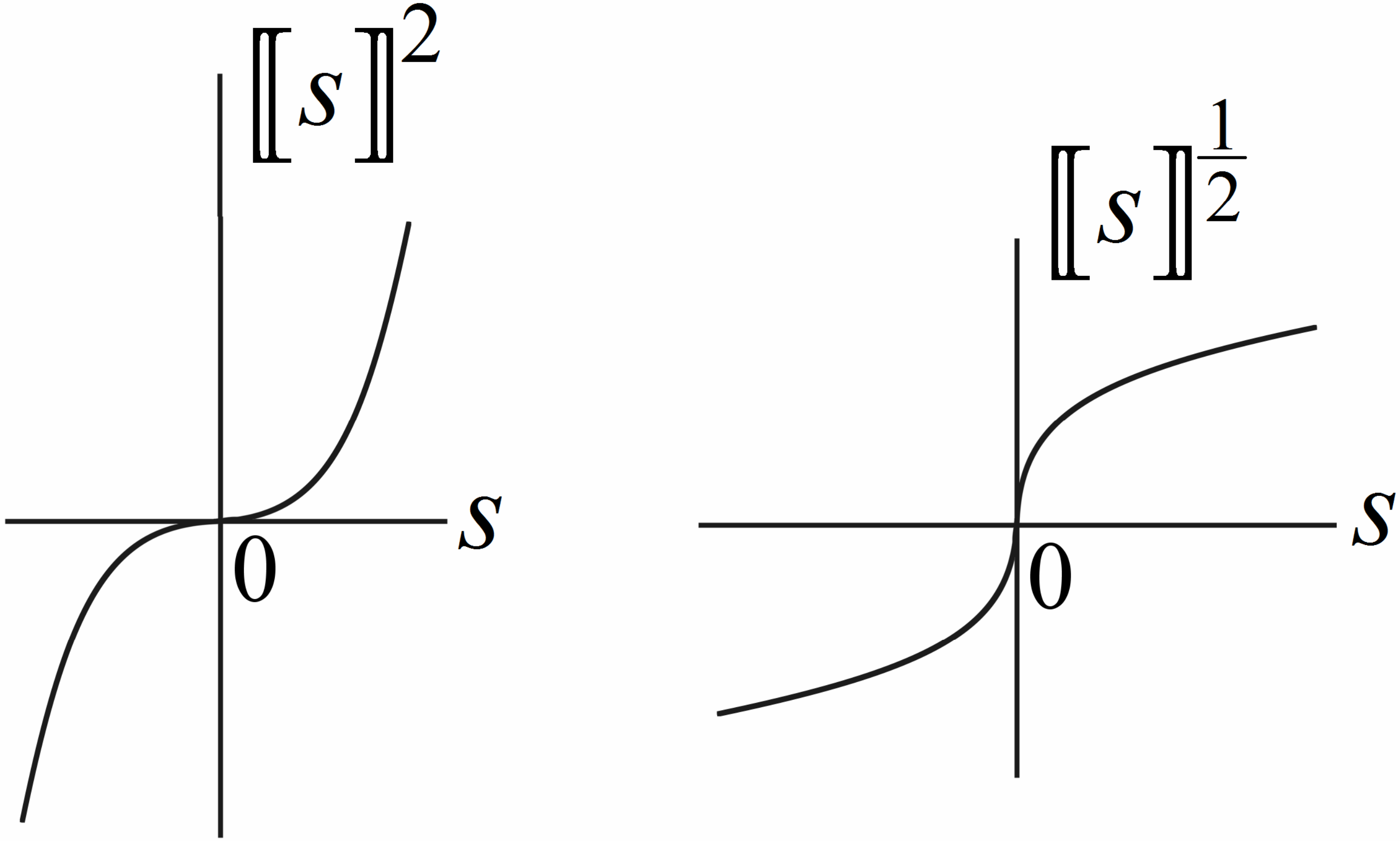


It includes exact robust differentiation of any order and robustness to small sampling/model noises, delays and disturbances (also singular).

Special power functions

(standard notation)

$$\lfloor s \rceil^{\gamma} = [\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s$$



The following controllers exactly robustly and in finite time provide for

$$\sigma \equiv 0$$

for the simplest model

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

Simplest *r*-SM controllers

(Ding, Levant, Li, Automatica 2016)

$$[\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s, \qquad \forall\, d > 0,\ \exists \beta_0, \ldots, \beta_{n-2} > 0$$

Relay-polynomial homogeneous *r*-SMC

$$u = -\alpha \ \operatorname{sign}\left[[\![\sigma^{(r-1)}]\!]^{\frac{d}{1}} + \beta_{n-2} [\![\sigma^{(r-2)}]\!]^{\frac{d}{2}} + \cdots + \beta_0 [\![\sigma]\!]^{\frac{d}{r}} \right]$$

Quasi-continuous polynomial homogeneous *r*-SMC

$$u = -\alpha \ \frac{[\![\sigma^{(r-1)}]\!]^{\frac{d}{1}} + \beta_{n-2} [\![\sigma^{(r-2)}]\!]^{\frac{d}{2}} + \cdots + \beta_0 [\![\sigma]\!]^{\frac{d}{r}}}{|\sigma^{(r-1)}|^{\frac{d}{1}} + \beta_{n-2} |\sigma^{(r-2)}|^{\frac{d}{2}} + \cdots + \beta_0 |\sigma|^{\frac{d}{r}}}$$

Quasi-continuous control

$$u = U(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$$

is called quasi-continuous (quasi-smooth), provided it remains a continuous (smooth) function whenever

$$(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \neq (0, 0, ..., 0)$$

Example: $u = -\alpha \dfrac{[\![\sigma^{(r-1)}]\!]^{\frac{d}{1}} + \beta_{n-2}[\![\sigma^{(r-2)}]\!]^{\frac{d}{2}} + \cdots + \beta_0 [\![\sigma]\!]^{\frac{d}{r}}}{|\sigma^{(r-1)}|^{\frac{d}{1}} + \beta_{n-2}|\sigma^{(r-2)}|^{\frac{d}{2}} + \cdots + \beta_0 |\sigma|^{\frac{d}{r}}}$

$d > kr \Rightarrow$ quasi k-smooth

List of controllers, $d = r$

$r = 1,2,3,4,5$

1. $u = -\alpha \ \operatorname{sign} \sigma,$
2. $u = -\alpha \ \operatorname{sign}([\![\dot\sigma]\!]^2 + \sigma),$
3. $u = -\alpha \ \operatorname{sign}(\ddot\sigma^3 + [\![\dot\sigma]\!]^{\frac{3}{2}} + \sigma),$
4. $u = -\alpha \operatorname{sign}([\![\dddot\sigma]\!]^4 + 2[\![\ddot\sigma]\!]^2 + 2[\![\dot\sigma]\!]^{\frac{4}{3}} + \sigma),$
5. $u = -\alpha \operatorname{sign}\left([\![\sigma^{(4)}]\!]^5 + 6[\![\dddot\sigma]\!]^{\frac{5}{2}} + 5[\![\ddot\sigma]\!]^{\frac{5}{3}} + 3[\![\dot\sigma]\!]^{\frac{5}{4}} + \sigma\right).$

α is to be taken sufficiently large.

Quasi-continuous controllers, *d* = *r*

1. $u = -\alpha \ \operatorname{sign}\sigma,$

2. $u = -\alpha \ \dfrac{[\![\dot\sigma]\!]^2 + \sigma}{\dot\sigma^2 + |\sigma|},$

3. $u = -\alpha \ \dfrac{\ddot\sigma^3 + [\![\dot\sigma]\!]^{\frac{3}{2}} + \sigma}{|\ddot\sigma|^3 + |\dot\sigma|^{\frac{3}{2}} + |\sigma|},$

4. $u = -\alpha \dfrac{[\![\dddot\sigma]\!]^4 + 2[\![\ddot\sigma]\!]^2 + 2[\![\dot\sigma]\!]^{\frac{4}{3}} + \sigma}{\dddot\sigma^4 + 2\ddot\sigma^2 + 2\dot\sigma^{\frac{4}{3}} + |\sigma|},$

5. $u = -\alpha \dfrac{[\![\sigma^{(4)}]\!]^5 + 6[\![\dddot\sigma]\!]^{\frac{5}{2}} + 5[\![\ddot\sigma]\!]^{\frac{5}{3}} + 3[\![\dot\sigma]\!]^{\frac{5}{4}} + \sigma}{\sigma^{(4)}|^5 + 6|\dddot\sigma|^{\frac{5}{2}} + 5|\ddot\sigma|^{\frac{5}{3}} + 3|\dot\sigma|^{\frac{5}{4}} + |\sigma|}.$

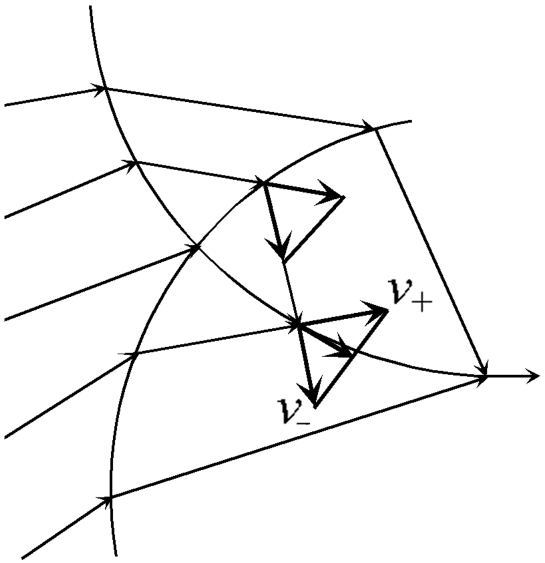
Infinitely many families (Levant 2017)

quasi-continuous controllers (Levant 2005):

***r* = 2:** $$u = -\alpha \frac{\dot{\sigma} + |\sigma|^{1/2}\,\mathrm{sign}\,\sigma}{|\dot{\sigma}| + |\sigma|^{1/2}}$$

***r* = 3:** $$u = -\alpha \frac{\ddot{\sigma} + 2\dfrac{(\dot{\sigma} + |\sigma|^{2/3}\,\mathrm{sign}\,\sigma)}{(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}}}{|\ddot{\sigma}| + 2(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}}$$

Discontinuous Differential Equations



Filippov Definition

$$\dot{x} = f(x) \iff \dot{x} \in F(x)$$

$x(t)$ is an absolutely continuous function

$$F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu N = 0} \text{convex\_closure } f(O_\varepsilon(x) \setminus N)$$

Filippov DI: $F(x)$ is non-empty, convex, compact, upper-semicontinuous.

Theorem (Filippov 1960-1970): $\Rightarrow$ Solutions exist for Filippov DIs, and for any locally bounded Lebesgue-measurable $f(x)$.

Non-autonomous case: $\dot{t} = 1$ is added.

When switching imperfections (delays, sampling errors, etc) tend to zero **usual solutions**

uniformly converge to Filippov solutions

*n*th-order differentiation problem

Parameters of the problem: $n \in \mathbb{N}, L > 0$

Measured input: $f(t) = f_0(t) + \eta(t), \quad |\,\eta\,| < \varepsilon$

f_0 ,η, ε are unknown,

$\eta(t)$ - Lebesgue-measurable function,

known: $\quad |f_0^{(n+1)}(t)| \le L$

(or |Lipschitz constant of $f_0^{(n)}| \le L$)

The goal:

real-time estimation of $\dot{f}_0(t), \ddot{f}_0(t), ..., f_0^{(n)}(t)$

Optimal differentiation

$$f(t) = f_0(t) + \eta(t), \ |\eta(t)| \le \varepsilon, \ \varepsilon \text{ is unknown}$$

$$f_0 \in \mathrm{Lip}_{\mathbb{R}_+}(n, L), \ |f_0^{(n+1)}(t)| \le L$$

A differentiator is **asymptotically optimal**, if in the steady state for $i = 0, 1, \ldots, n$

$$|z_i(t) - f_0^{(i)}(t)| \le \gamma_i L^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}} = \gamma_i L \left(\frac{\varepsilon}{L}\right)^{\frac{n+1-i}{n+1}},$$

(the Kolmogorov-like asymptotics)

The best worst-case error (Levant, Livne, Yu, 2017):

$$\sup |z_i(t) - f_0^{(i)}(t)| \in [1, \tfrac{\pi}{2}] \cdot 2^{\frac{i}{n+1}} L^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}}.$$

Example: $f(t) = \sin t$, $n = 5$, $L = 1$

The Kolmogorov constant $K_{5,5} = 1.505$

$$\varepsilon = 10^{-6},\ |\text{error of } f^{(5)}| \geq 1.5 \cdot 2^{\frac{5}{6}} \varepsilon^{\frac{1}{6}} > 0.2$$

Computer round-up error:

$$\varepsilon = 5 \cdot 10^{-16},\ |\text{error of } f^{(5)}| > 0.0075$$

It cannot be improved!

Differentiator (Levant 1998, 2003)

$$\dot{z} = D_n(z, f(t), L), \quad |f^{(n+1)}| \le L$$

$$\dot{z}_0 = -\tilde{\lambda}_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1,$$

$$\dot{z}_1 = -\tilde{\lambda}_{n-1} L^{\frac{1}{n}} \lfloor z_1 - \dot{z}_0 \rceil^{\frac{n-1}{n}} + z_2,$$

$$\dots$$

$$\dot{z}_{n-1} = -\tilde{\lambda}_1 L^{\frac{1}{2}} \lfloor z_{n-1} - \dot{z}_{n-2} \rceil^{\frac{1}{2}} + z_n,$$

$$\dot{z}_n = -\tilde{\lambda}_0 L \,\mathrm{sign}(z_n - \dot{z}_{n-1}), \quad z_i - f_0^{(i)} \to 0.$$

$\{\tilde{\lambda}_n\}$ = 1.1, 1.5, 2, 3, 5, 7, 10, 12, … for $n \le 7$

Differentiator: non-recursive form

$$\dot{z}_0 = -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1,$$

$$\dot{z}_1 = -\lambda_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} + z_2,$$

$$\ldots$$

$$\dot{z}_{n-1} = -\lambda_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} + z_n,$$

$$\dot{z}_n = -\lambda_0 L \ \mathrm{sign}(z_0 - f(t)), \quad z_i - f_0^{(i)} \to 0.$$

$$\lambda_0 = \tilde{\lambda}_0, \ \lambda_n = \tilde{\lambda}_n, \ \lambda_j = \tilde{\lambda}_j \lambda_{j+1}{}^{j/(j+1)}$$

Differentiator parameters

n	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

Asymptotically optimal accuracy

In the presence of the noise with the magnitude ε, and sampling with the step τ: $\exists \mu_j \geq 1$

$$| z_j - f_0^{(j)} | \leq \mu_j L \rho^{n+1-j}, \quad \rho = \max(\tau, \left(\frac{\varepsilon}{L}\right)^{\frac{1}{n+1}})$$

$$\varepsilon = \tau = 0 \ \Rightarrow$$

in a finite time $z_i \equiv f^{(i)}, i = 0,\ldots,n$

Universal controller for any RD *r*

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u$$

$$u = -\alpha\Psi_r(z),$$

$$z = D_{r-1}(z, \sigma, L)$$

$L \geq C + \alpha K_M$, α is sufficiently large

Accuracy: |noise| $\leq \varepsilon$, sampling step $\leq \tau$

$$|\sigma^{(j)}| \leq \nu_j \rho^{r-j}, \quad \rho = \max(\tau, \varepsilon^{\frac{1}{r}}),$$

$\tau = \varepsilon = 0 \Rightarrow \sigma \equiv 0$ in finite time

EXAMPLES

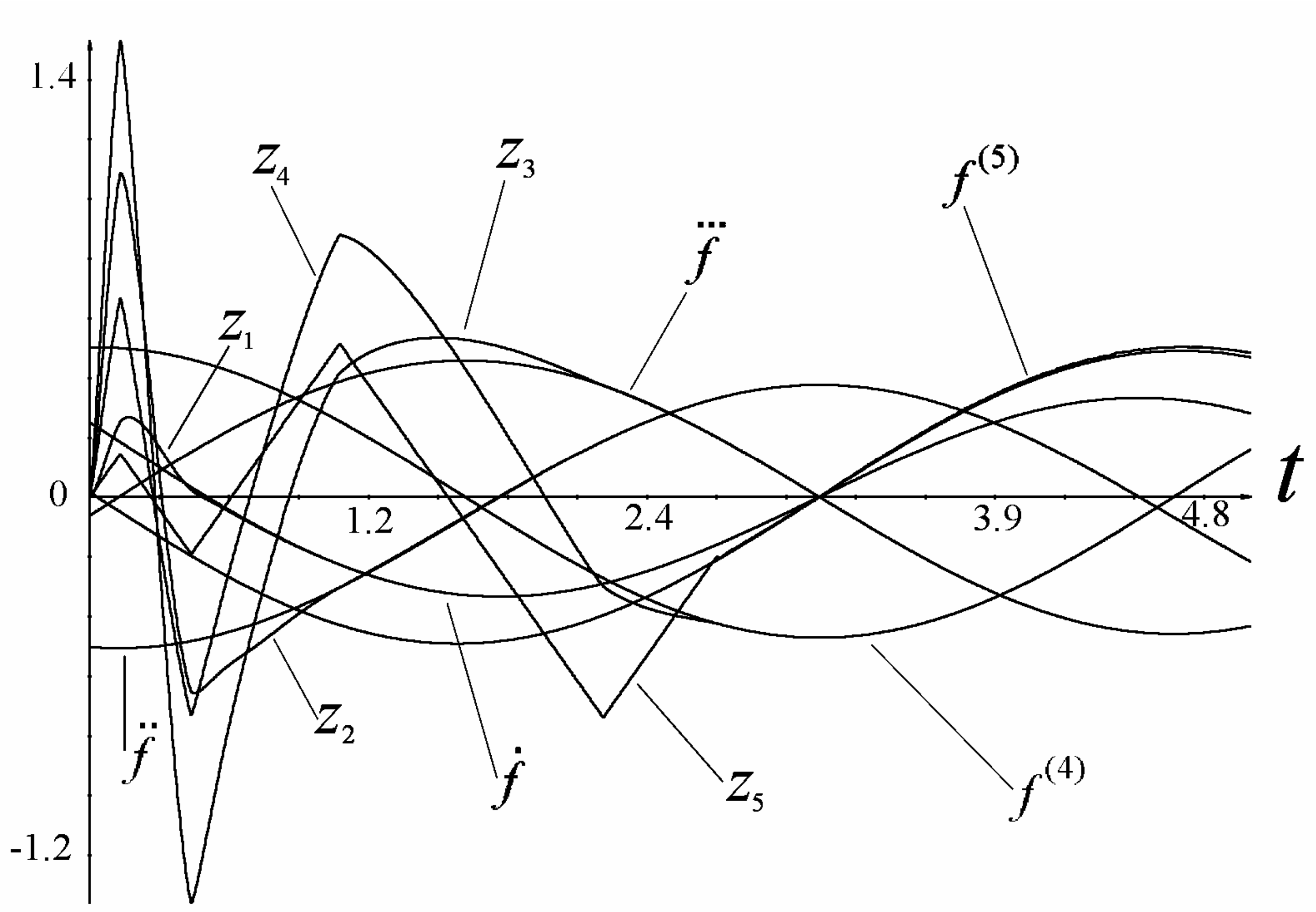
5th-order differentiator, $|f^{(6)}| \leq L$.

5 4 3 2 1

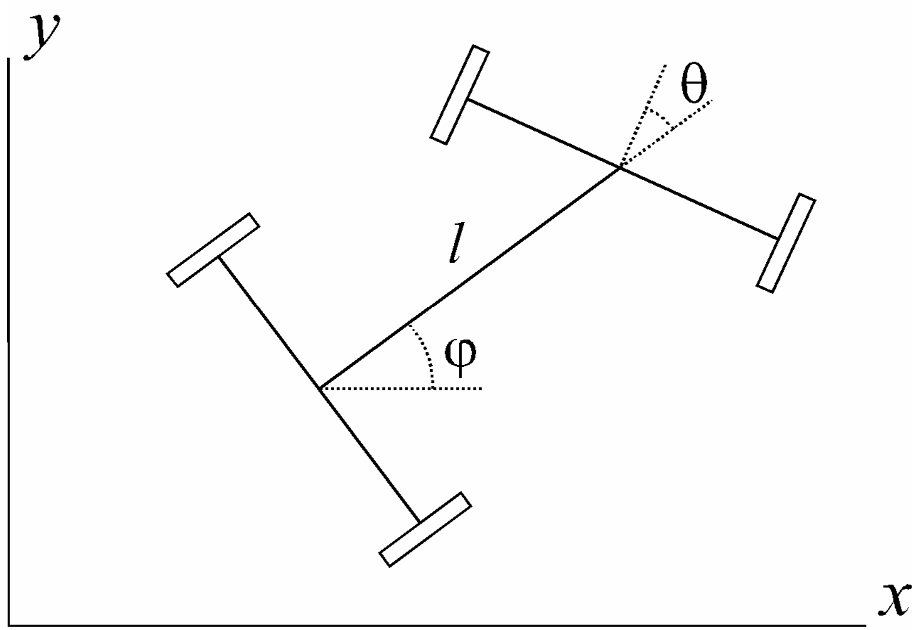
$$
\begin{aligned}
\dot{z}_0 &= -12L^{\frac{1}{6}} [\![z_0 - f(t)]\!]^{\frac{5}{6}} + z_1, \\
\dot{z}_1 &= -8L^{\frac{1}{5}} [\![z_1 - \dot{z}_0]\!]^{\frac{4}{5}} + z_2, \\
\dot{z}_2 &= -5L^{\frac{1}{4}} [\![z_2 - \dot{z}_1]\!]^{\frac{3}{4}} + z_3, \\
\dot{z}_3 &= -3L^{\frac{1}{3}} [\![z_3 - \dot{z}_2]\!]^{\frac{2}{3}} + z_4, \\
\dot{z}_4 &= -1.5L^{\frac{1}{2}} [\![z_4 - \dot{z}_3]\!]^{\frac{1}{2}} + z_5, \\
\dot{z}_5 &= -1.1L \operatorname{sign}(z_5 - \dot{z}_4)
\end{aligned}
$$

5th-order differentiation

$$f(t) = \sin 0.5t + \cos 0.5t, \quad L = 1.1$$



Example: car control



$\dot{x} = V \cos \varphi,\ \dot{y} = V \sin \varphi,$

$\dot{\varphi} = (V/l)\tan \theta,$

$\dot{\theta} = u$

RD = 3

x, y are measured.

The task: real-time tracking $y = g(x)$

$V = const = 10$ m/s = 36 km/h, $l = 5$ m,

$x = y = \varphi = \theta = 0$ at $t = 0$

Solution: $\sigma = y - g(x)$, $r = 3$

3-sliding controller (N°3), $\alpha = 2$, $L = 100$

3-sliding car control

$\sigma = y - g(x)$.

Simulation: $g(x) = 10 \sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$.

The controller:

$u = 0, \ 0 \le t < 1,$

$u = -2[s_2 + 2\,(|s_1| + |s_0|^{2/3})^{-1/2}(s_1 + |s_0|^{2/3} \operatorname{sign} s_0)] \,/\, [|s_2| + 2\,(|s_1| + |s_0|^{2/3})^{1/2}],$

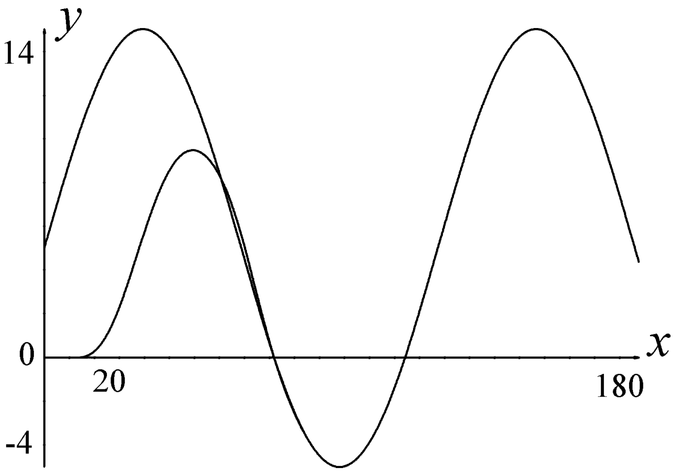
Differentiator: $\dot{s} = D_2(s, \sigma, 100)$, $L = 100$:

$\dot{s}_0 = -\,9.28\,|s_0 - \sigma|^{2/3} \operatorname{sign}(s_0 - \sigma) + s_1,$

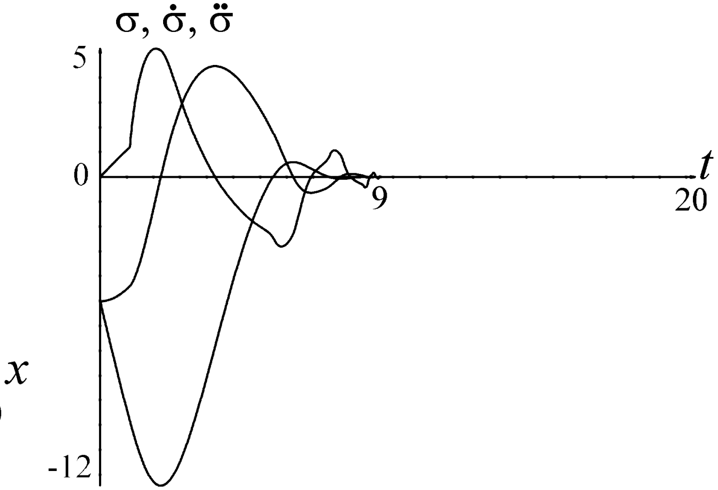
$\dot{s}_1 = -\,15\,|s_1 - \dot{s}_0|^{1/2} \operatorname{sign}(s_1 - \dot{s}_0) + s_2,$

$\dot{s}_2 = -\,110 \operatorname{sign}(s_2 - \dot{s}_1),$

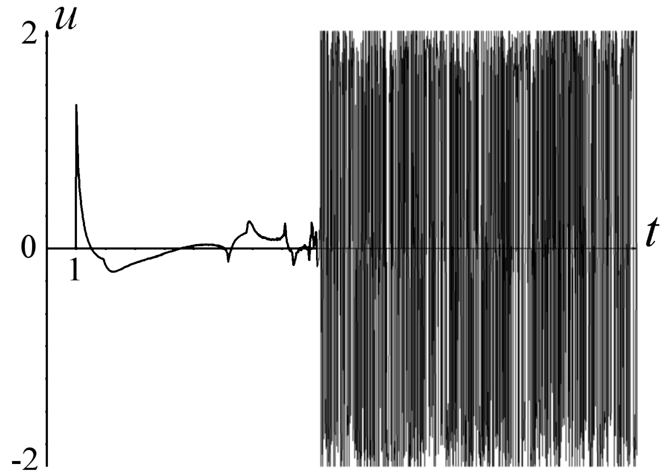
3-sliding car control



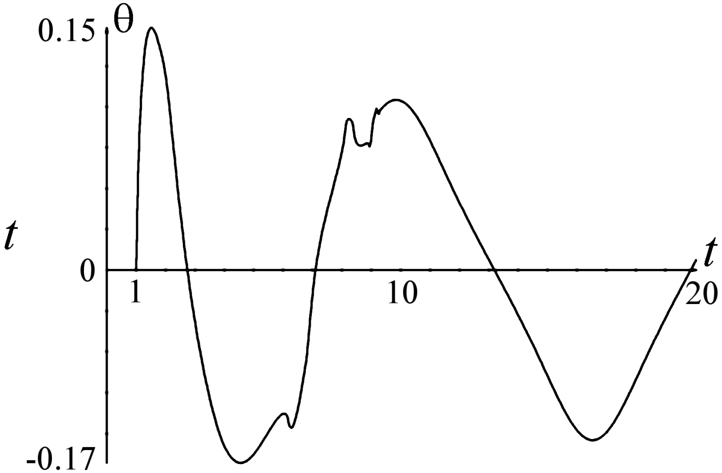
a. Car trajectory



b. 3-sliding deviations



c. Steering-angle derivative

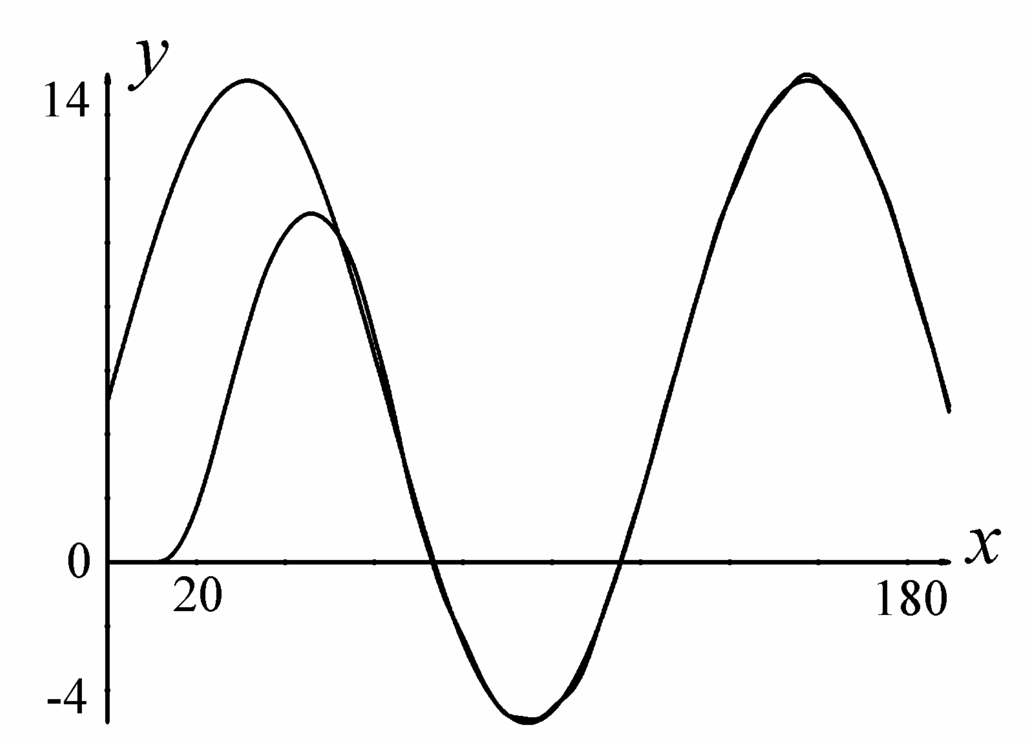


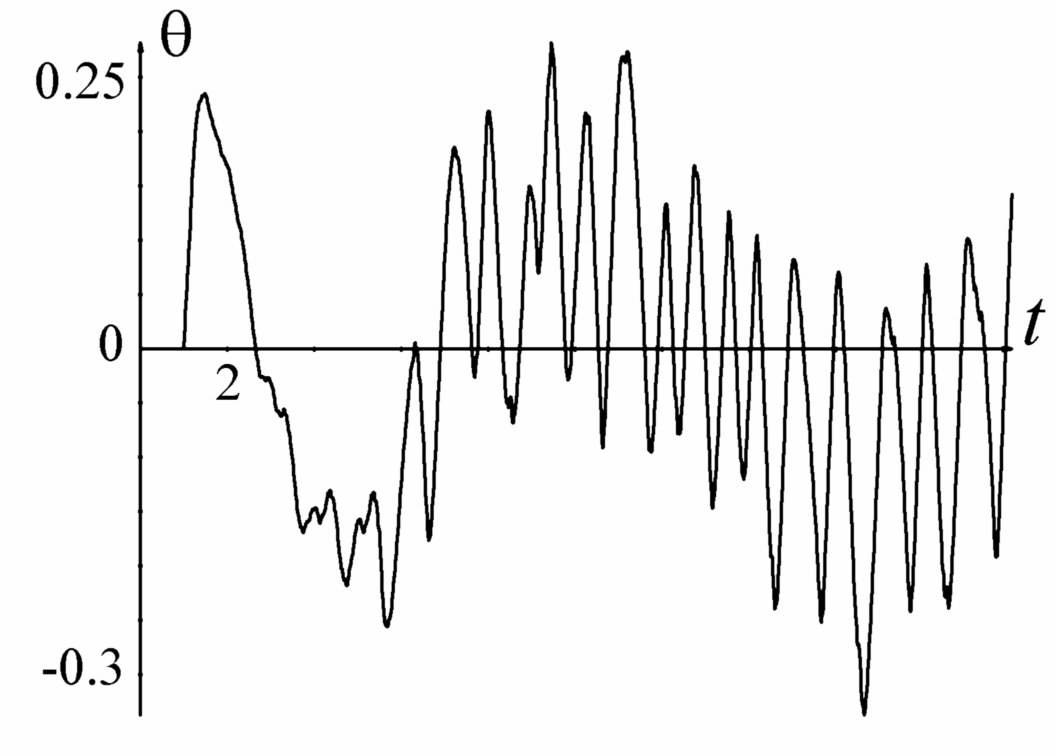
d. Steering angle

$$\tau = 10^{-4} \Rightarrow \quad |\sigma| \le 5.4\cdot 10^{-7}, \quad |\dot{\sigma}| \le 2.5\cdot 10^{-4}, |\ddot{\sigma}| \le 0.04$$

$$\tau = 10^{-5} \Rightarrow \quad |\sigma| \le 5.6\cdot 10^{-10}, \quad |\dot{\sigma}| \le 1.4\cdot 10^{-5}, |\ddot{\sigma}| \le 0.004$$

Input noise magnitude ε = 0.1m , $0 \le t \le 20$



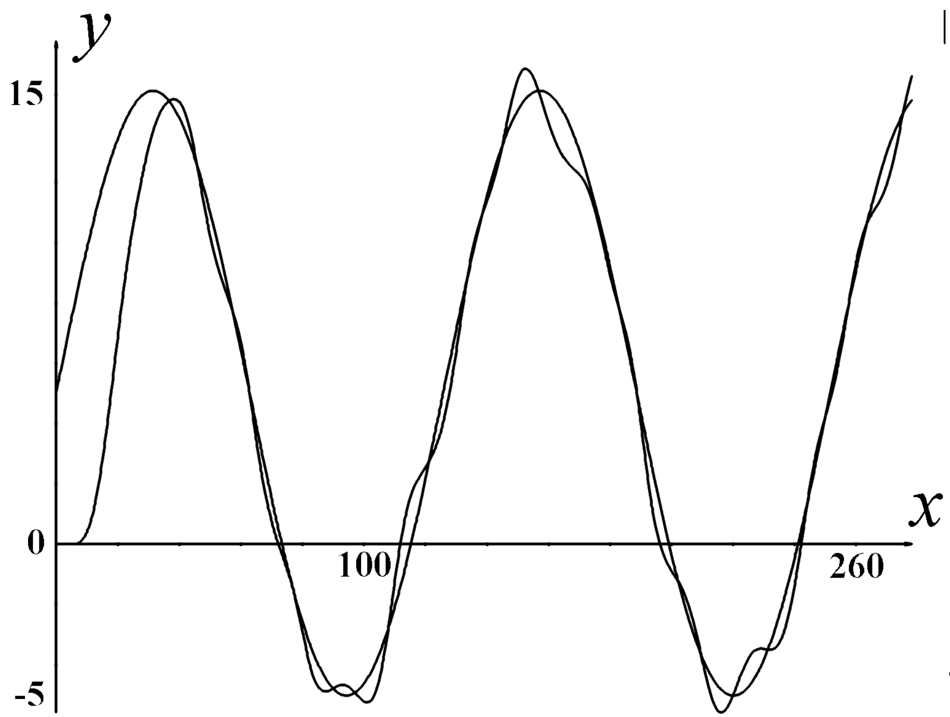


Car trajectory

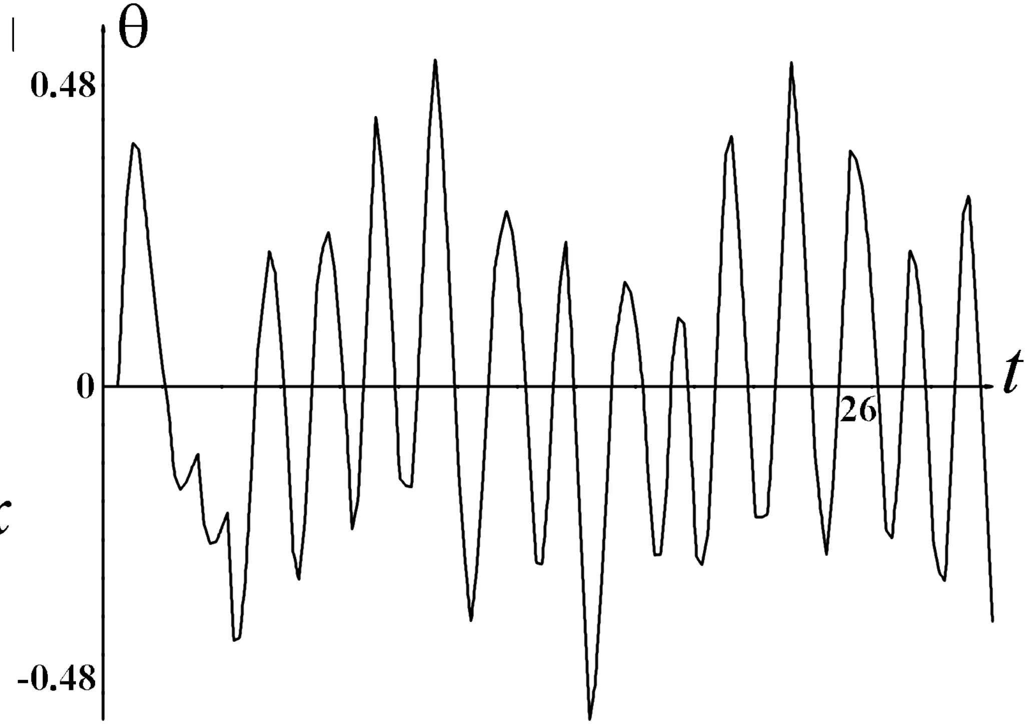
Steering angle

$$\tau = 10^{-5}, \; |\sigma| \le 0.2m,$$

Sampling step τ = 0.2s, ε = 0.1m, $0 \le t \le 30$



Car trajectory



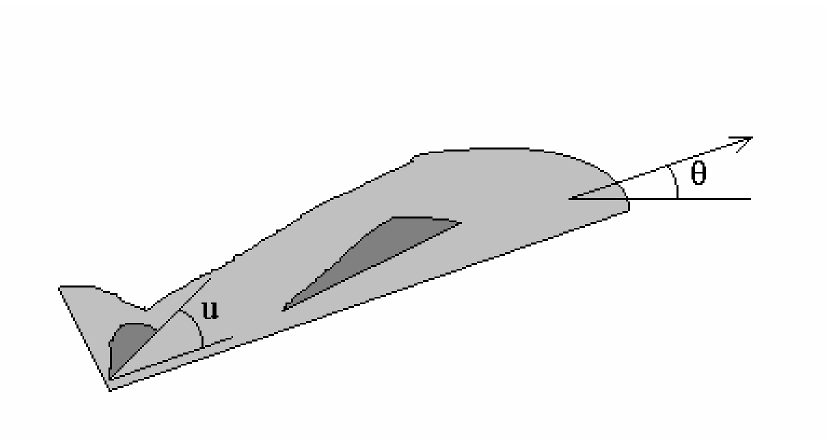
Steering angle

$$|\sigma| \le 1.2m, |\dot{\sigma}| \le 2.9m/s, |\ddot{\sigma}| \le 8.9m/s^2$$

Example: practical pitch control

Levant, Pridor, Gitizadeh, Yaesh, Ben-Asher, 2000

Pitch Control, Delilah (IMI, 1994-98)



Problem statement. A non-linear process is given by a set of 42 linear approximations

$$\tfrac{d}{dt}(x,\theta,q)^t = G(x,\theta,q)^t + Hu,\ q = \dot{\theta},$$

$$x \in \mathbf{R}^3,\ \theta,\ q,\ u \in \mathbf{R},$$

x_1, x_2 -velocities, x_3 - altitude

The Task: $\theta \to \theta_c(t)$, $\theta_c(t)$ is given in real time.

G and *H* are not known properly

Sampling Frequency: 64 Hz, Measurement noises

Actuator: delay and discretization.

dθ/dt does not depend explicitly on *u* (relative degree 2)

Primary Statement:

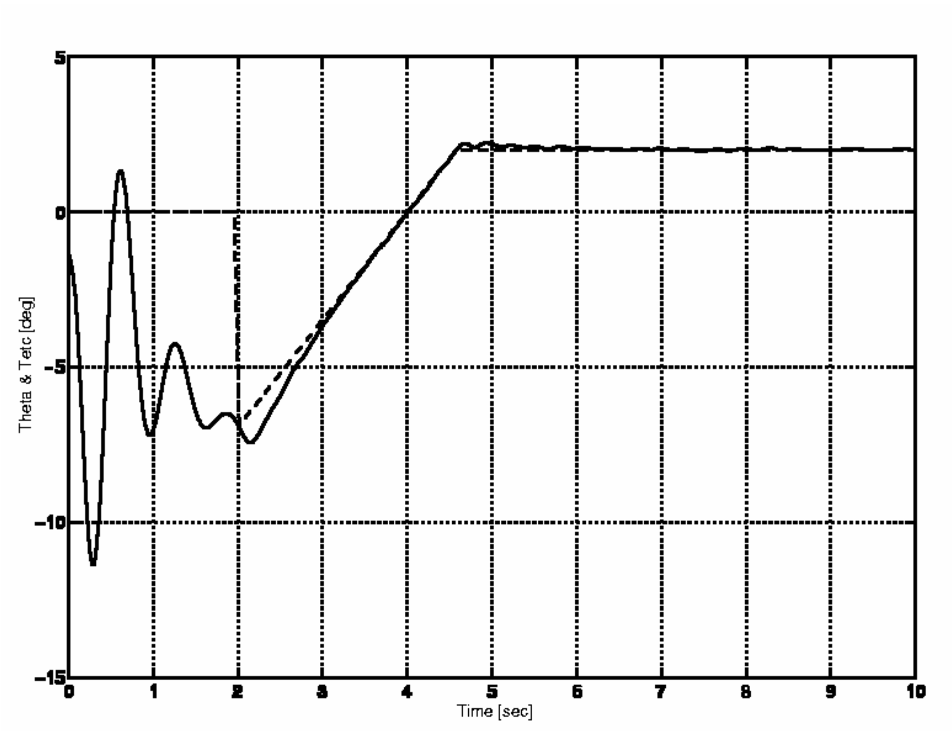
Available: θ, θ_c, Dynamic Pressure and Mach.

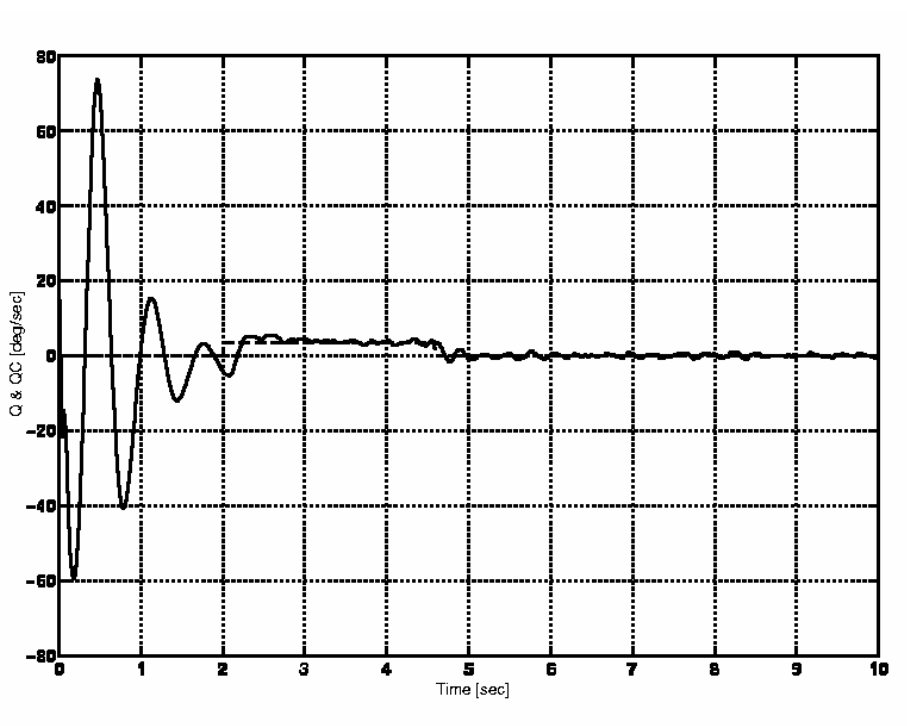
Main Statement: also $\dot{\theta}$, $\dot{\theta}_c$ are measured

The idea: keeping $5(\theta - \theta_c) + (\dot{\theta} - \dot{\theta}_c) = 0$ in 2-sliding mode

(asymptotic 3-sliding)

Flight Experiments

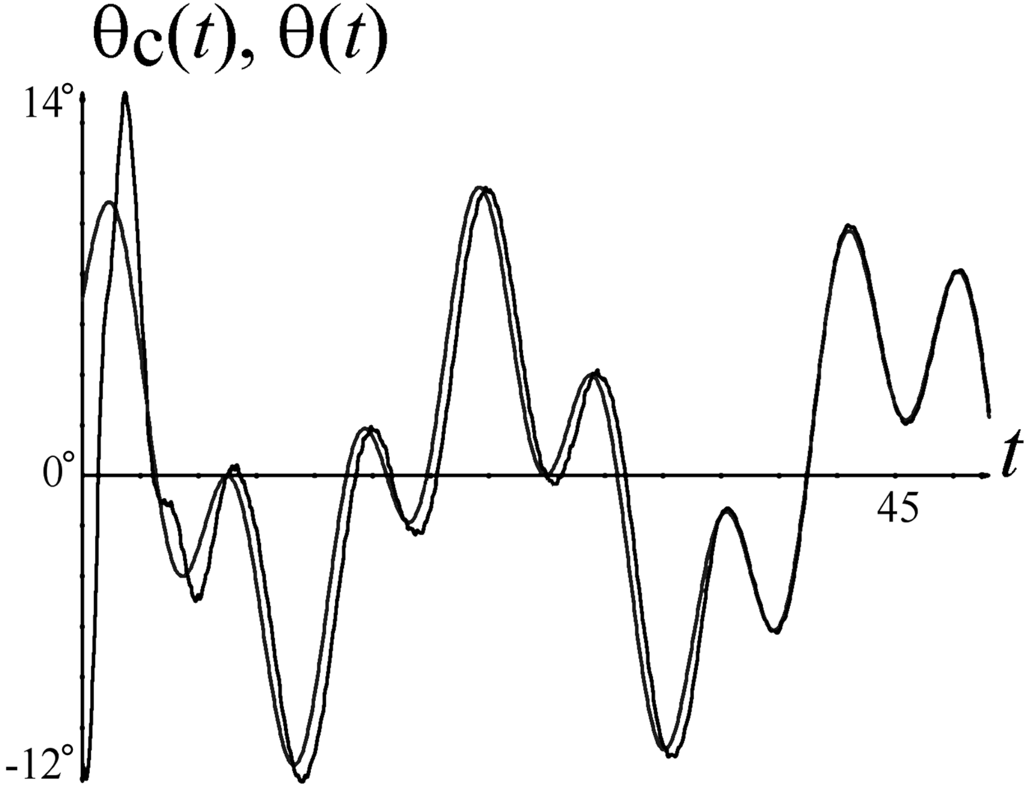


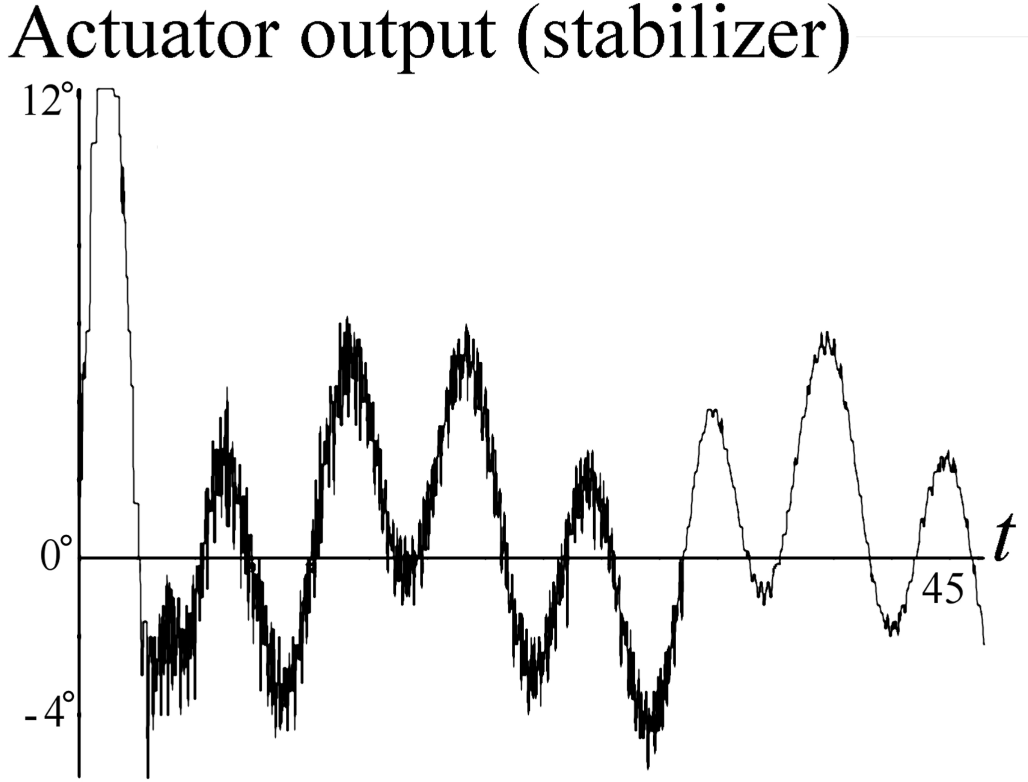


$\theta_c(t), \theta(t)$

$\dot{\theta}_c = q_c(t), \dot{\theta} = q(t)$

Actuator (server-stepper) output





Switch from Linear (H_∞) control to 3-SM control

Practical Relative Degree
PRD

NO MODEL AT ALL

Practical Relative Degree Definition

Nothing is known on the system.

$r \in \mathbb{N}$ is called the PRD, if $\exists \lambda_\sigma = 1$ or -1:

$\exists \varepsilon, \delta_t, \alpha_M, \alpha_m, L, L_m > 0, \alpha_m \le \alpha_M, L_m \le L$,:

1. For any (measurable) $u(t)$, $|u-u_0| \le U_M$:

   Output: $\tilde{\sigma} = \sigma + \eta$, $|\eta| \le \varepsilon$,

   $\sigma^{(r-1)} \in \mathrm{Lip}(L)$

2. For $\omega = \lambda_\sigma \sigma$:

   If $\forall t \ge t_0$

   $\alpha_M \ge u(t) - u_0 \ge \alpha_m \quad (-\alpha_M \le u(t) - u_0 \le -\alpha_m)$,

   then $\forall t \ge t_0 + \delta_t$:

   $\omega^{(r)} \ge L_m \quad (\omega^{(r)} \le -L_m)$

Naming

u_0 is the *reference input*,

in the following $u_0 = 0$

λ_σ is the *influence direction* parameter,

in the following $\lambda_\sigma = 1$

δ_t is the *delay* parameter

ε is the *approximation* parameter.

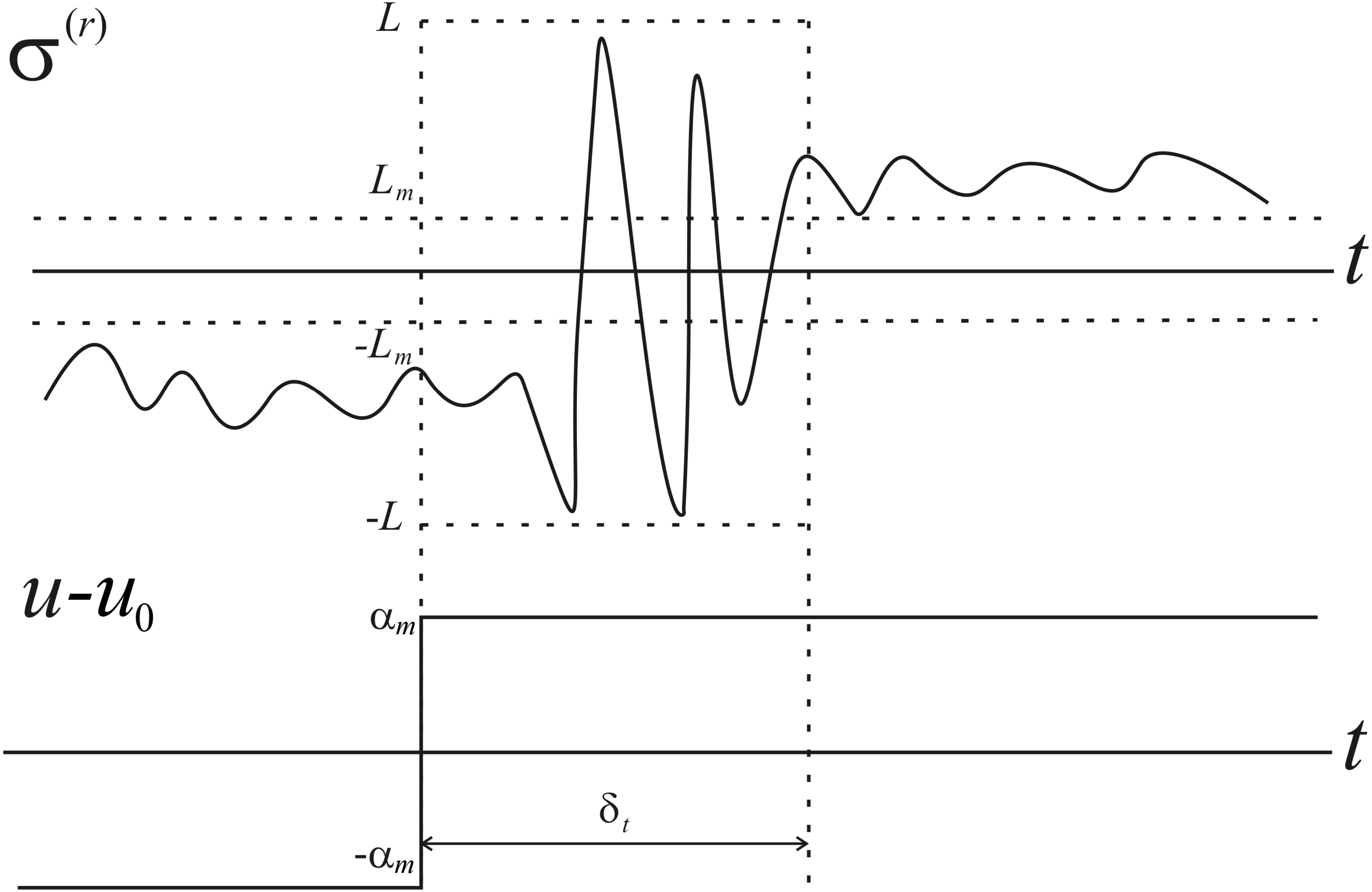
Local Practical Relative Degree Definition

$\exists\ t_1, t_2, T, t_1 < t_2, \delta_t < T$, such that

requirement 1 is true over the time interval $[t_1, t_2 + T]$;

requirement 2 is true for each $t_0 \in [t_1, t_2]$ over $[t_0, t_0 + T]$.

Graphical interpretation



Remarks

The function σ does not necessarily need to have any real meaning. It can be just an output of some smoothing filter.

Keeping $\sigma \equiv 0$ is not possible under these conditions.

Control

$$u = -\alpha \Psi_r(z), \;\; \dot{z} = D_{r-1}(z, \sigma, L),$$
$$\alpha_m \le \alpha \le \alpha_M$$

Differentiator parameters λ_i are properly chosen

Theorem. $\exists$ $\beta_1, \ldots, \beta_{r\text{-}1}$ (coefficients of the r-SM homogeneous controller):

Accuracy: $\sigma = O(\max[\varepsilon, \delta_t^{\,r}])$

Continuous controller

based on any quasi-continuous controller

$$u = -\alpha\Phi(\|z\|_h)\Psi_r(z)$$

(SM regularization)

$$\Phi(\| z \|_h) = \begin{cases} 1 \text{ with } \| z \|_h > \gamma \max[\varepsilon, \delta_t^r], \\ \dfrac{1}{\max[\varepsilon, \delta_t^r]} \| z \|_h^2 \text{ with } \| z \|_h \leq \gamma \max[\varepsilon, \delta_t^r], \end{cases}$$

$$\| z \|_h^2 = z_0^{2/r} + z_1^{2/(r-1)} + ... + z_{r-1}^2$$

The accuracy is the same.

Simulation

Perturbed car model

$\dot{x} = V\cos\phi, \dot{y} = V\sin\phi,$

$\ddot{\phi} = -4\,\mathrm{sign}(\phi - \varphi) - 6\dot{\phi}, \quad \Rightarrow \quad$ Rel. degree does not exist!

$\dot{\varphi} = \frac{V}{\Delta}\tan\theta, \qquad \dot{\theta} = \zeta_1,$

Actuator: input u, output ζ_1

$$\ddot{\zeta}_1 = -100(2(\zeta_1 - u) + 0.01\dot{\zeta}_1)^3 - 100(\zeta_1 - u) - 2\dot{\zeta}_1,$$

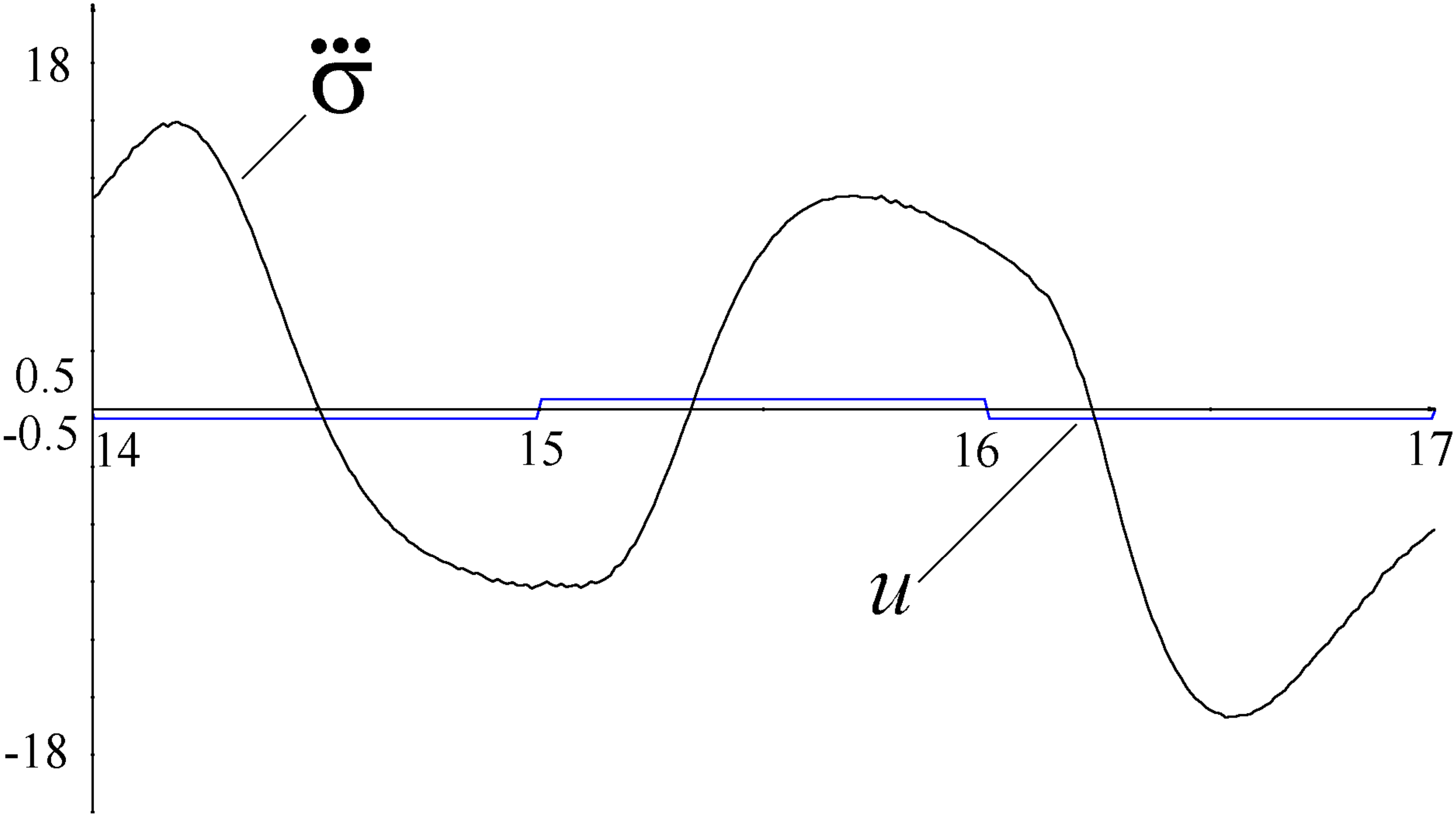
Sensor: $\tilde{\sigma} = \zeta_2 + 0.01\dot{\zeta}_2 - g(x) + \eta(t)$, η is a noise, $|\eta| \le 0.01$.

$$\dddot{\zeta}_2 = -100(\zeta_2 - y) - 2\dot{\zeta}_2 - 0.02\ddot{\zeta}_2,$$

$\zeta_2 = -10$, $\dot{\zeta}_2 = 2000$, $\ddot{\zeta}_2 = -80000$, $\zeta_1 = \dot{\zeta}_1 = \phi = \dot{\phi} = 0$ at $t = 0$,

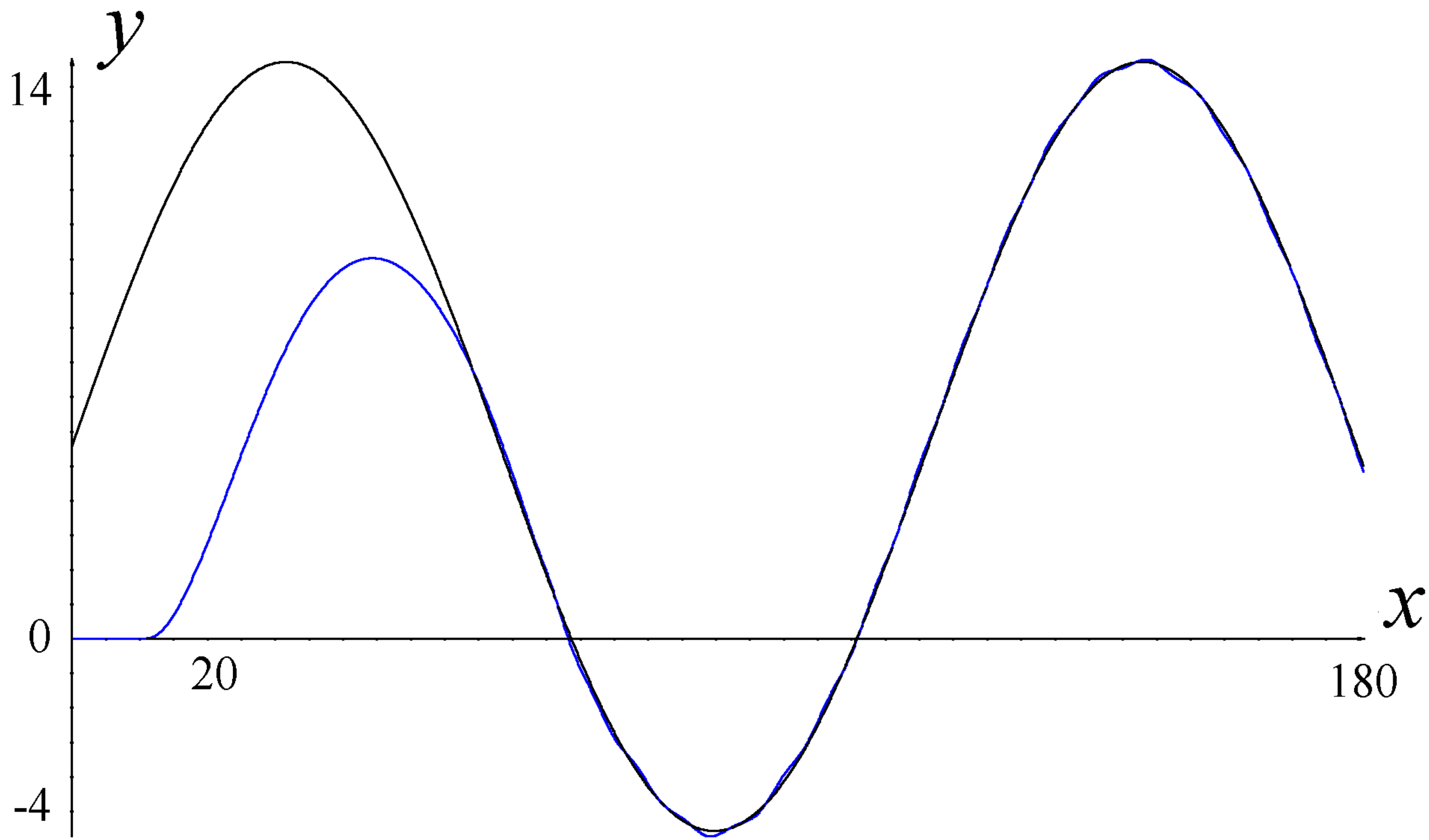
If the system were smooth the new RD were 10

Practical rel. degree = 3



Differentiator of the order 3 is used with $L = 100$.

System performance



$|\underline{\sigma}| \leq 0.16$

APPLICATION

Blood Glucose Control

High-Order Sliding-Mode Control of Blood Glucose Concentration via Practical Relative Degree Identification

CDC-ECC 2011

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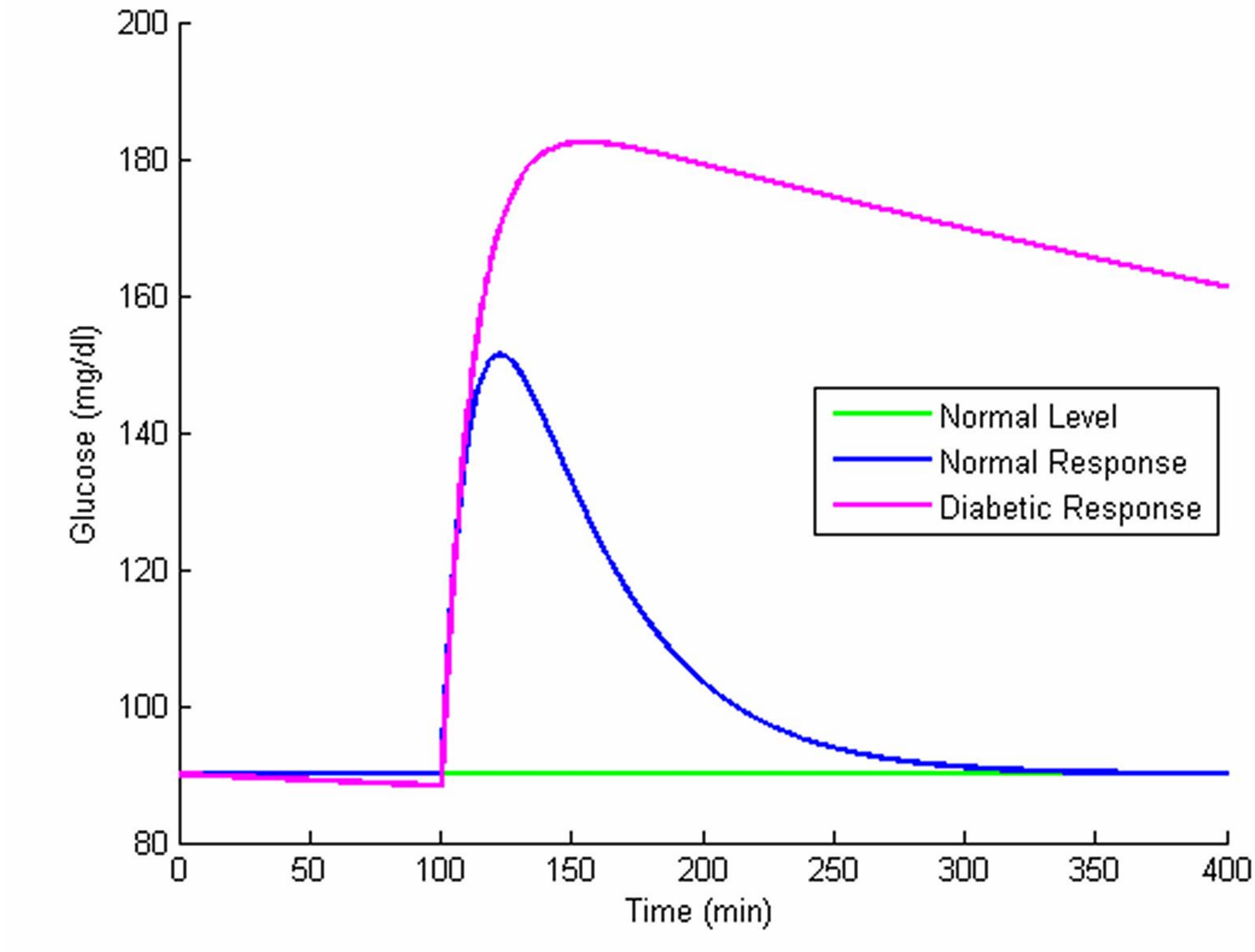
Body reaction to glucose concentration increase



Different models

Model	RD	No. States
Bergman	3	3
Candas-Radziuk	3	4
Cobelli	3	7
Hovorka	5	8
Dalla Man	5	8
Sorensen	5	18

- Output: blood glucose
- Input: insulin

The simplest model

BeM Bergman et al., 1979

$$\dot{B}_1 = -p_1[B_1 - G_b] - B_1 B_2 + \dot{D}(t)$$
$$\dot{B}_2 = -p_2 B_2 + p_3[B_3 - I_b]$$
$$\dot{B}_3 = -n[B_3 - I_b] + \gamma[B_1 - h]^+ u(t)$$

Table: Variables

Variable	Description
B_1	Blood glucose concentration (Output)
B_2	Effect of insulin on glucose uptake
B_3	Blood insulin concentration

Sorensen model

$$\dot{S}_1 = \frac{1}{V_H^G}(-Q_H^G S_1 + Q_L^G S_2 + S_7 - F_{RBGU})$$

$$\dot{S}_2 = \frac{2}{V_L^G}(Q_A^G S_1 + Q_G^G S_6 - Q_L^G S_2 + f_{HGP} S_8 - f_{HGU} S_3)$$

$$\dot{S}_3 = \frac{1}{\tau_1}(2\tanh(0.55 S_4^N) - S_3)$$

$$\dot{S}_4 = \frac{2}{V_L^I}(Q_A^I S_5 + Q_G^I S_{10} - Q_L^I S_4 - F_{LIC})$$

$$\dot{S}_5 = \frac{1}{V_H^I}(Q_L^I S_4 - Q_H^I S_5 + S_9 + u(t))$$

$$\dot{S}_6 = \frac{Q_G^G}{V_G^G}(S_1 - S_6) + \frac{1}{V_G^G}(F_{MEAL} - R_{GGU})$$

$$\dot{S}_7 = Q_K^G \dot{G}_K + G_P^G \dot{G}_{PV} + Q_B^G \dot{G}_{BV}$$

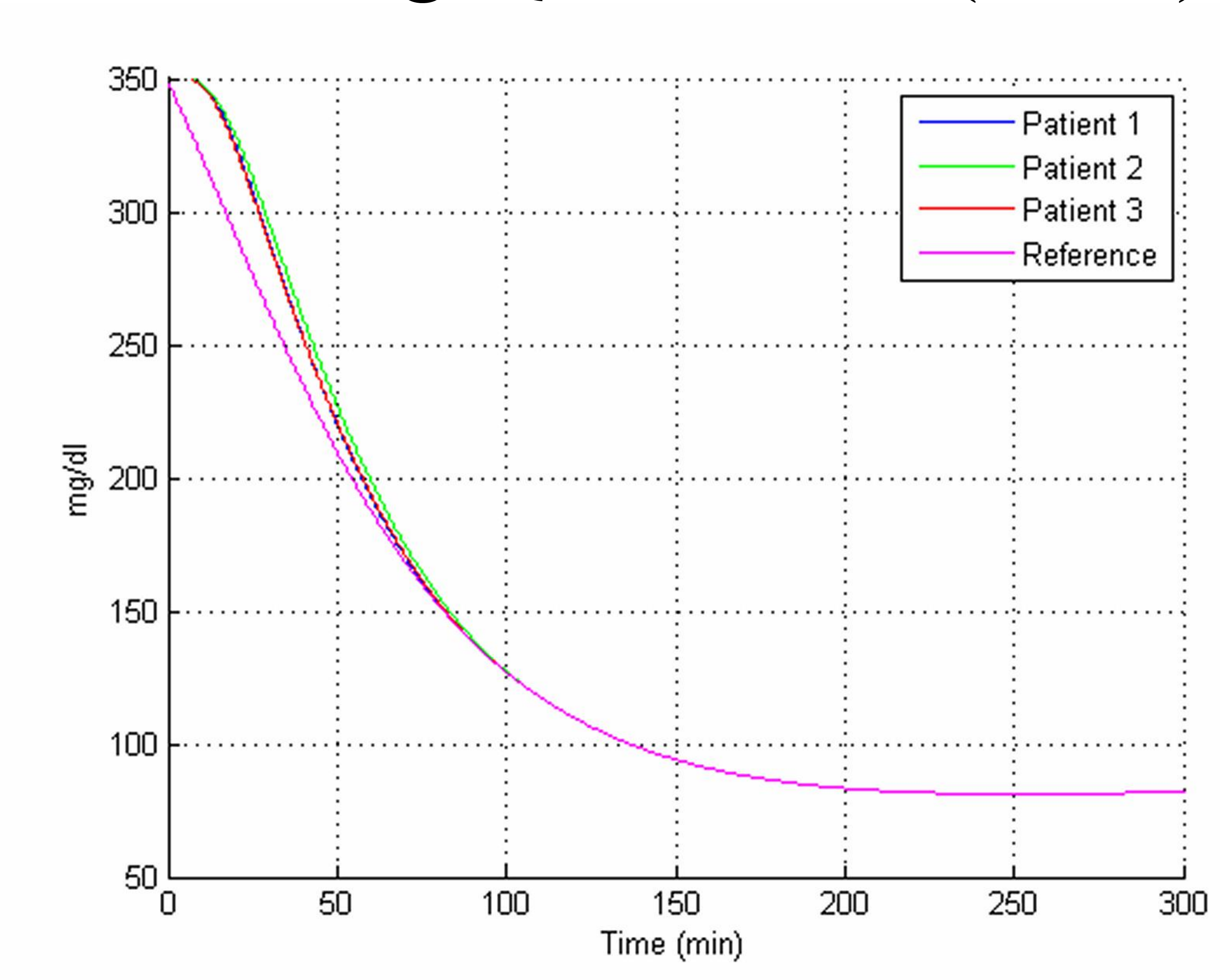
$$\dot{S}_8 = \frac{1}{\tau_1}(1.21 - 1.14\tanh[1.66(S_4^N - 0.89)] - S_8)$$

$$\dot{S}_9 = Q_B^I \dot{I}_B + Q_K^I \dot{I}_K + Q_P^I \dot{I}_{PV}$$

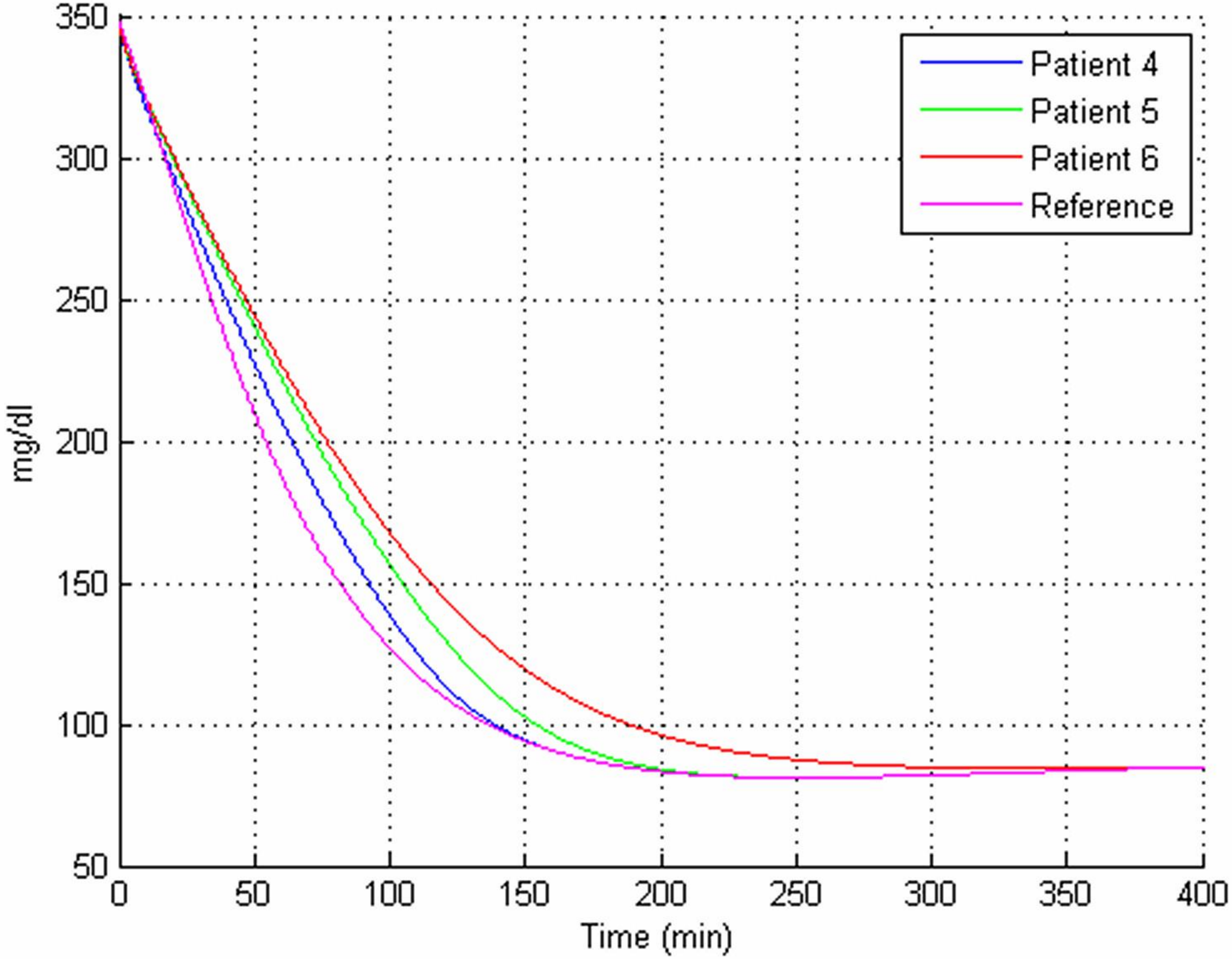
$$\dot{S}_{10} = \frac{Q_G^I}{V_G^I}(S_5 - S_{10})$$

$$\dot{S}_{11} = \frac{1}{V_C}(F_{PCR} - F_{MCC} S_{11}^N)$$

3-sliding QC control (BeM)

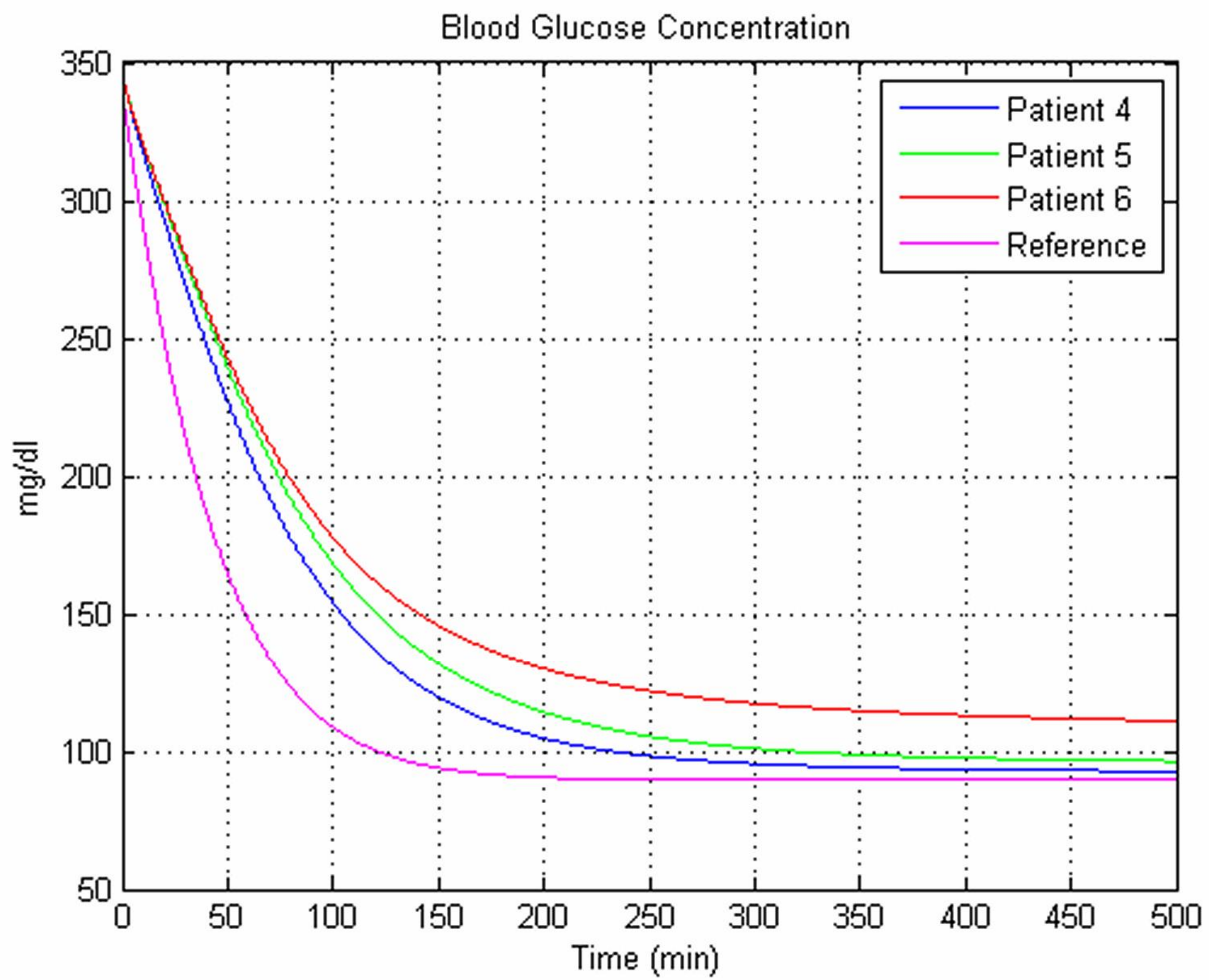


3-sliding QC control (SoM)

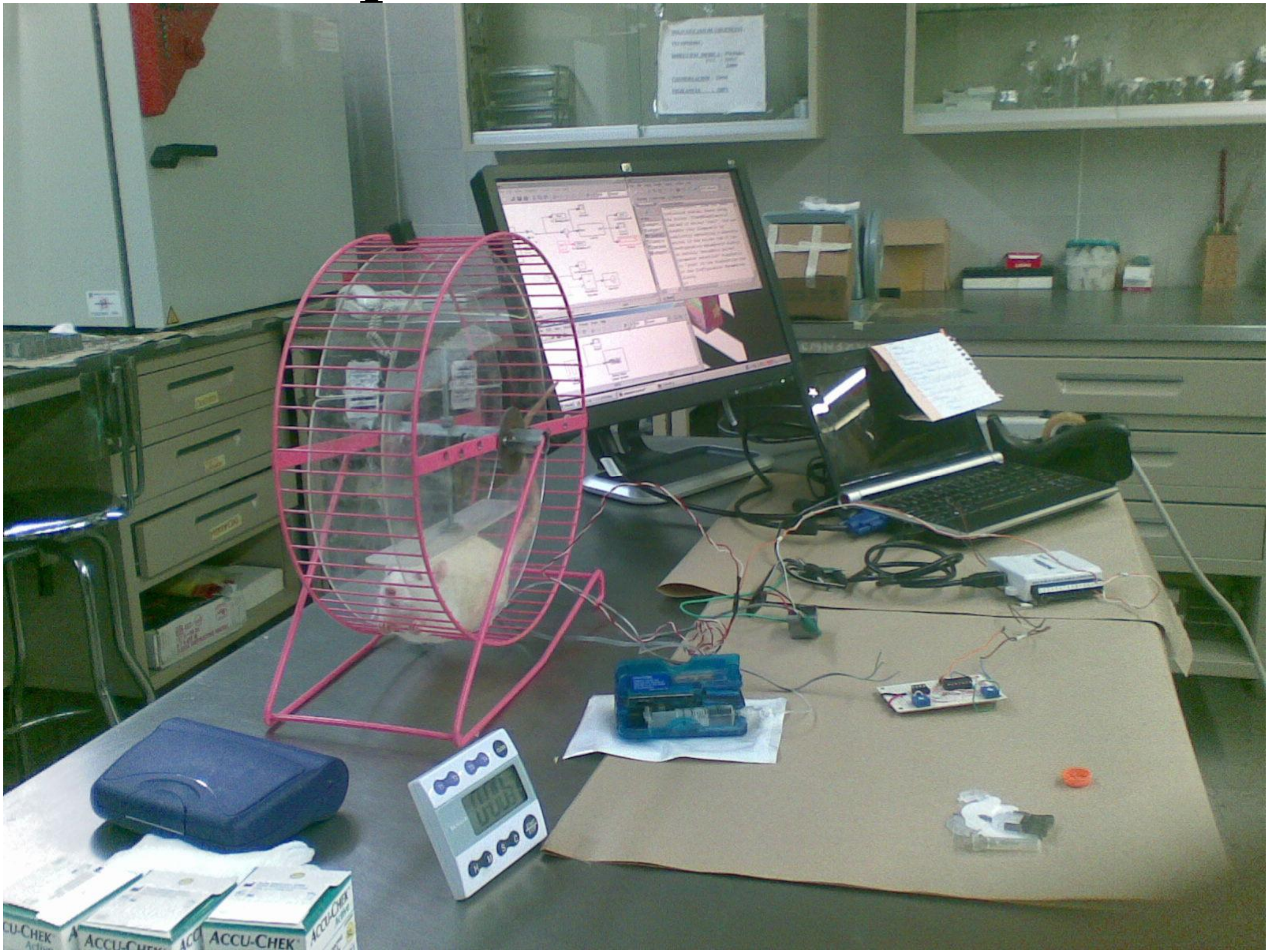


The same parameters

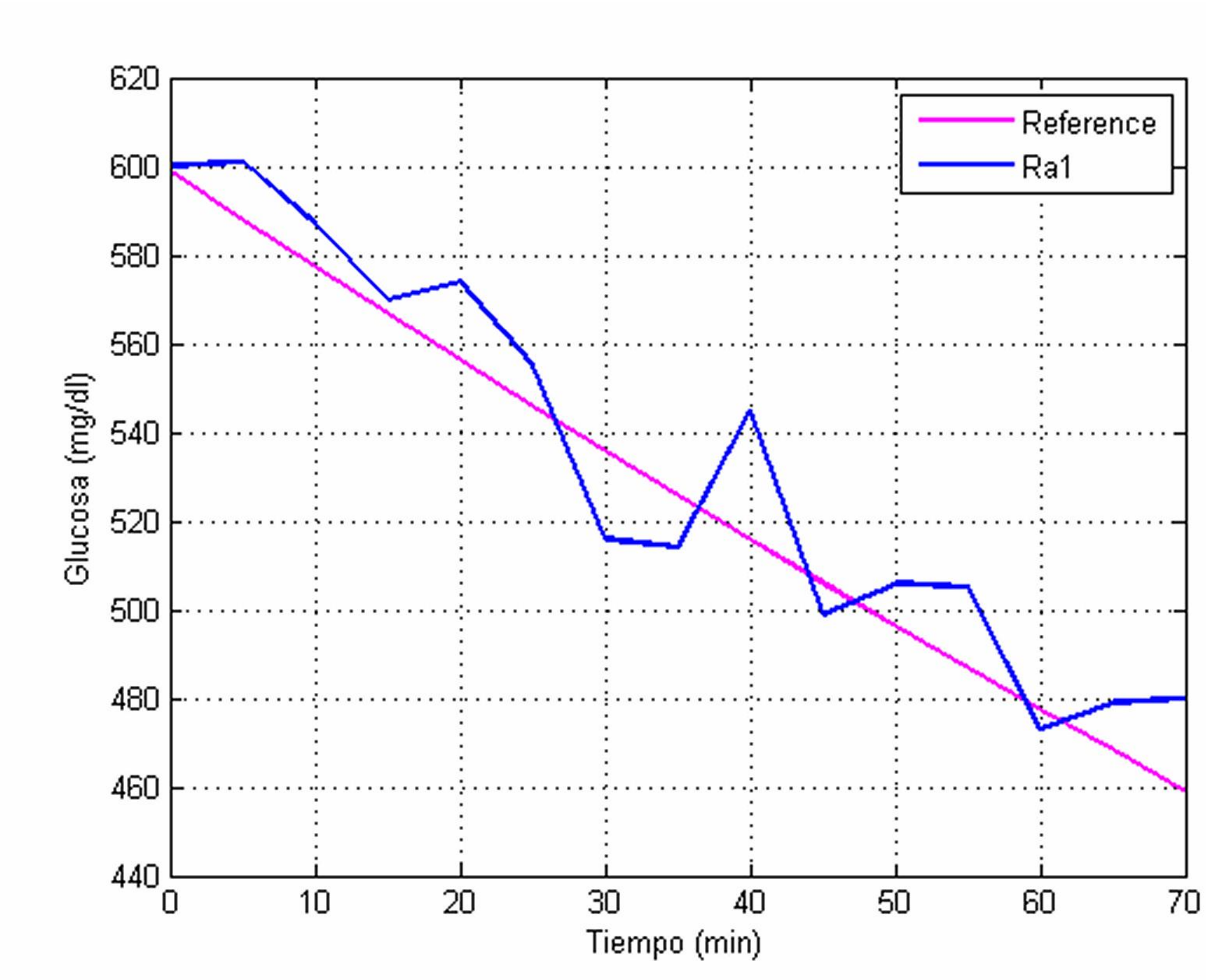
PID control (SoM)



Experiments on rats

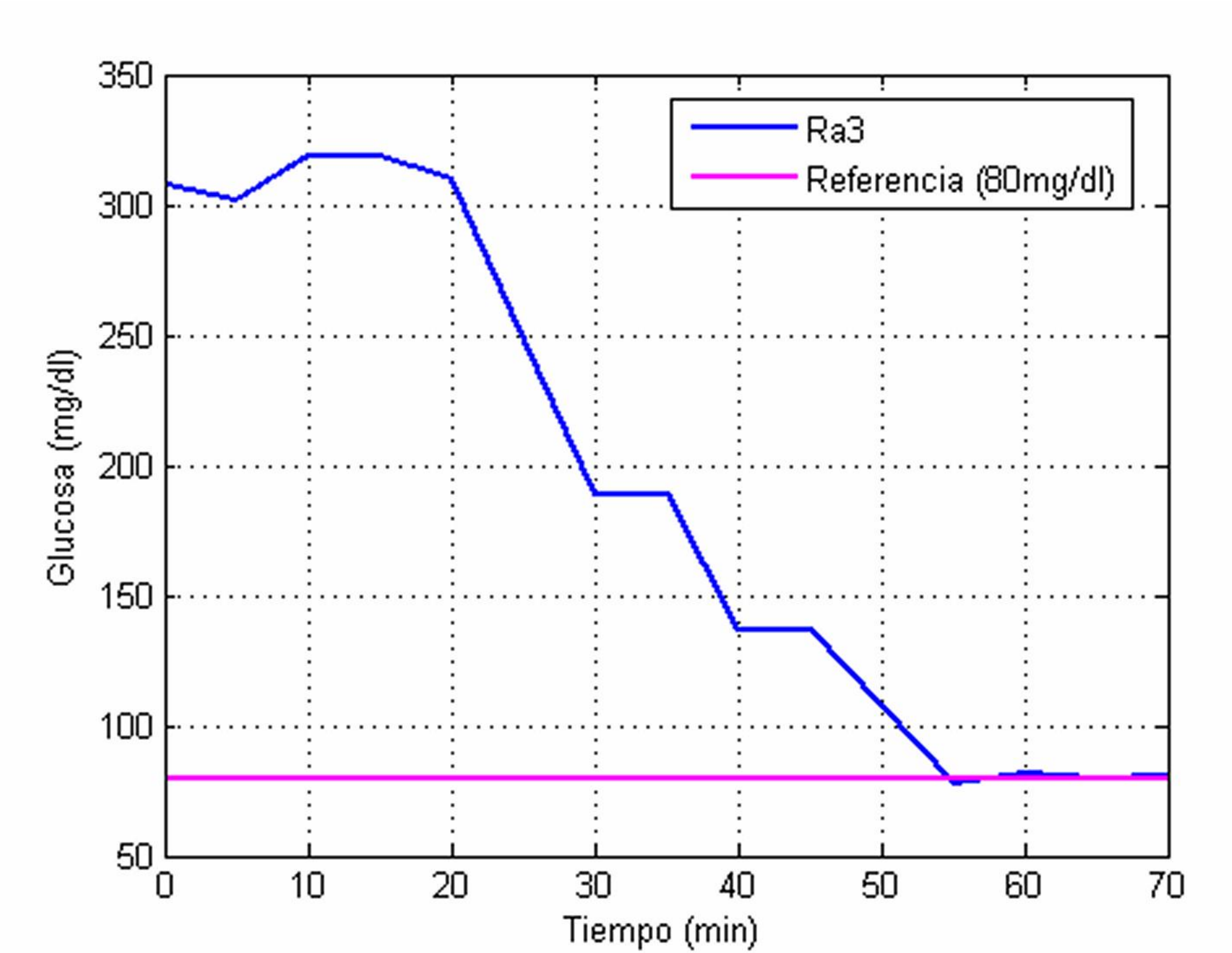


Rat 1



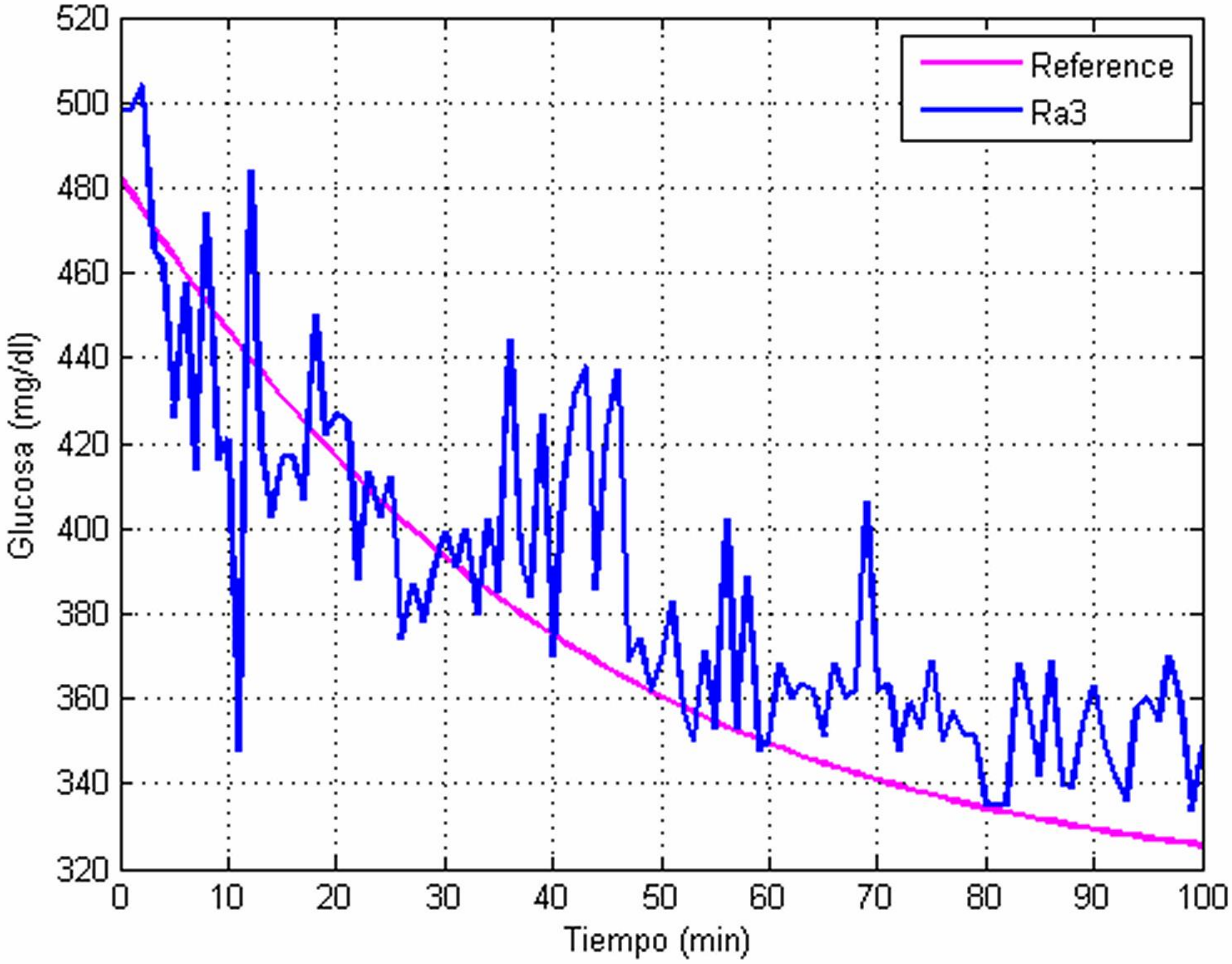
620
600
580
560
540
520
500
480
460
440
0
10
20
30
40
50
60
70
Glucosa (mg/dl)
Tiempo (min)
Reference
Ra1

Rat 2



Ra3
Referencia (80mg/dl)
Glucosa (mg/dl)
Tiempo (min)
350
300
250
200
150
100
50
0
10
20
30
40
50
60
70

Rat 3



Reference
Ra3
Glucosa (mg/dl)
Tiempo (min)
520
500
480
460
440
420
400
380
360
340
320
0
10
20
30
40
50
60
70
80
90
100

Conclusions

In practice the system relative degree is a design parameter.

Systems of uncertain nature can be effectively controlled, provided their practical relative degree is identified.

A system can have a few generalized PRDs! That is why the considered control is universal.

Hypothesis

Humans (and animals) have universal controllers embodied for PRD $\leq$ 2 (3?).

Thank you very much!