Sliding Mode Based Differentiation and Filtering

Arie Levant, *Senior Member, IEEE* and Xinghuo Yu, *Fellow, IEEE*

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***Abstract*—The proposed kth-order filter is based on sliding modes (SMs). It exactly estimates k input derivatives in the absence of noises, and is robust to noises of small magnitudes or having small average values. Estimation accuracy asymptotics are calculated. The filter is applied to the real-time accurate estimation of the equivalent control in SM control systems.**

***Index Terms*—Sliding-mode control, nonlinear filtering, estimation, uncertain systems, discrete event systems.**

I. Introduction

Sliding-mode (SM) control (SMC) is based on the exact keeping of a properly chosen output σ (the sliding variable) at zero by means of high-frequency switching control. SMC systems are accurate, and robust [6], [30], [32]. The main drawback is the chattering effect [7], [13], [33], [35], [30].

If the relative degree [16] of the output σ is r, the corresponding SM is called the rth-order SM (r-SM, high-order SM (HOSM)) [2], [5], [6], [11], [18], [26], [24], [25], [27]. The corresponding control appears in $\sigma^{(r)}$ and is discontinuous on the r-SM set $\sigma = \dot{\sigma} = ... = \sigma^{(r-1)} = 0$.

HOSM theory requires and develops robust exact differentiators [3], [18], [25] which in finite time (FT) estimate derivatives $f, ..., f^{(k)}$, provided $|f^{(k+1)}(t)| \le L$ holds for some known L. Noises of small magnitude are filtered out in an asymptotically optimal way [18], [22]. *The new filter/differentiator proposed in this paper also filters out noises of small average values*. It is still homogeneous, asymptotically optimal, and only requires the knowledge of L.

We demonstrate the filter application by solving the classic SMC problem of the equivalent-control [30] estimation. The equivalent control u_{eq} is the value of control providing for the equality $\sigma^{(r)} = 0$ in an r-SMC system. It is used to diminish the chattering [4], [10], [15], [29], [31], [34] and for observation and identification purposes [8], [9], [14], [28].

A. Levant is with the Department of Applied Mathematics, Tel-Aviv University, Tel-Aviv, 69978, Israel (e-mail: levant@post.tau.ac.il).

X. Yu is with the School of Engineering, RMIT University, Melbourne, VIC 3001, Australia (e-mail: x.yu@rmit.edu.au).

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Since 1970s equivalent control is estimated by Utkin's method [30] exploiting the low-pass filter $\dot{z} = -\alpha(z - u(t))$ of the switching control u keeping $\sigma^{(r-1)} \approx 0$. Provided α is properly chosen, z estimates u_{eq} with a good accuracy under the conditions that $\sup|\sigma^{(r-1)}(t)|$ is small, u and $\dot{u}_{eq}$ are bounded. These and some other conditions appear in [30] and *are proved to be unremovable* in this paper.

A number of the cited papers assume that a good or even exact estimation of u_{eq} is available in real time. Unfortunately that assumption is difficult to satisfy. Indeed, Utkin's method [30] requires $\sup|\sigma^{(r-1)}(t)| \le \varepsilon$ to hold for some known ε. Moreover accuracy optimization implies $\alpha(\varepsilon) \to \infty$ as $\varepsilon \to 0$, but $\sup|z - u_{eq}|$ approximates the switching component magnitude for each fixed $\varepsilon > 0$ and sufficiently large α.

Only approximate SMs (real SMs [30]) allow evaluation of u_{eq}, since *in the exact SM $\sigma \equiv 0$ the control u features infinite-frequency switching, i.e. ceases to be a function of time (Fig. 1b)*. The best possible result is that the estimation be asymptotically exact, i.e. exact in the limit when $\sup|\sigma^{(r-1)}(t)| \to 0$, whereas the filter parameters remain fixed.

Let $|u_{eq}^{(k+1)}| \le L$, $k \ge 0$. Our new kth-order filter/differentiator directly "differentiates" the chattering control $u(t)$ producing asymptotically exact estimations of $u_{eq}, ..., u_{eq}^{(k)}$. The estimation accuracy asymptotics are calculated in the presence of discrete noisy sampling.

II. The proposed filter

A. Filtering signals sampled continuously (i.e. all the time)

Assumption 1. *The unknown function $f_0(t)$, $t \ge 0$, is available in real time by its noisy approximation $f(t) = f_0(t) + \eta(t) + \eta_c(t)$. It is known that the derivative $f_0^{(k)}$ exists and has the known Lipschitz constant $L > 0$. The first noise component $\eta(t)$ is Lebesgue-measurable and (essentially) bounded, i.e. $|\eta(t)| \le \delta$ for some unknown $\delta \ge 0$.*

Assumption 2. *The second noise component $\eta_c(t)$ is Lebesgue-measurable and approximately centralized at zero, i.e. it is integrable and the inequality $|\int_0^t \eta_c(s)ds| \le \varepsilon$ holds for any $t \ge 0$ and some unknown $\varepsilon \ge 0$.*

The problem is to accurately restore $f_0, \dot f_0, ..., f_0^{(k)}$ in finite time (FT). We call $k \geq 0$ the order of the problem and the corresponding filter. The stated problem turns into the standard differentiation problem [18] if $\eta_c \equiv 0$.

Following [6], [25] denote $\lfloor \omega \rceil^\gamma = |\omega|^\gamma \operatorname{sign} \omega$ for any $\omega, \gamma \in \mathbb{R}, \omega \neq 0$. Let also $\lfloor \omega \rceil^0 = \operatorname{sign} \omega,\ \forall \gamma > 0: \lfloor 0 \rceil^\gamma = 0$. Then the proposed filter of the kth order is

$$\begin{aligned}
&\dot z_{-1} = -\tilde\lambda_{k+1} L^{\frac{1}{k+2}} \lfloor z_{-1} \rceil^{\frac{k+1}{k+2}} + z_0 - f(t),\\
&\dot z_0 = -\tilde\lambda_k L^{\frac{2}{k+2}} \lfloor z_{-1} \rceil^{\frac{k}{k+2}} + z_1,\\
&\ldots\\
&\dot z_{k-1} = -\tilde\lambda_1 L^{\frac{k+1}{k+2}} \lfloor z_{-1} \rceil^{\frac{1}{k+2}} + z_k,\\
&\dot z_k = -\tilde\lambda_0 L \lfloor z_{-1} \rceil^0,
\end{aligned} \tag{1}$$

where $\tilde\lambda_i > 0$, $i = 0, 1, ..., k+1$. The solutions are understood in the Filippov sense. Here z_i estimates $f_0^{(i)}$ for $i \geq 0$, z_{-1} is an auxiliary variable. It is reasonable to take $z_{-1}(0) = 0$.

Similarly to [18] (1) can be rewritten in the recursive form

$$\begin{aligned}
&\dot z_{-1} = v_{-1} - f(t),\\
&\qquad v_{-1} = -\lambda_{k+1} L^{\frac{1}{k+2}} \lfloor z_{-1} \rceil^{\frac{k+1}{k+2}} + z_0,\\
&\dot z_0 = v_0 = -\lambda_k L^{\frac{1}{k+1}} \lfloor z_0 - v_{-1} \rceil^{\frac{k}{k+1}} + z_1,\\
&\ldots\\
&\dot z_{k-1} = v_{k-1} = -\lambda_1 L^{\frac{1}{2}} \lfloor z_{k-1} - v_{k-2} \rceil^{\frac{1}{2}} + z_k,\\
&\dot z_k = v_k = -\lambda_0 L \lfloor z_k - v_{k-1} \rceil^0.
\end{aligned} \tag{2}$$

where $\tilde\lambda_0 = \lambda_0$, $\tilde\lambda_{k+1} = \lambda_{k+1}$, and $\tilde\lambda_j = \lambda_j \tilde\lambda_{j+1}^{j/(j+1)}$, $j = k, k-1, \ldots, 1$. Hence, filter (1) has one internal variable more than the kth-order differentiator [18] and differs from the differentiator [18] of the order $k+1$ by its entry point.

Theorem 1. *For any $\lambda_0 > 1$ there exists an infinite sequence $\{\lambda_i\}$, $\lambda_i > 0$, $i = 0, 1, ...,$ chosen recursively sufficiently large in the list order, such that under assumptions 1,2 for any k, $\varepsilon, \delta \geq 0$ the kth-order filter* (1) *in FT provides for the accuracy*

$$\begin{aligned}
&|z_{-1}| \leq \mu_{-1} L \varpi^{k+2},\ \varpi = \max[(\tfrac{\varepsilon}{L})^{\frac{1}{k+2}}, (\tfrac{\delta}{L})^{\frac{1}{k+1}}]\\
&|z_i - f_0^{(i)}| \leq \mu_i L \varpi^{k-i+1},\ i = 0, ..., k,
\end{aligned} \tag{3}$$

*for any initial values. For each k the coefficients $\mu_i > 0$ only depend on the filter parameters λ_i, $i \leq k+1$. In particular the filter converges in FT to **exact** derivatives for $\varepsilon = \delta = 0$.*

The sequence λ_j is the same as for the standard $(k+1)$th-order differentiator [18]. In particular, the starting numbers of $\Lambda = \{\lambda_j\}_{j=0}^\infty = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$ are sufficient for $k \leq 6$ [22].

In the following proofs we denote $W(t) = \int_0^t \eta_c(s) ds$ (it becomes a sum in the next subsection), $\omega_{-1} = (z_{-1} + W)/L$, $\omega_i = (z_i - f_0^{(i)})/L$ for $i = 0, ..., k$.

Proof of Theorem 1. Add $\dot W = \eta_c(t)$ to the both sides of the equation for z_{-1} of (1) and subtract $f_0^{(i+1)}$ from the both sides of the equations for z_i, $i = 0, ..., k$. Now using $|f_0^{(k+1)}| \leq L$, $z_{-1} = L\omega_{-1} - W$, $|W| \leq \varepsilon$, and dividing by L obtain the differential inclusion (DI)

$$\begin{aligned}
&\dot\omega_{-1} \in -\tilde\lambda_{k+1} \lfloor \omega_{-1} + \tfrac{\varepsilon}{L}[-1,1] \rceil^{\frac{k+1}{k+2}}\\
&\qquad\qquad + \tfrac{\delta}{L}[-1,1] + \omega_0,\\
&\dot\omega_0 \in -\tilde\lambda_k \lfloor \omega_{-1} + \tfrac{\varepsilon}{L}[-1,1] \rceil^{\frac{k}{k+2}} + \omega_1,\\
&\dot\omega_1 \in -\tilde\lambda_{k-1} \lfloor \omega_{-1} + \tfrac{\varepsilon}{L}[-1,1] \rceil^{\frac{k-1}{k+2}} + \omega_2,\\
&\ldots\\
&\dot\omega_k \in -\tilde\lambda_0 \lfloor \omega_{-1} + \tfrac{\varepsilon}{L}[-1,1] \rceil^0 + [-1,1].
\end{aligned} \tag{4}$$

Increase the uncertainties using $\varepsilon \leq L\varpi^{k+2}$, $\delta \leq L\varpi^{k+1}$. The resulting DI

$$\begin{aligned}
&\dot\omega_{-1} \in -\tilde\lambda_{k+1} \lfloor \omega_{-1} + \varpi^{k+2}[-1,1] \rceil^{\frac{k+1}{k+2}}\\
&\qquad\qquad + \varpi^{k+1}[-1,1] + \omega_0,\\
&\dot\omega_0 \in -\tilde\lambda_k \lfloor \omega_{-1} + \varpi^{k+2}[-1,1] \rceil^{\frac{k}{k+2}} + \omega_1,\\
&\dot\omega_1 \in -\tilde\lambda_{k-1} \lfloor \omega_{-1} + \varpi^{k+2}[-1,1] \rceil^{\frac{k-1}{k+2}} + \omega_2,\\
&\ldots\\
&\dot\omega_k \in -\tilde\lambda_0 \lfloor \omega_{-1} + \varpi^{k+2}[-1,1] \rceil^0 + [-1,1].
\end{aligned} \tag{5}$$

is homogeneous of the degree -1 with the weights $\deg \omega_j = k - j + 1$, $j = -1, 0, ..., k$, and $\deg \varpi = 1$ [21]. Here ϖ measures the intensity of the homogeneous disturbance [6], [21]. It is FT stable for $\varpi = 0$ [18], thus for arbitrary $\varpi \geq 0$ obtain the required accuracy [21]. □

Remark 1. The proved accuracy (3) coincides with the accuracy of the standard differentiator [18] for $\varepsilon = 0$, i.e. the new differentiator (1) is asymptotically optimal [22]. The proved robustness to noises with small averages significantly extends the integral input-to-state stability feature [6].

It is similarly proved that moving the term $-f(t)$ from the first line of (1) to the equation on z_i, $i = 1, .., k-1$, one also obtains new alternative asymptotically-optimal homogeneous differentiators of lower orders.

B. Discrete filter for signals sampled at discrete times

Discrete sampling can destroy the centralization of the noise η_c at 0. Indeed, one can easily get a constant signal instead of a switching signal ± 1. Thus, Assumption 2 is to be replaced with its discrete version.

Assumption 3. *The input function f is sampled at the instants $t_0, t_1, ...,$ $t_0 = 0$, $0 < t_{j+1} - t_j = \tau_j \leq \tau$. The noise $\eta_c(t_j)$*

is approximately centralized at zero, i.e. $|\sum_{s=0}^{j}\eta_c(t_s)\tau_s| \leq \varepsilon$ *holds for any* $j \geq 0$ *and some unknown* $\varepsilon \geq 0$.

Denote filter (1) by $\dot{z} = F_{k,\Lambda}(z, f, L)$, where $z \in \mathbb{R}^{k+2}$. Then the following is the discrete version of filter (1) fitting computer-based applications:

$$z(t_{j+1}) = z(t_j) + F_{k,\Lambda}(z(t_j), f(t_j), L)\tau_j + T_k(z(t_j), \tau_j), \tag{6}$$

Here $T_k(z(t_j), \tau_j) \in \mathbb{R}^{k+1}$ is the vector of Taylor-like terms,

$$\begin{pmatrix} T_{k,-1} \\ T_{k,0} \\ \ldots \\ T_{k,i} \\ \ldots \\ T_{k,k-2} \\ T_{k,k-1} \\ T_{k,k} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2!}z_2(t_j)\tau_j^2 + \ldots + \frac{1}{k!}z_k(t_j)\tau_j^k \\ \ldots \\ \sum_{s=i+2}^{k} \frac{1}{(s-i)!} z_s(t_j)\tau_j^{s-i} \\ \ldots \\ \frac{1}{2!}z_k(t_j)\tau_j^2 \\ 0 \\ 0 \end{pmatrix}. \tag{7}$$

In particular $T_0(z,\tau) = 0 \in \mathbb{R}^2$, $T_1(z,\tau) = 0 \in \mathbb{R}^3$.

The identically equivalent recursive form of filter (6) is

$$z(t_{j+1}) = z(t_j) + V_{k,\Lambda}(z(t_j), f(t_j), L)\tau_j + T_k(z(t_j), \tau_j), \tag{8}$$

where $V_k(z(t_j)) = (v_{-1}, \ldots, v_k)^T \in \mathbb{R}^{k+2}$ is defined in (2).

Theorem 2. *Let* $\{\lambda_i\}$, $i = 0, 1, \ldots$, *be chosen as in Theorem 1. Then under assumptions 1,3 for any* $k, \varepsilon, \delta \geq 0$, $\tau > 0$ *the kth-order filter* (6) *in FT provides for the accuracy*

$$\begin{array}{l} |z_{-1}| \leq \mu_{-1}L\varpi^{k+2}, \ \varpi = \max[(\frac{\varepsilon}{L})^{\frac{1}{k+2}}, (\frac{\delta}{L})^{\frac{1}{k+1}}, \tau] \\ |z_i - f_0^{(i)}| \leq \mu_i L\varpi^{k-i+1}, \ i = 0, \ldots, k, \end{array} \tag{9}$$

at each t_j *for any initial values. For each* k *the coefficients* $\mu_i > 0$ *only depend on the filter parameters* λ_s, $s \leq k+1$.

Proof. Let $W(t_j) = \sum_{s=0}^{j-1}\eta_c(t_s)\tau_s$, $W(0) = 0$. Adding $W(t_{j+1})$ to the both sides of the equation for z_{-1} in (6), subtracting the Taylor expansion

$$f_0^{(i)}(t_{j+1}) = \sum_{s=i}^{k} \frac{f_0^{(s)}(t_j)}{(s-i)!}\tau_j^{s-i} + \frac{\theta_i}{(k-i+1)!}\tau_j^{k-i+1},$$

$|\theta_i| \leq L$, from the both sides of the equation for z_i, $i \geq 0$, and dividing by L, similarly to the previous proof get a homogeneous *discrete* system with the weights $\deg t = \deg\tau = \deg\tau_j = 1$, $\deg\omega_i = k+1-i$, $\deg\varepsilon = k+2$, $\deg\delta = k+1$. Its solutions approximate solutions of the undisturbed DI (4). The rest of the proof is similar to [23] and uses [21]. $\square$

III. Estimation of equivalent control in SMC

A. Sliding-mode control framework

Consider a smooth dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \ x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \tag{10}$$

with the vector sliding variable $\sigma(t,x) \in \mathbb{R}^m$, closed by the discontinuous feedback

$$u = U(t,x). \tag{11}$$

Let the system have the vector relative degree $r = (r_1, \ldots, r_m)$ [16]. Denoting $\sigma^{(r)} = (\sigma_1^{(r_1)}, \ldots, \sigma_m^{(r_m)})^T$, obtain

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{12}$$

where $\det g(t,x) \neq 0$ [16]. The functions $h(t,x)$ and $g(t,x)$ are smooth and usually unknown in SMC.

The function u_{eq},

$$u_{eq}(t,x) = -g^{-1}(t,x)h(t,x), \tag{13}$$

which satisfies $\sigma^{(r)}(t,x,u_{eq}(t,x)) = 0$ is called the *equivalent control* [30]. Therefore, (12) is rewritten as

$$\sigma^{(r)} = g(t,x)(u - u_{eq}(t,x)). \tag{14}$$

Due to the discontinuity of $U(t,x)$ when $\sigma \equiv 0$, the solutions of the closed-loop system are understood in the Filippov sense [12], and the corresponding motion $\sigma \equiv 0$ is a sliding-mode (SM) motion of the order r (r-SM) [18].

Since the system dynamics (10) are nowhere involved, for simplicity we often omit the argument $x(t)$ and, for example, write $u_{eq}(t)$ and $\sigma(t)$ instead of $u_{eq}(t,x(t))$ and $\sigma(t,x(t))$.

Remark 2. The Filippov dynamics of the r-SM $\sigma \equiv 0$ are limit motions on the discontinuity set of $U(t,x)$ obtained when some switching imperfections Π gradually vanish, $\Pi \to 0$ [12]. Since $\lim_{\Pi\to 0} u(t)$ does not exist (Fig. 1b), *the control signal does not exist as a function of time* t *and cannot be filtered in the ideal SM.* The very requirement of the control signal to be available in the SM makes the SM only approximate, $\sigma \approx 0$.

In the approximate SM (*real* SM [30]) the average value of the control $u(t)$ approximates the equivalent control [12], [30]. The value of the equivalent control is very important, since it allows chattering attenuation and SM adaptation by canceling the term h in (12) [4], [10], [15], [29], [31], [34], and is useful in observation [8], [9], [14], [28].

Below we estimate the function $u_{eq}(t)$ and its time derivatives $\dot{u}_{eq}, \ldots, u_{eq}^{(k)}$ in real time, provided the approximate r-SM is kept and the applied control $u(t)$ is available.

B. Equivalent control: conventional estimation

The following are the slightly generalized standard assumptions by Utkin [30] for the estimation of u_{eq}. *Only two numbers k, L appearing below are needed for the application of filter* (1). The classic equivalent-control estimation method [30] also requires the knowledge of the SM accuracy ε.

Assumption 4. *The actual control $u(t)$ entering* (12) *is a Lebesgue-measurable bounded function of time. From the starting moment $t = 0$ a real SM holds keeping $||\sigma^{(r-1)}|| \le \varepsilon$, where $r - 1 = (r_1 - 1, r_2 - 1, \ldots, r_m - 1)$.*

Assumption 5. *The vector input $u(t) \in \mathbb{R}^m$ of the system* (12) *is available in real time by its Lebesgue-measurable approximation $\tilde{u}(t)$, $||\tilde{u} - u|| \le \delta$. The input u and the function u_{eq} are uniformly bounded, $||u|| \le U_M$, $||u_{eq}(t, x(t))|| \le U_M$. The equivalent control* (13) *has k total time derivatives along the trajectory, the last one, $u_{eq}^{(k)}(t, x(t))$, being Lipschitzian with the known Lipschitz constant L, $||u_{eq}^{(k+1)}|| \le L$, $L > 0$. The matrix $\dot{g}(t, x(t), u(t)) = g'_t + g'_x(a + bu)$ is bounded, $||\dot{g}|| \le D_g$; also $\det g(t, x(t)) \ne 0$, $||g^{-1}|| \le C_g$.*

Assumptions 4, 5 accept $\varepsilon = 0$, since there exists a function $u(t)$ which keeps $\sigma \equiv 0$. Remark 2 is still valid. Boundedness of u_{eq} and $u_{eq}^{(k+1)}$ implies the boundedness of $\dot{u}_{eq}, ..., u_{eq}^{(k)}$ [17]. Due to (13) the additional boundedness of g would lead to the boundedness of h as well.

The assumption that the control $u(t)$ actually entering (12) is available in real time is important and *non-trivial*, since only it contains the information on u_{eq}. In practice it may require $\tilde{u}(t)$ to be produced by a sensor at the actuator output.

The standard method [30] applied today in SMC was proposed by Utkin in 1970s and suggests application of the completely decoupled low-pass filter

$$\alpha^{-1}\dot{z}_u + z_u = \tilde{u}(t),\ z_u(0) = 0,\ z_u \in \mathbb{R}^m,\ \alpha > 0. \qquad (15)$$

Its vector output z_u tracks $u_{eq}(t)$. The solutions are understood in the Caratheodory or Filippov sense [12].

The main result here belongs to Utkin [30], but it has never been formulated in a complete mathematical form. The auxiliary lemma ([30], p.23) that is usually cited in that context is inexactly formulated in spite of accurate proof calculations. The following theorem formulates Utkin's result and extends it to the noisy control sampling (for $\delta > 0$) and the relative degrees different from $(1, ..., 1)$.

Theorem 3. *Let $k = 0$, then under assumptions 4, 5 filter* (15) *provides for the inequalities*

$$\begin{aligned} ||z_u(t) - u_{eq}(t)|| &\le e^{-\alpha t}(\sqrt{m}U_M + C_g\varepsilon) \\ &\quad + \sqrt{m}L\alpha^{-1} + C_g^2 D_g\varepsilon + 2C_g\alpha\varepsilon + \sqrt{m}\delta, \\ ||z_u(t) - u_{eq}(t)|| &\le 2\sqrt{m}(U_M + \delta). \end{aligned} \qquad (16)$$

The proof of the first inequality is similar to the proof of the mentioned Utkin's lemma [30] and is omitted. The second one trivially follows from $|\tilde{u}_l| \le U_M + \delta$, $\tilde{u} = (\tilde{u}_1, ..., \tilde{u}_m)^T$. Rewrite (16) as

$$\begin{aligned} &||z_u(t) - u_{eq}(t)|| = \\ &\quad O(1)e^{-\alpha t} + O(L/\alpha) + O(\min[\alpha\varepsilon, U_M]) + O(\delta) \end{aligned} \qquad (17)$$

for large α and small ε, δ.

We say that $\phi(w) = O(\psi(w))$ is *strictly* $O(\psi(w))$ as $w \to w_0$, $w_0 \in \mathbb{R}_\infty^{n_w}$, $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty, -\infty\}$, if in a vicinity of w_0 a) the equality $\psi(w) = 0$ implies $\phi(w) = 0$, and b) $|\phi/\psi|$ is bounded and *separated* from zero whenever $\psi(w) \ne 0$.

Proposition 1. *1. Under Assumptions 4, 5 for $k = 0$ the accuracy* (17) *is unimprovable, i.e. for some systems all $O(\cdot)$ in* (17) *are strict. The worst-case error satisfies* $\limsup ||z_u(t) - u_{eq}(t)|| \ge U_M$ *for $\alpha \to \infty$, $t \to \infty$ and any fixed $\varepsilon > 0$.*
2. Removal of each one of boundedness conditions on $\dot{u}_{eq}$, $\dot{g}$ or g^{-1} destroys the uniform asymptotics (17)*. By that we mean that for some $\delta \ge 0$, $\rho_0 > 0$ and any $\alpha, \varepsilon > 0$ there are system* (12) *and $u(t)$ satisfying the respectively modified assumptions such that for any $t_0 > 0$ and $z_u(0)$ the inequality $||z_u(t) - u_{eq}(t)|| \ge \rho_0$ holds for some $t > t_0$.*

Proof. The formal proof consists of 4 proper examples. It is enough to take $m = 1$.

1. To prove Statement 1 one takes (10), (12), $u(t)$ in the form $\dot{\sigma} = -U_M\cos(Lt) + u$, $u = U_M\cos(Lt) + U_M\cos(\frac{U_M}{\varepsilon}t)$, $\tilde{u} = u + \delta$, $\sigma(0) = 0$.

2. The boundedness of $\dot{u}_{eq}$ is removed in the system $\dot{\sigma} = -2\sin^2(\frac{1}{2\varepsilon}t) + u$, $\tilde{u} = u = 1$, $\sigma(0) = 0$. $||z_u - u_{eq}|| \ge 0.5$ is observed for some t in spite of fixed $U_M = C_g = D_g = 2$.

Let $\delta = 0$ and $[w]$ be the maximal integer not exceeding w. Consider $\dot{\sigma} = g(t)u(t)$, $\sigma(0) = 0$, where $u(t) = 1 + 2 \cdot (-1)^{[t/T]}$, $T > 0$. Let $g \in \gamma[1, 5]$, $\gamma > 0$, and define $\dot{g} = -10\gamma T^{-1}\operatorname{sign}(u\sigma)$ with saturation/stopping at γ and 5γ, $g(0) = 3\gamma$. It is easy to check that $|\sigma| \le \varepsilon = 3\gamma T$ is kept for all $t \ge 0$.

Now taking $T = 1/N$, $N = 1, 2, ...$, $\gamma = 1$, obtain a counterexample for unbounded $\dot{g}$. If also $\gamma = 1/N$ obtain an example with bounded $\dot{g}$, but unbounded $1/g$. In all cases $u_{eq} \equiv 0$, filter (15) has the same predefined input, and z_u

infinitely many times crosses the mean value 1 of u. □

No choice of α makes filter (15) exact. Too small values of α lead to low accuracies, too large values lead to the control-switching-magnitude-order errors. Nevertheless, filter (15) has been used for arbitrary-order approximate differentiation [14].

A reasonable strategy is taking α proportional to $\sqrt{L/\varepsilon}$. The main problem of that strategy is that the accuracy ε of the sliding mode is required to be available.

C. Equivalent control: asymptotically exact estimation

The proposed equivalent-control estimator gets the form

$$\begin{aligned} &\dot{z}_l = F_{k,\Lambda}(z_l, \tilde{u}_l, L),\ l = 1, ..., m; \\ &z_l = (z_{l,-1}^T, ..., z_{l,k}^T)^T \in \mathbb{R}^{k+2}. \end{aligned} \tag{18}$$

Here the output $z_{l,i}$ approximates $u_{eq,l}^{(i)}$, $i = 0, 1, ..., k$, $z_{l,-1}$ are auxiliary internal variables.

In the following the time needed to converge to the exact values of u_{eq} and its derivatives for $\varepsilon = \delta = 0$ is called *the transient time*.

Theorem 4. *Under Assumptions 4, 5 for any $\varepsilon, \delta \geq 0$ the kth-order filter* (18) *in finite time provides for the accuracy*

$$\begin{aligned} &|z_{l,i} - u_{eq,l}^{(i)}| \leq \mu_{li} L \rho^{k-i+1},\ \rho_1 = \varepsilon[3C_g + C_g^2 D_g] \\ &\rho = \max\{(\tfrac{\rho_1}{L})^{\frac{1}{k+2}}, (\tfrac{\delta+\rho_1}{L})^{\frac{1}{k+1}}\}, \\ &i = 0, ..., k,\ l = 1, ..., m, \end{aligned} \tag{19}$$

for any initial values. The coefficients $\mu_{li} > 0$ only depend on the filter parameters $\lambda_0, ..., \lambda_{k+1}$. The transient time is uniformly bounded for the initial value $z(0) = 0$, i.e. in that case it depends only on $U_M, L, \lambda_0, ..., \lambda_{k+1}$.

In particular, the observer is FT exact if $\varepsilon = \delta = 0$. The theorem proof uses the following technical lemma.

Lemma 1. *Let $\gamma > 0$ and consider the auxiliary equation*

$$\dot{w}_e = u(t) - u_{eq}(t) - \gamma w_e,\ w_e(0) = 0 \in \mathbb{R}^m. \tag{20}$$

Then for any $t \geq 0$ (20) *provides for*

$$||w_e|| \leq \rho_\gamma = \varepsilon[3C_g + \gamma^{-1} C_g^2 D_g]. \tag{21}$$

Proof. Taking (14) into account and integrating by parts get

$$\begin{aligned} w_e = e^{-\gamma t} &\int_0^t e^{\gamma s} g^{-1}(s) \sigma^{(r)}(s) ds = \\ &g^{-1}(t)\sigma^{(r-1)}(t) - g^{-1}(0)\sigma^{(r-1)}(0)e^{-\gamma t} \\ &- e^{-\gamma t} \int_0^t [\gamma e^{\gamma s} g^{-1} - e^{\gamma s} g^{-1} \dot{g} g^{-1}] \sigma^{(r-1)}(s) ds, \end{aligned}$$

where the argument s of $g^{-1}, \dot{g}$ is omitted. Hence,

$$\begin{aligned} ||w_e|| &\leq 2C_g\varepsilon + \varepsilon[C_g + \gamma^{-1} C_g^2 D_g][1 - e^{-\gamma t}] \\ &\leq \varepsilon[3C_g + \gamma^{-1} C_g^2 D_g]. \end{aligned}$$

□

Proof of Theorem 4. Apply Lemma 1 and rewrite

$$\tilde{u} = u_{eq} + (u - u_{eq} - \gamma w_e) + (\gamma w_e + \tilde{u} - u).$$

Introduce the "noises" $\eta_{c,l} = u_l - u_{eq,l} - \gamma w_{el}$, $\eta_l = \tilde{u}_l - u_l + \gamma w_{el}$, $|\eta_l| \leq \delta + \gamma \rho_\gamma$. Applying Theorem 1 component-wise and taking into account $||(1, ..., 1)|| = \sqrt{m}$ obtain the stated accuracy (19), but for $\rho_1 = \rho_\gamma$. That accuracy estimation is true for any $\gamma > 0$. Taking $\gamma = 1$ obtain the theorem. □

The accuracy asymptotics of the novel kth-order filter are better than those of the classical filter (15). Indeed, consider systems satisfying Assumptions 4, 5 for both filter orders 0 and $k \geq 0$, and fix all corresponding assumption parameters. Let $z_u(0) = 0$, $z(0) = 0$. Then for any $\alpha > 0$

$$\lim_{\rho \to 0} \frac{\limsup_{t\to\infty} ||z_0(t) - u_{eq}(t)||}{\limsup_{t\to\infty} ||z_u(t) - u_{eq}(t)||} = 0,$$

where $z_0 = (z_{1,0}, ..., z_{m,0})^T$, and the upper limits are taken over all systems satisfying the assumptions. The formula follows from the first statement of Proposition 1 and (19).

Consider the overall system (10), (11), (18) with the ideal SM $\sigma \equiv 0$ and $\tilde{u} = u$. Then filter (18) becomes a part of the overall Filippov dynamics, and its solution is understood differently than in Theorem 4.

In the SM the signal $u(t)$ is undefined (Remark 2). To find the solution one formally replaces $\tilde{u}, u$ with u_{eq} in (10), (11), (12), (18) (the equivalent-control principle [30], [12]). Since $\varepsilon = \delta = 0$, Theorem 4 now implies $z_{l,i}(t) \equiv u_{eq,l}^{(i)}(t)$ in FT along the Filippov solutions. Since Filippov solutions are limits of real-SM solutions [12], it still only means that $z_{l,i} - u_{eq,l}^{(i)} \to 0$ as various switching imperfections vanish.

Example 1. Homogeneous SMC is convenient for testing filter (18) due to the availability of the SM accuracy [19] (see Section IV). Let the feedback (11) be homogeneous r-SM control [20]. Then, if each σ_l is sampled with error not exceeding $\varepsilon_{w,l} \geq 0$ and sampling-time intervals not exceeding $\tau_w \geq 0$, the control [20] in FT provides for the SM accuracy

$$|\sigma_l^{(i)}| \leq \nu_{li} \tilde{\rho}^{r_l - i},\ \tilde{\rho} = \max_l [\tau_w, \max_l \varepsilon_{wl}^{\frac{1}{r_l}}]. \tag{22}$$

where $i = 0, 1, ..., r_l - 1$, $l = 1, 2, ..., m$ [21]. Thus $||\sigma^{(r-1)}|| \leq \tilde{\nu}\tilde{\rho}$ for some $\tilde{\nu} > 0$. Correspondingly Theorems

3 and 4 for sufficiently small ε_{wi}, τ_w imply that

1. under Assumptions 4, 5 with $k = 0$ and $\tilde{u} = u$ (i.e. $\delta = 0$) filter (15) provides for the accuracy

$$||z_u(t) - u_{eq}(t)|| = O(1)e^{-\alpha t} + O(L/\alpha) + O(\alpha\tilde{\rho}); \quad (23)$$

2. under Assumptions 4, 5 with $\tilde{u} = u$ (i.e. $\delta = 0$) filter (18) provides for the accuracy

$$|z_{l,i} - u_{eq,l}^{(i)}| \leq \mu_{li} L^{\frac{1+i}{k+2}} \tilde{\rho}^{\frac{k-i+1}{k+2}}, \ i = 0, 1, ..., k, \quad (24)$$

for some constants $\mu_{li} > 0$ depending only on the parameters of the SMC system, and the parameters Λ of the filter.

D. Discretization of filters

Nowadays filters (15), (18) and controller (11) are usually implemented by computer technique. Then both $\tilde{u}(t)$ and $u(t)$ become piece-wise constant. As previously let the sampling instants be $t_j = t_0, t_1, ..., t_0 = 0, t_{j+1} - t_j = \tau_j > 0$.

Direct integration of (15) over $t \in [t_j, t_{j+1}]$ results in

$$z(t_{j+1}) = e^{-\alpha\tau_j} z(t_j) + (1 - e^{-\alpha\tau_j})\tilde{u}(t_j), \quad (25)$$

which can be used instead of (15) in that case.

Since the filters are decoupled, it is enough to consider the scalar case $m = 1$. The corresponding filter naturally takes the form (6) and (8) with $\tilde{u}$ substituted for f. *Discrete filters ready for immediate use are presented in Section IV for $k = 0, 1$.* The following theorem is proved in the same way as Theorem 2 using Lemma 1 as in Theorem 4.

Theorem 5. *Under the conditions of Theorem 4 let $\tau_j \leq \tau$ for some constant $\tau > 0$. Then the multi-input version of the discrete filter* (6) *provides for the accuracy*

$$\begin{aligned}
&|z_{l,i} - u_{eq,l}^{(i)}| \leq \mu_{li} L \rho^{k-i+1}, \ i = 0, ..., k, \ l = 1, ..., m, \\
&\rho = \max\{[\tfrac{\rho_1}{L}]^{\frac{1}{k+2}}, [\tfrac{\delta+\rho_1+\rho_2}{L}]^{\frac{1}{k+1}}, \tau\}, \\
&\rho_1 = \varepsilon[3C_g + C_g^2 D_g], \ \rho_2 = \tau[2L^{\frac{1}{k+1}} U_M^{\frac{k}{k+1}} + 2U_M + \rho_1],
\end{aligned} \quad (26)$$

at the sampling instants t_j. The constants $\mu_{li} > 0$ are determined by the filter parameters Λ.

Continuous-time estimation accuracies (23), (24) (Example 1) and Theorem 5 imply that, provided the sampling instants of the filters and the system coincide, $\tau = \tau_w$, the discrete filters have the same asymptotic accuracies (23) and (24).

IV. Simulation

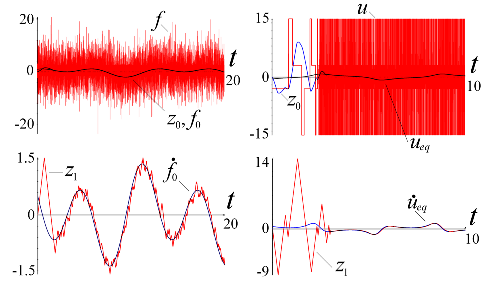
The following are novel discrete filters/differentiators (6) of the most practical orders $k = 0, 1$. The 0th order filter extracts the basic component f_0 of f and has the equations

$$\begin{aligned}
&z_{-1}(t_{j+1}) = z_{-1}(t_j) + \\
&\quad (-1.5L^{\frac{1}{2}} \lfloor z_{-1}(t_j) \rceil^{\frac{1}{2}} + z_0(t_j) - f(t_j))\tau_j, \\
&z_0(t_{j+1}) = z_0(t_j) - 1.1L \,\mathrm{sign}(z_{-1}(t_j))\tau_j.
\end{aligned} \quad (27)$$

The 1st-order filter for estimating $f_0, \dot{f}_0$ is

$$\begin{aligned}
&z_{-1}(t_{j+1}) = z_{-1}(t_j) + \\
&\qquad (-2L^{\frac{1}{3}} \lfloor z_{-1}(t_j) \rceil^{\frac{2}{3}} + z_0(t_j) - f(t_j))\tau_j, \\
&z_0(t_{j+1}) = z_0(t_j) + \\
&\qquad (-2.12L^{\frac{2}{3}} \lfloor z_{-1}(t_j) \rceil^{\frac{1}{3}} + z_1(t_j))\tau_j, \\
&z_1(t_{j+1}) = z_1(t_j) - 1.1L \,\mathrm{sign}(z_{-1}(t_j))\tau_j.
\end{aligned} \quad (28)$$

In the following we take equal sampling steps $\tau = \tau_j$. Initial values are zeroed at $t = 0$, $z(0) = 0$.



a. Filtering noisy signal   b. Filtering chattering SM control

Figure 1. Convergence of the 1st-order filters for $\tau = 10^{-6}$: a. filtering/differentiating a signal corrupted by the Gaussian noise with dispersion 5; b. filtering/differentiating the chattering SM control (29).

Filtering signals. Simulation shows that the novel differentiator demonstrates practically the same performance and accuracy as the standard differentiator [18] for $k = 0, 1, ..., 5$ in the presence of bounded (small) noises.

Consider the signal $f(t) = f_0(t) + \eta(t)$ where $f_0(t) = \sin 0.5t + \cos t$, and $\eta(t_j)$ is a Gaussian random variable with the standard deviation 5 and the mean 0. Due to the central-limit theorem such noise in practice satisfies Assumptions 1, 3. Results of filtering the signal f for $k = 1$, $L = 2$ and the sampling step $\tau = 10^{-6}$ are shown in Fig. 1a. The accuracies $|z_0 - f_0| \leq 0.05$, $|z_1 - \dot{f}_0| \leq 0.36$ are obtained.

Estimating the equivalent control. Consider a simple SMC system (Example 1) with the twisting controller [18]

$$\ddot{\sigma} = \cos t + (2 + \sin(2t))u, \ u = -9 \,\mathrm{sign}\, \sigma - 6 \,\mathrm{sign}\, \dot{\sigma}. \quad (29)$$

Obviously Assumptions 4, 5 hold here for any $k = 0, 1, ...$ In particular $|\dot{u}_{eq}|, |\ddot{u}_{eq}| \leq L = 30$. Let $f = \tilde{u} = u$, $\delta = 0$. The initial values $\sigma(0) = 5$, $\dot{\sigma}(0) = -3$ are taken. The system enters the SM at $t = 2.45$. Let $z(0) = 0, z_u(0) = 0$. The 1st-order filter converges at 3.05 (Fig. 1b). The conventional filter

(25) with $\alpha = 30$ and the new filter of the 0th order converge practically simultaneously at 2.5.

The accuracies are calculated as the corresponding maximal deviations for $t \in [5, 10]$. Due to (22) in the steady state $\varepsilon = \sup|\dot{\sigma}|$ is roughly proportional to τ [21]. According to Example 1 the 0th-order filter (27) provides for the accuracy $z_0 - u_{eq} = O(\tau^{1/2})$. The conventional filter (15) (or (25)) has the same optimal accuracy $z_u - u_{eq} = O(\tau^{1/2})$, provided α is kept proportional to $\tau^{-1/2}$. The 1st-order filter (28) has the better accuracy $z_0 - u_{eq} = O(\tau^{2/3})$, $z_1 - \dot{u}_{eq} = O(\tau^{1/3})$.

Consider the classical filter (25). First let the sampling step be $\tau = 10^{-4}$. System (29) keeps the accuracies $|\sigma| \leq 4.9 \cdot 10^{-6}$ and $|\dot{\sigma}| \leq \varepsilon = 0.013$. The roughly best accuracy $|z - u_{eq}| \leq 0.16$ is obtained for $\alpha = 10$. The accuracies obtained for $\alpha = 30$ and $\alpha = 100$ are 0.17 and 0.47 respectively (Fig. 2).

Let now the sampling step be $\tau = 10^{-5}$. System (29) keeps the accuracies $|\sigma| \leq 5.0 \cdot 10^{-8}$ and $|\dot{\sigma}| \leq \varepsilon = 0.0013$, which corresponds to the standard asymptotics (22). The best accuracy of filter (25) is expected for $\alpha \approx \sqrt{10^{-4}/10^{-5}} \cdot 10 \approx 30$. And indeed, the accuracies obtained for $\alpha = 10, 30, 1000$ are $0.12, 0.066, 0.47$ respectively (Fig. 2). Note that for each τ the error becomes large for large α (Fig. 2).

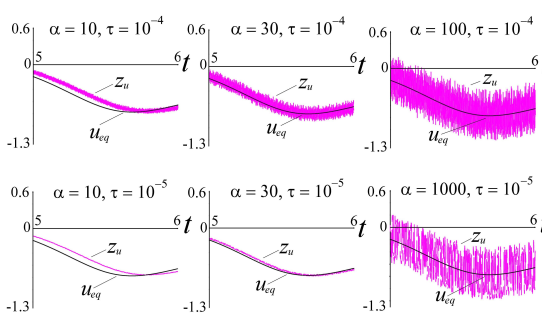


Figure 2. Performance of the classic filter (15) over the interval $[5, 6]$. The roughly best performance for $\tau = 10^{-4}$ is obtained for $\alpha = 10$, while the value $\alpha = 30$ is the best for $\tau = 10^{-5}$.

The 0th-order filter (27) with $L = 30$ yields the accuracies $|z_0 - u_{eq}| \leq 0.085$ and $|z_0 - u_{eq}| \leq 0.021$ for $\tau = 10^{-4}$ and $\tau = 10^{-5}$ respectively, while the 1st-order filter demonstrates the accuracies $|z_0 - u_{eq}| \leq 0.04$, $|z_1 - \dot{u}_{eq}| \leq 0.65$ for $\tau = 10^{-4}$ and $|z_0 - u_{eq}| \leq 0.008$, $|z_1 - \dot{u}_{eq}| \leq 0.29$ for $\tau = 10^{-5}$ according to Example 1. One does not need to adjust the filter parameters with respect to ε or τ (Fig. 3).

The obtained accuracies of the 0-order filters (27) and (25) are indeed of the same order, provided the conventional filter parameter is adjusted as $\alpha = O(\tau^{-1/2})$. Comparison of the graphs is shown in Fig. 4. Note the chattering of the linear filter output.

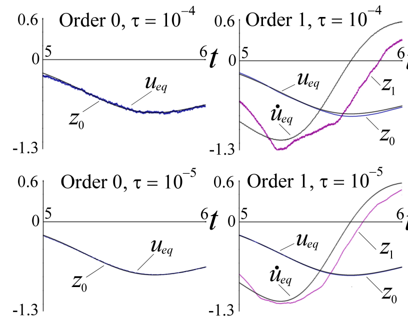


Figure 3. Performance of the nonlinear filters (27) and (28) over the interval $[5, 6]$ for $\tau = 10^{-4}, 10^{-5}$.

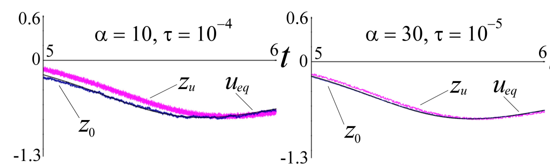


Figure 4. Comparison of the classic filter (15) with optimaly chosen $\alpha(\tau)$ and the output z_u, and the nonlinear filter (27) of the 0th order with fixed parameters and the output z_0 over the interval $[5, 6]$ for $\tau = 10^{-4}, 10^{-5}$.

The general performance of all the considered filters does not depend on the SM order. Indeed, the asymptotics (17), (19) do not depend on r. The estimation errors are not centralized at 0 in Figs 2-4, since they are determined by the limit orbits of the discrete-SM steady-state dynamics [1], [33], [35].

V. Conclusions

A novel FT-exact homogeneous differentiator is proposed which is based on the ideas of [18], while being robust not only to small noises, but also to any noises approximately centralized at zero. In particular, in simulation it has been shown to successfully reject large Gaussian white noises. The only needed information for the kth-order differentiation still is a rough Lipschitz constant L of the kth input derivative.

The proposed differentiator is both homogeneous and asymptotically optimal [22] in the presence of bounded noises. The simulation shows that the obtained accuracies and performance are also practically identical to those of the standard differentiators [18]. Thus, one comes to the conclusion that the new differentiator overcomes its predecessor [18].

The novel kth-order differentiator is applied to solving the classical SMC problem of the equivalent control estimation. It directly "differentiates" the chattering SM control, filters out the chattering component and produces the estimations of the equivalent control u_{eq} and its k derivatives, provided $|u_{eq}^{(k+1)}| \leq L$ holds.

The very nature of that SMC problem requires the SM to be *approximate*, since in the ideal r-SM $\sigma \equiv 0$ the switching control u is not a function of time and cannot be processed. But the output of our filter converges to the exact equivalent control u_{eq} as the switching imperfections and the SM error ε vanish. Our filter is asymptotically exact in that sense.

The main alternative for the equivalent control estimation is the classic linear filter (15) by Utkin [30]. No value of its only parameter α makes it asymptotically exact. The needed conditions [30] are the same as for our differentiator, but α is to be a function of the usually unknown SM accuracy ε. Moreover, $\alpha \to \infty$ makes the worst-case estimation error approach the switching-control-component magnitude. We list the conditions by Utkin for the application of filter (15), calculate its accuracy (Theorem 3), and prove that none of these conditions can be removed (Proposition 1).

For any order $k \geq 0$ and $\alpha > 0$ for sufficiently small ε our kth-order filter provides for the worst-case error lower than the linear filter (15). The higher k the better the accuracy.

We believe that the new differentiator/filter will prove its effectiveness in signal processing. A number of well-known adaptation and observation methods based on equivalent-control estimations will benefit from the proposed asymptotically-exact method.

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