Uncertain disturbances' attenuation by homogeneous MIMO sliding mode control and its discretization

Arie Levant and Miki Livne

School of Mathematical Sciences, University of Tel-Aviv, Israel
levant@post.tau.ac.il, miki.livne@gmail.com

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Abstract

Design of Multi-Input Multi-Output (MIMO) Homogeneous Sliding Modes (HSMs) for uncertain dynamic systems is considered. The resulting closed-loop systems feature all well-known standard properties of Single-Input Single-Output (SISO) HSM systems. Introduction of robust exact differentiators produces homogeneous output-feedback controllers. The ultimate asymptotic accuracy of SISO HSM technique is proved to be preserved in the MIMO and SISO cases, if the discrete-time implementation is based on the one-step Euler integration.

1 Introduction

Sliding-mode (SM) control is an effective technique to control systems under heavy uncertainty conditions. The main idea is to remove the uncertainty permanently keeping some properly chosen constraint functions (sliding variables) at zero. The method is based on high-frequency (theoretically infinite frequency) control switching, and the respective modes are called SMs. SMs are accurate and insensitive to matched disturbances [10, 30], and are usually established in finite time.

The control switching produces possibly dangerous system vibrations (the so-called chattering effect), which are considered the main drawback of the method [10, 12, 30]. Also, standard SMs [10, 30] are based on the relay control and sliding variables of relative degree 1.

High order sliding modes (HOSMs) [5, 15, 17, 18, 20, 27, 29] hide the switching in the higher derivatives of the sliding variables and remove the relative-degree restriction, while preserving the finite-time transient to the sliding mode. The relative degree of the sliding variable has become the main parameter of the HOSM application. Artificially increasing the relative degree, one produces arbitrarily smooth control and removes the dangerous high-energy chattering [4, 5, 20, 29]. Such controllers directly solve the control problem, if the sliding variable is a tracking error. Another important application of SMs is the robust finite-time-exact differentiation and observation [6, 7, 14, 16, 17, 29, 30, 31].

Let the system be understood in the Filippov sense [11] and $\sigma_1, \dots, \sigma_m$ be its scalar outputs. Recall [15, 17] that if the system is closed by some possibly-dynamical discontinuous feedback, the successive total time derivatives σ_i, $\dot{\sigma}_i$, ..., $\sigma_i^{(r_i-1)}$, $i = 1, \dots, m$, are continuous functions of the closed-system state-space variables; and the r-sliding set $\sigma_i = ... = \sigma_i^{(r_i-1)} = 0$, $i = 1, \dots, m$, is a non-empty integral set, $r = (r_1, \dots, r_m)$, then the motion on the set is said to be in r-sliding (rth-order sliding) mode. The vector $r = (r_1, \dots, r_m)$ is called the sliding order. The standard sliding mode [10, 30] is of the first order (σ_i are continuous, and $\dot{\sigma}_i$ are discontinuous, $r = (1, \dots, 1)$).

The asymptotic accuracy of the r-SM was analyzed in [15]. The accuracy is called asymptotic, since it is calculated as $O(\gamma)$, where γ is an infinitesimal function. It is shown there that the best possible asymptotic accuracy with the sampling time interval $\tau > 0$ is $\sigma_i^{(j)} = O(\tau^{r_i-j})$, $j = 0, 1, \dots, r_i - 1$. On the other hand, the homogeneity technique [18] indeed provides for this accuracy at least for the SISO case. Moreover, the accuracy is preserved, if the derivatives are estimated by homogeneous differentiators [17]. More exactly with the sampling accuracy $\varepsilon > 0$ of the σ-measurements the asymptotic accuracy $\sigma_1^{(j)} = O(\max(\tau^{r_1-j}, \varepsilon^{(r_1-j)/r_1}))$ is obtained (SISO case, $m = 1$, $r = (r_1)$). Unfortunately, this result is restricted to the ideal case, when the system is described by the Filippov differential equations with zero-hold measurements.

Most known results on HOSM control were obtained for Single-Input Single-Output (SISO) systems. In the case of Multi-Input Multi-Output (MIMO) systems the case of the well-defined relative degree corresponds to the non-singularity of the matrix of partial derivatives of higher-order total output derivatives explicitly containing controls with respect to controls (the high-frequency gain matrix). One mostly needs to know this matrix exactly or with high precision to use HOSM controls, since it allows exact decoupling of such a MIMO system into SISO subsystems of relative degrees r_i.

The paper [9] deals with the case when the matrix is uncertain, at the same time providing for the finite-time convergence to the sliding mode in the general MIMO case. The nominal value of the above matrix of partial derivatives is assumed available in [9] and the deviation from the nominal value is bounded. The control is based on the integral first-order SM, while in the integral SM the system dynamics is finite-time stable and is taken from [8]. Thus, one cannot provide for the arbitrarily fast convergence of the resulting integral SM dynamics [8]. Moreover, since the control combines 1-SM dynamics with the special homogeneous dynamics [8], the combined dynamics are not homogeneous. That causes the loss of the ultimate accuracy of r-sliding homogeneous control [18].

The MIMO control [21] is simpler, preserves SM homogeneity and the respective accuracy and allows arbitrarily fast convergence rate. It also supposes the availability of the above nominal matrix of partial derivatives. This paper presents and further develops results of [21], while the nominal matrix is assumed known up to a bounded positive factor. In that way the system is equipped with SM-based differentiators [17] yielding finite-time exact robust estimations of the output derivatives. The asymptotic accuracy of the obtained output-feedback control is estimated, and is shown to be the standard ultimate accuracy of homogeneous SM control [18] extended to the MIMO case, i.e. $\sigma_i^{(j)} = O(\max(\tau^{r_i-j}, \varepsilon^{(r_i-j)/r_i}))$, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$.

As it has been already mentioned, the above standard accuracy [18] is obtained under the assumption that, whereas the sampling is performed at discrete times, the system itself evolves in the continuous time. In practice controllers and observers are based on real-time computer calculations. Thus, formally the computer-based implementation of the HOSM controllers and differentiators requires that the dynamic parts of controllers/differentiators be integrated with infinitesimally small integration step, and the control be continuously fed to the system. This significantly complicates the implementation, and even may increase the chattering, when sampling periods are uneven or long. Indeed, in such a case the differentiator input is a piece-wise constant function, featuring zero derivatives. Respectively, the differentiator outputs are in a persistently renewed transient to zero.

The natural way of the computer-based implementation is to keep the outputs of observers and integrators constant between the measurements, and to apply the one-step Euler-integration method in the dynamic parts of the controllers and observers. The approach remains the same also with variable sampling intervals. The resulting hybrid system requires special analysis. The approach is for the first time formulated and established in this paper for MIMO and SISO systems. A special case is the case of the standard chattering attenuation procedure, when the system control input is built as the integral of an auxiliary control. In such a way the vector relative degree components are increased by one. The inserted discrete integrators are also based on the Euler approximation and produce piece-wise constant functions.

It is known that the Euler discretezation of the standard differentiators [15] lacks their homogeneity and their standard asymptotic accuracy [23, 26]. Hence, one would expect the accuracy deterioration also in the case of the differentiator output-feedback application. The sudden and important result of this research is that in the closed-loop system the above asymptotic system accuracy is proved to be preserved in all considered cases.

2 Preliminaries. SISO HOSM control, homogeneity notions

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)\,u, \quad \omega = \omega(t,x), \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}$ is the control, $\omega : \mathbf{R}^{n+1} \to \mathbf{R}$ and a, b are unknown smooth functions, n itself can be also uncertain. Informally, the control task is to keep the real-time measured output ω as small as possible. All differential equations are understood in the Filippov sense [11] in order to allow discontinuous controls.

In order to simplify the presentation, the input u, and the relative degree r have the same notation both in the SISO and MIMO cases, when they turn out to be vectors.

The relative degree r of system (1) is assumed to be constant and known. It means [13] that for the first time the control explicitly appears in the rth total time derivative of ω, i.e.

$$\omega^{(r)} = \tilde{h}(t,x) + \tilde{g}(t,x)u, \tag{2}$$

where $\tilde{h}(t,x)$, $\tilde{g}(t,x)$ are some unknown smooth functions, $\tilde{g} \neq 0$. According to the standard HOSM control approach, let

$$0 < K_m \leq \tilde{g}(t,x) \leq K_M, \quad |\tilde{h}(t,x)| \leq C, \tag{3}$$

for some K_m, K_M, $C > 0$. Also assume that solutions of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control $u(t,x)$.

In practice the operational region of any plant is inevitably bounded. In that case conditions (3) hold locally, in which case the results can be respectively reformulated [17].

Obviously, (2) and (3) imply the differential inclusion

$$\omega^{(r)} \in [-C, C] + [K_m, K_M]u, \tag{4}$$

and the problem is reduced to the stabilization of (4). Here and further a binary operation of two sets produces the set of all possible binary operations of their elements, a number (vector) is treated in that context as a one-element set.

A bounded feedback control

$$u = U_r(\omega, \dot{\omega}, ..., \omega^{(r-1)}), \tag{5}$$

is constructed, such that all solutions of (4), (5) converge in finite time to the origin $\omega = \dot{\omega} = ... = \omega^{(r-1)} = 0$. The function U_r is a locally-bounded Borel-measurable function. Thus, substituting any Lebesgue-measurable estimations of $\omega, \dot{\omega}, ..., \omega^{(r-1)}$ obtain a Lebesgue measurable control. At the next step, the lacking derivatives are real-time evaluated, producing an output-feedback controller.

Note that here and further the right-hand side of any closed-loop differential inclusion is minimally enlarged providing for its compactness, convexity and upper-semicontinuity [18].

It is easy to see that the function U_r is to be discontinuous at the r-sliding set $\omega = \dot{\omega} = ... = \omega^{(r-1)} = 0$ [18, 19]. Some other properties of the controller (5) are described below.

A function $f : \mathbf{R}^n \to \mathbf{R}$ (respectively a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field $f : \mathbf{R}^n \to \mathbf{R}^n$) is called *homogeneous of the degree* $q_s \in \mathbf{R}$ *with the dilation* [2] $d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1}x_1, \kappa^{m_2}x_2, ..., \kappa^{m_n}x_n)$, and the *weights* m_1, ..., $m_n > 0$, if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q_s}f(d_\kappa x)$ holds (respectively, $F(x) = \kappa^{-q_s}d_\kappa^{-1}F(d_\kappa x)$, or $f(x) = \kappa^{-q_s}d_\kappa^{-1}f(d_\kappa x)$). The non-zero homogeneity degree q_s of a vector field can always be scaled to ± 1 by an appropriate proportional change of the weights m_1, ..., m_n.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot{x} = f(x)$ (differential inclusion $\dot{x} \in F(x)$) with respect to the combined time-coordinate transformation $(t, x) \mapsto (\kappa^{-q_s}t, d_\kappa x)$, where $-q_s$ might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as

$$\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^{-q_s} t)} \in F(d_\kappa x).$$

Suppose that feedback (5) imparts homogeneity properties to the closed-loop inclusion (4), (5). Due to the term $[-C, C]$, the right-hand side of (5) can only have the homogeneity degree 0 with $C \neq 0$. Scaling the system homogeneity degree to -1, achieve that the homogeneity weights of t, ω , $\dot{\omega}$, ..., $\omega^{(r-1)}$ are 1, r, r - 1, ..., 1 respectively. This homogeneity is called the *r-sliding homogeneity* [14]. The inclusion (4), (5) is called *r-sliding homogeneous* if for any $\kappa > 0$ the combined time-coordinate transformation

$$(t, \omega, \dot{\omega}, \ldots, \omega^{(r-1)}) \mapsto (\kappa t, \kappa^r \omega, \kappa^{r-1}\dot{\omega}, \ldots, \kappa\omega^{(r-1)}) \tag{6}$$

preserves the closed-loop inclusion (4), (5).

Transformation (6) transfers (4), (5) into

$$\frac{d^r(\kappa^r \omega)}{d(\kappa t)^r} = \frac{d^r \omega}{dt^r} \in [-C, C] + [K_m, K_M] U_r(\kappa^r \omega, \kappa^{r-1}\dot{\omega}, \ldots, \kappa\omega^{(r-1)}).$$

Thus, the r-sliding homogeneity condition is

$$U_r(\kappa^r \omega, \kappa^{r-1}\dot{\omega}, \ldots, \kappa\omega^{(r-1)}) \equiv U_r(\omega, \dot{\omega}, \ldots, \omega^{(r-1)}). \tag{7}$$

Respectively, controller (5) is called *r-sliding homogeneous*, if the identity (7) holds for any positive κ and any arguments. Also the corresponding r-sliding mode $\omega \equiv 0$ is called homogeneous in that case. In particular, the relay controller is 1-sliding homogeneous, as well as the corresponding sliding mode. Since the control is locally bounded, due to (7) it is also globally bounded.

Let $\beta_{1,r}, \ldots, \beta_{r-1,r}$ be some predefined positive coefficients, and α be the chosen control magnitude. Then some r-sliding homogeneous controllers of the form

$$u = -\alpha \Psi_{r-1,r}(\omega, \dot{\omega}, \ldots, \omega^{(r-1)}) \tag{8}$$

are provided by the following recursive procedures. The procedure

$$\begin{array}{l} \varphi_{0,r} = \omega,\ N_{0,r} = |\omega|,\ \Psi_{0,r} = \text{sign}\ \omega; \\ \varphi_{j,r} = \omega^{(j)} + \beta_{j,r} N_{j-1,r}^{(r-j)/(r+1-j)} \Psi_{j-1,r}, \\ N_{j,r} = \left|\omega^{(j)}\right| + \beta_{j,r} N_{j-1,r}^{(r-j)/(r+1-j)}, \\ \Psi_{j,r} = \frac{\varphi_{j,r}}{N_{j,r}},\ j = 0, 1, \ldots, r, \end{array} \tag{9}$$

produces the controllers, called quasi-continuous [19, 25], for the resulting control is continuous everywhere except the r-sliding set $\omega = \dot{\omega} = \ldots = \omega^{(r-1)} = 0$. The following simple generalization is a new result.

Theorem 1. *Let $N_{j,r}$ in* (9) *be any positive-definite r-sliding homogeneous function of $\omega, \dot\omega, \ldots, \omega^{(j)}$ of the weight $r-j$. Let also $\Psi_{j-1,r}$ in the definitions of $\varphi_{j,r}$ be replaced with $\zeta_{j-1,r}(\Psi_{j-1,r})$, $j = 1, \ldots, r$, where $\zeta_{j-1,r}(s)$ is any strictly increasing continuous function, $\zeta_{j-1,r}(0) = 0$. Then* (9) *still produces a quasi-continuous r-sliding homogeneous finite-time stable controller, provided the coefficients $\beta_{1,r}, \ldots, \beta_{r-1,r}$ are chosen sufficiently large in the list order.*

The proof is similar to [25]. It is easy to see that $|\zeta_{1,r}(\Psi_{1,r})| \le \xi_1$ defines a finite-time stable r-sliding homogeneous differential inclusion for any sufficiently small $\xi_1 > 0$. A recursion step follows: for sufficiently small $\xi_j > 0$ with sufficiently large $\beta_{j,r}$ the homogeneous differential inclusion $|\zeta_{j,r}(\Psi_{j,r})| \le \xi_j$ provides for the finite time establishment of the inequality $|\zeta_{j-1,r}(\Psi_{j-1,r})| \le \xi_{j-1}$ in the space $\omega, \ldots, \omega^{(j-1)}$. The fact that with sufficiently large α system (4) provides in finite time for $|\zeta_{r-1,r}(\Psi_{r-1,r})| \le \xi_{r-1}$ finishes the proof.

Let $d \ge r$. Another well-known family of SM controllers, called embedded SM controllers [17], is provided by the procedure

$$\begin{array}{l} \varphi_{0,r} = \omega,\ N_{0,r} = |\omega|^{1/r},\ \Psi_{0,r} = \operatorname{sign}\omega; \\ \varphi_{j,r} = \omega^{(j)} + \beta_{j,r} N_{j-1,r}^{r-j} \Psi_{j-1,r}, \Psi_{j,r} = \operatorname{sign}\varphi_{j,r}, \\ N_{j,r} = \left(|\omega|^{d/r} + |\dot\omega|^{d/(r-1)} + \ldots + |\omega^{(j-1)}|^{d/(r+1-j)}\right)^{1/d}. \end{array} \tag{10}$$

Any positive coefficients can be taken in the definition of $N_{j,r}$. The proof is the same as in [15].

A number of quasi-continuous SM controllers (9), (8) with tested coefficients is listed below for $r = 1, 2, 3, 4$. It is enough to adjust only the parameter α in order to control any system (1), (3) of the corresponding relative degree.

1. $u = -\alpha \operatorname{sign}\omega$,
2. $u = -\alpha\, (\dot\omega + |\omega|^{1/2} \operatorname{sign}\omega)/(|\dot\omega| + |\omega|^{1/2})$,
3. $u = -\alpha\, [\ddot\omega + 2(|\dot\omega| + |\omega|^{2/3})^{-1/2})(\dot\omega + |\omega|^{2/3} \operatorname{sign}\omega)]/$
   $[|\ddot\omega| + 2(|\dot\omega| + |\omega|^{2/3})^{1/2})]$,
4. $\varphi_{3,4} = \dddot\omega + 3[\ddot\omega + (|\dot\omega| + 0.5|\omega|^{3/4})^{-1/3}(\dot\omega + 0.5|\omega|^{3/4} \operatorname{sign}\omega)]$
   $[|\ddot\omega| + (|\dot\omega| + 0.5|\omega|^{3/4})^{2/3}]^{-1/2}$,
   $N_{3,4} = |\dddot\omega| + 3[|\ddot\omega| + (|\dot\omega| + 0.5|\omega|^{3/4})^{2/3}]^{1/2},\ u = -\alpha\varphi_{3,4}/N_{3,4}$.

Note that the same coefficients $\beta_{j,r}$ can be used for embedded controllers with $r \le 4$. Other constructions of homogeneous HOSM controllers and the choice of parameters are considered in [18, 25]. It is further assumed that

$\beta_{1,r}, \ldots, \beta_{r-1,r}$ are always properly chosen, which means that the differential equation $\varphi_{r-1,r} = 0$ is finite-time stable [25].

Any r-sliding homogeneous controller can be combined with an $(r-1)$th-order differentiator [17] producing an output feedback controller. Its applicability in this case is possible due to the boundedness of $\omega^{(r)}$ implied by the boundedness of the feedback function U_r in (5).

Let the input signal $f(t)$ be a function consisting of a bounded Lebesgue-measurable noise with unknown features, and of an unknown base signal $f_0(t)$, whose k_dth derivative has a known Lipschitz constant $L > 0$. The following differentiator provides for the estimations z_j of the derivatives $f_0^{(j)}$, $j = 0, \ldots, k_d$:

$$\begin{array}{l}
\dot{z}_0 = -\tilde{\lambda}_{k_d} L^{1/(k_d+1)} |z_0 - \omega|^{k_d/(k_d+1)} \operatorname{sign}(z_0 - f(t)) + z_1, \\
\dot{z}_1 = -\tilde{\lambda}_{k_d-1} L^{1/k_d} |z_1 - \dot{z}_0|^{(k_d-1)/k_d} \operatorname{sign}(z_1 - \dot{z}_0) + z_2, \\
\ldots \\
\dot{z}_{k_d-1} = -\tilde{\lambda}_1 L^{1/2} |z_{k_d-1} - \dot{z}_{k_d-2}|^{1/2} \operatorname{sign}(z_{k_d-1} - \dot{z}_{k_d-2}) + z_{k_d}, \\
\dot{z}_{k_d} = -\tilde{\lambda}_0 L \operatorname{sign}(z_{k_d} - \dot{z}_{k_d-1}).
\end{array} \tag{11}$$

The parameters $\tilde{\lambda}_i$ of differentiator (11) are chosen in advance for each k_d. An infinite sequence of parameters $\tilde{\lambda}_i$ can be built, valid for any k_d [17]. In particular, one can choose $\tilde{\lambda}_0 = 1.1$, $\tilde{\lambda}_1 = 1.5$, $\tilde{\lambda}_2 = 2$, $\tilde{\lambda}_3 = 3$, $\tilde{\lambda}_4 = 5$, $\tilde{\lambda}_5 = 8$ [19], which is enough for $k_d \leq 5$. In the absence of noises the differentiator provides for the exact estimations in finite time. With discrete-time sampling time period $\tau > 0$ and the maximal possible sampling error $\varepsilon \geq 0$ the accuracy $z_j - f_0^{(j)} = O(\max(\tau^{k_d+1-j}, \varepsilon^{(k_d+1-j)/(k_d+1)}))$ is provided.

Equations (11) can be rewritten in the standard (non-recursive) dynamic-system form

$$\begin{array}{lll}
\dot{z}_0 & = & -\lambda_{k_d} L^{1/(k_d+1)} |z_0 - f(t)|^{k_d/(k_d+1)} \operatorname{sign}(z_0 - f(t)) + z_1, \\
\dot{z}_1 & = & -\lambda_{k_d-1} L^{2/(k_d+1)} |z_0 - f(t)|^{(k_d-1)/(k_d+1)} \operatorname{sign}(z_0 - f(t)) + z_2, \\
 & & \ldots \\
\dot{z}_{k_d-1} & = & -\lambda_1 L^{k_d/(k_d+1)} |z_0 - f(t)|^{1/(k_d+1)} \operatorname{sign}(z_0 - f(t)) + z_{k_d}, \\
\dot{z}_{k_d} & = & -\lambda_0 L \operatorname{sign}(z_0 - f(t)).
\end{array} \tag{12}$$

It is easy to see that $\lambda_0 = \tilde{\lambda}_0$, $\lambda_{k_d} = \tilde{\lambda}_{k_d}$, and $\lambda_j = \tilde{\lambda}_j {\lambda_{j+1}}^{j/(j+1)}$, $j = k_d - 1, k_d - 2, \ldots, 1$.

Assuming that the sequence $\tilde{\lambda}_j$, $j = 0, 1, \ldots$, is the same over the whole paper, denote both (11) and (12) by the equality $\dot{z} = D_{k_d}(z, f, L)$. Incorporating the $(r-1)$th order differentiator into the feedback equations, obtain the output-feedback r-sliding controller

$$u = U_r(z), \ \dot{z} = D_{r-1}(z, \omega, L), \tag{13}$$

where $L \geq C + K_M \sup |U_r|$. Obviously, provided (4), (5) is finite-time stable, the output-feedback controller (13) ensures the finite-time establishment of the r-sliding mode $(\omega, \dot{\omega}, \ldots, \omega^{(r-1)}) = 0$. Moreover [18], if (5) is r-sliding homogeneous, and ω is measured with the sampling accuracy $\varepsilon > 0$, then the asymptotic accuracy $\omega^{(j)} = O(\max(\tau^{r-j}, \varepsilon^{(r-j)/r}))$ is obtained.

3 MIMO SM control

Once more consider dynamic system (1),

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t,x), \tag{14}$$

but let now σ and u be vectors, $\sigma : \mathbf{R}^{n+1} \to \mathbf{R}^m$, $u \in \mathbf{R}^m$. As previously, a, b, σ are assumed smooth. The task is to stabilize the output σ at 0.

The system is assumed to have the vector relative degree $r = (r_1, ..., r_m)$, $r_i > 0$. It means that the successive total time derivatives $\sigma_i^{(j)}$, $j = 0, 1, ..., r_i - 1$, $i = 1, ..., m$, do not contain controls, and can be used as new coordinates [13]. Respectively, instead of (4) obtain a vector equation,

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{15}$$

where $\sigma^{(r)}$ denotes $(\sigma_1^{(r_1)}, ..., \sigma_m^{(r_m)})^T$, the functions h, and g are unknown and smooth. The function g is a nonsingular matrix.

Let g be represented in the form $g = K\bar{g}$, where $K > 0$ defines the "size" of the matrix g, and $\bar{g}$ defines the matrix "direction". In the scalar case $m = 1$, $\bar{g} = \pm 1$. The following assumption directly generalizes (3).

Assumption 1. *The scalar function $K(t,x)$ is bounded, a nominal "direction" matrix $G(t,x)$ is assumed nonsingular and available in real time, so that*

$$g(t,x) = K(t,x)(G(t,x) + \Delta g(t,x)), \;\; \left\|\Delta g G^{-1}\right\|_1 \leq p < 1, \; 0 < K_m \leq K \leq K_M. \tag{16}$$

Here Δg is the uncertain deviation of $\bar{g}$ from G, and the norm $\|\cdot\|_1$ of the matrix $A = (a_{ij})$ is defined as $\|A\|_1 = \max\limits_i \sum\limits_j |a_{ij}|$. The estimation G can be any Lebesgue-measurable function, K_m, K_M, p are known constants.

Assumption 2. *The uncertain vector function h is supposed to satisfy the estimation*

$$\|h(t,x)\| \leq H(t,x), \;\; H(t,x) \geq h_0 > 0. \tag{17}$$

where $H(t,x)$ is some strictly positive locally bounded Lebesgue-measurable function available in real time, h_0 is known. It is also assumed that trajectories of (14) are infinitely extendible in time for any Lebesgue-measurable control with uniformly bounded $||gu||/H$.

Note that the functions $G(t,x)$ and $H(t,x)$ are assumed directly available along the current trajectory $(t,x(t))$ in real time, but this does not necessarily mean that $x(t)$ is assumed available, or the functions are given by some formulas. For example, due to extensive wind-tunnel experiments the aerodynamic characteristics of an aircraft are usually available as approximate functions of the observable dynamic pressure and altitude.

3.1 Control design

Introduce a virtual control v,

$$v = G(t,x)u. \tag{18}$$

Then dynamics (15) takes the form

$$\sigma^{(r)} = h(t,x) + K(t,x)(I + \Delta g(t,x)G^{-1}(t,x))v, \tag{19}$$

where I is the unit matrix.

Introduce the notation $\vec{\sigma}_i = (\sigma_i, \ldots, \sigma_i^{(r_i-1)})$, $\vec{\sigma} = (\vec{\sigma}_1, \ldots, \vec{\sigma}_m)$. Choose the components of $v = (v_1, \ldots, v_m)^T$ in the form

$$v_i = -\alpha H(t,x)\,\mathrm{sat}(\eta_i \Psi_{r_i-1,r_i}(\vec{\sigma}_i)), i = 0, 1, , m, \tag{20}$$

where $-\Psi_{r_i-1,r_i}$ is an embedded (10) or quasi-continuous (9) r_i-sliding homogeneous controller (8), $\alpha > 0$, $\mathrm{sat}(s) = \min[1, \max(-1, s)]$ is the saturation function. The parameter $\eta_i \geq 1$ is further chosen as a function of r_i. Note that $\mathrm{sat}(s)$ can be replaced here by any function $\zeta_i(s)$, where ζ_i is any continuous strictly growing function, satisfying $\zeta_i(-s) = -\zeta_i(s)$.
Choice of η_i. In the case of quasi-continuous controllers, if

$$|\varphi_{r_i-1,r_i}| \leq \frac{2p}{1-p}\beta_{i,r_i-1}N^{1/2}_{r_i-2,r_i}$$

is a finite-time stable differential inclusion in the state space $\sigma_i, \dot{\sigma}_i, \ldots, \sigma_i^{(r_i-2)}$, then $\eta_i = 1$ can be taken. Otherwise, $\eta_i > 1$ is taken sufficiently large, so that the differential inclusion

$$|\varphi_{r_i-1,r_i}| \leq \frac{2\eta_i^{-1}}{1-\eta_i^{-1}}\beta_{i,r_i-1}N^{1/2}_{r_i-2,r_i}$$

is finite-time stable, and $\eta_i^{-1} < p$. Such $\eta_i > 1$ always exists with properly chosen coefficients β_{ij}, $j = 1, 2, \ldots, r_i - 1$, and can be found in advance [19]. In the case of embedded controllers one can always take $\eta_i = 1$, for it does not influence the control.

Theorem 2. *With sufficiently large $\alpha > 0$ and η_i control* (20) *provides for the finite-time establishment of the r-sliding mode $\sigma \equiv 0$.*

Remark. Note that with uniformly bounded H one can also claim that the r-sliding mode is uniformly finite-time stable, i.e. there exists a uniform upper convergence time bound $T(R)$ for solutions starting in the region $||\vec{\sigma}|| \leq R$, and for any $\delta_0 > 0$ there exists $\delta_1 > 0$, such that at any moment the inequality $||\vec{\sigma}|| \leq \delta_1$ implies $||\vec{\sigma}|| \leq \delta_0$ from this moment and forever.

Theorem 3. *Let $H(t, x) = const > 0$. Let $\sigma_i^{(j)}$ be measured with sampling noises not exceeding ε_{ij} and with the sampling intervals not exceeding $\tau > 0$. Then with sufficiently large α and η_i the feedback* (20) *provides for the accuracy $|\sigma_i^{(j)}| \leq \mu_{ij} \left(\max[\tau, \max_{i,j} \varepsilon_{ij}^{1/(r_i - j)}] \right)^{r_i - j}$, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$, with some constant $\mu_{ij} > 0$.*

3.2 Output-feedback control

Suppose that $H(t, x)$ is continuous. It follows from (16)-(17) that $|\sigma_i^{(r_i)}| \leq (\alpha K_M(1 + p) + 1)H(t, x)$. Thus [22], the needed values of the derivatives can be obtained globally and in finite time by means of the robust exact differentiators (11) of the orders $r_1 - 1, \ldots, r_m - 1$ with the variable functional parameter

$$L(t, x) = k_L(\alpha K_M(1 + p) + 1)H(t, x), \tag{21}$$

where $k_L \geq 1$ is an arbitrary coefficient, usable in practical realization. Respectively, the output feedback obtains the form

$$\begin{aligned} & v_i = -\alpha H(t, x)\,\mathrm{sat}(\eta_i \Psi_{r_i - 1, r_i}(z_i)),\ \dot{z}_i = D_{r_i - 1}(z_i, \sigma_i, L), \\ & i = 1, 2, \ldots, m. \end{aligned} \tag{22}$$

Theorem 4. *Let $H(t, x) = const > 0$. Then with sufficiently large α and η_i the output-feedback control* (21), (22) *provides for the finite-time establishment of the r-sliding mode $\sigma \equiv 0$. In the presence of sampling noises of σ_i not exceeding ε with the sampling intervals not exceeding $\tau > 0$ the accuracy $|\sigma_i^{(j)}| \leq \mu_{ij} \max(\varepsilon^{(r_i - j)/r_i}, \tau^{r_i - j})$, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$, is established with some constant $\mu_{ij} > 0$.*

Consider now the general case. Let L be differentiable and the logarithmic derivative $\dot{L}/L$ be uniformly bounded, then the differentiation is robust, and in the presence of a Lebesgue-measurable sampling noise with the magnitude $\varepsilon L(t,x)$ and the sampling interval τ the accuracy $|z_{ij} - \sigma_i^{(j)}|/L(t,x) = O\left(\max(\varepsilon^{(r_i-j)/r_i}, \tau^{r_i-j})\right)$ is obtained [22].

The initial errors of the differentiator should be sufficiently small, i.e. $||z_i - \vec{\sigma}_i||/L(t,x)$ should be less than some constant independent of L. Choosing larger L the region of acceptable initial differentiation errors can be made arbitrarily large. In the absence of noises the equalities $z_i = \vec{\sigma}_i$ are established in finite time [17]. The convergence criterion [1] allows to detect the end of the transient in the presence of noises and in real time. Then the function L can be abruptly decreased to a less value. Another way is to evaluate the initial approximate derivative values using finite differences. In practice the implementation will always require some rough upper initial estimation of $||\vec{\sigma}||$. In the following theorem the differentiator transient is assumed finished.

Theorem 5. *Let $H(t,x)$ be continuous and differentiable with uniformly bounded $\dot{H}/H$. Then with sufficiently large α and η_i the output-feedback control* (21), (22) *provides for the finite-time establishment of the r-sliding mode $\sigma \equiv 0$. Let the sampling noises of σ_i not exceed $\varepsilon H(t,x)$ and the sampling intervals not exceed $\tau > 0$. Let also the initial conditions belong to some compact region. Then there exists the time instant t_0 (the end of the transient), such that for any $t_2 > t_1 \geq t_0$ the accuracy $|\sigma_i^{(j)}| \leq \mu_{ij} \max(\varepsilon^{(r_i-j)/r_i}, \tau^{r_i-j})$, $j = 0, 1, \ldots, r_i - 1$, $i = 1, 2, \ldots, m$, is kept over the steady-state time interval $[t_1, t_2]$ with sufficiently small ε, τ and constant $\mu_{ij} > 0$. In general coefficients μ_{ij} depend on the steady-state time interval $[t_1, t_2]$.*

3.3 Chattering attenuation

High-frequency switching of the control in the SM $\sigma \equiv 0$ causes potentially-dangerous vibrations called chattering [30, 10, 3]. One of the standard ways to suppress high-energy vibrations is to artificially increase the relative degree [15, 4, 20]. Let the new control be $\dot{u}$. Differentiating (15) obtain $\sigma^{(r+\vec{1})} = h_e(t,x,u) + g(t,x)\dot{u}$, where $\vec{1} = (1, \ldots, 1) \in \mathbf{R}^m$. Applying the transformation

$$\dot{u} = G^{-1}(t,x)v_e, \tag{23}$$

obtain

$$\sigma^{(r+\vec{1})} = h_e(t,x,u) + K(t,x)(I + \Delta g(t,x)G^{-1}(t,x))v_e, \tag{24}$$

where h_e is some uncertain smooth function.

Note that with $\vec{\sigma} \equiv 0$ the control satisfies $\sigma^{(r)} = 0$. Hence, (15) yields $u = u_{eq}(t, x) = -g^{-1}(t, x)h(t, x)$, where u_{eq} is some another uncertain smooth vector function. Thus, in the vicinity of the $(r + \vec{1})$-sliding mode u is close to $u_{eq}(t, x)$ and therefore locally bounded. Denote $\vec{\sigma}_{ei} = (\sigma_i, \ldots, \sigma_i^{(r_i)})$, $\vec{\sigma}_e = (\vec{\sigma}_{e1}, \ldots, \vec{\sigma}_{em})$.

Assumption 3. *The function $h_e(t, x, u)$ is uniformly bounded in the region $||\vec{\sigma}_e|| \leq \delta_e$ for some $\delta_e > 0$,*

$$||h_e(t, x, u)|| \leq C_e,\ C_e > 0 \text{ with } ||\vec{\sigma}_e|| \leq \delta_e. \tag{25}$$

The above assumption is at least locally always true due to the smoothness of h_e. In the following we do not consider the global control, but restrict ourselves to the local chattering attenuation. Let

$$\begin{array}{l} v_{ei} = -\alpha_e C_e \operatorname{sat}(\eta_i \Psi_{r_i, r_i+1}(z_{ei})),\ \dot{z}_{ei} = D_{r_i}(z_{ei}, \sigma_i, L_e), \\ L_e = k_L(\alpha_e K_M(1 + p) + 1)C_e,\ i = 1, 2, \ldots, m. \end{array} \tag{26}$$

Theorem 6. *Let the initial values of $\vec{\sigma}_e$ be sufficiently close to zero and the differentiators be initialized by zero initial conditions. Then after a finite time the $(r+\vec{1})$-sliding mode $\sigma \equiv 0$ is established. In the presence of sampling noises of σ_i not exceeding ε with the sampling intervals not exceeding $\tau > 0$ the accuracy $|\sigma_i^{(j)}| \leq \mu_{ij} \max(\varepsilon^{(r_i - j+1)/(r_i+1)}, \tau^{r_i+1-j})$, $j = 0, 1, \ldots, r_i$, $i = 1, 2, \ldots, m$, is established with some constant $\mu_{ij} > 0$.*

If the function $h_e(t, x, u)$ has a functional bound, the Theorem is to be reformulated similarly to Theorem 5.

Note that the closed-loop dynamic systems are nowhere replaced by discrete dynamics in the above Theorems. The discretization issues are considered in Section 5.

4 Proofs of Theorems 2 - 6

Obviously, (15) - (20) imply the differential inclusions

$$\sigma_i^{(r_i)} \in H(t, x)\,([-1, 1] - \alpha[K_m(1 - p), K_M(1 + p)] \operatorname{sat}(\eta_i \Psi_{r_i - 1, r_i}(\vec{\sigma}_i))) \tag{27}$$

valid for each $i = 1, \ldots, m$.

Proof of Theorem 2. Prove that (27) implies the finite-time convergence to the partial r_i-sliding mode $\vec{\sigma}_i = 0$. The proof should be performed separately for quasi-continuous (9) and embedded (10) SM controllers. In the case of the embedded SM controller the proof is especially simple. Indeed, Ψ_{r_i-1,r_i} only accepts the values ± 1. Thus, solutions of (27) satisfy

$$\sigma_i^{(r_i)} \in -[K_m(1-p)\alpha - 1, K_M(1+p)\alpha + 1]H(t,x)\Psi_{r_i-1,r_i}(\vec{\sigma}_i),$$

which is finite-time stable with sufficiently large α according to the gain-function robustness property of the controller (i.e., roughly speaking, multiplication of the control by any function larger than 1 does not destroy the finite-time convergence [24]).

Consider the quasi-continuous controller (9). The presented proof is a modified proof from [25]. The following simple technical Lemma plays important role in the sequel.

Lemma 1. *Let* $A, B \geq 0$, $|\theta| \leq 1$, $0 \leq \xi < 1$. *Then the inequality* $\frac{|A+B\theta|}{A+B} \leq \xi$ *implies that* $|A + B\theta| \leq \frac{2\xi}{1-\xi}B$.

Proof. Obviously, the inequality implies that $B > 0$. Divide the denominator and the nominator by B. Let $\tilde{A} = A/B$. It is enough to prove that

$$|\tilde{A} + \theta|/(\tilde{A} + 1) \leq \xi \tag{28}$$

implies that $|\tilde{A} + \theta| \leq \frac{2\xi}{1-\xi}$. Indeed, if $\tilde{A} \leq \frac{1+\xi}{1-\xi}$ then (28) implies

$$|\tilde{A} + \theta| \leq \left(\frac{1+\xi}{1-\xi} + 1\right)\xi = \frac{2\xi}{1-\xi}.$$

Now suppose that $\tilde{A} > \frac{1+\xi}{1-\xi}$. Then

$$\frac{|\tilde{A} + \theta|}{\tilde{A} + 1} = \frac{|\tilde{A} + 1 + \theta - 1|}{\tilde{A} + 1} \geq 1 - \frac{2}{\tilde{A} + 1} > 1 - \frac{2}{\frac{1+\xi}{1-\xi} + 1} = \xi,$$

and we come to contradiction. □

Fix a valid combination β_{j,r_i}, $j = 1, \ldots, r_i - 1$, of the parameters of (9) used in (20). Note that $N_{r_i-1,r_i}(\vec{\sigma}_i)$ is positive definite, i.e. $N_{r_i-1,r_i} = 0$ iff $\vec{\sigma}_i = 0$. As well as φ_{r_i-1,r_i}, it is also an r_i-sliding homogeneous function of the weight 1. On the other hand, $\Psi_{r_i-1,r_i} = \varphi_{r_i-1,r_i}/N_{r_i-1,r_i}$ is homogeneous of the weight 0, and $|\Psi_{r_i-1,r_i}| \leq 1$.

Define the point set $\Omega(\xi) = \{\vec{\sigma}_i | \, |\Psi_{r_i-1,r_i}(\vec{\sigma}_i)| \leq \xi\}$ for some fixed ξ, $\xi < 1$. As follows from Lemma 1 the points of $\Omega(\xi)$ satisfy the inequality

$$\left|\sigma_i^{r_i-1} + \beta_{r_i-1,r_i} N^{1/2}_{r_i-2,r_i}(\vec{\sigma}_i)\Psi_{r_i-2,r_i}(\vec{\sigma}_i)\right| \leq \frac{2\xi}{1-\xi}\beta_{r_i-1,r_i} N^{1/2}_{r_i-2,r_i}(\vec{\sigma}_i). \tag{29}$$

In other words, $|\varphi_{r_i-1,r_i}| \leq \frac{2\xi}{1-\xi}\beta_{r_i-1,r_i}N^{1/2}_{r_i-2,r_i}$. Note that β_{j,r_i} are chosen in such a way that $\varphi_{r_i-1,r_i}(\vec{\sigma}_i) = 0$ defines a finite-time stable r_i-sliding homogeneous differential equation in the space $\sigma_i, \dot{\sigma}_i, \ldots, \sigma_i^{(r_i-2)}$ [25]. Respectively, with sufficiently small ξ inequality (29) corresponds to a finite-time stable homogeneous differential inclusion [18]. Fix such a value of ξ.

Obviously, it is enough to prove that with sufficiently large α the trajectories of (27) in finite time enter the region $\Omega(\xi)$ to stay there forever. Let $\eta_i \geq \xi^{-1}$, then outside of $\Omega(\xi)$ we have $\mathrm{sat}(\eta_i\Psi_{r_i-2,r_i}) = \pm 1$. That in its turn implies that

$$\sigma_i^{(r_i)} \in -[K_m(1-p)\alpha - 1, K_M(1+p)\alpha + 1]H(t,x)\,\mathrm{sign}\,\Psi_{r_i-1,r_i}(\vec{\sigma}_i). \quad (30)$$

is kept outside of $\Omega(\xi)$. It is proved in [25] that with any sufficiently large $\gamma > 0$ the relation $\sigma_i^{(r_i)}\,\mathrm{sign}(\Psi_{r_i-2,r_i}(\vec{\sigma}_i)) \leq -\gamma$, provides for the entrance into $\Omega(\xi)$ in finite time, and for its invariance. Fix such γ. Hence, due to (30), the inequality $(K_m(1-p)\alpha - 1)h_0 > \gamma$ implies the finite-time convergence to the r-sliding mode. □

Proof of the Remark to Theorem 2, Theorems 3, 4 and 6. Let H be bounded. Substitute an appropriate segment for H in (27) and get an r-sliding homogeneous inclusion. Obviously, the above proof provides for the existence of an upper estimation of the convergence time to zero for trajectories starting in a unit ball [25]. Thus, the obtained inclusion is finite-time stable [18]. In particular, with the constant function H inclusion (27) becomes r-sliding homogeneous and finite-time stable. The Theorems now follow from [18]. □

Proof of Theorem 5. Fix some starting point (t_*, x_*) for the system trajectory. Let first the measurements be exact and sampled continuously in time. Starting from some moment $\hat{t}_0 > t_*$ all trajectories starting in a small vicinity of (t_*, x_*) enter the r-SM $\vec{\sigma} = 0$. Over any sufficiently small time interval in the r-SM the function can be considered almost constant: $H(t, x(t)) \in [H_m, H_M]$. Then the trajectory satisfies the inclusion

$$\sigma_i^{(r_i)} \in [H_m, H_M]\left([-1,1] - \alpha[K_m(1-p), K_M(1+p)]\,\mathrm{sat}(\eta_i\Psi_{r_i-1,r_i}(\vec{\sigma}_i))\right),$$

which is homogeneous and finite-time stable [18]. The Theorem now follows from Theorem 3 and the compactness of the region of initial conditions. □

5 Discretization of HOSMS

Note that the results of this section are new both in the SISO and MIMO cases. Implementation of HOSM controllers usually requires the control values to be calculated by computers using discrete sampling and fed to the systems at discrete time instants. In the case, when the controller does not involve its own dynamics, in particular, if all needed derivatives are directly sampled, the resulting system is adequately described by variable sampling noises and delays, and by Theorem 3.

The situation changes when the output feedback is applied, which incorporates a dynamic observer. Let the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$. It is known [23, 26] that the error dynamics of the discrete dynamics $\dot{z}_f = D_{k_d}(z_f, f(t), L)$ loses its homogeneity, if the differentiator is replaced by its Euler approximation $z_f(t_{k+1}) = z_f(t_k) + D_{r-1}(z_f(t_k), \omega(t_k), L)\tau_k$. The reason is that the error $z_f(t_k) - \vec{f}(t)$, $\vec{f} = (f, \dot{f}, \ldots, f^{(k_d)})$, combines both discrete and continuous-time variables. As a result, the accuracy of all derivatives accept $z_{f,0} - f$ is proportional to τ with $\tau_k = \tau$ and is worth with variable τ_k. Therefore, one would expect that the accuracy of the output-feedback r-SM control also deteriorates. We prove that it is not the case.

5.1 SISO case

Differentiating (2) obtain

$$\omega^{(r+1)} = \tilde{h}_e(t, x, u) + \tilde{g}(t, x)\dot{u}. \tag{31}$$

Assumption 4. *The functions $\tilde{h}_e$ and $\tilde{h}'_x b + \tilde{g}'_x bu$ are bounded in a vicinity of the $(r+1)$-sliding mode $\omega \equiv 0$,*

$$|\tilde{h}_e(t, x, u)| \leq \tilde{C}_e,\ |\tilde{h}'_x(t, x)b(t, x) + \tilde{g}'_x(t, x)b(t, x)u| \leq \tilde{C}_{1e}. \tag{32}$$

Just as in the MIMO case this assumption is natural, since u is close to $u_{eq} = -\tilde{h}/\tilde{g}$ in the vicinity of the $(r+1)$-sliding mode. Let now the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$, and the sampling error be $\nu(t)$, $|\nu(t)| \leq \varepsilon$. In other words the applied feedback control (13) at the moment $t \in [t_k, t_{k+1})$ is calculated at the moment t_k and gets the form

$$u(t) = U_r(z(t_k)),\ z(t_k) = z(t_{k-1}) + \tau_{k-1} D_{r-1}(z(t_{k-1}), \omega(t_{k-1}), L).$$

Theorem 7. *Discretization does not destroy the asymptotic closed-system accuracy* [18] *in the SISO case under the standard assumptions. Under assumption 4 the same is true for the chattering attenuation procedure with sufficiently small τ.*

Note that the deterioration of the accuracy has been avoided, since in the following proof the coordinates z_j, $\omega^{(j)}$ are treated as independent homogeneous coordinates stabilizing at zero. In the case of the discrete general differentiation the accuracy deteriorates, since the inputs do not tend to zero, and the discrete differentiation errors cannot be anymore considered as homogeneous coordinates.

Proof. Recall that in the SISO case $r \in \mathbf{N}$, $u, \omega \in \mathbf{R}$, and the closed system dynamics satisfies the finite-time stable r-sliding homogeneous differential inclusion

$$\omega^{(r)} \in [-C, C] + [K_m, K_M] U_r(z),\ \dot{z} = D_{r-1}(z, \omega, L), \tag{33}$$

where $z \in \mathbf{R}^r$. The homogeneity degree is -1 and the homogeneity weights are $\deg \omega^{(j)} = \deg z_j = r - j$, $j = 0, 1, \ldots, r-1$.

Let the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$, and the sampling error be $\nu(t)$, $|\nu(t)| \leq \varepsilon$. The discretization yields the system

$$\begin{aligned} &\omega^{(r)} \in [-C, C] + [K_m, K_M] U_r(z(t_k)),\ t \in [t_k, t_{k+1}]; \\ &z(t_{k+1}) = z(t_k) + \tau_k D_{r-1}(z(t_k), \omega(t_k) + \nu(t_k), L). \end{aligned} \tag{34}$$

Note that $t \in [t_k, t_{k+1}]$ implies $t_k \in [t - \tau, t]$. Thus, any solution $(\vec{\omega}(t), z(t_k))$, $t \in [t_k, t_{k+1}]$, of (34) can be regarded as a solution of the continuous-time differential inclusion with variable delays,

$$\begin{aligned} &\omega^{(r)} \in [-C, C] + [K_m, K_M] U_r(z(t - \tau[0, 1])), \\ &\dot{z}(t) \in D_{r-1}(z(t - \tau[0, 1]), \omega((t - \tau[0, 1]) + \varepsilon[-1, 1], L), \end{aligned} \tag{35}$$

in the sense that the component $\vec{\omega}(t)$ is the same, whereas the component $z(t)$ takes on the same values at the sampling moments t_k. That solution is obviously indefinitely extendable in time.

Note that (35) is (33) with delays and noises. According to [18] all its solutions defined over sufficiently large time interval in finite time enter some invariant compact set of the dimensions $\omega^{(j)} = O([\max(\tau, \varepsilon^{1/r})]^{r-j})$ and $z_j = O([\max(\tau, \varepsilon^{1/r})]^{r-j})$. This proves the first part of the Theorem.

Consider now the chattering attenuation procedure. The control u is added as a new coordinate. The closed-loop system takes on the form

$$\omega^{(r)} = \tilde{h}(t, x) + \tilde{g}(t, x) u,\ \dot{u} = U_{r+1}(z),\ \dot{z} = D_r(z, \omega, L), \tag{36}$$

where $z \in \mathbf{R}^{r+1}$. Using $|\tilde{h}_e| \leq \tilde{C}_e$ in a vicinity of the $(r+1)$-SM, obtain a finite-time stable $(r+1)$-sliding homogeneous inclusion

$$\omega^{(r+1)} \in [-\tilde{C}_e, \tilde{C}_e] + [K_m, K_M] U_{r+1}(z),\ \dot{z} = D_r(z, \omega, L) \tag{37}$$

with the homogeneity degree -1 and the homogeneity weights $\deg \omega^{(j)} = \deg z_j = r + 1 - j$, $j = 0, 1, \ldots, r$.

Let now the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$, and the sampling error be $\nu(t)$, $|\nu(t)| \leq \varepsilon$. Discretization replaces (36) with

$$\begin{array}{l} \omega^{(r)} = \tilde{h}(t,x) + \tilde{g}(t,x)u(t_k),\ t \in [t_k, t_{k+1}], \\ u(t_{k+1}) = u(t_k) + \tau_k U_{r+1}(z(t_k)), \\ z(t_{k+1}) = z(t_k) + \tau_k D_r(z(t_k), \omega(t_k) + \nu(t_k), L). \end{array} \tag{38}$$

Define $u(t) = u(t_k) + (t - t_k)U_{r+1}(z(t_k))$ between the sampling instants t_k and t_{k+1}, producing a Lipschitzian control. It does not affect the process, since only the values $u(t_k)$ are fed to the system. Introduce the notation $w_j = \omega^{(j)}$, $j = 0, 1, \ldots, r-1$, $w_r = \tilde{h}(t,x) + \tilde{g}(t,x)u(t)$, and let $\alpha = \sup |U_{r+1}|$. Thus,

$$\dot{w}_{r-1} = \tilde{h}(t,x) + \tilde{g}(t,x)u(t_k) = w_r - \tilde{g}(t,x)(t - t_k)U_{r+1}(z(t_k)). \tag{39}$$

Differentiating w_r obtain

$$\begin{array}{l} \dot{w}_r = \tilde{h}'_t(t,x) + \tilde{h}'_x(t,x)(a(t,x) + b(t,x)u(t_k)) + \tilde{g}'_t(t,x)u(t) + \\ \quad \tilde{g}'_x(t,x)(a(t,x) + b(t,x)u(t_k))u(t) + \tilde{g}(t,x)U_{r+1}(z(t_k)) = \\ \quad \tilde{h}_e(t,x,u(t)) + \tilde{g}(t,x)U_{r+1}(z(t_k)) \\ \quad - (\tilde{h}'_x(t,x)b(t,x) + \tilde{g}'_x(t,x)b(t,x)u(t))(t - t_k)U_{r+1}(z(t_k)). \end{array} \tag{40}$$

Let $\tilde{C}_{2e} > \tilde{C}_e + \tau\tilde{C}_{1e}\alpha$. Taking into account (39) and (40), obtain from (38) that

$$\begin{array}{l} \dot{w}_0 = w_1, \ldots, \dot{w}_{r-2} = w_{r-1}, \\ \dot{w}_{r-1} \in w_r + \tau[-K_M, K_M]\alpha, \\ \dot{w}_r \in [-\tilde{C}_{2e}, \tilde{C}_{2e}] + [K_m, K_M]U_{r+1}(z(t - \tau[0,1])), \\ \dot{z} \in D_r(z(t - \tau[0,1]), \omega(t - \tau[0,1]) + \varepsilon[-1,1], L). \end{array} \tag{41}$$

The produced differential inclusion is $(r+1)$-sliding homogeneous and finite-time stable with $\tau = \varepsilon = 0$, provided C_{2e} is sufficiently close to C_e [18]. It is also invariant with respect to the time - parameter - coordinate transformation

$$\begin{array}{l} (t, \tau, \omega, \ldots, \omega^{(r)}, z_0, z_1, \ldots, z_r) \mapsto \\ \qquad (\kappa t, \kappa\tau, \kappa^{r+1}\omega, \kappa^r\dot{\omega}, \ldots, \kappa\omega^{(r)}, \kappa^{r+1}z_0, \kappa^r z_1, \ldots, \kappa z_r). \end{array}$$

The end of the proof follows from the lemmas of [26]. □ □

5.2 MIMO case

Like in the SISO case we need an additional assumption.

Assumption 5. *The functions h_e and $h'_x b + g'_x bu$ are bounded in a vicinity of the $(r+\vec{1})$-sliding mode $\vec{\omega}_e \equiv 0$,*

$$\begin{array}{l} ||h_e(t,x,u)|| \leq C_e,\ 0 < K_{em} \leq ||g(t,x)|| \leq K_{eM}, \\ ||h'_x(t,x)b(t,x) + g'_x(t,x)b(t,x)u|| \leq C_{1e}. \end{array} \qquad (42)$$

Theorem 8. *In the MIMO case under the assumptions of Theorem 4 the discretization does not destroy the stated asymptotic closed-system accuracy. Under additional assumption 5 and with sufficiently small τ the discrete chattering-attenuation procedure does not destroy the accuracy stated in Theorem 6.*

Mark that the Theorem assumes that the proposed homogeneous SM control (18), (20) is applied. The proof is actually the same as for the previous theorem, due to the effective decoupling of the system.

6 Simulation Results

6.1 MIMO car control

Consider a simple MIMO ("bicycle") model of car control [28]

$$\begin{array}{l} \dot{x} = V\cos\varphi,\ \dot{y} = V sin\varphi,\ \dot{\varphi} = \frac{V}{\Delta}\tan\theta, \\ \dot{V} = \mu_1 T_{net}(V,\rho) - \mu_2 V^2 - \mu_3 R_x,\ \mu_3 R_x = \varepsilon(1-\cos(5\theta)), \\ \mu_1 T_{net}(V,\rho) = (2.5\sin\rho - 0.7)(1 - 0.001(V-9)^2), \\ \dot{\theta} = u_1, \dot{\rho} = u_2, \end{array}$$

where x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, V is the longitudinal velocity, Δ is the length between the two axles and θ is the steering angle (i.e. the first real input) (Fig. 1), $T_{net}(V,\rho)$ is the net combustion torque of the engine, ρ is the throttle angle (i.e. the second real input), $\rho \in [0, \pi/2]$, R_x is the rolling resistance of the tires. Parameters $\mu_2 = 0.005$, $\Delta = 5$m were taken. For simplicity brakes are not applied. Usually T_{net} is available as a table function of the engine angle velocity and ρ. It is presented here by some regression roughly approximating the data from [28], Fig. 9-6. The rolling resistance is voluntarily represented here by a function, corresponding to some mechanical car damage, $\varepsilon = 0.1$.

The task is to steer the car from a given initial position and speed to the trajectory $y = y_c(x)$, and $V = V_c(t)$, where $g(x)$, y and $V_c(t)$ are assumed to be available in real time.

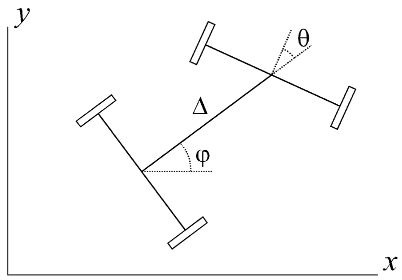

Figure 1: The car model.

Define $\sigma_1 = y - y_c(x)$, $\sigma_2 = V - V_c(t)$. The initial values are $V = 5m/s$, $x = y = \varphi = \rho = \theta = 0$ at $t = 0$, $y_c(x) = 10\sin(0.05x) + 5$, $V_c(t) = 9 + \sin(0.5t)$.

The choice $H =$ const is natural here. The relative degree of the undisturbed system is (3,2) and the quasi-continuous 3,2-sliding controllers solve the problem. The controller magnitude α, the parameters η_1, and η_2 are conveniently found by simulation. The differentiator parameters are taken deliberately large, in order to provide for better performance in the presence of measurement errors. It was taken $\eta_1 = \eta_2 = 1$, $\alpha = 20$, the differentiator parameters are $L = 80$, $\tilde\lambda_0 = 1.1$, $\tilde\lambda_1 = 1.5$, $\tilde\lambda_2 = 2$. The control is applied starting from $t = 1$ in order to provide some time for the differentiators' convergence.

The resulting output-feedback controller is

$$v_1 = -20\frac{s_2 + 2(|s_1| + |s_0|^{2/3})^{-1/2}(s_1 + |s_0|^{2/3}\,\mathrm{sign}\ s_0)}{|s_2| + 2(|s_1| + |s_0|^{2/3})^{1/2}},\ \dot s = D_2(s, \sigma_1, 80),$$
$$v_2 = -20(w_1 + |w_0|^{1/2}\,\mathrm{sign}\ w_0)]/(|w_1| + |w_0|^{1/2}), \qquad \dot w = D_1(w, \sigma_2, 80).$$

In order to define G calculate the matrix

$$g = \begin{bmatrix} \dddot\sigma'_{1u_1} & \dddot\sigma'_{1u_2} \\ \ddot\sigma'_{2u_1} & \ddot\sigma'_{2u_2} \end{bmatrix},$$
$$\dddot\sigma'_{1,u_1} = \frac{V^2}{l}\frac{\cos\varphi + y'_c\sin\varphi}{\cos^2\theta} - 5\varepsilon V\sin(5\theta)(\sin\varphi - y'_c\cos\varphi),$$
$$\dddot\sigma'_{1,u_2} = 2.5\cos\rho(1 - 0.001(V-9)^2)(\sin\varphi - y'_c\cos\varphi),$$
$$\ddot\sigma'_{2,u_1} = -5\varepsilon V\sin(5\theta), \ddot\sigma'_{2,u_2} = 2.5\cos\rho\,(1 - 0.001(V-9)^2);$$

and, taking into account that the mechanical damage should be considered unknown, define $G = \begin{bmatrix} \frac{V^2}{l}\frac{\cos\varphi}{\cos^2\theta} & 0 \\ 0 & 2.5\cos\rho \end{bmatrix}$, $K = 1$.

The nominal matrix G is calculated each 5 seconds starting from $t = 1$. For this sake $V^2\cos\varphi = V\dot x$ is calculated using two more differentiators of the second order producing estimations of $\dot{x}$ and $\dot{y}$, the stirring angle (the first input θ) is considered available. Differentiators' parameter $L = 80$ is taken.

The matrix G turns out to be singular at $\rho = \pi/2$, which corresponds to the fully open gas. Therefore ρ is artificially saturated at $\rho = \pi/3$.

The integration was carried out by the Euler method, the sampling step being equal to the integration step $\tau = 10^{-4}$. The tracking accuracies $|\sigma_1| \leq 1.01 \cdot 10^{-8}$, $|\dot{\sigma}_1| \leq 1.12 \cdot 10^{-4}$, $|\ddot{\sigma}_1| \leq 0.0205$, $|\sigma_2| \leq 3.13 \cdot 10^{-6}$, $|\dot{\sigma}_2| \leq 0.0091$ were obtained. After the sampling step has changed to $\tau = 10^{-3}$ the accuracies $|\sigma_1| \leq 0.946 \cdot 10^{-5}$, $|\dot{\sigma}_1| \leq 1.6 \cdot 10^{-3}$, $|\ddot{\sigma}_1| \leq 0.207$, $|\sigma_2| \leq 2.66 \cdot 10^{-4}$, $|\dot{\sigma}_2| \leq 0.0845$ were obtained, which corresponds to the asymptotics stated in Theorem 8.

The car trajectory, 3-sliding tracking errors σ_1, $\dot{\sigma}_1$, $\ddot{\sigma}_1$, 2-sliding tracking errors σ_2, $\dot{\sigma}_2$, and the velocity tracking graph are shown in Fig. 2. Controls u_1, u_2, throttle angle ρ and steering angle θ, as well as nonzero elements of the nominal matrix G are presented in Fig. 3. It is seen that $u_2 \equiv 0$ when ρ comes to the saturation level $\pi/3$.

For comparison calculate matrices G and g.

At $t = 7$ get $g = \begin{bmatrix} 14.45 & -0.06 \\ 1.51 & 2.23 \end{bmatrix}$, $G = \begin{bmatrix} 13.68 & 0 \\ 0 & 1.46 \end{bmatrix}$.

At t = 13 get $g = \begin{bmatrix} 17.63 & 1.8 \cdot 10^{-5} \\ -2.24 & 1.81 \end{bmatrix}$, $G = \begin{bmatrix} 12.4 & 0 \\ 0 & 2.07 \end{bmatrix}$.

6.2 Discretization accuracy asymptotics

Consider a simple second-order system

$$\begin{array}{l} \dot{x}_0 = \sin(t) + x_1, \\ \dot{x}_1 = \cos(x_0) + u(t_k), \quad \sigma = x_0, \\ u(t_{k+1}) = u(t_k) - \tau_k \alpha \Psi_{2,3}(z_0(t_k), z_1(t_k), z_2(t_k)), \\ z(t_{k+1}) = z(t_k) + \tau_k D_2(z(t_k), \sigma(t_k), L), \end{array} \tag{43}$$

where the 3-sliding embedded controller is taken

$$\Psi_{2,3}(z) = \operatorname{sign}\left(z_2 + 2(|z_1|^3 + |z_0|^2)^{1/6} \operatorname{sign}(z_1 + (|z_0|^{2/3} \operatorname{sign} z_0))\right).$$

The continuous-time part of system (43) was integrated by the Euler method with the integration step 10^{-4} and initial values $x_0(0) = 10$, $x_1(0) = 5$. The discrete-time subsystem in (43) has the parameters $\tilde{\lambda}_0 = 1.1$, $\tilde{\lambda}_1 = 1.5$, $\tilde{\lambda}_2 = 2$, $L = 16$, $\alpha = 8$ and $z_0(0) = z_1(0) = z_2(0) = 0$.

Take random positive sampling steps τ_k, uniformly distributed in the segment $[10^{-4}, \tau]$. The stabilization of σ, $\dot{\sigma}$, $\ddot{\sigma}$ with $0 < \tau_k \leq \tau = 0.01$ is

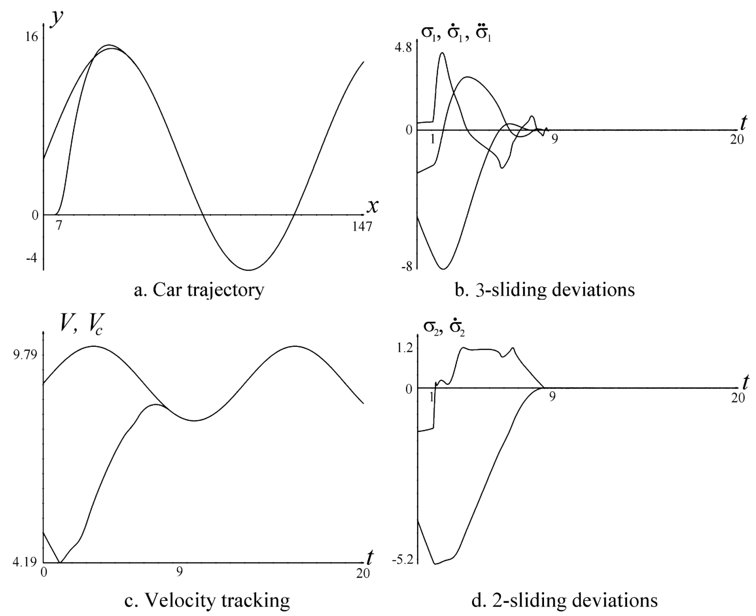

Figure 2: Quasi-continuous (3,2)-sliding car control.

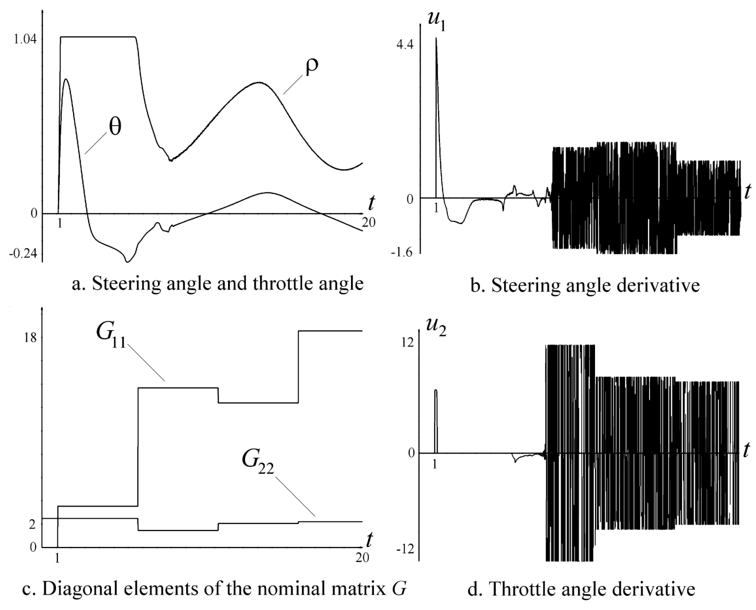

Figure 3: Car controls and the nominal control matrix elements.

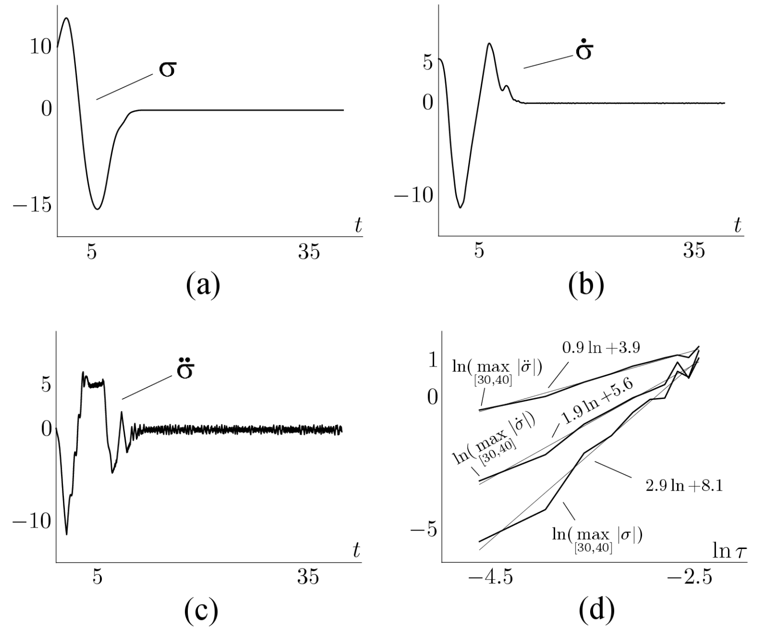

Figure 4: (a), (b), (c) Stabilization of $\sigma, \dot{\sigma}, \ddot{\sigma}$ with $\tau = 0.01$; (d) asymptotics of $\sigma, \dot{\sigma}, \ddot{\sigma}$ for the maximal sampling steps $\tau = 0.01, 0.02, ..., 0.1$.

respectively demonstrated in Figs. 4a, b, and c. Now let the maximal sampling step τ take values $0.01, 0.02, ..., 0.1$. Logarithmic plots of $\max_{[30,40]} |\sigma^{(i)}|$, $i = 0, 1, 2$, together with the corresponding best-fitting lines $2.9 \ln \tau + 8.1$, $1.9 \ln \tau + 5.6$ and $0.9 \ln \tau + 3.9$ are shown in Fig. 4d. According to Theorem 7, the worst-case accuracy orders correspond to the slope values 3, 2, 1, respectively. Thus, the simulation results are in good compliance with the theory.

7 Conclusions

A simple robust MIMO homogeneous SM control is proposed, preserving the main properties of SISO homogeneous SM control including its high asymptotic accuracy. The required conditions are a direct generalization of the SISO case.

It is proved that the well-known standard asymptotic accuracy of the output-feedback homogeneous SM control is preserved, when the observer is realized as a discrete-time system based on the Euler integration. Moreover, the accuracy is also preserved, if, as a part of the artificial relative degree increase, the discrete dynamics includes a discrete Euler integrator.

These results are also true in the MIMO case, provided the proposed MIMO homogeneous SM control is applied.

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