Robust exact real-time differentiation

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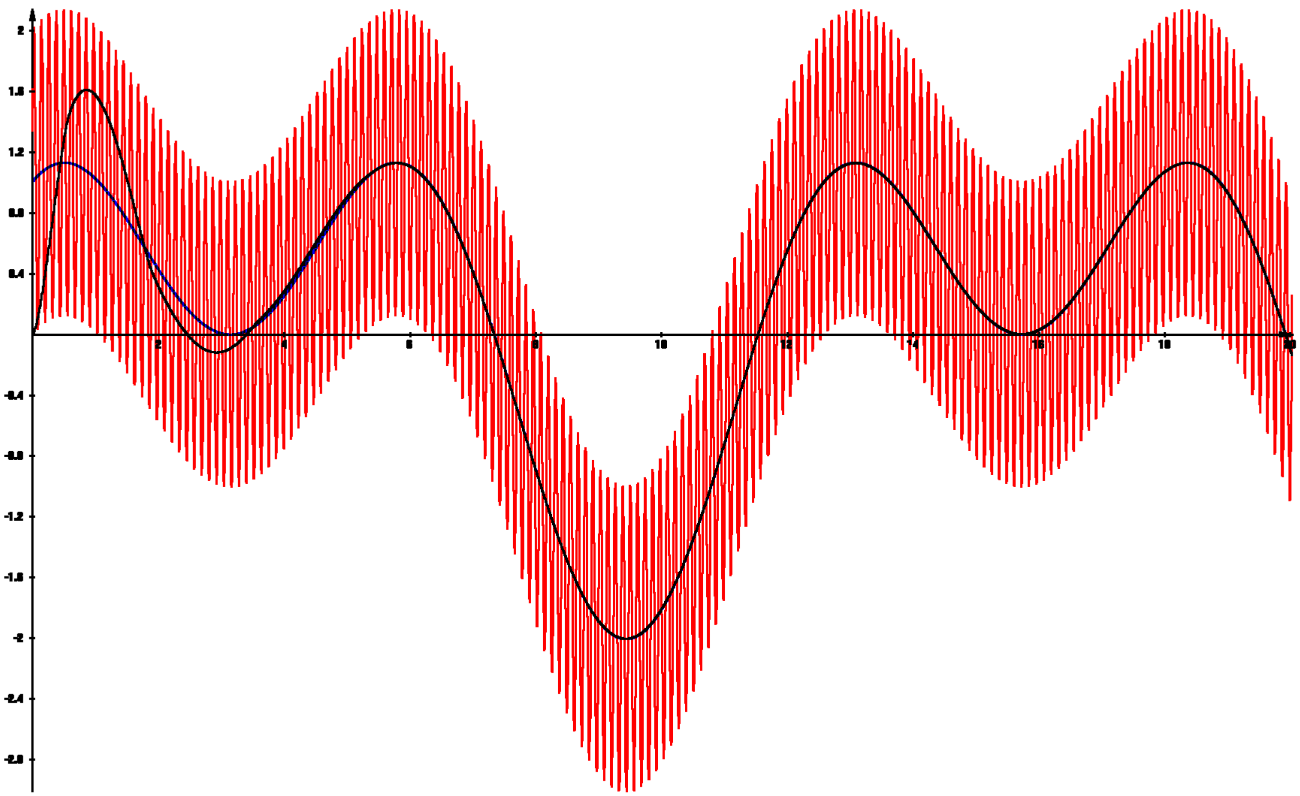


Differentiation problematics:

Division by zero: $$f'(t) = \lim_{\tau \to 0} \frac{f(t+\tau) - f(t)}{\tau}$$

Let $f(t) = f_0(t) + \eta(t)$, $\eta(t)$ - noise

$$\frac{f(t+\tau) - f(t)}{\tau} = \frac{\Delta f}{\tau} + \frac{\Delta \eta}{\tau}, \quad \frac{\Delta \eta}{\tau} \in (-\infty, \infty)$$

Differentiation is an ancient **ill-posed** problem: from the times of Newton and Leibnitz

Usually one needs to detect and neglect the noises.

Exact differentiation in real time is impossible:

1. Because of the noises (infinite error!)
2. Philosophically: close future prediction

Still it is sometimes possible:

If we know that the signal is constant, then the derivative is 0. Very robust to noises. ☺

*n*th order differentiator

is any algorithm, producing *n*+1 outputs for any measurable function *f* :

$$f(t) \mapsto (z_0(t), z_1(t), \ldots, z_n(t)).$$

The outputs are considered as derivative estimations

$$z_i(t) \cong f^{(i)}(t)$$

when derivatives exist.

We differentiate **in real time**.

Some approaches

1. Numeric differentiation (splines, divided differences, etc.)

$$Df(t) = F(f(t_1), \ldots, f(t_n))$$

Advantages: exact on certain functions (polynomials, *etc*)

Drawbacks: actual division by zero with $t_i \to t$

2. Fourier / Laplace transform, **high harmonics neglection**

Integral transformations: Fliess, Mboup, 2008

Advantages: robustness. **Drawbacks:** not exact

3. Linear filters, High Gain Observers (Khalil 1996) with transfer function $\frac{P_m(p)}{Q_n(p)} \approx p,\, m \le n.$ $\quad \frac{p}{(0.01p+1)^2} \approx p$

Advantages: Fast. **Drawbacks:** not exact, sensitive to noises

4. Nonlinear filtering

Emelyanov, Golembo, Utkin, Solovijov 1970-80s
Yu 1990
Sliding-mode (SM) based differentiators
Neither exact, nor robust

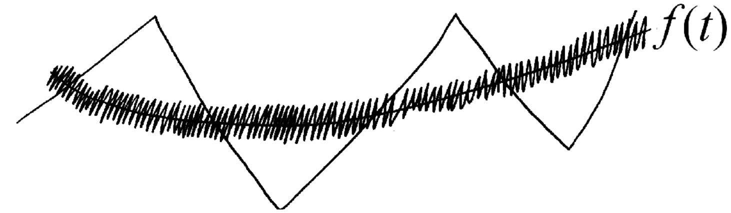
High-Gain Nonlinear observers: Han et al, 1994, 2009, Wang et al, 2007.

Advantages: Fast. **Drawbacks:** not exact, sensitive to noises

Levant 1998(1st order), 2003(any order)
Sliding-mode (SM) based homogeneous differentiator
Exact and robust, optimal accuracy in some sense
Angulo, Bartolini, Barbot, Efimov, Fridman, Koch, Moreno, Perruquetti, Pisano, Polyakov, …
Some exact, some not, fixed-time convergence, etc.

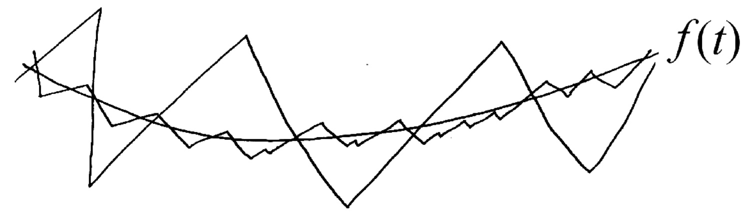
Robust differentiation problem

Unbounded derivatives



Bounded 1st derivatives

$$|\dot{\hat{f}}| \le L$$



Bounded 2nd derivatives

$$|\ddot{\hat{f}}| \le L$$

Theorem (Arzela): Bounded functions with constant-L-Lipschitzian derivative of the order n constitute a compact set in C.

"Solution": The closest function $\hat{f}$ & its derivatives!

The problem statement:

Input: $f(t) = f_0(t) + \eta(t)$, $t \geq 0$**,** $f_0^{(n)} \in \mathrm{Lip}(L)$
$|\eta(t)| \leq \varepsilon$**,** ε is unknown, $L > 0$
$\eta(t)$ is Lebesgue-measurable

Task: real-time finite-time estimation of

$$f_0(t),\ \dot{f}_0(t),\ \ldots,\ f_0^{(n)}(t).$$

The estimations are to continuously depend on ε and to be exact for $\varepsilon = 0$.

Best worst differentiation error

(Levant, Livne, Yu, IFAC 2017)

For some $t_0 = t_0(L, \varepsilon) > 0$, for $t \geq t_0$ get that

if f, f_0 satisfy $f^{(n)}, f_0^{(n)} \in \mathrm{Lip}(L)$, then

$$\max_{f, f_0} \sup_{t \geq t_0} | f^{(i)}(t) - f_0^{(i)}(t) | = K_{ni} (2L)^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}}.$$

$K_{i,n}$ is the Kolmogorov constant (1939),

Thus, for any measurable noise $\eta = f - f_0$

$$\max_{f, f_0} \sup_{t \geq t_0} | \widehat{f_0^{(i)}}(t) - f_0^{(i)}(t) | \geq K_{ni} (2L)^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}}.$$

Kolmogorov constants

$$1 \le K_{ni} < \pi / 2$$

n	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
1	1.41421					
2	1.04004	1.44225				
3	1.08096	1.09545	1.48017			
4	1.04426	1.11665	1.11942	1.49631		
5	1.04298	1.08001	1.14520	1.14280	1.50892	
6	1.03451	1.07289	1.10472	1.16471	1.15137	1.51748

For $n = 1$ get $K_{1,1} = \sqrt{2} \;\Rightarrow$

$$\max_{f, f_0} \sup_{t \ge t_0} | \hat{\dot{f}}_0(t) - \dot{f}_0(t) | \ge 2\sqrt{L\varepsilon}$$

Optimal differentiation

$$f(t) = f_0(t) + \eta(t),\ |\eta(t)| \le \varepsilon,\ t \ge 0$$

ε is unknown, $|f_0^{(n+1)}(t)| \le L$

A differentiator is **asymptotically optimal**, if for some $t_0 = t_0(L, \varepsilon) > 0$, for any $t \ge t_0$ get for $i = 0, 1, \ldots, n$

$$|\widehat{f_0^{(i)}}(t) - f_0^{(i)}(t)| \le \gamma_{ni} L^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}} = \gamma_{ni} L \left(\tfrac{\varepsilon}{L}\right)^{\frac{n+1-i}{n+1}},$$

The worst possible error in the *i*th derivative is never less than $K_{ni}(2L)^{\frac{i}{n+1}} \varepsilon^{\frac{n+1-i}{n+1}}$, i.e. $\gamma_{ni} \ge K_{ni} 2^{\frac{i}{n+1}}$, $\gamma_{1,1} \ge 2$.

Conclusions in advance

In spite of the ill-posedness of the differentiation problem

1. Real-time exact robust nth-order differentiation is possible if an upper bound L for $| f_0^{(n+1)} |$ is known.
2. It is exact in the absence of noises, features optimal error asymptotics in the presence of bounded noises, and filters out unbounded noises which are small in average.
3. It is easily realized, since does not require hard computations.

Types of developed differentiators

1. Homogeneous differentiator (1998, 2003)
2. Hybrid differentiator (variable $L(t)$, 2014, 2018)
3. Differentiators producing smooth outputs: $\frac{d}{dt}\widehat{f_0^{(i)}} = \widehat{f_0^{(i+1)}}$ ($\dot{z}_i = z_{i+1}$) (2010)
4. Discrete differentiators (2015, 2017)
5. Filtering differentiators (2017, 2018)

The covered topics: 1 and partially 2, 4, 5

Control theory point of view:

Observation with unknown input and noise

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_3,$$
$$\ldots$$
$$\dot{x}_{n+1} = v$$

$$\begin{cases} f(t) = x_1(t) + \eta(t) \\ v = f_0^{(n+1)}(t), \quad |v| \leq L \end{cases}$$

High Gain Observer (Khalil, ~1996)

$$\dot{z}_0 = -\lambda_n \alpha (z_0 - f(t)) + z_1,$$

$$\dot{z}_1 = -\lambda_{n-1} \alpha^2 (z_0 - f(t)) + z_2,$$

...

$$\dot{z}_{n-1} = -\lambda_1 \alpha^n (z_0 - f(t)) + z_n,$$

$$\dot{z}_n = -\lambda_0 \alpha^{n+1} (z_0 - f(t)).$$

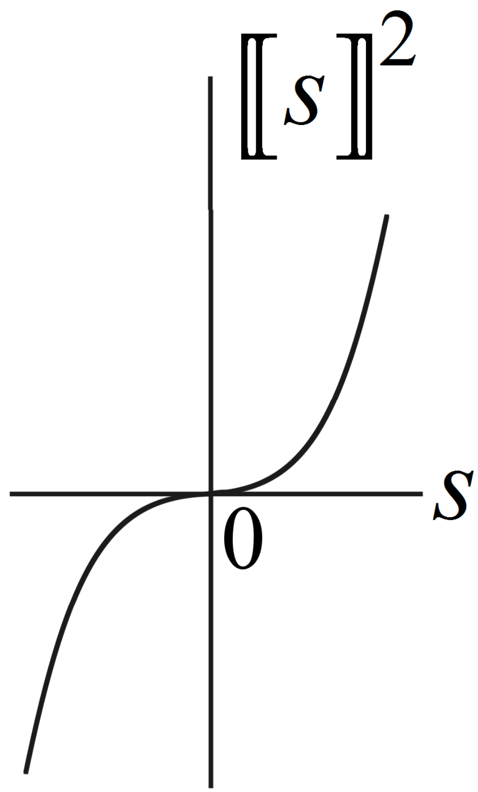
$\alpha >> \text{L},\ z_i - f_0^{(i)} = O(L / \alpha^{n+1-i})$

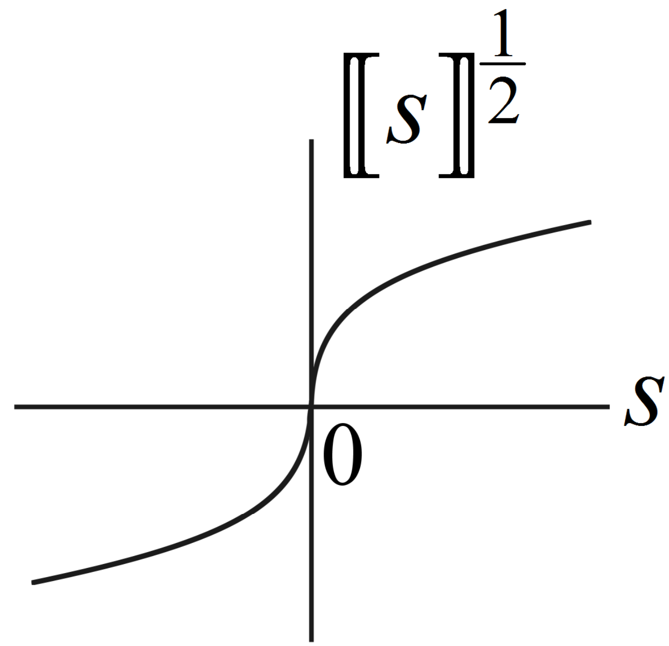
$s^{n+1} + \lambda_n s^n + ... + \lambda_0 s$ is Hurwitz

Special power functions

(standard notation)

$$\operatorname{sig}^{\gamma} s = \lfloor s \rceil^{\gamma} = [\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s$$





Homogeneous SM differentiator

(Levant 1998, 2003)

$$\dot z_0 = -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1,$$

$$\dot z_1 = -\lambda_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} + z_2,$$

$$\dots$$

$$\dot z_{n-1} = -\lambda_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} + z_n,$$

$$\dot z_n = -\lambda_0 L \ \mathrm{sign}(z_0 - f(t)), \quad z_i - f_0^{(i)} \to 0.$$

Hypothesis (Koch, Reichhartinger et al, 2018):

$s^{n+1} + \lambda_n s^n + \dots + \lambda_0 s$ is to be Hurwitz

Recursive form

$$\dot z_0 = -\tilde\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1,$$

$$\dot z_1 = -\tilde\lambda_{n-1} L^{\frac{1}{n}} \lfloor z_1 - \dot z_0 \rceil^{\frac{n-1}{n}} + z_2,$$

$$\dots$$

$$\dot z_{n-1} = -\tilde\lambda_1 L^{\frac{1}{2}} \lfloor z_{n-1} - \dot z_{n-2} \rceil^{\frac{1}{2}} + z_n,$$

$$\dot z_n = -\tilde\lambda_0 L \ \mathrm{sign}(z_n - \dot z_{n-1}), \quad z_i - f_0^{(i)} \to 0.$$

$\{\tilde\lambda_n\} = 1.1,\ 1.5,\ 2,\ 3,\ 5,\ 7,\ 10,\ 12,\ \dots$ for $n \le 7$

$$\lambda_0 = \tilde\lambda_0, \ \ \lambda_n = \tilde\lambda_n, \ \ \lambda_j = \tilde\lambda_j \lambda_{j+1}^{\ j/(j+1)}$$

Lyapunov function: Moreno, 2017

Theorem (2003). A sequence $\{\tilde{\lambda}_k\}$ is build, which is valid for all $n \geq 0$: for each $n \geq 0$ one sufficiently large value $\tilde{\lambda}_n$ is added, $\tilde{\lambda}_0 > 1$. In finite time (FT) get

$$| z_i - f_0^{(i)} | \leq \gamma_i L \left(\frac{\varepsilon}{L} \right)^{\frac{n+1-i}{n+1}}$$

$\{\tilde{\lambda}_n\}$= 1.1, 1.5, 2, 3, 5, 7, 10, 12, … for $n \leq 7$ (2017)

also

$\{\tilde{\lambda}_n\}$= 1.1, 1.5, 2, 3, 5, 8, … for $n \leq 5$ (2005)

and

$\{\tilde{\lambda}_n\}$= 1.1, 1.5, 3, 5, 8, 12, … for $n \leq 5$ (2003)

are valid.

Differentiator parameters

$$\lambda_0 = \tilde{\lambda}_0, \quad \lambda_n = \tilde{\lambda}_n, \quad \lambda_j = \tilde{\lambda}_j \lambda_{j+1}{}^{j/(j+1)}$$

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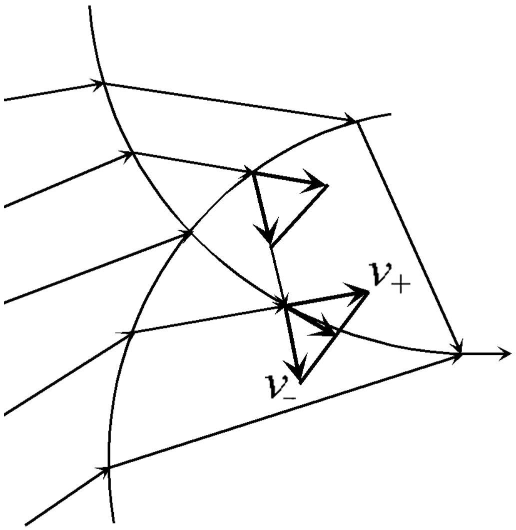
n	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

Discontinuous Differential Equations



Filippov Definition

$$\dot{x} = f(x) \;\Leftrightarrow\; \dot{x} \in F(x)$$

$x(t)$ is an absolutely continuous function

$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu N=0} \text{convex\_closure } f(O_\varepsilon(x) \setminus N)$$

Non-autonomous case: $\dot{t} = 1$ is added.

When switching imperfections (delays, sampling errors, etc) tend to zero, **usual solutions uniformly converge to Filippov solutions**

Asymptotically optimal accuracy

In the presence of the noise with the magnitude ε, and sampling with the step τ: $\exists \mu_j \geq 1$

$$|z_j - f_0^{(j)}| \leq \gamma_j L \rho^{n+1-j}, \;\; \rho = \max(\tau, \left(\frac{\varepsilon}{L}\right)^{\frac{1}{n+1}})$$

$$\varepsilon = \tau = 0 \;\Rightarrow$$

$$\text{in finite time get } z_i \equiv f^{(i)}, i = 0,\ldots,n$$

Homogeneity of error dynamics

Denote $\sigma_i = (z_i - f_0^{(i)}) / L$, $\eta = 0$, then $\frac{f_0^{(n+1)}}{L} \in [-1,1]$

$$
\begin{aligned}
\dot{\sigma}_0 &= -\lambda_n \left[\!\left[\sigma_0 \right]\!\right]^{\frac{n}{n+1}} + \sigma_1, \\
\dot{\sigma}_1 &= -\lambda_{n-1} \left[\!\left[\sigma_0 \right]\!\right]^{\frac{n-1}{n+1}} + \sigma_2, \\
&\dots \\
\dot{\sigma}_{n-1} &= -\lambda_1 \left[\!\left[\sigma_0 \right]\!\right]^{\frac{1}{n+1}} + \sigma_n, \\
\dot{\sigma}_n &\in -\lambda_0 \, \mathrm{sign}\, \sigma_0 + [-1,1].
\end{aligned}
$$

Invariance:

$$\forall \kappa > 0,\ (t, \sigma_0, \sigma_1, \dots, \sigma_n) \mapsto (\kappa t, \kappa^{n+1}\sigma_0, \kappa^n \sigma_1, \dots, \kappa\sigma_n) \ .$$

After transformation κ is cancelled:

$$\frac{d\kappa^{n+1}\sigma_0}{d\kappa t} = -\tilde{\lambda}_n \left[\!\left[\kappa^{n+1}\sigma_0\right]\!\right]^{\frac{n}{n+1}} + \kappa^n\sigma_1,$$

$$\frac{d\kappa^{n}\sigma_1}{d\kappa t} = -\tilde{\lambda}_{n-1} \left[\!\left[\kappa^{n+1}\sigma_0\right]\!\right]^{\frac{n-1}{n+1}} + \kappa^{n-1}\sigma_2,$$

$$\ldots$$

$$\frac{d\kappa^{2}\sigma_{n-1}}{d\kappa t} = -\tilde{\lambda}_1 \left[\!\left[\kappa^{n+1}\sigma_0\right]\!\right]^{\frac{1}{n+1}} + \kappa\sigma_n,$$

$$\frac{d\kappa\sigma_n}{d\kappa t} \in -\tilde{\lambda}_0 \operatorname{sign}(\kappa^{n+1}\sigma_0) + [-1,1].$$

Convergence proof idea

(Levant 2001-2005)

Contraction:

Trajectories starting in the unit ball in FT gather in a smaller ball around zero.

homogeneity+contraction → FT collapse to zero

Accuracy

$$\dot\sigma_0 \in -\tilde\lambda_n \left[\!\left[\sigma_0(t-[0,\tau]) + [-\tfrac{\varepsilon}{L},\tfrac{\varepsilon}{L}] \right]\!\right]^{\frac{n}{n+1}} + \sigma_1,$$

$$\dot\sigma_1 \in -\tilde\lambda_{n-1} \left[\!\left[\sigma_0(t-[0,\tau]) + [-\tfrac{\varepsilon}{L},\tfrac{\varepsilon}{L}] \right]\!\right]^{\frac{n-1}{n+1}} + \sigma_2,$$

$$\dots$$

$$\dot\sigma_{n-1} \in -\tilde\lambda_1 \left[\!\left[\sigma_0(t-[0,\tau]) + [-\tfrac{\varepsilon}{L},\tfrac{\varepsilon}{L}] \right]\!\right]^{\frac{1}{n+1}} + \sigma_n,$$

$$\dot\sigma_n \in -\tilde\lambda_0 \operatorname{sign}(\sigma_0(t-[0,\tau]) + [-\tfrac{\varepsilon}{L},\tfrac{\varepsilon}{L}]) + [-1,1].$$

Invariancy: $(\varepsilon,\tau) \mapsto (\kappa^{n+1}\varepsilon, \kappa\tau),$

$$(t,\sigma_0,\sigma_1,\dots,\sigma_n) \mapsto (\kappa t, \kappa^{n+1}\sigma_0, \kappa^n\sigma_1,\dots,\kappa\sigma_n)$$

Hybrid differentiator (Levant, Livne IJC2018)

$$|f_0^{(n+1)}(t)| \le L(t), \; |\dot{L}/L| \le M$$

$$\dot{z}_0 = -\lambda_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} - \mu_n M(z_0 - f(t)) + z_1,$$

$$\dot{z}_1 = -\lambda_{n-1} L^{\frac{1}{n}} \lfloor z_1 - \dot{z}_0 \rceil^{\frac{n-1}{n}} - \mu_{n-1} M(z_1 - \dot{z}_0) + z_2,$$

...

$$\dot{z}_{n-1} = -\lambda_1 L^{\frac{1}{2}} \lfloor z_{n-1} - \dot{z}_{n-2} \rceil^{\frac{1}{2}} - \mu_1 M(z_{n-1} - \dot{z}_{n-2}) + z_n,$$

$$\dot{z}_n = -\lambda_0 L \operatorname{sign}(z_n - \dot{z}_{n-1}) - \mu_0 M(z_n - \dot{z}_{n-1})$$

$\lambda_i =$ 1.1 1.5 2 3 5 7 10 12 ...

$\mu_i =$ 2 3 4 7 9 13 19 23 ...

Hybrid differentiator becomes “standard” homogeneous for μ_i =0, and turns into the standard HGO by Khalil for $\lambda_i = 0$, $M >> L$:

$$
\begin{aligned}
\dot{z}_0 &= -\mu_n M(z_0 - f(t)) + z_1, \\
\dot{z}_1 &= -\mu_{n-1}\mu_n M^2(z_0 - f(t)) + z_2, \\
&\ldots \\
\dot{z}_{n-1} &= -\mu_1\mu_2 \ldots \mu_n M^n(z_0 - f(t)) + z_n, \\
\dot{z}_n &= -\mu_0\mu_1 \ldots \mu_n M^{n+1}(z_0 - f(t))
\end{aligned}
$$

Non-recursive form

$$\varphi_i(t,s) = \lambda_{n-i} L(t)^{\frac{1}{n-i+1}} \, |s|^{\frac{n-i}{n-i+1}} \ \text{sign}\, s + \mu_{n-i} M s$$

$$\begin{aligned}
\dot{z}_0 &= -\varphi_0(t, z_0 - f(t)) + z_1, \\
\dot{z}_1 &= -\varphi_1(t, \varphi_0(t, z_0 - f(t))) + z_2, \\
&\dots \\
\dot{z}_n &= -\varphi_n(t, \varphi_{n-1}(\dots(t, \varphi_0(t, z_0 - f(t)))\dots)).
\end{aligned}$$

It is not convenient to use!

Non-recursive form, *n* = 1

(particular case)

Moreno 2009,2017

$$\dot{z}_0 = -\lambda_1 L^{\frac{1}{2}} \lfloor z_0 - f(t) \rceil^{\frac{1}{2}} - \mu_1 M (z_0 - f(t)) + z_1,$$

$$\dot{z}_1 = -\lambda_0 L \ \mathrm{sign}(z_0 - f(t)) - \mu_0 \lambda_1 L^{\frac{1}{2}} M \lfloor z_0 - f(t) \rceil^{\frac{1}{2}} - \mu_0 \mu_1 M^2 (z_0 - f(t))$$

J. Moreno has found a Lyapunov function (2017).

Theorem (2017): There exists a sequence $\{\lambda_k, \mu_k\}$ valid for all $n \geq 0$. One can start with any $\lambda_0 > 1$, $\mu_0 > 1$. For each $n \geq 0$ one chooses arbitrary $\theta_n > 0$ and adds one pair (λ_n, μ_n) with any sufficiently large λ_n and $\mu_n = \theta_n \lambda_n > 1$.

The accuracy is the same, but for ***sufficiently small*** noise magnitude $\hat{\varepsilon}$, $\left|\frac{\eta(t)}{L(t)}\right| \leq \hat{\varepsilon}$, and sampling interval τ

$$| z_j - f_0^{(i)} | \leq \gamma_i L \rho^{n+1-i}, \;\; \rho = \max(\tau^{n+1-i}, \hat{\varepsilon}^{\frac{n+1-i}{n+1}})$$

up to $n = 7$

$$\lambda_i = 1.1 \quad 1.5 \quad 2 \quad 3 \quad 5 \quad 7 \quad 10 \quad 12 \; \ldots$$

$$\mu_i = 2 \quad 3 \quad 4 \quad 7 \quad 9 \quad 13 \quad 19 \quad 23 \; \ldots$$

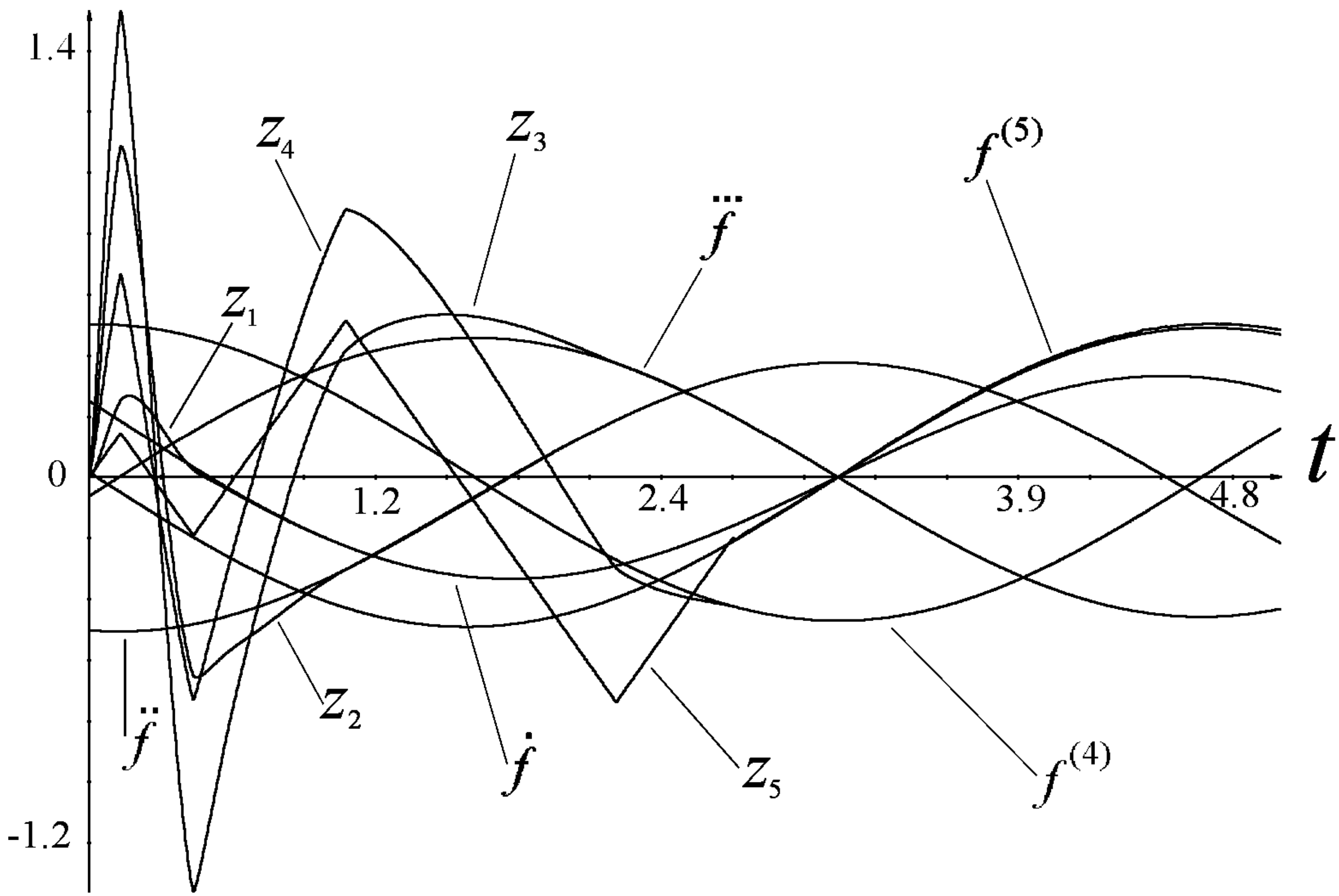
5th-order differentiator, $|f^{(6)}| \le L$.

5 4 3 2 1

$$
\begin{aligned}
\dot{z}_0 &= -12L^{\frac{1}{6}} [\![z_0 - f(t)]\!]^{\frac{5}{6}} + z_1, \\
\dot{z}_1 &= -8L^{\frac{1}{5}} [\![z_1 - \dot{z}_0]\!]^{\frac{4}{5}} + z_2, \\
\dot{z}_2 &= -5L^{\frac{1}{4}} [\![z_2 - \dot{z}_1]\!]^{\frac{3}{4}} + z_3, \\
\dot{z}_3 &= -3L^{\frac{1}{3}} [\![z_3 - \dot{z}_2]\!]^{\frac{2}{3}} + z_4, \\
\dot{z}_4 &= -1.5L^{\frac{1}{2}} [\![z_4 - \dot{z}_3]\!]^{\frac{1}{2}} + z_5, \\
\dot{z}_5 &= -1.1L \,\mathrm{sign}(z_5 - \dot{z}_4)
\end{aligned}
$$

5th-order differentiation

$$f(t) = \sin 0.5t + \cos 0.5t, \quad L = 1$$



The **worst-case** accuracy

$$\sup |z_n - f^{(n)}| \geq K_{nn} 2^{\frac{n}{n+1}} L^{\frac{n}{n+1}} \varepsilon^{\frac{1}{n+1}}, \; K_{n,n} \approx 1.5$$

$$n = 5, \; L = 1, \; \varepsilon = 10^{-6}, \; \text{error of } f^{(5)} > 0.3$$

Digital round up: $\varepsilon = 5 \cdot 10^{-16}$

$$n = 5: \; \text{error} \; \sim 0.01; \quad n = 6: \; \text{error} \; \sim 0.02$$

It is bad, but it cannot be improved!

Discretization

In reality the differentiator is realized by computers as a discrete system processing a sampled signal produced by continuous dynamics.
In order to preserve the accuracy special discretization is required.

Matlab ODE (Runge-Kutta) solvers destroy the accuracy!!

Differentiator Euler integration

Theorem. The simplest one-Euler-step discretization with the constant step τ leads to the accuracies

$$| z_0(t_{k,j}) - f_0(t_{k,j}) | \le \gamma_0 L \rho^{n+1}; \ \rho = \max[\varepsilon^{\frac{1}{n+1}}, \tau]$$

$$| z_i(t_{k,j}) - f_0^{(i)}(t_{k,j}) | \le \gamma_i L \rho^{n+1-i} + iL\tau D_{i+1}, i = 1,\ldots,n,$$

$$| f_0^{(i)} | / L \le D_i, \ \ D_{n+1} = 1$$

Variable sampling step asymptotics are worse.

The accuracies are also proportional to maximal lower derivatives.

Homogeneous discrete differentiator

3rd order differentiator (Livne, Levant 2015):

$$z_0(t_{k+1}) = z_0(t_k) - \tau_k \lambda_3 L^{1/4} \left| z_0(t_k) - f(t_k) \right|^{3/4} \operatorname{sign}(z_0(t_k) - f(t_k)) + \tau_k z_1(t_k) + \frac{\tau_k^2}{2!} z_2(t_k) + \frac{\tau_k^3}{3!} z_3(t_k),$$

$$z_1(t_{k+1}) = z_1(t_k) - \tau_k \lambda_2 L^{2/4} \left| z_0(t_k) - f(t_k) \right|^{2/4} \operatorname{sign}(z_0(t_k) - f(t_k)) + \tau_k z_2(t_k) + \frac{\tau_k^2}{2!} z_3(t_k),$$

$$z_2(t_{k+1}) = z_2(t_k) - \tau_k \lambda_1 L^{3/4} \left| z_0(t_k) - f(t_k) \right|^{1/4} \operatorname{sign}(z_0(t_k) - f(t_k)) + \tau_k z_3(t_k),$$

$$z_3(t_{k+1}) = z_3(t_k) - \tau_k \lambda_0 L \operatorname{sign}(z_0(t_k) - f(t_k)).$$

The original theoretical accuracy is restored

Hybrid discrete 3rd-order differentiator

Recursive form

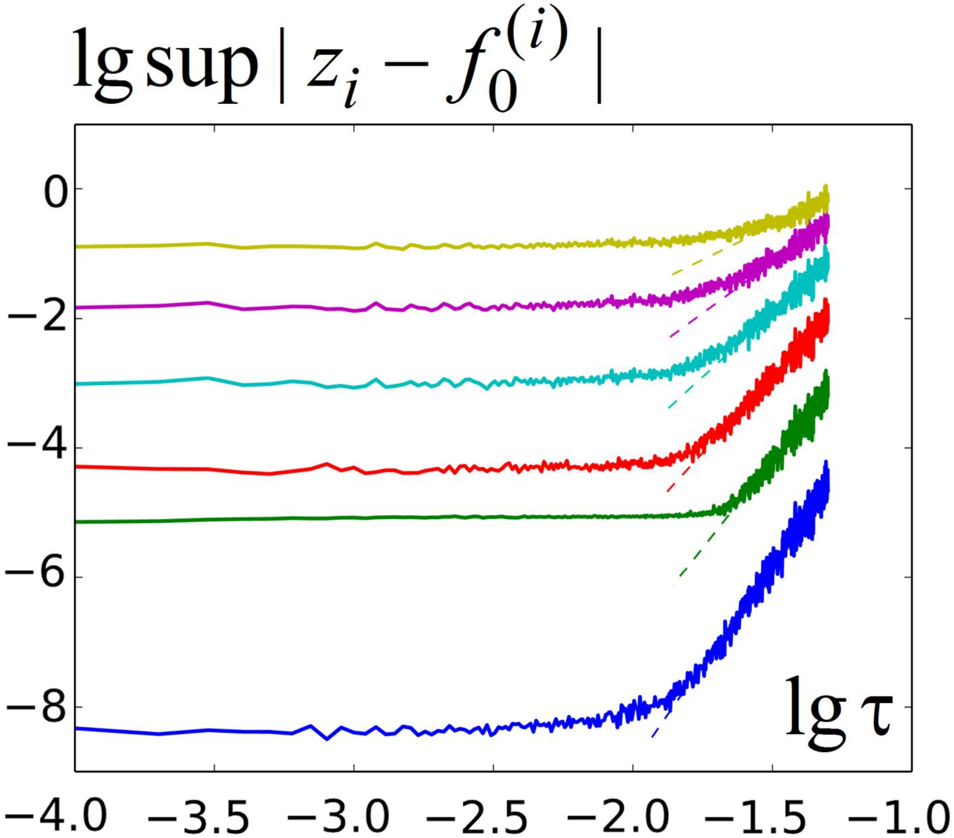
$$\begin{aligned}
z_0(t_{k+1}) &= z_0(t_k) + v_0\tau_k + \tfrac{1}{2}z_2(t_k)\tau_k^2 + \tfrac{1}{6}z_3(t_k)\tau_k^3,\\
v_0 &= -3L(t_k)^{1/4}|z_0(t_k) - f(t_k)|^{3/4}\ \mathrm{sign}(z_0(t_k) - f(t_k))\\
&\qquad -7M(z_0(t_k) - f(t_k)) + z_1(t_k),
\end{aligned}$$

$$\begin{aligned}
z_1(t_{k+1}) &= z_1(t_k) + v_1\tau_k + \tfrac{1}{2}z_3(t_k)\tau_k^2,\\
v_1 &= -2L(t_k)^{1/3}|z_1(t_k) - v_0(t_k)|^{2/3}\ \mathrm{sign}(z_1(t_k) - v_0(t_k))\\
&\qquad -4M(z_1(t_k) - v_0(t_k)) + z_2(t_k),
\end{aligned}$$

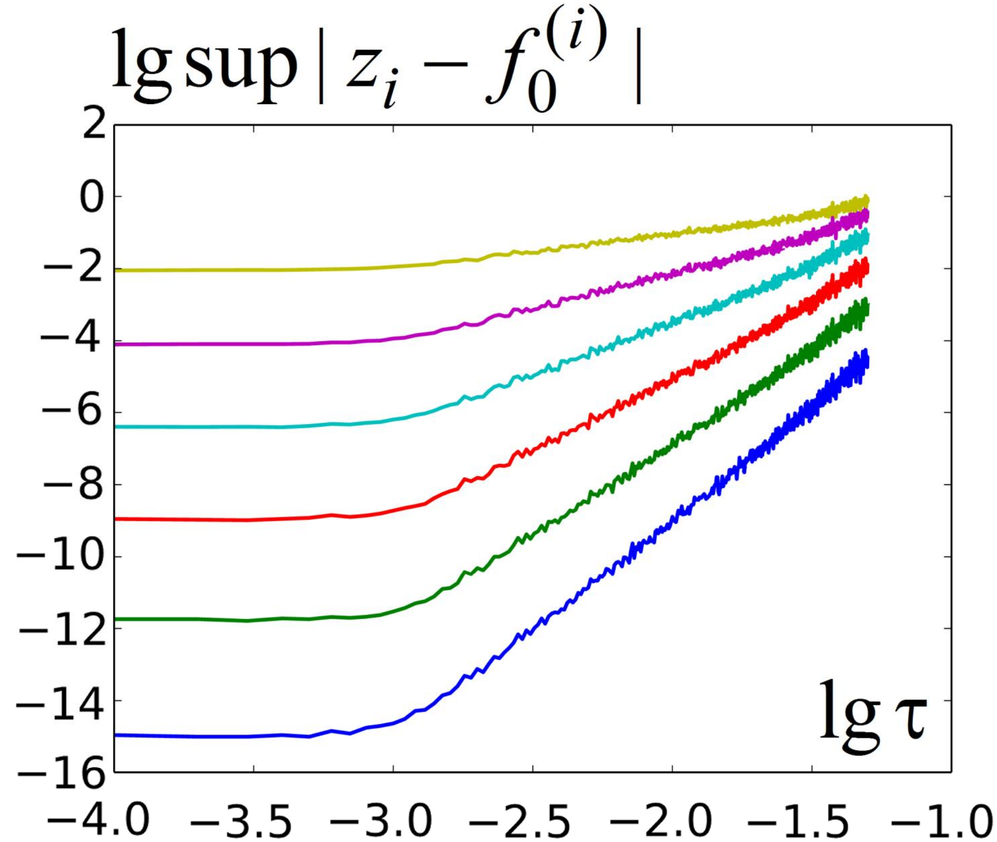
$$\begin{aligned}
z_2(t_{k+1}) &= z_2(t_k) + v_2\tau_k,\\
v_2 &= -1.5L(t_k)^{1/2}|z_2(t_k) - v_1(t_k)|^{1/2}\ \mathrm{sign}(z_2(t_k) - v_1(t_k))\\
&\qquad -3M(z_2(t_k) - v_1(t_k)) + z_3(t_k),
\end{aligned}$$

$$\begin{aligned}
z_3(t_{k+1}) &= z_3(t_k) - 1.1L(t_k)\ \mathrm{sign}(z_3(t_k) - v_2(t_k))\tau_k\\
&\qquad -2M(z_3(t_k) - v_2(t_k))\tau_k.
\end{aligned}$$

Asymptotics for randomal sampling steps (Hybrid differentiator, $M = 1$)



a: No Taylor·terms

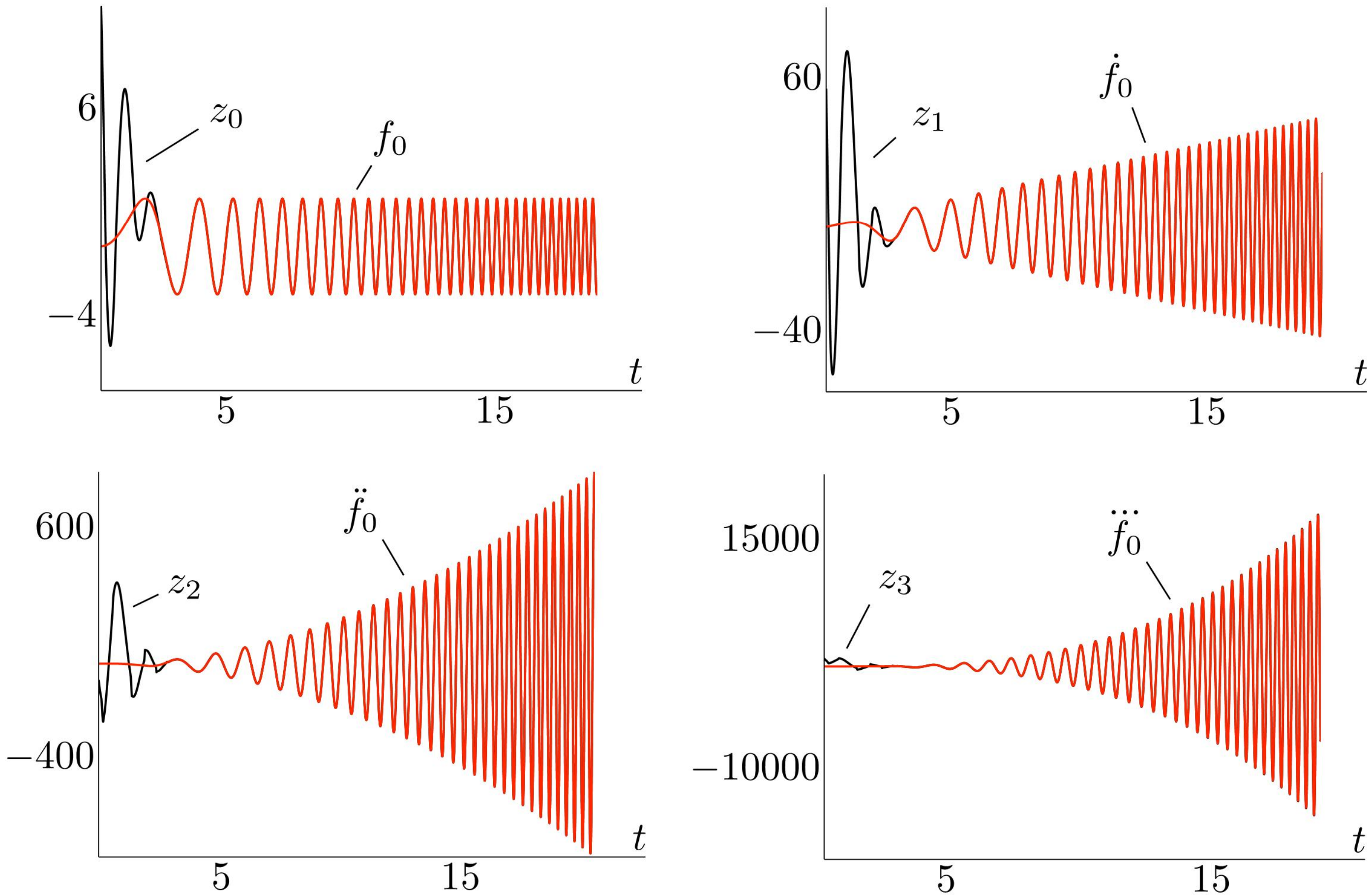


b: With Taylor·terms

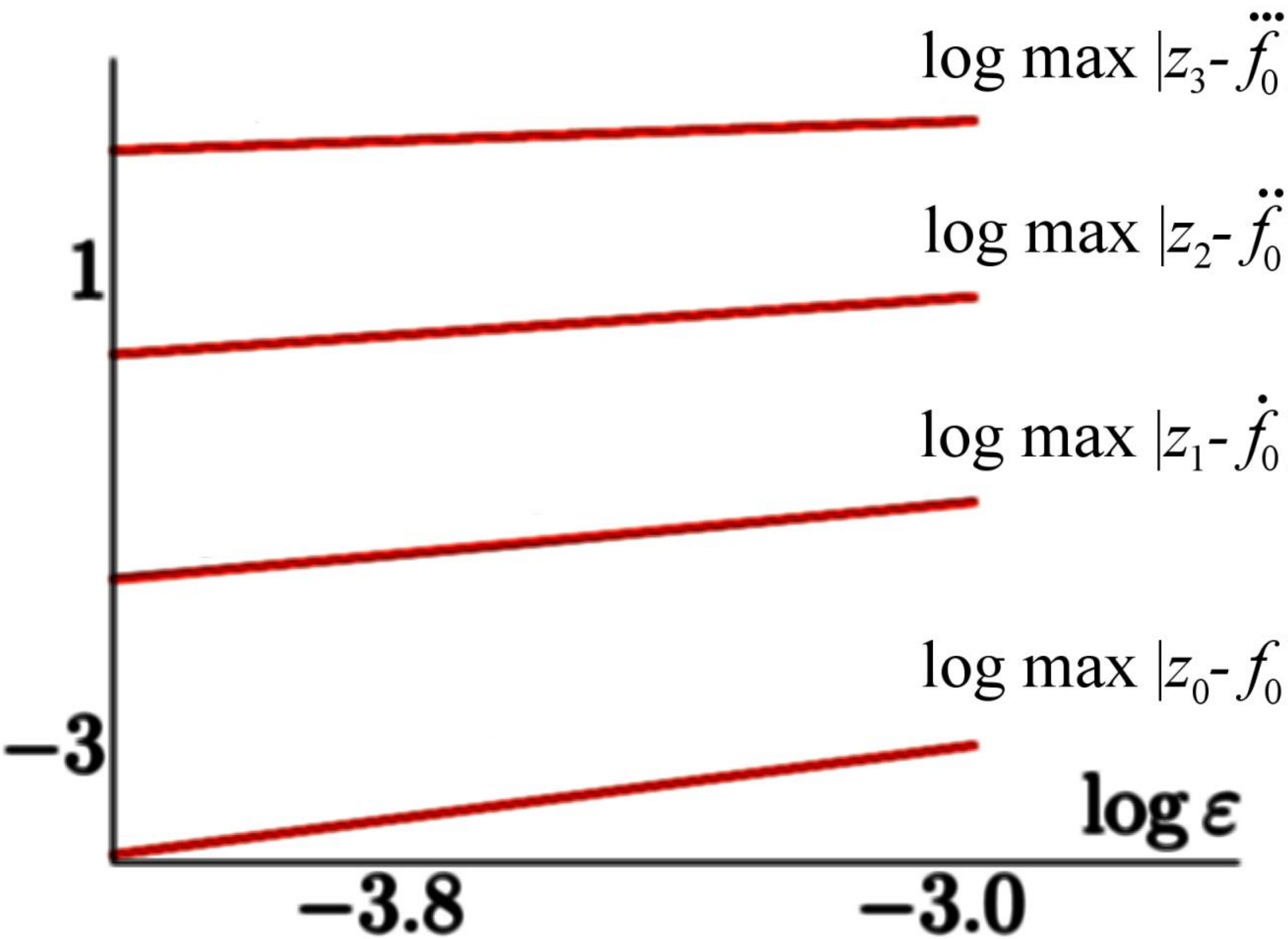
Example:

$$f(t) = f_0(t) + \eta(t),\ f_0(t) = 2\sin\left(\tfrac{1}{2}t^2\right),\ |\eta(t)| \leq \varepsilon,$$

$$z(0) = (10,50,-70,800),\quad L(t) = 2t^4 + 12t^2 + 6,\ M = 2$$



Asymptotics, n = 3:



slopes ~ theory

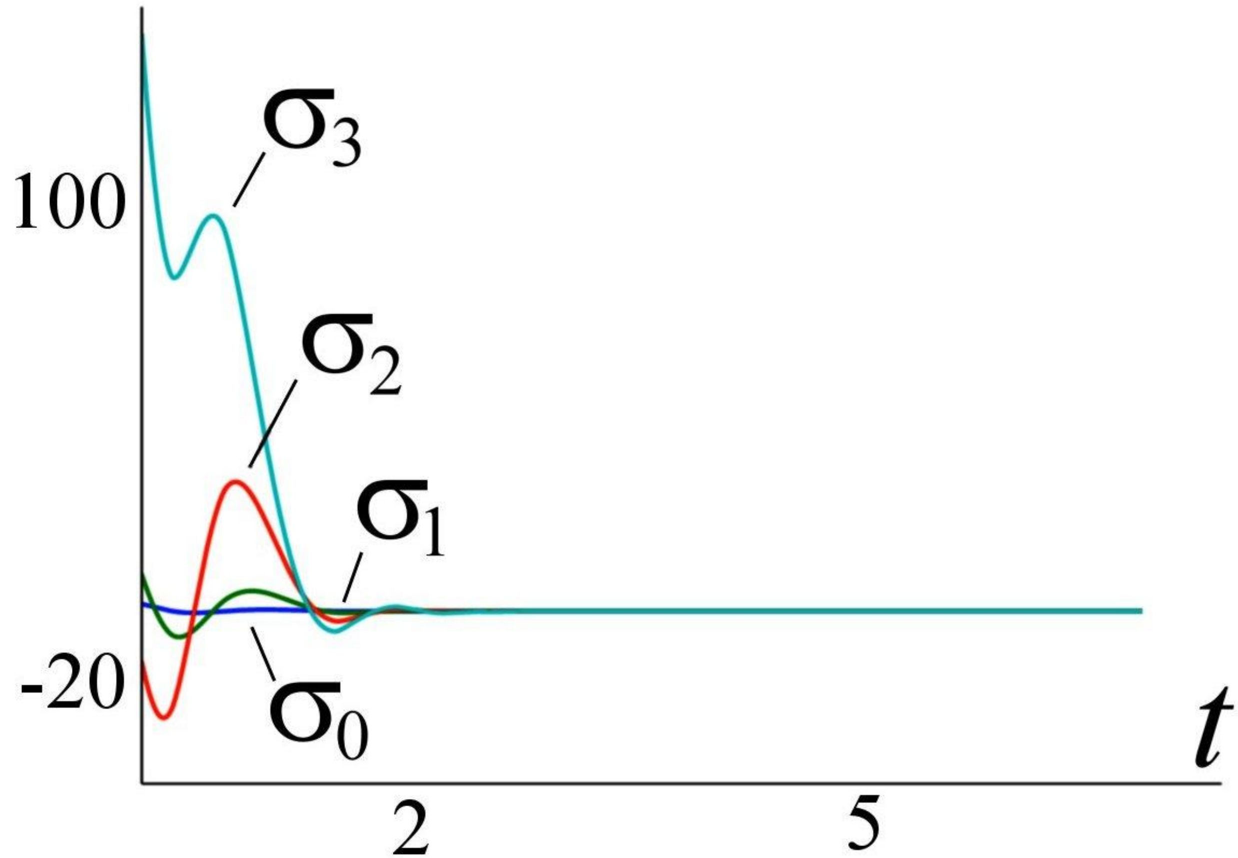
0.25~0.25

0.49~0.5

0.67~0.75

0.95~1

Error Dynamics: $\sigma_i = (z_i - f_0^{(i)}) / L$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \le (7 \cdot 10^{-12}, 8 \cdot 10^{-8}, 6 \cdot 10^{-4}, 2.8)$$
$$(|\sigma_0|, |\sigma_1|, |\sigma_2|, |\sigma_3|) \le (1 \cdot 10^{-15}, 2 \cdot 10^{-11}, 1 \cdot 10^{-7}, 5 \cdot 10^{-4})$$

Filtering differentiators

Levant, VSS 2018

Levant, Yu, IEEETAC 2018

Filtering order

A function η(t), $\eta : [0,\infty) \rightarrow \mathbb{R}$, has
the **filtering order** $k \geq 0$ ($\eta \in \mathrm{Fltr}_k(\delta)$) if

1. ν is **locally essentially bounded**,
2. there exists bounded $\xi(t)$ which satisfies

$$\xi^{(k)} = \eta, \ \operatorname*{ess\,sup}_{t \geq 0} |\, \xi(t) \,| \leq \delta \ .$$

Any such number δ is called the ***k*th-order integral magnitude** of η(t).

$$\eta \in \mathrm{Fltr}_0(\delta) \leftrightarrow |\, \eta(t) \,| \leq \delta$$

The problem statement

1. The input: $f(t) = f_0(t) + \eta(t)$, $f_0 \in \mathrm{Lip}_{\mathbb{R}_+}(n_d, L)$, $|f_0^{(n_d+1)}(t)| \le L$
2. The noise: $\eta(t) = \eta_0(t) + \ldots + \eta_{n_f}(t)$ (not unique!).

   Each η_k is of the *k*th filtering order with the integral magnitude $\delta_k \ge 0$, $k = 0, 1, \ldots, n_f$.

Remark: η_0 is just a usual bounded noise.

Task: to restore $f_0(t)$, $\dot{f}_0(t)$, …, $f_0^{(n_d)}(t)$ robustly and exactly for $\delta_0 = \ldots = \delta_{n_f} = 0$.

n_dth-order filtering differentiator, of the filtering order n_f

$$
n_f \begin{cases}
\dot{w}_1 = -\lambda_{n_d+n_f} L^{\frac{1}{n_d+n_f+1}} [\![w_1]\!]^{\frac{n_d+n_f}{n_d+n_f+1}} + w_1, \\
\quad \ldots \\
\dot{w}_{n_f} = -\lambda_{n_d+1} L^{\frac{n_f}{n_d+n_f+1}} [\![w_1]\!]^{\frac{n_d+1}{n_d+n_f+1}} + z_0 - f(t),
\end{cases}
$$

$$
n_d \begin{cases}
\dot{z}_0 = -\lambda_{n_d} L^{\frac{n_f+1}{n_d+n_f+1}} [\![w_1]\!]^{\frac{n_d}{n_d+n_f+1}} + z_1, \\
\quad \ldots \\
\dot{z}_{n_d-1} = -\lambda_1 L^{\frac{n_d+n_f}{n_d+n_f+1}} [\![w_1]\!]^{\frac{1}{n_d+n_f+1}} + z_{n_d}, \\
\dot{z}_{n_d} = -\lambda_0 L \operatorname{sign} w_1.
\end{cases}
$$

Theorem: accuracy

1. no noise $\Rightarrow$ in FT $z_i \equiv f^{(i)}, \; i = 0,\dots,n_d$

2. noises of the integral magnitudes δ_k, sampling step τ: $\exists \gamma_i \geq 1$

$$|z_i - f_0^{(i)}| \leq \gamma_i L \rho^{n_d + 1 - i},$$

$$\rho = \max\left[\left(\frac{\delta_0}{L}\right)^{\frac{1}{n_d+1}}, \left(\frac{\delta_1}{L}\right)^{\frac{1}{n_d+2}}, \dots, \left(\frac{\delta_{n_f}}{L}\right)^{\frac{1}{n_d+n_f+1}}, \tau\right],$$

It is the optimal asymptotics!

(i.e. optimal in δ_0 obtained for $\delta_1 = \dots = \delta_{n_f} = 0$)

Applicability of filtering differentiators

$$0 \le t_1 - t_0 \le T \Rightarrow \left| \int_{t_0}^{t_1} \eta(s)ds \right| \le \delta$$

$$\Rightarrow \ \eta \in \mathrm{Fltr}_0(\delta / T) + \mathrm{Fltr}_1(2\delta)$$

Moreover

$$\forall k \ge 1: \quad \mathrm{Fltr}_0 \subset \mathrm{Fltr}_0 + \mathrm{Fltr}_1 \subset \mathrm{Fltr}_0 + \mathrm{Fltr}_k$$

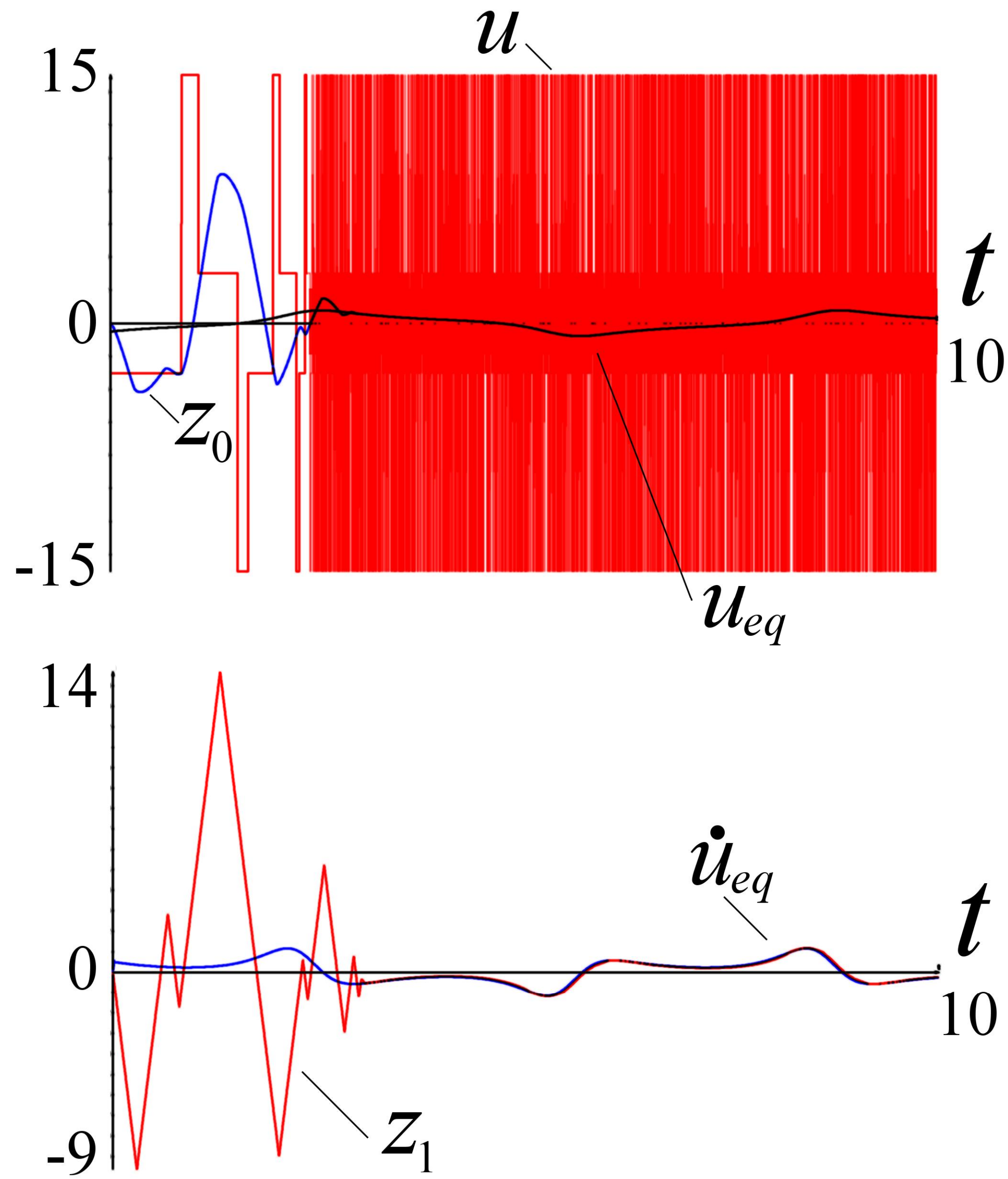
Thus the higher-filtering-order differentiators are applicable instead of the first-filtering-order ones.

Simulation

Equivalent control extraction

Levant, Yu, 2018

$$n_f = n_d = 1,$$
$$|\ddot{u}_{eq}| \le L$$



3rd-order differentiation, $n_d = 3$

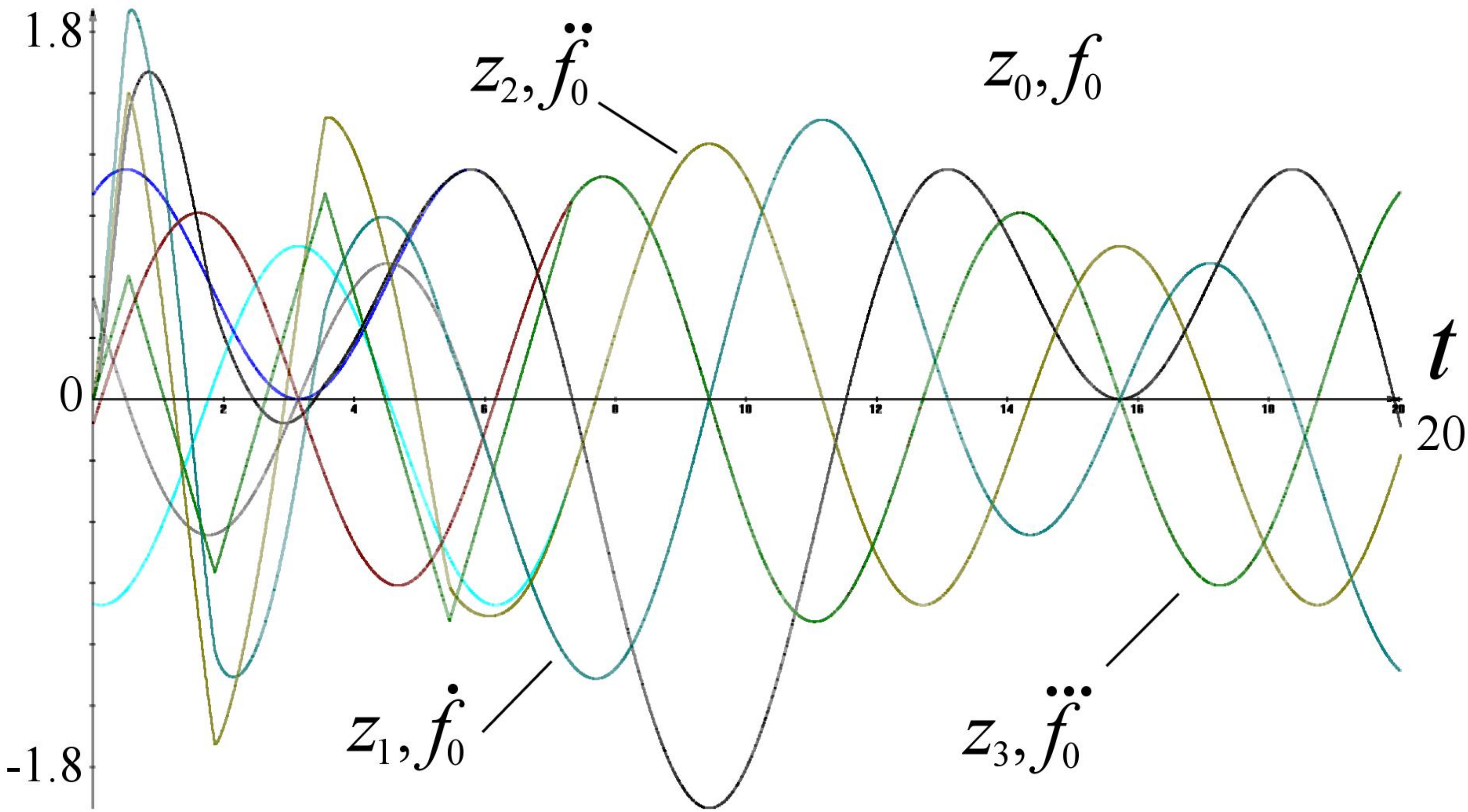
$$f(t) = \sin(0.5t) + \cos t + \eta(t),$$

$$\eta(t) = \cos(10000t + 1.2370851422),$$

$n_d = 3,$ $L = 1.1,$ $\tau = 0.0001$

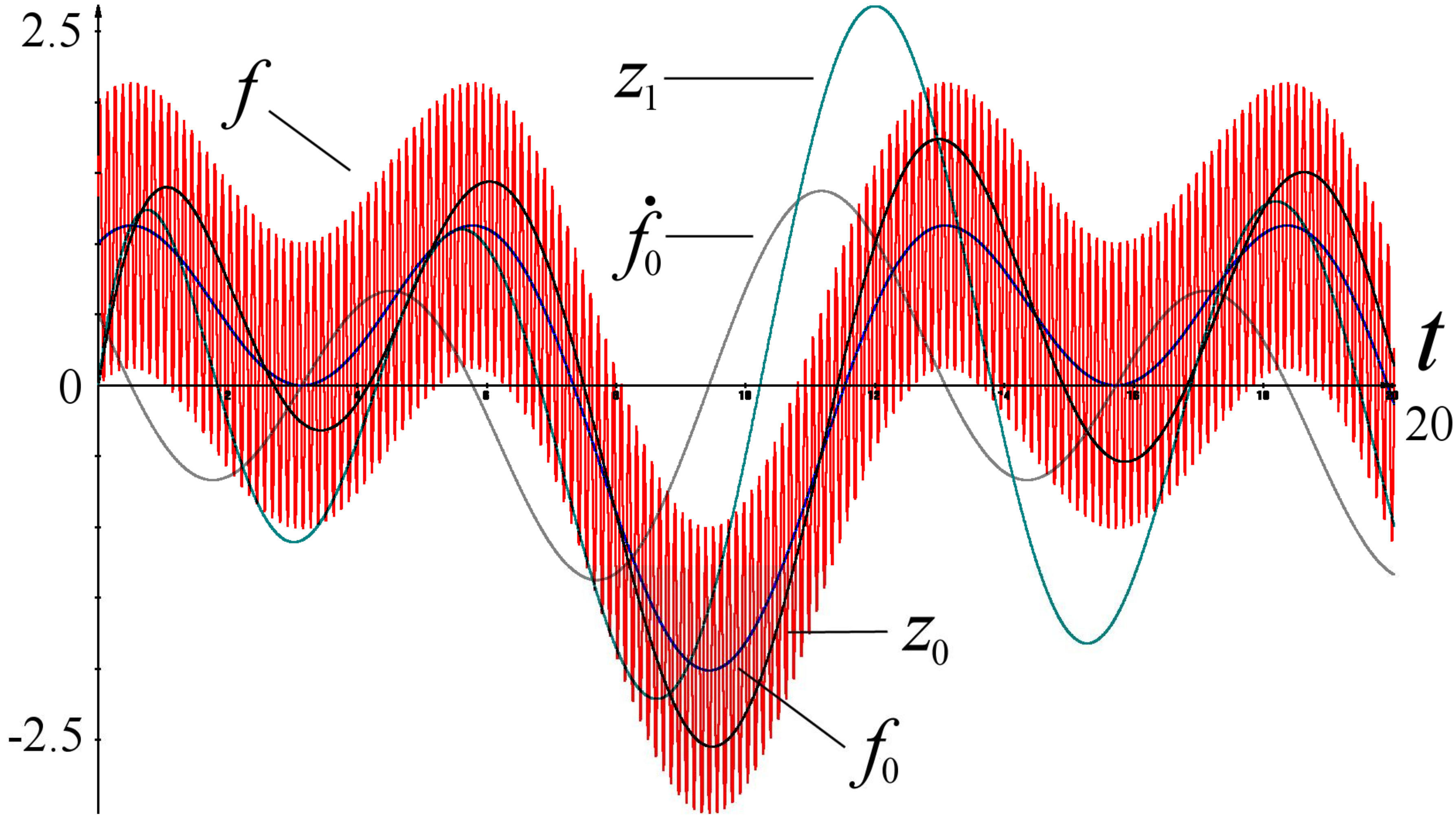
$N_f = 0$: standard differentiator (2003)
$N_f > 0$: filtering differentiator (2018)

3rd-order differentiation, $n_f = 1$, η=0



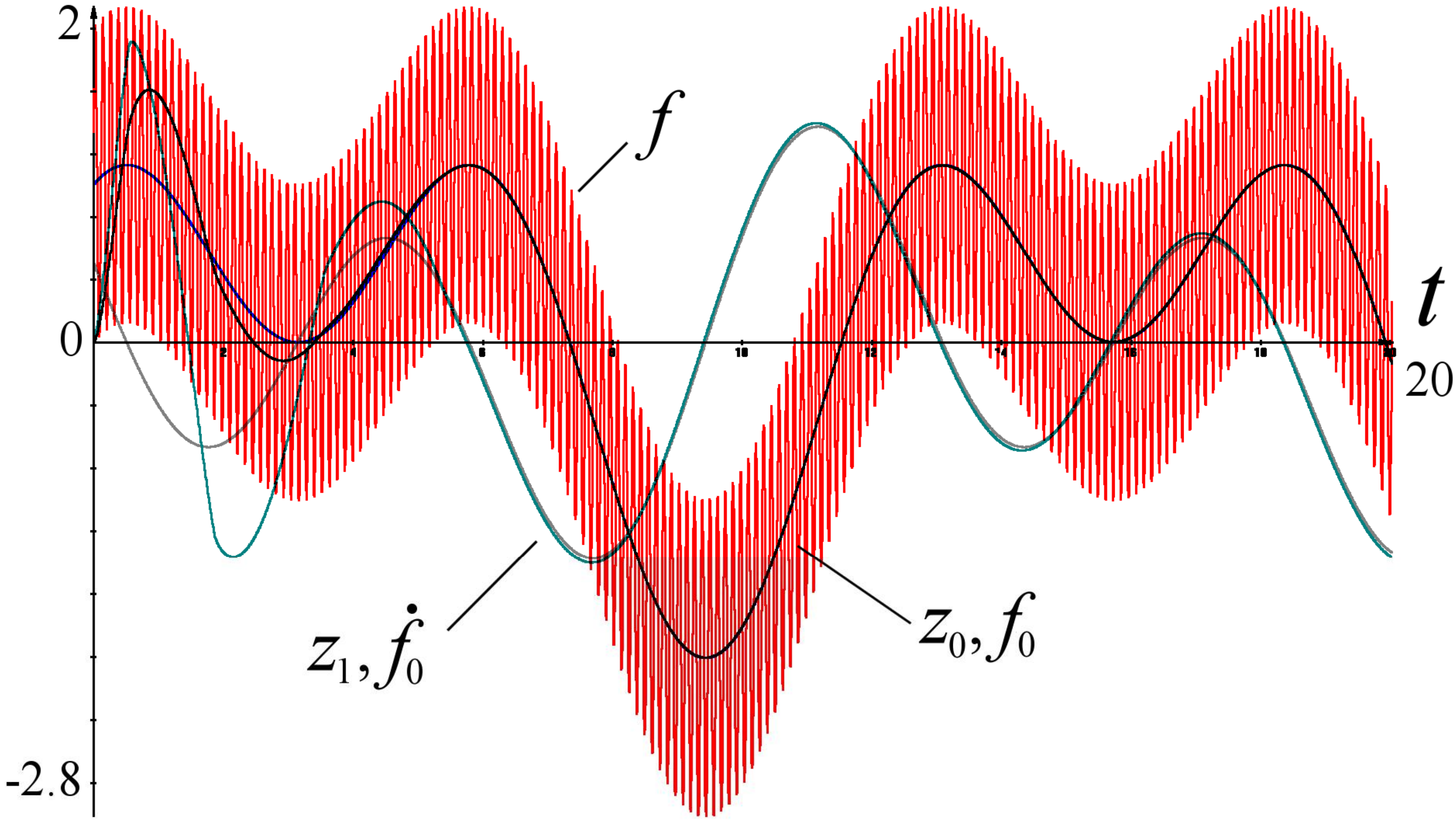
$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \leq (4 \cdot 10^{-15}, 4 \cdot 10^{-11}, 2 \cdot 10^{-7}, 7 \cdot 10^{-4})$$

3rd-order differentiation, $n_f = 0$ (standard, 2003)



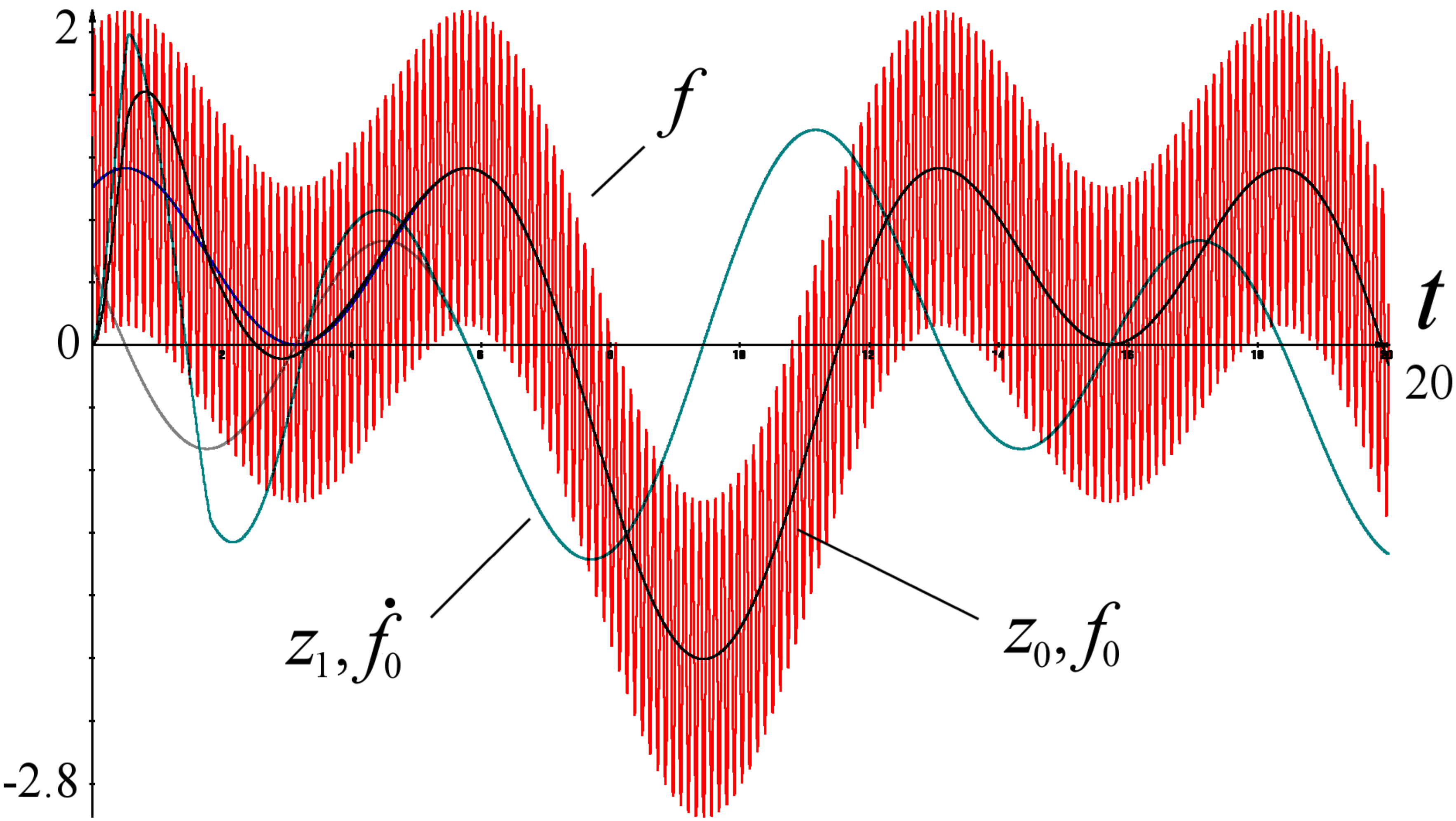
$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \leq (0.6, 1.6, 2.1, 1.4)$$

3rd-order differentiation, $n_f = 1$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \le (0.003, 0.04, 0.22, 0.66)$$

3rd-order differentiation, $n_f = 2$

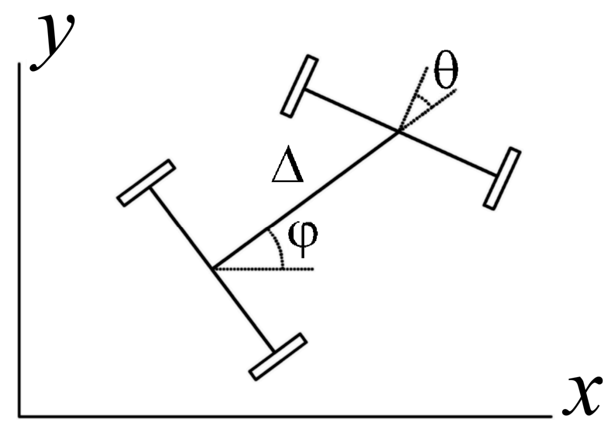


$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \le (0.0001, 0.003, 0.04, 0.28)$$

Car control with Gaussian noise

$$\dot{x} = V\cos(\varphi), \quad \dot{y} = V\sin(\varphi)$$

$$\dot{\varphi} = \frac{V}{\Delta}\tan\theta, \quad \dot{\theta} = u,$$



Measurements:

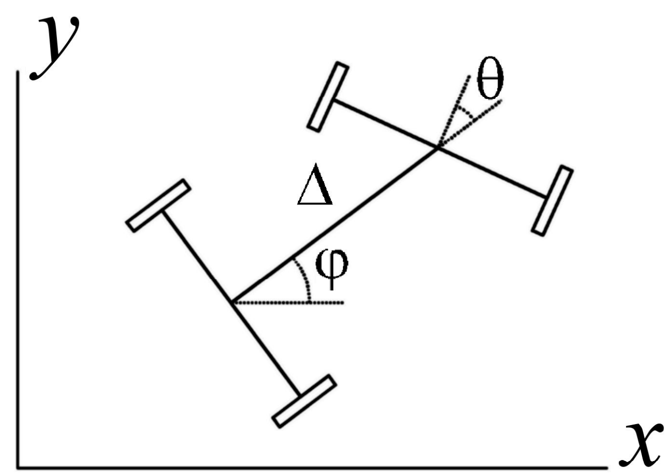
$$s = y(t) - g(x(t)) + \eta(t), \ \eta(t) = N(0, 0.75) \text{ in meters},$$

$$u = -2\frac{z_2 + 2(|z_1| + |z_0|^{\frac{2}{3}})^{-\frac{1}{2}}(z_1 + z_0^{\frac{2}{3}}\operatorname{sign} z_0)}{|z_2| + 2(|z_1| + |z_0|^{\frac{2}{3}})^{\frac{1}{2}}}, \ n_d = 2.$$

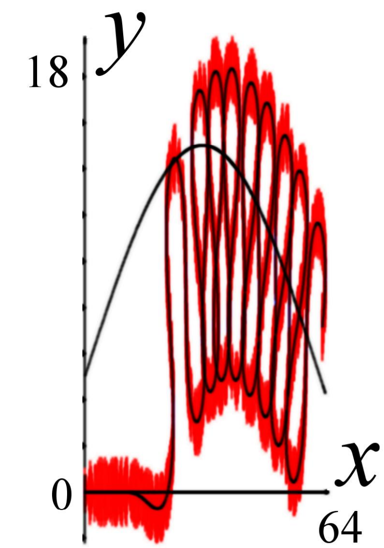
Car control with the noise

$\eta = N(0, 0.75\text{m})$

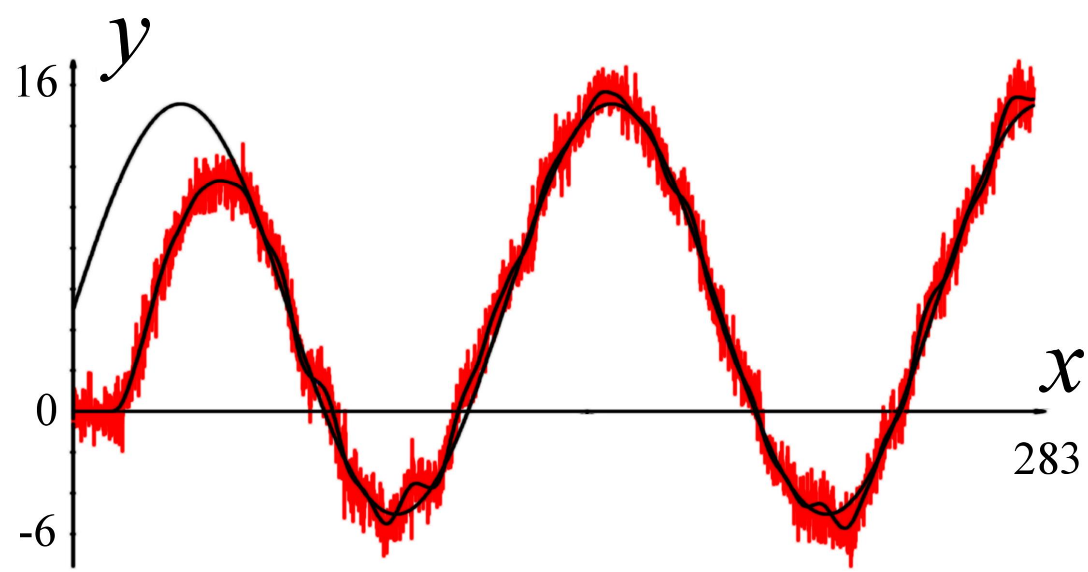
$\tau = 0.001s$



a. Car model



b. $n_f = 0$



c. $n_f = 1$

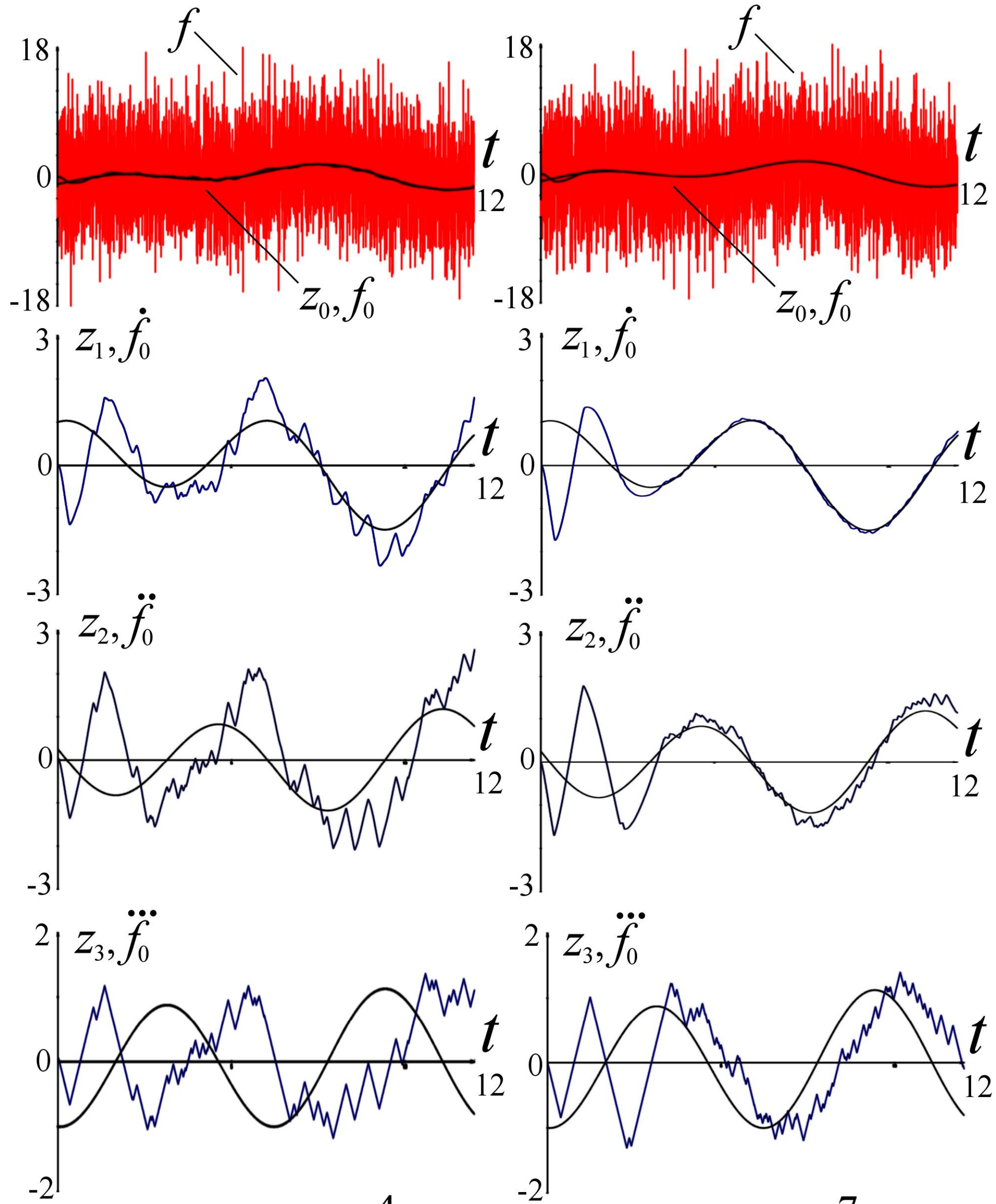
3rd-order differentiation

$$n_d = 3, \quad n_f = 2$$

$$f_0 = \sin t - \cos 0.5t$$

$$L = 2$$

$$\eta = N(0,5)$$



$\tau = 10^{-4}$ $\tau = 10^{-7}$

Thank you very much for your attention!