Robust exact filtering differentiators

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Abstract

Filtering differentiators are capable both of rejecting large noises and exactly differentiating smooth signals. Even unbounded noises are rejected provided their local average is small. A special type constitute tracking filtering differentiators producing smooth outputs being derivatives one of another. Discrete filter versions are proposed, and the accuracy asymptotics are studied in the presence of noises and discrete sampling. Extensive computer simulation confirms the theoretical results.
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1. Introduction

Filtering noisy signals is an old problem, and differentiation is its special type. While the first only extracts the basic signal, the second also provides its derivatives. The problem is ill posed, since one cannot distinguish between the noise and the unknown signal to be extracted and processed.

The classic approach classifies the signals by their Fourier expansions. The high-frequency components are considered noises, and the approximate differentiation is only achieved for signals of a bounded frequency range. The corre-

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sponding filters are widely and successfully applied in signal processing, observation and output-feedback control [1, 2, 3]. Generaly speaking, their efficiency in smooth output-feedback control is due to the reduction of the asymptotic stability to the stability of the system linearization. In particular, the control creates an artificial system equilibrium, and becomes constant or (after the feedback linearization [4]) vanishes at it.

Sliding-mode (SM) control (SMC) is used to control systems which cannot be stabilized by smooth control due to significant uncertainties [5, 6, 7]. Correspondingly the observation requirements are higher, since closed-loop systems cannot be linearized at the stabilization point. The differentiation is considered a classic problem of SMC [8, 9]. SMC inevitably produces small high frequency vibrations [10], [11] often called chattering. The corresponding output vibrations contain important information on the system state and cannot be ignored in observation.

A system output f_0 is naturally classified by the lowest order of its total time derivative containing control, i.e. by its relative degree r [4]. High-order SMs (HOSMs) were intentionally developed for the regulation of outputs with high relative degrees r [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Exact finite-time (FT) stabilization of the output f_0 is performed by the so-called r-th order SMC (r-SMC) and requires the knowledge of the output derivatives up to the order $r-1$. Thus, the exact $(r-1)$th-order differentiation is essential for the HOSM control (HOSMC) theory [24, 25, 26, 27, 28, 29, 18, 30, 31]. In the r-SMC framework one can assume that $f_0^{(r)}$ is a bounded high-frequency switching signal. Reformulate that problem in a general form for $r = n + 1$.

The sampled signal $f(t)$ inevitably contains a noise $\eta(t)$, i.e. $f(t) = f_0(t) + \eta(t)$. Correspondingly, the problem is to evaluate $f_0(t), \dot{f}_0(t), ..., f_0^{(n)}(t)$ robustly and in real time, under the condition that $|f_0^{(n+1)}(t)| \leq L$. The available numeric information consists of $L > 0$ and $n \in \mathbb{N}$. Derivatives are to be exact for $\eta = 0$. The problem has been solved in [32, 18].

Thus, "standard" HOSM-based differentiators [18] produce the outputs z_i,

$i = 0, ..., n$, which in FT converge to exact derivatives $f_0^{(i)}$. They exactly differentiate small high-frequency output components of f_0, but still are very robust to noises (see Section 5). Moreover, the accuracy asymptotics in the presence of bounded noises, $|\eta| \leq \varepsilon$, have been shown to be optimal with respect to ε [32, 33] (see Section 2.2).

Nevertheless, one would like to provide a feature similar to linear filters' ability to reject large high-frequency signal components. That feature together with the linearity of a filter (or the ideal differentiation) immediately exclude the filter robustness to small noises, but it becomes possible for *nonlinear* FT-exact filters. The aim is achieved by the recently proposed filtering differentiators capable of filtering out unbounded noises with small average values, while preserving *all* advantageous features of the "standard" SM-based differentiators.

The simplest form of the filtering differentiator has been proposed and applied in [34] to the extraction of the equivalent control from the SMC. The general form of the filtering differentiator has been proposed at the conference [35]. Paper [35] contains no proofs, is concentrated on the SM observation lemma and feedback applications, and unfortunately has a few inaccuracies.

Differentiators [18] produce estimations $z_i \approx f_0^{(i)}$ which themselves are differentiable only once. In particular, in the presence of even small noises $\dot{z}_i \neq z_{i+1}$, and one cannot evaluate $\frac{d}{dt}\dot{f}_0^2$ as $2z_1z_2$. The issue is resolved by the tracking differentiators introduced in [36, 37] for the practical-relative-degree identification [38, 39]. The tracking differeniators are unified to a standard form and extended to the filtering tracking differentiators in this paper.

A paper on filtering differentiators submitted in parallel to CDC' 2019 contains significantly less theoretical results and has no proofs. The simulation part of the present paper is completely different from the CDC paper, whereas its theoretic results are much more comprehensive.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$, $a \diamond B = \{a\} \diamond B$. A function of a set is the set of function values on this set. The norm $||x||$ stands for the standard Euclidian norm of x, $||x||_\infty =$

$\max|x_i|$, $||x||_{h\infty} = \max_{i=1,...,n} |x_i|^{1/(n+1-i)}$ for $x \in \mathbb{R}^n$. $B_\varepsilon = \{x \mid ||x|| \le \varepsilon\}$, $\mathbb{R}_+ = [0, \infty)$; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$.

2. Preliminaries: homogeneous differentiation

Recall that solutions of the differential inclusion (DI)

$$\dot{x} \in \Phi(x),\ \Phi(x) \subset T_x\mathbb{R}^{n_x}, \tag{1}$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all t. Here $T_x\mathbb{R}^{n_x}$ denotes the tangent space to $\mathbb{R}^{n_x}$ at $x \in \mathbb{R}^{n_x}$.

We call the DI (1) *Filippov DI*, if the vector-set field $\Phi(x) \subset T_x\mathbb{R}^{n_x}$ is non-empty, compact and convex for any x, and Φ is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $\Phi(x)$ from the set $\Phi(y)$ tends to zero, as $x \to y$.

Filippov DIs feature existence, extendability etc. of solutions, but not their uniqueness [40]. The Filippov definition [40] replaces a differential equation (DE) $\dot{x} = \varphi(x)$ with the Filippov DI (1), $\Phi = K_F[\varphi]$, where

$$K_F[\varphi](x) = \bigcap_{\mu_L N = 0} \bigcap_{\delta > 0} \overline{co}\, \varphi((x + B_\delta)\backslash N). \tag{2}$$

Here $\overline{co}$, μ_L stand for the convex closure and the Lebesgue measure respectively, (2) defines the celebrated Filippov procedure.

2.1. Weighted homogeneity basics

Introduce the weights $m_1, ..., m_{n_x} > 0$ of the coordinates $x_1, ..., x_{n_x}$ in $\mathbb{R}^{n_x}$, $\deg x_i = m_i$, and the dilation [41]

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}),$$

where $\kappa \ge 0$. Recall [41] that a function $g : \mathbb{R}^{n_x} \to \mathbb{R}^m$ is said to have the homogeneity degree (HD) (weight) $q \in \mathbb{R}$, $\deg g = q$, if the identity $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^{n_x}\backslash\{0\}$ and $\kappa > 0$.

Consider the combined time-coordinate transformation

$$(t, x) \mapsto (\kappa^{-q} t, d_\kappa x), \quad \kappa > 0, \tag{3}$$

where the number $-q \in \mathbb{R}$ might be interpreted as the weight of t. The DI $\dot{x} \in \Phi(x)$ and the vector-set field $\Phi(x)$ are called homogeneous of the HD q, if the DI is invariant with respect to (3). The following is the corresponding formal definition.

Definition 1. [19] A vector-set field $\Phi(x) \subset T_x\mathbb{R}^{n_x}$ (DI $\dot{x} \in \Phi(x)$), $x \in \mathbb{R}^{n_x}$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $\Phi(x) = \kappa^{-q} d_\kappa^{-1} \Phi(d_\kappa x)$ holds for any $x \neq 0$ and $\kappa > 0$.

A system of DEs $\dot{x}_i = \varphi_i(x)$, $i = 1, ..., n_x$, is a particular case of DI, when the set $\Phi(x)$ contains only one vector $\varphi(x)$. Then Definition 1 is reduced to $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg \varphi_i$ [41]. Note that if φ is discontinuous, the DE is equivalent to the corresponding homogeneous Filippov DI (1).

Note that the weights $-q, m_1, ..., m_{n_x}$ are defined up to proportionality. The sign of the HD determines many properties of DIs.

Any continuous positive-definite function of the HD 1 is called a *homogeneous norm*. In particular $||x||_h = \max_i |x_i|^{1/m_i}$ is such a norm.

It is proved in [19] that if the HD q of the DI (1) is negative then it is asymptotically stable iff it is FT stable. Moreover, there exist such $\mu_1, ..., \mu_{n_x} > 0$ that in the presence of a maximal delay $\tau \geq 0$ and noises of the magnitudes $\varepsilon_i \geq 0$, $i = 1, 2, ..., n_x$, each extendable-in-time solution of the disturbed DI

$$\dot{x} \in \Phi\left(x(t - \tau[0, 1]) + [-\varepsilon_1, \varepsilon_1] \times ... \times [-\varepsilon_{n_x}, \varepsilon_{n_x}]\right)$$

starting from some time satisfies the inequalities

$$|x_i| \leq \mu_i \rho^{m_i},\ \rho = \max[||\varepsilon||_h, \tau^{-q}] = \max[\varepsilon_1^{1/m_1}, ..., \varepsilon_n^{1/m_{n_x}}, \tau^{-q}]. \tag{4}$$

2.2. SM-based homogeneous differentiation

Let the set of all functions $\mathbb{R}_+ \to \mathbb{R}$, whose nth derivative has the Lipschitz constant $L > 0$, be denoted by $\mathrm{Lip}_n(L)$. Following [18] the differentiators are required to be exact on $\mathrm{Lip}_n(L)$ after a FT transient.

Assumption 1. *The input signal $f(t)$, $t \geq 0$, is assumed to have the form $f(t) = f_0(t) + \eta(t)$, consisting of an unknown basic signal $f_0 \in \mathrm{Lip}_n(L)$ and a Lebesgue-measurable noise $\eta(t)$.*

Assumption 2. *The noise $\eta(t)$ is bounded, $|\eta| \leq \varepsilon_0$. Whereas L is assumed known, $\varepsilon_0 \geq 0$ remains unknown.*

Differentiation problem. [[**32, 18**]] The problem is to evaluate the derivatives $f_0^{(i)}(t)$, $i = 0, 1, ..., n$, in real time, robustly with respect to small noises $\eta(t)$, and exactly in their absence.

Theorem 1 ([33])**.** *For any $t_0 > 0$ there exists such $\varepsilon_* > 0$ that for any ε_0, $0 < \varepsilon_0 \leq \varepsilon_*$, and any $f_0, f_1 \in \mathrm{Lip}_n(L)$ the inequality $\sup_{t\geq 0} |f_1(t) - f_0(t)| \leq \varepsilon_0$ implies the inequalities*

$$\sup_{t\geq t_0} |f_1^{(i)}(t) - f_0^{(i)}(t)| \leq K_{i,n}(2L)^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}, \; i = 0, 1, ..., n. \tag{5}$$

Here $K_{i,n}$, $K_{i,n} \in [1, \pi/2]$, are the Kolmogorov constants [42]. For each ε_0, i these inequalities become equalities for certain functions f_0, f_1.

Let $z_0(t), z_1(t), ..., z_n(t)$ be the outputs of some differentiator estimating $f_0(t), \dot{f}_0(t), \ldots, f_0^{(n)}(t)$ for the input $f_0 \in \mathrm{Lip}_n(L)$, and let $z_i(t) \equiv f_0^{(i)}(t)$ for $t \geq t_0$. Then, defining $f = f_1 = f_0 + \eta$, $\eta = f_1 - f_0$, under Assumptions 1, 2 obtain from Theorem 1 that the best worst-case accuracy to be kept from some moment $t_* > t_0$ is $\sup_{f_0,\eta} \sup_{t\geq t_*} |z_i - f_0^{(i)}| \geq K_{i,n}(2L)^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}$. In particular, since $K_{1,1} = \sqrt{2}$, get $\sup_{f_0,\eta} \sup_{t\geq t_*} |z_1 - \dot{f}_0| \geq 2\sqrt{L\varepsilon_0}$ for the first-order differentiation.

A differentiator is called **asymptotically optimal**, if for some constants $\mu_i > 0$ under Assumptions 1, 2 it in FT provides the accuracy $|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L^{\frac{i}{n+1}} \varepsilon_0^{\frac{n+1-i}{n+1}}$, $i = 0, 1, ..., n$, for all inputs, noises and $\varepsilon_0 \geq 0$. Obviously, $\mu_i \geq K_{i,n} 2^{\frac{i}{n+1}} > 2^{\frac{i}{n+1}}$.

The following is the asymptotically-optimal differentiator [18] in its so-called non-recursive form:

$$
\begin{array}{rcl}
\dot{z}_0 & = & -\tilde{\lambda}_n L^{\frac{1}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n}{n+1}} + z_1, \\
\dot{z}_1 & = & -\tilde{\lambda}_{n-1} L^{\frac{2}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{n-1}{n+1}} + z_2, \\
 & & \ldots \\
\dot{z}_{n-1} & = & -\tilde{\lambda}_1 L^{\frac{n}{n+1}} \lfloor z_0 - f(t) \rceil^{\frac{1}{n+1}} + z_n, \\
\dot{z}_n & = & -\tilde{\lambda}_0 L \operatorname{sign}(z_0 - f(t)).
\end{array}
\tag{6}
$$

Here and further all differential equations are understood in the Filippov sense (2) [40]. Differentiator (6) is further called **standard**.

Differentiator (6) is called homogeneous due to its homogeneous error dynamics. Let $\sigma_i = (z_i - f^{(i)})/L$. Now subtracting $f^{(i+1)}$ from the both sides of the equation for z_i, dividing by L and taking into account $f^{(n+1)}/L \in [-1, 1]$ in the last equation, obtain the FT stable differential inclusion (DI)

$$
\begin{array}{rcl}
\dot{\sigma}_0 & = & -\tilde{\lambda}_n \lfloor \sigma_0 \rceil^{\frac{n}{n+1}} + \sigma_1, \\
\dot{\sigma}_1 & = & -\tilde{\lambda}_{n-1} \lfloor \sigma_0 \rceil^{\frac{n-1}{n+1}} + \sigma_2, \\
 & & \ldots \\
\dot{\sigma}_{n-1} & = & -\tilde{\lambda}_1 \lfloor \sigma_0 \rceil^{\frac{1}{n+1}} + \sigma_n, \\
\dot{\sigma}_n & \in & -\tilde{\lambda}_0 \operatorname{sign}(\sigma_0) + [-1, 1].
\end{array}
\tag{7}
$$

Here and further in all DIs the function $\operatorname{sign}(\cdot)$ is replaced with $K_F[\operatorname{sign}(\cdot)](\cdot)$ (see (2)). The homogeneity of (7) is due to its invariance with respect to the transformation $\sigma_i \mapsto \kappa^{n+1-i}\sigma_i$, $t \mapsto \kappa t$ for $\kappa > 0$ (Section 2.1). Obviously the HD is -1.

Parameters $\tilde{\lambda}_i$ are most easily calculated using the parameters $\lambda_0, ..., \lambda_n$ of the differentiator recursive form [18]: $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_n = \lambda_n$, and $\tilde{\lambda}_j = \lambda_j \tilde{\lambda}_{j+1}^{j/(j+1)}$, $j = n-1, n-2, \ldots, 1$. An infinite sequence of parameters $\vec{\lambda} = \{\lambda_0, \lambda_1, ...\}$ can be built [18], providing coefficients $\tilde{\lambda}_i$ of (6) for all natural n. In particular, $\vec{\lambda} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$ suffice for $n \leq 7$ [43, 33]. The corresponding parameters $\tilde{\lambda}_i$ are listed in Table 1.

The differentiator accuracy follows from the homogeneity of dynamics (7). Due to (4) in the presence of discrete measurements with the maximal sampling

Table 1: Parameters $\tilde{\lambda}_0, \tilde{\lambda}_1, ..., \tilde{\lambda}_n$ of differentiator (6) for $n = 0, 1, ..., 7$

0	1.1							
1	1.1	1.5						
2	1.1	2.12	2					
3	1.1	3.06	4.16	3				
4	1.1	4.57	9.30	10.03	5			
5	1.1	6.75	20.26	32.24	23.72	7		
6	1.1	9.91	43.65	101.96	110.08	47.69	10	
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12

time interval $\tau > 0$ differentiator (6) in FT provides the accuracy

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \le \mu_i L \rho^{n+1-i}, \ i = 0, 1, ..., n, \\ &\rho = \max[(\varepsilon_0/L)^{1/(n+1)}, \tau] \end{aligned} \tag{8}$$

for some $\mu_i > 0$ [18]. Here the case $\tau = 0$ formally corresponds to continuous sampling. The same accuracy asymptotics (with different constants μ_i) is maintained by properly discretized differentiator [44, 43].

3. Filtering differentiators

Filtering differentiators are developed to reject some large noises, while preserving *all properties* of the standard differentiators (6). The rejectable noises are required to be small in average. Two main examples are the noises of the form $\gamma \cos(\omega t)$, $\omega >> 1$, and a random discretely-sampled signal of the zero mean value (mostly in Section 4). The latter example is not considered mathematically in this paper, but provides a lot of practical examples (Section 5).

3.1. "Standard" filtering differentiators

Introduce the number $n_f \ge 0$ which is further called *the filtering order*. The following differentiator [35] extends the standard differentiator (6) and is further called **the standard filtering differentiator**:

$$
\begin{aligned}
\dot{w}_1 &= -\tilde{\lambda}_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+n_f}{n+n_f+1}} + w_2, \\
&\quad \ldots \\
\dot{w}_{n_f-1} &= -\tilde{\lambda}_{n+2} L^{\frac{n_f-1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+2}{n+n_f+1}} + w_{n_f}, \\
\dot{w}_{n_f} &= -\tilde{\lambda}_{n+1} L^{\frac{n_f}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n+1}{n+n_f+1}} + z_0 - f(t), \\
\dot{z}_0 &= -\tilde{\lambda}_n L^{\frac{n_f+1}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{n}{n+n_f+1}} + z_1, \\
&\quad \ldots \\
\dot{z}_{n-1} &= -\tilde{\lambda}_1 L^{\frac{n+n_f}{n+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{n+n_f+1}} + z_n, \\
\dot{z}_n &= -\tilde{\lambda}_0 L \ \mathrm{sign}(w_1).
\end{aligned}
\tag{9}
$$

In order to keep the standard differentiator (6) as a particular case, it is assumed that for $n_f = 0$ the first n_f equations disappear, and $w_1 = z_0 - f(t)$ is formally substituted for w_1. In the case $n = 0$ only the equation for z_0 remains in the lower part.

For example, the filter of the filtering order $n_f = 2$ and the differentiation order $n = 0$ gets the form

$$
\begin{aligned}
\dot{w}_1 &= -2\, L^{\frac{1}{3}} \lfloor w_1 \rceil^{\frac{2}{3}} + w_2, \\
\dot{w}_2 &= -2.12 L^{\frac{2}{3}} \lfloor w_1 \rceil^{\frac{1}{3}} + z_0 - f(t), \\
\dot{z}_0 &= -1.1 L \,\mathrm{sign}\, w_1.
\end{aligned}
\tag{10}
$$

where the parameters $\tilde{\lambda}_0 = 1.1, \tilde{\lambda}_1 = 2.12, \tilde{\lambda}_2 = 2$ are taken from the row $n + n_f = 2$ of table 1. Its output z_0 estimates the component f_0 of the noisy signal f, $|\dot{f}_0| \le L$.

Introduce a short notation for (9):

$$
\begin{aligned}
\dot{w} &= \Omega_{n,n_f}(w, z_0 - f, L, \vec{\lambda}), && \text{for } n_f > 0, \\
w_1 &= z_0 - f && \text{for } n_f = 0, \\
\dot{z} &= D_{n,n_f}(w_1, z, L, \vec{\lambda});
\end{aligned}
\tag{11}
$$

Any valid parametric sequence $\vec{\lambda}$ introduced in Section 2.2 can be used. The particular sequence listed there generates table 1. In the sequel for brevity we omit the case $n_f = 0$ in (11).

In the following one can consider the signal $\cos(\omega t)$, $\omega >> 1$, as a simplest-form noise to be rejected by the filter. Note that $|\int_0^t \cos(\omega t)dt| = |\sin(\omega t)|/\omega <<$ 1. The next definition generalizes these properties.

Definition 2. A function $\nu(t)$, $\nu : [0,\infty) \to \mathbb{R}$, is called globally filterable, or a *signal of the (global) filtering order* $k \geq 0$, if ν is a locally integrable Lebesgue-measurable function, and there exists a uniformly bounded locally absolutely-continuous solution $\xi(t)$, $\xi : [0,\infty) \to \mathbb{R}$, of the equation $\xi^{(k)} = \nu$. Any number exceeding $\sup|\xi(t)|$ is called a *kth-order (global) integral magnitude of* ν.

Assumption 3. *Extend Assumption 2 assuming that the input noise signal is comprised of $n_f + 1$ components, $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, where each η_k, $k = 0, ..., n_f$, is a signal of the global filtering order k and the kth-order integral magnitude $\varepsilon_k \geq 0$. Components $\eta_1, ..., \eta_{n_f}$ are possibly unbounded.*

The noise of Assumption 2 is of the 0th filtering order and the 0th order integral magnitude ε_0. The standard differentiator (6) corresponds to $n_f = 0$ and is known to be robust with respect to the noises $\eta = \eta_0$ of the filtering order 0 [18].

The following theorem shows that differentiator (9) of the filtering order n_f is FT exact and robust with respect to *possibly unbounded* noises of the filtering orders not exceeding n_f. Moreover, its asymptotic optimality feature is preserved and corresponds to the case $\varepsilon_1 = ... = \varepsilon_{n_f} = 0$.

Theorem 2. *Under Assumptions 1 and 3 differentiator* (9) *in FT provides for the accuracy*

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L \rho^{n+1-i},\ i = 0, 1, ..., n, \\ &|w_1(t)| \leq \mu_{w1} L \rho^{n+n_f+1}, \end{aligned} \tag{12}$$

$$\rho = \max[(\tfrac{\varepsilon_0}{L})^{\frac{1}{n+1}}, (\tfrac{\varepsilon_1}{L})^{\frac{1}{n+2}}, ..., (\tfrac{\varepsilon_{n_f}}{L})^{\frac{1}{n+n_f+1}}], \tag{13}$$

where $\{\mu_i\}$, $\{\mu_{w1}\}$ only depend on the choice of $\{\lambda_l\}$, $l = 0, ..., n + n_f$.

Estimations of w_k, $k > 1$, depend on the noises. In particular, similar inequalities $|w_k| \leq \mu_{wk} L \rho^{n+n_f+2-k}$ are maintained for $\eta_2 = ... = \eta_{n_f} = 0$. Here and further the proofs appear in the Appendix.

The standard differentiator (6) is included here for $n_f = 0$. Note that increasing the differentiator filtering order n_f while keeping the same noise filtering orders does not change the differentiation accuracy asymptotics provided by the theorem.

Example 1. The noise $\eta = \gamma\cos(\omega_* t)$ features any global filtering order $k \ge 0$ with the integral magnitude γ for $k = 0$ and $2\gamma/{\omega_*}^k$ for $k > 0$. It means that it has at least $n_f + 1$ trivial expansions of the form $\eta = \eta_k$ for any $k = 0, 1, ..., n_f$ and other components equal 0. It follows from Theorem 2 that the estimation (12) holds for each corresponding value of $\rho = \rho_k$, $k \le n_f$, i.e. for

$$\rho = \min\{(\tfrac{\gamma}{L})^{\frac{1}{n+1}}, \min_{k=1,2,...,n_f}[(2\tfrac{\gamma}{L})^{\frac{1}{n+k+1}}{\omega_*}^{-\frac{k}{n+k+1}}]\}.$$

Obviously, ρ decreases for growing n_f approaching $\lim\rho = 1/\omega_*$, provided $\omega_*^{n+1} > L/\gamma$, $\omega_* > 1$. Also see the simulation in Section 5.

Other simple examples of the signals of global filtering orders are arbitrary-order derivatives of periodic functions (see Section 5.1). $\square$

Signals of the same filtering order obviously constitute linear spaces. Moreover, let $\nu_1, \nu_2, ...$ be the sequence of the signals $\mathbb{R}_+ \to \mathbb{R}$ of the same global filtering order k, with the bounded solutions $\xi_1, \xi_2, ...$ and the integral magnitudes $b_1, b_2, ...$ Then, provided the series $\sum_j \xi_j^{(l)}$ uniformly converge over any bounded interval for $l = 0, 1, ..., k$ and $\sum_j b_j$ converges, the series $\sum_j \eta_j$ is also a signal of the kth filtering order of the integral magnitude $\sum_j b_j$.

The proof of the above claim is trivial. In particular, Example 1 establishes application of Theorem 2 for filtering out smooth periodic high-frequency noises.

A noise can have uniformly small integrals over intervals of bounded length, but still have no global filtering order. Further we show that also such noises can be filtered out. The following definition formalizes that property.

Definition 3. A locally integrable Lebesgue-measurable function $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called *locally filterable* if there exists an integer $k > 0$ (the local filtering order) and some numbers $T > 0$, $a_0, a_1, ..., a_{k-1} \ge 0$, such that for any $t_1 \ge 0$ there exists a solution $\xi(t)$, $t \in [t_1, t_1 + T]$, of the equation

$\xi^{(k)}(t) = \nu(t)$ which satisfies $|\xi^{(l)}(t)| \le a_l$ for $l = 0, 1, ..., k-1$. Numbers a_l are called the *local* $(k-l)$*th-order integral magnitudes of* ν. Signals of local filtering order 0 are trivially defined as bounded signals of the magnitude a_0.

Note that *each signal of a positive local filtering order* $k > 0$ *trivially possesses any other local positive filtering order.*

Obviously, also locally filterable signals constitute a linear space. Contrary to the signals of global filtering orders locally filterable signals are relatively easily recognized.

Example 2. The function $\cos(\omega_* t \cdot \ln\ln(t+2))$, $\omega_* > 0$, is locally filterable. Indeed, fix any $T > 0$. Let $t_1 \le t_2 \le t_1 + T$, then due to the Lagrange theorem

$$\ln\ln(t+2) = \ln\ln(t_1+2) + \tfrac{1}{(c+2)\ln(c+2)}(t-t_1)$$

for any $t \in [t_1, t_1+T]$ and some $c \in [t_1, t]$. Thus, the calculations

$$\begin{aligned}
&\cos(\omega_* t\ln\ln(t+2)) = \cos(\omega_* t\ln\ln(t_1+2)) \\
&-2\sin(\tfrac{1}{2}\omega_* t[\ln\ln(t+2) + \ln\ln(t_1+2)])\sin(\tfrac{1}{2}\omega_* t[\ln\ln(t+2) - \ln\ln(t_1+2)]), \\
&|\omega_* t[\ln\ln(t+2) - \ln\ln(t_1+2)| \le \tfrac{\omega_*(t-t_1)}{\ln(t_1+2)}\tfrac{t}{t_1+2}
\end{aligned}$$

prove the first local filtering order of the signal. Indeed,

$$\begin{aligned}
&|\textstyle\int_{t_1}^{t_2}\cos(\omega_* t\ln\ln(t+2))dt| = \left|\int_{t_1}^{t_2}\cos(\omega_* t\ln\ln(t_1+2))dt\right| + R(t_1,t_2,T); \\
&|\textstyle\int_{t_1}^{t_2}\cos(\omega_* t\ln\ln(t+2))dt| \le \frac{2}{\omega_*\ln\ln(t_1+2)} + R(t_1,t_2,T),
\end{aligned}$$

where the function $R(t_1,t_2,T)$ is uniformly bounded for any fixed $T > 0$, $0 \le t_2 - t_1 \le T$ and $t_1 \ge 0$, since $R = O(\frac{\omega_* T^2}{\ln(t_1+2)})$ as $t_1 \to \infty$. $\Box$

The following lemma shows that filtering differentiators can be applied when the noises are only locally filterable.

Lemma 1. *Any signal* $\nu(t)$ *of the* ***local*** *filtering order* $k \ge 0$ *from the Definition 2 can be represented as* $\nu = \eta_0 + \eta_1 + \eta_k$, *where* η_0, η_1, η_k *are signals of the (global) filtering orders* $0, 1, k$.

Fix any number $\rho_0 > 0$. *Then, provided* $\rho \le \rho_0$ *holds for* $\rho = ||a||_{h\infty} = \max[a_0^{1/k}, a_1^{1/(k-1)}, ..., a_{k-1}]$, *the integral magnitudes of the signals* η_0, η_1, η_k *are* *calculated as $\gamma_0\rho/T$, $\gamma_1\rho$, $\gamma_k\rho^k$ respectively, where the constants $\gamma_0, \gamma_1, \gamma_k > 0$ only depend on k and ρ_0. In particular, in the important case $k = 1$ get $\rho = a_0$, $\nu = \eta_0 + \eta_1$, and independently of ρ_0 get $\gamma_0 = 1$, $\gamma_1 = 2$, i.e. $|\eta_0| \leq a_0/T$, and the first-order integral magnitude of η_1 is $2a_0$.*

Note that due to Lemma 1 and Theorem 2 filtering differentiator (9) and the tracking filtering differentiator to be introduced further, preserve their practical stability for *any* locally filterable noises. The noise from Definition 3 is practically rejected if $||a||_{h\infty}$ is small.

Remark 1. *Lemma 1 provides sufficient conditions for the validity of Assumption 3 and shows that the noise representation $\eta = \eta_0 + ... + \eta_{n_f}$ is never unique. Since the accuracy estimation* (12) *holds for any such noise representation, the resulting accuracy inevitably corresponds to the best possible one.*

Example 3 (Application of Lemma 1)**.** Return to Example 2. The noise $\eta = \cos(\omega_* t \ln\ln(t+2))$ is bounded, and therefore is of the 0th global filtering order. That means that the filtering differentiator (9) of the filtering order $n_f = 0$ (i.e. the standard differentiator) keeps its practical stability in its presence, though probably has bad accuracy.

The same signal $\eta(t)$ is also locally filterable. Its first order local integral magnitude is of the form $O(\frac{T^2}{\ln(t_1+2)})$, depends on T and tends to zero as the left end t_1 of the T-segment tends to infinity. According to the lemma, $\eta = \eta_0 + \eta_1$, where η_0, η_1 are of the global filtering orders 0 and 1, and the integral magnitudes $O(\frac{1}{T\omega_* \ln\ln(t_1+2)})$ and $O(\frac{1}{\omega_* \ln\ln(t_1+2)})$ respectively. Thus, by restarting the time counting at larger time values obtain from Theorem 2 that differentiator outputs (slowly) *asymptotically* converge to the exact derivatives for any $n_f \geq 1$.

The noise $\dot\eta$ is **unbounded**. It is globally filterable of the order 1, which means that in its presence differentiator keeps its stability for $n_f \geq 1$ providing for the accuracies corresponding to the parameter $\rho = O(1)$ of (13). It is also locally filterable of the order 2 with the magnitudes 1 and $O(\frac{1}{\omega_* \ln\ln(t_1+2)})$ of the order 1 and 2 respectively. But in that case Lemma 1 and Theorem 2 only promise the accuracy $O(1)$ for $n_f \geq 2$.

Similarly the noise $\eta^{(k)}$ is unbounded signal of the global filtering order k and requires $n_f \geq k$. Note that the **unbounded** noise $\varepsilon\eta^{(k)}$ will cause only small differentiation errors for $n_f \geq k$ and small $\varepsilon > 0$. Also see the simulation with unbounded noise in Section 5.1. □

Example 4. The signal $\eta = \cos(\omega t)$ is already a signal of any global filtering order (Example 1). Nevertheless, one can consider it as a locally filterable signal and apply the lemma. It is natural to suppose that the estimated accuracy will not improve.

Indeed, fixing any $T > 0$, ρ_0, and formally considering it as a signal of local filtering order k with the integral magnitudes $2/\omega^k, ..., 2/\omega$, $\rho = 2/\omega$, obtain $\eta = \eta_k + \eta_1 + \eta_0$, where the new noise components are of the global filtering orders $k, 1, 0$ and the integral magnitudes γ_k/ω^k, γ_1/ω, $\gamma_1/(\omega T)$ for some $\gamma_{0,1,k} > 0$. Then for sufficiently large ω Theorem 2 provides for the same accuracy asymptotics (up to the coefficients) as the original signal representation $\eta = \eta_k = \cos(\omega t)$. □

3.2. Tracking filtering differentiators

One of the basic requirements to differentiator is to produce smooth and consistent derivatives of the input. It becomes critical if, for example, such inconsistency may trigger undesired alerts. One may think of an aircraft transmitting its deliberately erroneous location and velocity in order to misguide hostile detection. The signals are to be consistent in order not to be disclosed by a radar equipped with a sanity mechanism.

Because of always present noises no differentiator can keep $z_0 - f_0(t) \equiv 0$. Due to the differentiator equations (9) in that case $\dot{z}_i \neq z_{i+1}$. Moreover, outputs $z_i(t)$ of differentiators (6) (or (9)) are not Lipschitzian due to the fractional powers on the right-hand sides of (9). Hence, the difference $\dot{z}_i - z_{i+1}$ can be large even in the case of small estimation errors.

Example 5. Consider a simple academic example. The motion of some object is described by a scalar differential equation on the coordinate $x \in \mathbb{R}$, which is measured with some noise. Derivatives $x^{(i)}$ are successfully estimated by the differentiator outputs z_i with a certain error due to noise.

Let $E = \dot{x}^2/2$ be the kinetic energy. Its dynamics is governed by the equation $\dot{E} = P(t)$ where $P(t)$ is the propelling power of an engine. The task is to estimate $P(t) = \dot{x}\ddot{x}$, $\dot{P}(t) = \ddot{x}^2 + \dot{x}\dddot{x}$.

One possible way is to differentiate $z_1^2/2$ once more using the same differentiator. It will inevitably exaggerate the noises, since the successive differentiator application destroys the optimal accuracy asymptotics (Section 2.2). Another way is to utilize the already available estimations $P(t) \approx z_1 z_2$, $\dot{P}(t) \approx z_2^2 + z_1 z_3$. Unfortunately the latter way is not available, since the differences $P(t) - z_1 z_2$, and especially $\dot{P}(t) - z_2^2 - z_1 z_3$ are very significant even for very small estimation errors $z_i - f_0^{(i)}$ (see example in Section 5.2). $\square$

One would like to ensure $z_i(\cdot) \in C^{n-i}$ and $\dot{z}_i \equiv z_{i+1}$, $i = 0, ..., n-1$, with z_n being only Lipschitzian.

The corresponding differentiators have been introduced for the identification of the practical relative degree in [36, 37, 33]. The task was to experimentally detect the number of the output derivative which most strongly responds to control jumps. One needs a sequence of consistent derivative's estimations for that end. The following tracking filtering differentiator contains a significantly *improved and unified version* of that tracking differentiator.

Let a homogeneous SMC $u = \alpha_n \psi_{n+1}(\sigma, \dot{\sigma}, ..., \sigma^n)$ in FT stabilize the DI $\sigma^{(n+1)} \in [-1, 1] + u$ for some $\alpha_n > 0$. Such controls are called $(n+1)$th-order SM ($(n+1)$-SM) controls, [15, 16, 17, 19, 45]. The $(n+1)$-SM homogeneity means that $\psi_{n+1}(\sigma, \dot{\sigma}, ..., \sigma^n) \equiv \psi_{n+1}(\kappa^{n+1}\sigma, \kappa^n \dot{\sigma}, ..., \kappa \sigma^n)$ holds for any $\kappa > 0$ and $\sigma, \dot{\sigma}, ..., \sigma^n \in \mathbb{R}$. We also require $|\psi_{n+1}| \leq 1$.

Then the tracking filtering differentiator is defined as

$$
\begin{aligned}
&\dot{w} = \Omega_{n,n_f}(w, \zeta_0 - z_0 + f(t), L + \alpha_n L, \vec{\lambda}),\\
&\dot{\zeta} = D_{n,n_f}(w_1, \zeta, L + \alpha_n L, \vec{\lambda}),\\
&\dot{z}_0 = z_1,\ ...,\ \dot{z}_{n-1} = z_n,\\
&\dot{z}_n = \alpha_n L \psi_{n+1}(\zeta/L).
\end{aligned} \tag{14}
$$

Theorem 3. *Under Assumptions 1, 3 differentiator* (14) *in FT provides for the accuracy asymptotics* (12), (13)*. Also* $|\zeta_i| \leq \mu_{\zeta i} L \rho^{n+n_f+1-i}$ *are kept in the steady state for* $i = 0, ..., n$*. The constants* μ_i*,* $\mu_{\zeta i}$*,* μ_{wk} *only depend on the choice of* $\{\lambda_l\}$*,* $l = 0, ..., n + n_f$*,* α_n *and* ψ_{n+1}*.*

The following are probably the simplest such r-SM controllers $u = \alpha_n \psi_{n+1}(\zeta)$ ready to use for $n \leq 4$. They are known as quasi-continuous rational homogeneous SM controllers [16]:

$$
\begin{array}{ll}
n = 0. & u = -1.5 \,\mathrm{sign}\, \zeta_0, \\
n = 1. & u = -1.5 \frac{\lfloor \zeta_1 \rceil^2 + \zeta_0}{\zeta_1^2 + |\zeta_0|}, \\
n = 2. & u = -4 \frac{\zeta_2^3 + \lfloor \zeta_1 \rceil^{\frac{3}{2}} + \zeta_0}{|\zeta_2|^3 + |\zeta_1|^{\frac{3}{2}} + |\zeta_0|}, \\
n = 3. & u = -7 \frac{\lfloor \zeta_3 \rceil^4 + 2\lfloor \zeta_2 \rceil^2 + 2\lfloor \zeta_1 \rceil^{\frac{4}{3}} + \zeta_0}{\zeta_3^4 + 2\zeta_2^2 + 2\zeta_1^{\frac{4}{3}} + |\zeta_0|}, \\
n = 4. & u = -20 \frac{\lfloor \zeta_4 \rceil^5 + 6\lfloor \zeta_3 \rceil^{\frac{5}{2}} + 5\lfloor \zeta_2 \rceil^{\frac{5}{3}} + 3\lfloor \zeta_1 \rceil^{\frac{5}{4}} + \zeta_0}{|\zeta_4|^5 + 6|\zeta_3|^{\frac{5}{2}} + 5|\zeta_2|^{\frac{5}{3}} + 3|\zeta_1|^{\frac{5}{4}} + |\zeta_0|}.
\end{array}
\tag{15}
$$

Note that the tracking differentiation is not an easy task. Indeed, it is natural to assume that z_i vibrates around its limit value $f_0^{(i)}(t)$. But each time as z_i gets its local extremum the next estimation z_{i+1} vanishes. Hence, on the first glance the estimation accuracy $z_{i+1} - f_0^{(i+1)}$ turns to be of the order of $|f_0^{(i+1)}|$ independently of noises and sampling periods. Therefore, it follows from Theorem 3 that z_i *is only allowed to have local extrema near the points where* $|f_0^{(i+1)}|$ *vanishes*, $i = 0, 1, ..., n - 1$. Also see the simulation, Fig. 2.

4. Discrete filtering differentiators

Assumption 3 is formulated in continuous-time notions including differentiation and integration. Unfortunately, in reality a filter is a discrete dynamic system obtaining discretely sampled input $f(t)$.

Let the sampling take place at the times $t_0, t_1, ...$, $t_0 = 0$, $0 < t_{j+1} - t_j = \tau_j$, $\lim_{j\to\infty} t_j = \infty$. The sampling steps are assumed bounded, $\tau_j \leq \tau$, though τ can be unknown.

It follows from the Nyquist-Shannon sampling rate principle that not all sampling-time sequences are admissible, since noises small in average under one sequence can be large under another.

Assumption 4. *The set of admissible sampling-time sequences contains sequences for any $\tau > 0$. In particular, the number of sequences is infinite.*

Notation. Denote $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$ for any sampled vector signal $\phi(t_j)$.

Definition 4. A discretely sampled signal $\nu : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *signal of the global sampling filtering order* $k \geq 0$ and *the global kth order integral sampling magnitude* $a \geq 0$ if for each admissible sequence t_j there exists a discrete vector signal $\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1}$, $j = 0, 1, ...,$ which satisfies the relations

$$\begin{aligned}&\delta_j \xi_i = \xi_{i+1}(t_j)\tau_j, \ i = 0, 1, ..., k-1, \\ &\xi_k(t_j) = \nu(t_j), \ |\xi_0(t_j)| \leq a.\end{aligned}$$

The following assumption on noises is the most convenient.

Assumption 5. *The sampled noise signal is comprised of $n_f + 1$ components, $\eta(t_j) = \eta_0(t_j) + \eta_1(t_j) + ... + \eta_{n_f}(t_j)$. The discretely sampled signals $\eta_l(t_j)$ are of the global sampling filtering order l and integral magnitude ε_l, $l = 0, 1, ..., n_f$. Components $\eta_1, ..., \eta_{n_f}$ possibly are unbounded.*

Assumption 5 is proved to hold for the steady-state SMC and $n_f = 1$ if the deviation of the SMC from the equivalent-control u_{eq} is considered the noise. That allows direct extraction of $u_{eq}, ..., u_{eq}^{(n)}$ by a filtering differentiator [34]. The assumption also often holds due to the statistical features of the noise (see [35], and Section 5.3).

The proposed homogeneous discretization of the standard filtering differentiator (11) has the form

$$\begin{aligned}\delta_j w = &\quad \Omega_{n,n_f}(w, z_0(t_j) - f(t_j), L, \vec{\lambda})\tau_j, \\ \delta_j z = &\quad D_{n,n_f}(w_1(t_j), z(t_j), L, \vec{\lambda})\tau_j + T_n(z(t_j), \tau_j),\end{aligned} \tag{16}$$

where the Taylor-like term $T_n \in \mathbb{R}^{n+1}$ is defined as

$$
\begin{aligned}
&T_{n,0} = \tfrac{1}{2!} z_2(t_j)\tau_j^2 + ... + \tfrac{1}{n!} z_n(t_j)\tau_j^n,\\
&T_{n,1} = \tfrac{1}{2!} z_3(t_j)\tau_j^2 + ... + \tfrac{1}{(n-1)!} z_n(t_j)\tau_j^{n-1},\\
&...\\
&T_{n,n-2} = \tfrac{1}{2!} z_n(t_j)\tau_j^2,\\
&T_{n,n-1} = 0,\ T_{n,n} = 0.
\end{aligned}
\tag{17}
$$

An example is provided in (27). The following is the proposed homogeneous discretization of the tracking differentiator (14):

$$
\begin{aligned}
&\delta_j w = \Omega_{n,n_f}(w, \zeta_0(t_j) - z_0(t_j) + f(t_j), L + \alpha_n L, \vec{\lambda})\tau_j,\\
&\delta_j \zeta = D_{n,n_f}(w_1(t_j), \zeta(t_j), L + \alpha_n L, \vec{\lambda})\tau_j,\\
&\delta_j z = \Psi_n(t_j)\tau_j + T_n(z(t_j), \tau_j),\\
&\Psi_n(t_j) = (z_1(t_j), ..., z_n(t_j), \alpha_n L \psi_{n+1}(\zeta(t_j)/L))^T.
\end{aligned}
\tag{18}
$$

Theorem 4. *Under Assumptions 1, 4, 5 discrete differentiators* (16) *and* (18) *provide the same accuracy* (12) *as Theorems 2 and 3 respectively, but for*

$$
\rho = \max[\tau, \max_{0 \le l \le n_f} (\tfrac{\varepsilon_l}{L})^{\frac{1}{n+l+1}}] \tag{19}
$$

Example 6. Fix any small $\varepsilon_l > 0$ and let the sampling step be constant and equal $\tau > 0$. Take any discrete signal $\xi_l(t_j)$, $|\xi_l(t_j)| \le \varepsilon_l$, $t_j = j\tau$, $j = 0, 1, ...$ For example $\xi_l(t_j) = (-1)^j \varepsilon$ can be taken.

Let $\xi_{i-1}(t_j) = \delta_j \xi_i / \tau$, $i = l, l-1, ..., 1$. Define $\eta_l(t_j) = \xi_0(t_j)$. It is a signal of the global sampling filtering order l and the lth-order integral sampling magnitude ε_l.

Obviously $\eta_l(t_j) = O(\varepsilon_l \tau^{-l})$ and is very large for small τ. Nevertheless, according to Theorem 4, for $l \le n_f$ its impact (19) on the accuracy of differentiators (16) and (18) is determined by $\varepsilon_l^{1/(n+l)}/L$ only. □

Definition 5. A discretely sampled signal $\nu(t_j)$ is said to be *locally filterable of the local sampling filtering order* $k > 0$ if there exist numbers $T > 0$, and $a_0, a_1, ..., a_{k-1} \ge 0$, such that for any sufficiently small τ, admissible sequence $\{t_j\}$, and any sampling instant $t_{j_0} \ge 0$ there exists a discrete vector signal

$$
\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1},\ j = j_0, j_0{+}1, ..., j_1,\ t_{j_1} \in [t_{j_0}{+}T, t_{j_0}{+}T{+}\tau],
$$

which satisfies the relations

$$\begin{aligned}
&\delta_j \xi_i = \xi_{i+1}(t_j)\tau_j,\ i = 0, 1, ..., k-1, \\
&\xi_k(t_j) = \nu(t_j), \\
&|\xi_i(t_j)| \le a_i,\ i = 0, 1, ..., k-1.
\end{aligned} \tag{20}$$

Numbers a_i are called the *local* $(k-i)$*th-order sampling integral magnitudes of* ν. Signals of local sampling filtering order 0 by definition are just bounded signals of the magnitude a_0.

The following lemma is analogous to Lemma 1 and allows application of Theorem 4 in the case of locally filterable noises.

Lemma 2. *Let all admissible sampling time sequences satisfy the condition* $\sup \tau_j / \inf \tau_j \le c_\tau$ *for some* $c_\tau > 0$*. Then any discretely sampled signal* $\nu(t_j)$ *of the* ***local*** *sampling filtering order* $k \ge 0$ *from the Definition 5 can be represented as* $\nu = \eta_0 + \eta_1 + \eta_k$*, where* η_0, η_1, η_k *are signals of the (global) sampling filtering orders* $0, 1, k$*.*

Fix any number $\rho_0 > 0$*. Then, provided* $\rho = ||a||_{h\infty} \le \rho_0$ *the sampling integral magnitudes of the signals* η_0, η_1, η_k *are calculated as* $\gamma_0 \rho / T$*,* $\gamma_1 \rho$*,* $\gamma_k \rho^k$ *respectively, where the constants* $\gamma_0, \gamma_1, \gamma_k > 0$ *only depend on* k *and* ρ_0*.*

In the important particular case $k = 1$ *the condition* $\sup \tau_j / \inf \tau_j \le$ *const is not needed and there is no dependence on* ρ_0*. In that case get* $\rho = a_0$*,* $\nu = \eta_0 + \eta_1$*, and independently of* ρ_0 *get* $\gamma_0 = 1$*,* $\gamma_1 = 2$*, i.e.* $|\eta_0| \le a_0/T$*, and the first-order integral sampling magnitude of* η_1 *is* $2a_0$*.*

Remark 2. *The case of the local filtering order* 1 *is the most important in both continuous and discrete cases, since it ensures the applicability of the filtering differentiators for any filtering order* $n_f \ge 1$*. Higher filtering orders* n_f *never destroy the accuracy asymptotics, but may improve them. This reasoning ignores the asymptotics coefficients, which requires checking different options for* n_f *if* τ *is relatively large.*

In general one needs very small sampling steps to reveal the small average value of the noise. The following strong assumption on some noise components extends Assumption 5. It is natural in filtering theory, since it guaranties the stability of the average value with respect to sampling.

Assumption 6. *The sampled noise is comprised of* n_f+1 *components,* $\eta(t_j) = \eta_0(t_j)+\eta_1(t_j)+...+\eta_{n_f}(t_j)$*, where* $\eta_l = \underline{\eta}_l+\bar{\eta}_l$, $l=0,...,n_f$*. The discretely sampled signals* $\underline{\eta}_l(t_j)$ *are of the sampling filtering order* l *and integral magnitude* $\underline{\varepsilon}_l$, $\bar{\eta}_0 = 0$*. Each noise component* $\bar{\eta}_l(t)$ *is of the (continuous-time global) filtering order* l *and* l*th-order integral magnitude* $\bar{\varepsilon}_l$*. It is also absolutely continuous with* $|\dot{\bar{\eta}}_l| \leq L_{\eta l}$ *for* $l=1,...,n_f$*.*

Naturally, $L_{\eta l}$ can be unknown and large.

Theorem 5. *Under Assumptions 1, 4, 6 discrete differentiators* (16) *and* (18) *provide the same accuracy* (12) *as Theorems 2 and 3 respectively, but for*

$$\rho = \max[\tau, \max_{0\leq l\leq n_f}(\tfrac{\underline{\varepsilon}_l}{L})^{\frac{1}{n+l+1}}, \max_{1\leq l\leq n_f}\max_{1\leq k\leq l+1}(\tfrac{L_{\eta l}}{L}(\tfrac{\bar{\varepsilon}_l}{L_{\eta l}})^{\frac{k}{l+1}})^{\frac{1}{n+k}}]. \tag{21}$$

Similarly to Remark 1, also here one does not need to check Assumption 6 in order to use the differentiator. Due to Lemma 2 all noises appearing in Examples of Section 3 are also filterable by the discrete filtering differentiators of both types.

Example 7. In continuation of Example 1 consider the input noise $\eta_l(t) = \gamma\cos\omega_* t$. Then $L_{\eta l} = \gamma\omega_*$, $\varepsilon_l = 2\gamma/\omega_*^l$. Suppose that $\eta = \eta_l$, $n_f \geq l$. Then from (21) obtain

$$\rho = \max[\tau, \max_{1\leq k\leq l}(\tfrac{\gamma 2^k}{L{\omega_*}^k})^{\frac{1}{n+k+1}}] = \max[\tau, \tfrac{2}{\omega_*}\max_{1\leq k\leq l}(\tfrac{\gamma{\omega_*}^{n+1}}{2^{n+1}L})^{\frac{1}{n+k+1}}].$$

Obviously, for $L \leq \gamma(\omega_*/2)^{n+1}$ get that $\rho \approx 2/\omega_*$ for large l and small τ. $\square$

Thus also here one can successfully apply the discrete filtering differentiators when the noises are only locally filterable. Also here one can arbitrarily increase the filtering order preserving the accuracy asymptotics optimality. Note that the coefficients (not the powers) of the accuracy asymptotics depend on n_f. One also has to take into account the influence of the digital round-up errors for high sampling rates and accuracies [33, 44].

5. Numeric experiments

Consider the input signal

$$f(t) = f_0(t) + \eta(t),\ \ f_0(t) = -0.4 \sin t + 0.8 \cos(0.8t). \tag{22}$$

5.1. Comparison with the "standard" differentiator

Let the noise be

$$\begin{aligned} &\eta(t) = \eta_1(t) + \eta_2(t),\\ &\eta_1 = \cos(10000t + 0.7791),\\ &\eta_2 = 0.0375 \sin^2(100t)\lfloor \cos 100t \rceil^{-1/2} - 0.075\lfloor \cos(100t)\rceil^{3/2}\\ &\quad\ = 5 \cdot 10^{-6} \tfrac{d^2}{dt^2} \lfloor \cos(100t)\rceil^{\frac{3}{2}} \end{aligned} \tag{23}$$

where η_1 is a high-frequency harmonic signal, η_2 is an unbounded signal of the filtering order 2 and the integral magnitude $5 \cdot 10^{-6}$. It is saturated at ± 1000. Obviously $|f_0^{(k)}| \leq 1$ for all $k \geq 2$.

The filtering differentiator (16) of the differentiation order $n = 2$ and the filtering order 3 is applied with $L = 1$, $\tau_j = \tau = 10^{-5}$ and zero initial conditions, $z(0) = 0$, $w(0) = 0$. The coefficients are taken from Table 1 from line $5 = 2 + 3$. The standard differentiator (6) with $n = 2$, $L = 1$ and zero initial values is taken for the comparison.

The performance of the differentiators is demonstrated in Fig. 1. The standard differentiator demonstrates its remarkable stability, but nothing more. The accuracy of the filtering differentiator for $t \in [35, 40]$ is provided by the component-wise inequality

$$\begin{aligned} &(|w_1|, |w_2|, |w_3|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq \\ &\qquad (7.8 \cdot 10^{-8}, 1.0 \cdot 10^{-5}, 8.1 \cdot 10^{-4}, 3.7 \cdot 10^{-3}, 0.030, 0.14). \end{aligned} \tag{24}$$

The following is its accuracy in the absence of noises for the comparison:

$$\begin{aligned} &(|w_1|, |w_2|, |w_3|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \leq \\ &\qquad (4.1 \cdot 10^{-28}, 9.8 \cdot 10^{-23}, 9.2 \cdot 10^{-18}, 3.8 \cdot 10^{-13}, 7.7 \cdot 10^{-9}, 8.3 \cdot 10^{-5}). \end{aligned} \tag{25}$$

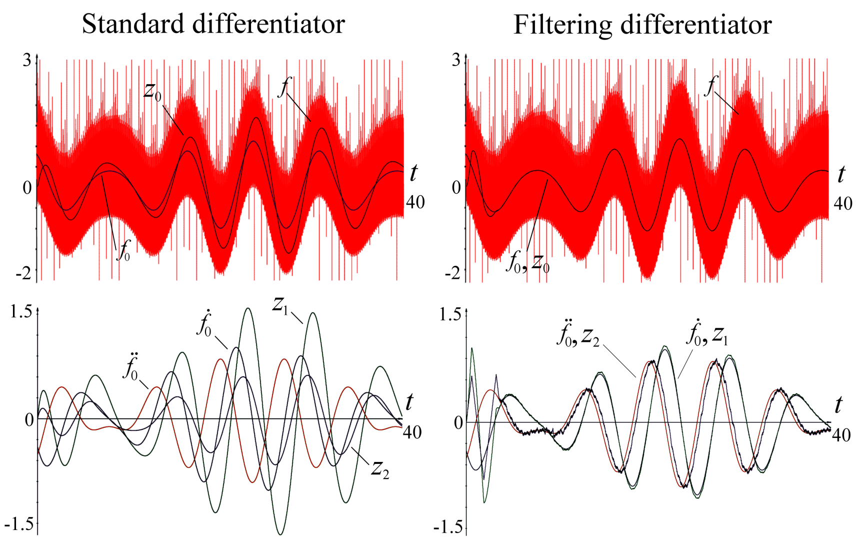

Figure 1: Comparison of differentiators (16), (17) with $n = 2$, $n_f = 3$, $L = 1$, and (6) with $n = 2$, $L = 1$, for $\tau = 10^{-5}$ and the input (22), (23). The graph of f is cut from above and below. Estimations of f_0, $\dot{f}_0$, $\ddot{f}_0$ are shown.

The simulation has shown that the unbounded deterministic noises like η_2 from (23) are the worst possible noises for the filtering differentiators. In fact removing η_1 almost does not affect the accuracy (24).

5.2. Comparison with the tracking differentiator

The tracking differentiator (14) naturally demonstrates slower convergence and lower accuracy. On the other hand it provides for the special feature: its derivative estimations are consistent.

Consider another two-components' noise

$$\begin{aligned} &\eta(t) = \eta_1(t) + \eta_2(t), \\ &\eta_1 = \cos(10000t + 0.7791),\ \eta_2 \in N(0, 0.5^2) \end{aligned} \tag{26}$$

where η_2 is normally distributed with the standard deviation 0.5. Apply the standard filtering differentiator (16), (17)

$$
\begin{aligned}
&\delta_j w_1 = \tau[-5L^{1/5} \lfloor w_1(t_j)\rceil^{4/5} + w_2(t_j)],\\
&\delta_j w_2 = \tau[-10.03L^{2/5} \lfloor w_1(t_j)\rceil^{3/5} + z_0(t_j) - f(t_j)],\\
&\delta_j z_0 = \tau[-9.30L^{3/5} \lfloor w_1(t_j)\rceil^{2/5} + z_1(t_j)] + \tfrac{\tau^2}{2} z_2(t_j),\\
&\delta_j z_1 = \tau[-4.57L^{4/5} \lfloor w_1(t_j)\rceil^{1/5} + z_2(t_j)],\\
&\delta_j z_2 = \tau[-1.1L\,\mathrm{sign}(w_1(t_j))],
\end{aligned}
\tag{27}
$$

where $n = n_f = 2$, $L = 1$, $\tau = 10^{-7}$. Recall that by definition $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$. For the comparison apply the tracking filtering differentiator (18) with the same parameters and SM controller number 2 from (15). Initial values are once more zeroed.

In order to reveal the consistency of its derivative estimations consider the differential identities from Example 5. Derivatives of $z_1^2/2$ approximating $E = \dot{f}_0^2/2$ are estimated by the auxiliary filtering differentiator with $n = 2$, $n_f = 1$ and $\tilde{L} = 400$, where $\tilde{L} > |E^{(4)}|$ is directly checked. If the estimations z_0, z_1, z_2 were consistent, $\dot{z}_0 = z_1$, $\dot{z}_1 = z_2$, the following identities would hold:

$$
\tfrac{d}{dt}(\tfrac{1}{2}z_1^2) - z_1 z_2 \equiv 0, \quad \tfrac{d^2}{dt^2}(\tfrac{1}{2}z_1^2) - z_1 z_3 - z_2^2 \equiv 0.
$$

The differentiator performances are demonstrated in Fig. 2. The comparison shows that the filtering tracking differentiator has worse characteristics, but it still performs better than the standard differentiator (6) in the presence of noises (compare with Fig. (23)). The resulting accuracy of the filtering differentiator (16) is provided by the component-wise inequality

$$
\begin{aligned}
&(|w_1|, |w_2|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le \\
&\qquad (7.5 \cdot 10^{-7}, 9.6 \cdot 10^{-5}, 1.6 \cdot 10^{-3}, 2.0 \cdot 10^{-2}, 0.13).
\end{aligned}
\tag{28}
$$

The accuracies of the tracking filtering differentiator (18) are

$$
\begin{aligned}
&(|w_1|, |w_2|, |\zeta_0|, |\zeta_1|, |\zeta_2|, |z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le \\
&\qquad (8.8 \cdot 10^{-7}, 1.2 \cdot 10^{-4}, 6.7 \cdot 10^{-2}, 0.18, 0.83, 0.066, 0.139, 0.74).
\end{aligned}
\tag{29}
$$

The above identities are significantly destroyed (Fig. 2) by the standard filtering differentiator in spite of its better accuracy (28). On the other hand both identities are kept with high accuracy of $1.5 \cdot 10^{-9}$, 0.001 respectively by its tracking counterpart. The latter accuracy is probably due to the estimation error by the applied auxiliary differentiator.

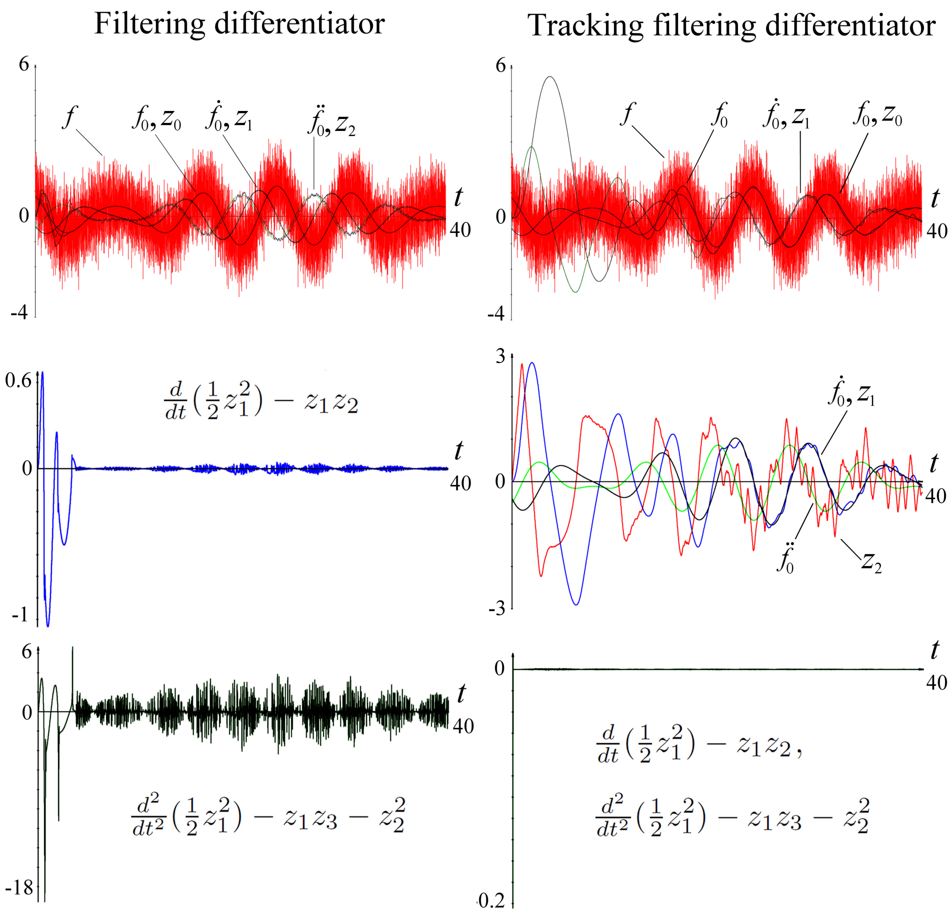

Figure 2: Comparison of differentiators (16) and (18) with $n = n_f = 2$, SMC (15), $L = 1$, $\tau = 10^{-7}$ for the input (22), (26). Estimations of f_0, $\dot f_0$ and $\ddot f_0$ are shown, and the identities' errors.

5.3. Comparison with the Kalman filter

Compare the standard Kalman filter approach to the filtering differentiator. Consider the sampled input

$$f(t_j) = f_0(t_j) + \eta(t_j),\ f_0(t_j) = \cos(5t_j),\ \eta(t_j) \in \mathcal{N}(0, \sigma^2),\ \sigma = 1,$$

with the sampling step $\tau = 10^{-7}$. The Kalman prediction and innovation equations are

$$\begin{aligned} \hat{x}_{j+1} &= \Phi_j \hat{x}_j, \\ y(t_j) &= f(t_j) - H\hat{x}_j, \end{aligned} \tag{30}$$

where $\hat{x}_j$ and $y(t_j)$ respectively are the estimation of $(f_0, \dot{f}_0, \ddot{f}_0)^T$ and the Kalman innovation. The state transition and the measurement models are

$$\Phi_j = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

respectively.

The covariance matrix of $\hat{x}_j$ is propagated in the standard fashion with the process-noise covariance matrix

$$Q_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tau \end{bmatrix},$$

and the Kalman update is correspondingly applied with the scalar measurement covariance matrix $R = \sigma^2$.

The term τ is introduced in Q_j for the model mismatch compensation. Enlarging this term weakens the model-based prediction and aggravates the inevitable performance deterioration in the presence of deterministic noises like (23).

The filtering differentiator (27) with $n = n_f = 2$, $L = 5^3 = 125$ is applied to the same input. The resulting filtering performance is presented in Fig. 3

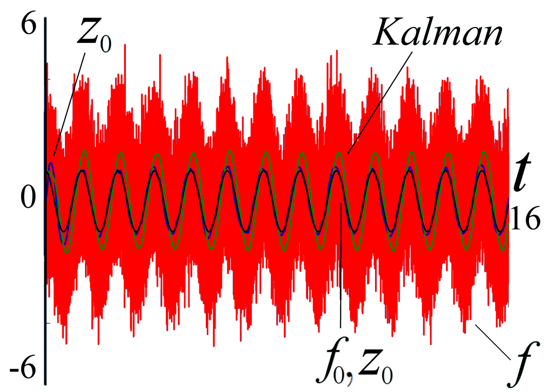

Figure 3: Comparison of the filtering differentiator (16) with $n = n_f = 2$, $L = 125$, $\tau = 10^{-7}$ with the Kalman filter (30).

6. Conclusions

The proposed homogeneous filtering differentiators are capable to filter out complicated noises of small average values. The filtering power of a differentiator is determined by its filtering order. The higher the filtering order the better the differentiation accuracy asymptotics in the presence of noises. The accuracy asymptotics are calculated and a proper homogeneous discretization is proposed.

The nth-order filtering differentiators and their discrete counterparts feature the same optimal accuracy asymptotics as their predecessors [18] in the presence of bounded noises. In particular they are exact in the absence of noises on the signals $f_0 \in \mathrm{Lip}_n(L)$ for the differentiator parameter $L > 0$.

The noise is assumed representable as a sum of a finite number of noises of different filtering orders, whereas noises of the filtering order 0 are just bounded measurable noises of any nature. The calculated accuracy estimation depends on that noise expansion. Since the expansion is not unique, and the differentiator does not "know" it, the actual accuracy corresponds to the best possible expansion *unknown to the "user"*.

The proposed homogeneous tracking differentiators and their filtering modifications are capable to yield the smooth derivative estimations $z_k \approx f_0^{(k)}$ satisfying the natural relation $\dot{z}_k = z_{k+1}$, which is advantageous in some practical applications. Having the same main features as the "standard" filtering differentiators, they have slower convergence and worse asymptotics coefficients.

Extensive simulation of proposed differentiators has shown their capability of filtering complicated large noises. Moreover, they seem to be decent rivals to the Kalman filter in coping with Gaussian noises. Nevertheless, rigorous analysis of SM-based differentiation in the presence of random input noises has never been done, and any conclusions in that aspect would be premature. Some qualitative analysis is available in [35].

Appendix .1. Proofs for the continuous-time case

Proof of Theorem 2. According to the filtering-order definition introduce the functions $\xi_k(t)$, $|\xi_k| \leq \varepsilon_k$, $\xi_k^{(k)}(t) = \nu_k(t)$, $k = 1, ..., n_f$. Let

$$\begin{aligned}
&\tilde{\omega}_1 = w_1 + \xi_{n_f}, \tilde{\omega}_2 = w_2 + \dot{\xi}_{n_f} + \xi_{n_f-1}, ..., \\
&\tilde{\omega}_{n_f} = w_{n_f} + \xi_{n_f}^{(n_f-1)} + ... + \dot{\xi}_2 + \xi_1; \\
&\tilde{\sigma}_i = z_i - f_0^i,\ i = 0, ..., n.
\end{aligned} \tag{.1}$$

Then $f = f_0 + \eta + \dot{\xi}_1 + ... + \xi_{n_f}^{(n_f)}$, and one can rewrite (9) in the form

$$\begin{aligned}
&\dot{\tilde{\omega}}_1 = -\tilde{\lambda}_{n+n_f} L^{\frac{1}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{n+n_f}{n+n_f+1}} + \tilde{\omega}_2 - \xi_{n_f-1}, \\
&\dot{\tilde{\omega}}_2 = -\tilde{\lambda}_{n+n_f-1} L^{\frac{2}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{n+n_f-1}{n+n_f+1}} + \tilde{\omega}_3 - \xi_{n_f-2}, \\
&\qquad ... \\
&\dot{\tilde{\omega}}_{n_f-1} = -\tilde{\lambda}_{n+2} L^{\frac{n_f-1}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{n+2}{n+n_f+1}} + \tilde{\omega}_{n_f} - \xi_2, \\
&\dot{\tilde{\omega}}_{n_f} = -\tilde{\lambda}_{n+1} L^{\frac{n_f}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{n+1}{n+n_f+1}} + \tilde{\sigma}_0 + \eta_0, \\
&\dot{\tilde{\sigma}}_0 = -\tilde{\lambda}_n L^{\frac{n_f+1}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{n}{n+n_f+1}} + z_1, \\
&\qquad ... \\
&\dot{\tilde{\sigma}}_{n-1} = -\tilde{\lambda}_1 L^{\frac{n+n_f}{n+n_f+1}} \lfloor \tilde{\omega}_1 - \xi_{n_f} \rceil^{\frac{1}{n+n_f+1}} + \tilde{\sigma}_n, \\
&\dot{\tilde{\sigma}}_n \in -\tilde{\lambda}_0 L \,\mathrm{sign}(\tilde{\omega}_1 - \xi_{n_f}) + [-L, L].
\end{aligned} \tag{.2}$$

Divide by L and denote $\omega_k = \tilde{\omega}_k/L$, $\sigma_k = \tilde{\sigma}_k/L$. Rewrite (.2) as the inclusion

$$
\begin{aligned}
&\dot{\omega}_1 \in -\tilde{\lambda}_{n+n_f} \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{n+n_f}{n+n_f+1}} + \omega_2 + \rho^{n+n_f}[-1,1],\\
&\dot{\omega}_2 \in -\tilde{\lambda}_{n+n_f-1} \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{n+n_f-1}{n+n_f+1}} + \omega_3 + \rho^{n+n_f-1}[-1,1],\\
&\qquad ...\\
&\dot{\omega}_{n_f-1} \in -\tilde{\lambda}_{n+2} \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{n+2}{n+n_f+1}} + \omega_{n_f} + \rho^{n+2}[-1,1],\\
&\dot{\omega}_{n_f} \in -\tilde{\lambda}_{n+1} \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{n+1}{n+n_f+1}} + \sigma_0 + \rho^{n+1}[-1,1],
\end{aligned}
\tag{.3}
$$

$$
\begin{aligned}
&\dot{\sigma}_0 \in -\tilde{\lambda}_n \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{n}{n+n_f+1}} + z_1,\\
&\qquad ...\\
&\dot{\sigma}_{n-1} \in -\tilde{\lambda}_1 \lfloor \omega_1 + \rho^{n+n_f+1}[-1,1] \rceil^{\frac{1}{n+n_f+1}} + \sigma_n,\\
&\dot{\sigma}_n \in -\tilde{\lambda}_0 \operatorname{sign}(\omega_1 + \rho^{n+n_f+1}[-1,1]) + [-1,1].
\end{aligned}
\tag{.4}
$$

which is the perturbation of the FT stable homogeneous error dynamics (7) of the standard $(n+n_f)$th-order differentiator (6) obtained by substituting $n+n_f$ for n. Obviously, $\deg \omega_k = n + n_f + 2 - k$, $\deg z_i = n + 1 - i$, $\deg t = -q = 1$, the HD is -1, $\deg \rho = 1$ is assigned.

It follows from [46] that $\sup |\sigma_i| \leq \mu_i \rho^{n+1-i}$, $\sup |\omega_k| \leq \hat{\mu}_{wk} \rho^{n+n_f+2-k}$ for some $\mu_i, \hat{\mu}_{wk} > 0$. Now the accuracy of z_i is directly obtained from these relations. The estimation of w_1 is obtained from $L\omega_1 = \tilde{\omega}_1 = w_1 + \xi_{n_f}$. Estimations of w_k, $k > 1$, can be similarly obtained from (.1) under additional assumptions on the noises. $\square$

Denote $\vec{\sigma} = (\sigma, ..., \sigma^{(n)})^T$. Then (.3), (.4) can be rewritten as

$$
\begin{aligned}
&\dot{\omega} \in \Omega_{n,n_f}(\omega + \vec{\rho}, \sigma_0 + \rho^{n+1}[-1,1], 1, \vec{\lambda}),\\
&\dot{\vec{\sigma}} \in D_{n,n_f}(\omega_1 + \rho^{n+n_f+1}[-1,1], \vec{\sigma}, 1, \vec{\lambda}) + h,\\
&\vec{\sigma} = (\sigma_0, ..., \sigma_n)^T,\ h = (0, ..., 0, [-1,1])^T \in \mathbb{R}^{n+1}\\
&\vec{\rho} = [-1,1](\rho^{n+n_f+1}, \rho^{n+n_f}, ..., \rho^{n+2})^T \in \mathbb{R}^{n_f}.
\end{aligned}
\tag{.5}
$$

Proof of Theorem 3. First consider the case without noises. Denote $\sigma = (z_0 - f_0)/L$, $\tilde{\zeta} = \zeta/L$, $\tilde{w} = w/L$ and rewrite (14) in the form

$$
\begin{aligned}
&\dot{\tilde{w}} = \Omega_{n,n_f}(\tilde{w}, \tilde{\zeta}_0 - \sigma, 1 + \alpha_n, \vec{\lambda}),\\
&\dot{\tilde{\zeta}} = D_{n,n_f}(\tilde{w}_1, \tilde{\zeta}, 1 + \alpha_n, \vec{\lambda}),\\
&\sigma^{(n+1)} \in \alpha_n \psi_{n+1}(\tilde{\zeta}) + [-1,1].
\end{aligned}
\tag{.6}
$$

It follows from $|\sigma^{(n+1)}| \leq 1 + \alpha_n$ and Theorem 2 that $\tilde{\zeta} \equiv \vec{\sigma}$ is ensured in FT. Now due to the choice of $\psi_{n+1}(\cdot)$ in FT get $\vec{\sigma} \equiv 0$. The observation that (.6) is homogeneous with the weights $\deg \tilde{\zeta}_i = \deg \sigma^{(i)} = n + 1 - i,\ i = 0, 1, ..., n,$ $\deg \tilde{w}_l = n + n_f + 2 - l,\ l = 1, ..., n_f$, and the HD -1, implies the FT stability of (.6) due to Section 2.1.

Now consider the presence of noises. Similarly to the proof of Theorem 2 introduce $\omega_1 = [w_1 + \xi_{n_f}]/L, ..., \omega_{n_f} = [w_{n_f} + \xi_{n_f}^{(n_f-1)} + ... + \dot{\xi}_2 + \xi_1]/L$, and obtain

$$\begin{aligned}
&\dot{\omega} \in \Omega_{n,n_f}(\omega + \vec{\rho}, \tilde{\zeta}_0 - \sigma + \rho^{n+1}[-1,1], 1 + \alpha_n, \vec{\lambda}),\\
&\dot{\tilde{\zeta}} \in D_{n,n_f}(\omega_1 + \rho^{n+n_f+1}[-1,1], \tilde{\zeta}, 1 + \alpha_n, \vec{\lambda}),\\
&\sigma^{(n+1)} \in \alpha_n \psi_{n+1}(\tilde{\zeta}) + [-1,1];\\
&\vec{\rho} = [-1,1](\rho^{n+n_f+1}, \rho^{n+n_f}, ..., \rho^{n+2})^T.
\end{aligned} \tag{.7}$$

The required accuracy is once more due to the above results from Section 2.1 and [46]. □

Proof of Lemma 1. Let ξ_j be a bounded solution of the equation $\xi_j^{(k)} = \nu(t)$ over the interval $t \in [jT, (j+1)T],\ j = 0, 1, 2, ...$ It is known that $|\xi_j^{(i)}| \leq a_i$, $i = 0, 1, ..., k-1$. The task is to build the global solution $\xi(t)$, and the global representation $\nu = \eta_0 + \eta_1 + \eta_k$, where η_0, η_1, η_k are of global filtering orders $0, 1, k$ respectively.

Introduce the vector function $\vec{\xi} = (\xi, \dot{\xi}, ..., \xi^{(k-1)})^T$, similarly $\vec{\xi}_j$ are defined. By definition $\dot{\vec{\xi}}_j = J_k \vec{\xi}_j + b\nu(t),\ t \in [(j-1)T, jT]$. Here $b = (0, ..., 0, 1)^T \in \mathbb{R}^k$ and J_k is the upper-triangular Jordan block with the zero diagonal.

One needs to combine functions $\vec{\xi}_j$ into one function $\vec{\xi}(t),\ t \geq 0$. It can be done by solving the equations

$$\begin{aligned}
&\dot{\vec{\xi}} = J_k \vec{\xi} + b(\nu(t) + u(t)),\\
&\vec{\xi}(jT) = \vec{\xi}_j(jT),\ j = 0, 1, ...,\\
&\nu = \eta_k + \tilde{\nu}, ; \eta_k = \nu + u,\ \tilde{\nu} = -u,
\end{aligned} \tag{.8}$$

where u is a virtual control. The lemma is proved if the signal $\tilde{\nu} = -u$ can be represented as a sum of two signals of the filtering orders 0 and 1 and the corresponding integral magnitudes proportional to ρ/T and $\rho = ||a||_{h\infty}$ respectively.

First consider the case $k=1$ ($k=0$ is trivial). Then $\rho = a_0$, $\xi_j : [jT,(j+1)T] \to \mathbb{R}$, $|\xi_j| \le a_0$, and taking $u(t) = (\xi_{j+1}((j+1)T) - \xi_j(jT))/T$ for $t \in [jT,(j+1)T]$ proves the Lemma.

Now consider the general case $k>0$. Without losing the generality assume $j=0$. Denote $x(t) = \vec{\xi}(t) - \vec{\xi}_j(t)$, $x = (x_1,...,x_k)^T$, $\Delta = \xi_1(T) - \xi_0(T)$. Then the problem is to find $u : [0,T] \to \mathbb{R}$ such that $x(0)=0$, $x(T)=\Delta$, while roughly speaking $x(t)$ components are to be of the order $\rho^k,...,\rho$, $\dot{x} = J_k x + bu(t)$, $t \in [0,T]$.

In its turn that problem is equivalent to the stabilization problem in the backward time $\hat{t}$

$$\begin{aligned} &\tfrac{d}{d\hat{t}}x = -J_k x - bu(\hat{t}),\ \hat{t} \in [0,T],\\ &x(0) = \Delta,\ x(T) = 0. \end{aligned} \tag{.9}$$

The further proof is based on the following lemma to be proved a bit later.

Lemma 3. *There are such constants $U_M, M > 0$ that for each $y_* \in \mathbb{R}^k$, $||y_*||_{h\infty} \le 2\rho_0$, and $T_* = ||y_*||_{h\infty}/(2\rho_0)$ there exists control $u_*(s)$, $s \in [0,T_*]$, $|u_*(s)| \le U_M$, which satisfies the equations*

$$\begin{aligned} &\tfrac{d}{ds}y = -J_k x - bu_*(s),\ s \in [0,T_*],\\ &y(0) = y_*,\ y(T_*) = 0,\ |u_*(s)| \le U_M,\\ &\max_s |y_i(s) - y_{*i}| \le M||y_*||_{h\infty}^{k+1-i},\ i = 1,2,...,k. \end{aligned} \tag{.10}$$

The differential equation in (.10) is homogeneous with the HD -1 and the weights $\deg y_1 = k, \deg y_2 = k-1, ..., \deg y_k = 1$ in the sense that for any control $u(s)$ the transformation

$$(s, y, u(\cdot)) \to (\kappa s, \kappa^k y_1, ..., \kappa y_k, u(\kappa\ \cdot))$$

preserves its solutions for any $\kappa > 0$.

Obviously $||\Delta||_{h\infty} \le ||2a||_{h\infty} \le 2\rho$. Apply the transformation $\hat{t} = Ts$, $x_i = T^{k+1-i}y_i$. Thus according to Lemma 3 system (.9) is stabilized in the time $T_* = ||\Delta||_{h\infty}/(2\rho_0)T \le T$ by the control $u = u_*(\hat{t})$. Define $u=0$ for $\hat{t} \in (T_*, T]$.

Return to the forward time. We have succeeded to build the global solution of (.8) keeping $|\vec{\xi}_i| \le |\vec{\xi}_{ji}(t) + x_{ji}(t)| \le \rho^i + 2^i M \rho^i$. Thus $|\xi| \le \gamma_k M \rho^k$, where

$\gamma_k = 1+2^k M$, $|\xi^{(k-1)}| \le \tilde\gamma_1 \rho$, where $\tilde\gamma_1 = 1+2M$, whereas $\tilde\eta = -u(t)$ is a signal of the local filtering order 1 and the filtering magnitude $\tilde\gamma_1\rho$.

Applying now the lemma for the case $k = 1$, which is already proved, represent $\tilde\eta = \eta_1 + \eta_0$. Here η_1 is a signal of the global filtering order 1 and the integral magnitude $\gamma_1\rho$, $\gamma_1 = 2\tilde\gamma_1 = 2 + 4M$. In its turn η_0 is a bounded noise of the magnitude $\gamma_0\rho/T$, $\gamma_0 = \tilde\gamma_1 = 1 + 2M$. □

Proof of Lemma 3. Rewrite the system in the form

$$\begin{array}{l} \frac{d}{ds}\theta = J_k\theta + bv, \\ v = \alpha\Psi(\theta),\ \theta(0) = \theta_0, \end{array} \tag{.11}$$

where $\theta_i = (-1)^{i+1} y_i$, $i = 1, ..., k$, $v = (-1)^k u_*$, the controller Ψ is to be chosen.

System (.11) is easily stabilized in FT by a proper homogeneous k-SM control $v = \alpha\Psi(\theta)$. For example a controller from the series (15) can be taken (with different coefficients) [16]. Its parameters can be easily adjusted so that the system stabilizes in the time not exceeding 1 for any initial conditions $||\theta_0||_{h\infty} \le 2\rho_0$.

Introduce the linear dilation transformation $d_\kappa : y \to (\kappa^k y_1, \kappa^{k-1} y_1, ..., \kappa y_k)^T$ for system (.11) which corresponds to the k-SM homogeneity condition $\forall \kappa > 0 : \Psi(d_\kappa \theta) = \Psi(\theta))$. The locally bounded function Ψ is always uniformly bounded [19].

Consider the set $\Omega_{2\rho} = \{\theta \in \mathbb{R}^k \mid ||\theta||_{h\infty} \le 2\rho_0\}$. Let $\Theta(\theta_0, s)$ be the solution of the Cauchy problem (.11) corresponding to the chosen control.

Choose the required constants as $U_M = \max_{\theta_0\in\Omega_{2\rho}} \max_{s\in[0,1]} |\Theta(\theta_0, s)|$, $M = \max_{\theta_0\in\Omega_{2\rho}} \max_{s\in[0,1]} ||\Theta(\theta_0, s)||_{h\infty}$. Now taking $\theta_* = (y_{*1}, -y_{*2}, ..., (-1)^{k-1} y_{*k})$, and $u_*(s) = (-1)^k \alpha\Psi(\Theta(\theta_0, s))$ obtain the statement of the Lemma. □

Appendix .2. Proofs for the discrete-time case

Proof of Theorem 4. Consider the case of differentiator (16). The case of the tracking differentiator is similar.

Similarly to the proof of Theorem 2 introduce $\xi_{l,k}(t_j)$ satisfying $\delta_j \xi_{l,k} = \xi_{l,k+1}\tau_j$ for $k = 0, 1, ..., l-1$, $\xi_{l,l}(t_j) = \eta_l(t_j)$, $|\xi_{l,0}(t_j)| \le \varepsilon_l$, $l = 1, 2, ..., n_f$.

Define

$$\begin{aligned}
\omega_1(t_j) = \quad & [w_1(t_j) + \xi_{n_f,0}(t_j)]/L, \\
\omega_2(t_j) = \quad & [w_2(t_j) + \xi_{n_f,1}(t_j) + \xi_{n_f-1,0}(t_j)]/L, \\
& ..., \\
\omega_{n_f}(t_j) = \quad & [w_{n_f}(t_j) + \xi_{n_f,n_f-1}(t_j) + ... + \xi_{2,1}(t_j) + \xi_{1,0}(t_j)]/L, \\
\sigma_i(t_j) = \quad & [z_i(t_j) - f_0^{(i)}(t_j)]/L.
\end{aligned}$$

Then (16) can be rewritten as

$$\begin{aligned}
\delta_j\omega = \quad & \tau_j\Omega_{n,n_f}(\omega(t_j) - d_f(t_j), \sigma_0(t_j) + \eta_0(t_j)/L, 1, \vec{\lambda}), \\
\delta_j\Sigma = \quad & \tau_j[D_{n,n_f}(\omega_1(t_j) - \xi_{n_f,0}(t_j), \Sigma(t_j), 1, \vec{\lambda}) + d(t_j)] \\
& \qquad + T_n(\Sigma(t_j), \tau_j) + R(t_j); \\
d_f(t_j) = \quad & (\xi_{n_f,0}(t_j), \xi_{n_f-1,0}(t_j), ..., \xi_{1,0}(t_j))^T, \\
d(t_j) = \quad & (0, ..., 0, -f_0^{(n+1)}(t_j))^T, \\
R(t_j) \in \quad & [-1,1](\tfrac{\tau_j^{n+1}}{(n+1)!}, ..., \tfrac{\tau_j^2}{(2)!}, 0, 0)^T,
\end{aligned} \tag{.12}$$

where the vector R contains the residual Taylor terms of $f_0^{(i)}$ of different orders.

In their turn discrete solutions of (.12) can be presented as piece-wise linear solutions of the retarded DI

$$\begin{aligned}
\dot{\omega} \in \quad & \Omega_{n,n_f}(\omega + \vec{\rho}, \sigma_0 + \rho^{n+1}[-1,1], 1, \vec{\lambda})|_{t=t_j}, \\
\dot{\Sigma} \in \quad & [D_{n,n_f}(\omega_1 + \rho^{n+n_f+1}[-1,1], \Sigma, 1, \vec{\lambda}) + h]|_{t=t_j} \\
& \qquad + T_n(\Sigma, \rho)|_{t=t_j}\rho^{-1} + \hat{R}; \\
\vec{\rho} = \quad & [-1,1](\rho^{n+n_f+1}, \rho^{n+n_f}, ..., \rho^{n+2})^T, \ t \in [t_j, t_{j+1}), \\
h = \quad & [-1,1](0, ..., 0, 1)^T, \ \hat{R} = [-1,1](\tfrac{\tau^n}{(n+1)!}, ..., \tfrac{\tau}{2!}, 0, 0)^T.
\end{aligned} \tag{.13}$$

Now the required accuracy once more follows from Section 2.1 and [46]. □

Proof of Lemma 2. The proof employs the proof idea of Lemma 1. Divide the time axis into intervals $[t_{j_l}, t_{j_{l+1}}]$ containing integer number N_l of sampling intervals, $j_{l+1} - j_l = N_l$ such that $t_{j_{l+1}} - t_{j_l} \le T < t_{j_{l+1}+1}$. In the following τ is assumed to be sufficiently small, which allows using the symbol $O(\tau)$.

There is a solution $\vec{\xi}_l(t_j) = (\xi_{l,0}(t_j), ..., \xi_{l,k-1}(t_j))^T \in \mathbb{R}^k$ of (20) defined over each such interval. A global solution $\vec{\xi}$ is to be built for the equation

$$\delta_j\vec{\xi} = \tau_j[J_k\vec{\xi}(t_j) + b\nu(t_j) + bu(t_j)], \tag{.14}$$

where J_k, b are as in (.8), u is the virtual control and

$$\nu(t_j) = \eta_k(t_j) + \tilde{\eta}(t_j),\ \tilde{\eta}(t_j) = -u(t_j),\ \eta_k(t_j) = \nu(t_j) + u(t_j).$$

In the case $k = 1$ get $\rho = a_0$, $\vec{\xi}_j$ is scalar, $|\xi_j| \leq a_0$, and taking $u(t_j) = (\xi_{l+1}(t_{j_{l+1}}) - \xi_l(t_{j_{l+1}}))/(t_{j_{l+1}} - t_{j_l})$ for $t_j \in [t_{j_l}, t_{j_{l+1}}]$ proves the Lemma.

Let now $k > 1$. Without losing the generality assume $l = 0$. Denote $x(t_j) = \vec{\xi}(t_j) - \vec{\xi}_0(t_j)$, $x = (x_1, ..., x_k)^T$, $\Delta = \xi_1(t_{j_1}) - \xi_0(t_{j_1})$. Then the problem is to find $u(t_j) \in \mathbb{R}$, $j \in [0, N_0]$ such that $x(t_0) = 0$, $x(t_{N_0}) = \Delta$, while roughly speaking $x_1(t_j), ..., x_k(t_j)$ are to be of the order $\rho^k, ..., \rho$, and $\delta_j x = J_k x(t_j) + bu(t_j)$.

In its turn that problem is equivalent to the stabilization problem in the backward time $\hat{t}_j = t_{N_0} - t_{N_0 - j}$

$$\begin{aligned} &\delta_j x = -J_k x(\hat{t}_j) - bu(\hat{t}_j),\ \hat{t}_j \in [0, N\tau], \\ &x(t_0) = \Delta,\ x(t_N) = 0. \end{aligned} \tag{.15}$$

Like in the proof of Lemma 1 apply the stabilization Lemma 3 FT stabilizing x from the value Δ in *continuous time* by continuous-time control $\hat{u}$. Let $\hat{u} = 0$ afterwards. The resulting control $\hat{u}(\hat{t})$ is in continuous backward time $\hat{t} \in [0, T]$. Recall that $t_{N_0} - t_0 \leq T < t_{N_0+1} - t_0 \leq t_{N_0} - t_0 + \tau$.

Discretize the obtained control by the simple zero-hold procedure. It turns into the standard Euler integration (still in the backward time) of the solution by Lemma 1 on the ending segment of the length not exceeding $T\rho/\rho_0$. Let $t_{\hat{j}}$ be the end point of that discrete solution. Naturally, $x(t_{N_0}) = \Delta$, $x(t_{\hat{j}}) \approx 0$.

Due to the homogeneity of system (.11) obtain [46]

$$\begin{aligned} &||x(t_{\hat{j}})||_{h\infty} \leq K\tau, \\ &||x(t_j)||_{h\infty} \leq (M+1)\rho/\rho_0,\ j = \hat{j}, \hat{j}+1, ..., j_{N_0}. \end{aligned} \tag{.16}$$

for some $K > 0$, where K depends only on the choice of Ψ [19]. The parameter M is defined in Lemma 3 and is voluntarily increased by 1 to take the discretization into account.

In order to close the difference apply the additional discrete control $u(t_j)$, $j = \hat{j}-1, \hat{j}-2, ..., \hat{j}-k$ at the additional k steps. Assign $x_{j,1} = 0$ for $j = 0, ..., \hat{j}$.

The equation $\delta_j x = J_k x(t_j)\tau_j + bu(t_j)$ is now easily solved for $u(t_j)$ and x_j in the backward direction for $j = \hat{j}-1, \hat{j}-2, ...$ It is easy to see that starting from $j = \hat{j} - k - 1$ all components of x vanish, and the equalities $x_j = 0$, $u(t_j) = 0$ hold for $j \leq \hat{j} - k - 1$. Moreover, it follows from (.16) that $|x_{j,i}| \leq \tilde{K}\tau^{k+1-i}$ for $j \leq \hat{j} - k - 1$ and some $\tilde{K}$ depending on K (proved by induction in k). It is the only place where the condition $\sup \tau_j / \inf \tau_j \leq c_\tau$ of the lemma is used. Thus the additional control $u(t_j)$, $j > \hat{j}$, and the corresponding solution $x(t_j)$ do not destroy the magnitude requirements.

The rest is the same as for Lemma 1: the constructed signal η_k is of the global sampling order k, whereas the found virtual control u is itself a locally filterable signal of the order 1. Now u is represented as the sum of a bounded signal η_0 and a signal η_1 of the global sampling filtering order 1 (see the case $k = 1$ above). □

Proof of Theorem 5. The noise components $\underline{\eta}_l$, $l = 0, ..., n_f$, are treated exactly as in the proof of Theorem 4. The idea is now to approximate the noise components $\bar{\eta}_l$, $l = 1, ..., n_f$, which are of the (non-discrete) filtering orders $1, ..., n_f$ by noises of the sampling filtering orders $1, ..., n_f$.

Consider one such component $\bar{\eta}_l$. Let $\bar{\xi}_l^{(l)} = \bar{\eta}_l$, $|\bar{\xi}_l| \leq \bar{\varepsilon}_l$, $|\bar{\xi}_l^{(l+1)}| \leq L_{\eta l}$. According to [33] starting from some moment inequalities $|\bar{\xi}_l^{(k)}| \leq \frac{\pi}{2} L_{\eta l}^{\frac{k}{l+1}} \varepsilon_l^{\frac{l+1-k}{l+1}}$ hold for $k = 0, ..., l+1$. Note that this can be rewritten as $|\bar{\xi}_l^{(k)}|/L \leq \gamma\rho^{n+l-k+1}$ for some $\gamma > 0$, which exactly corresponds to the weight $\deg \omega_{n_f+1-l+k} = n + n_f + 2 - (n_f + 1 - l + k)$, $k \leq l - 1$, from the proof of Theorem 4.

Similarly to the proofs of Theorems 4 and 2, introduce

$$
\begin{aligned}
\omega_1(t_j) = \quad & [w_1(t_j) + \underline{\xi}_{n_f,0}(t_j) + \bar{\xi}_{n_f}(t_j)]/L, \\
& ..., \\
\omega_{n_f}(t_j) = \quad & [w_{n_f}(t_j) + \underline{\xi}_{n_f,n_f-1}(t_j) + ... + \underline{\xi}_{2,1}(t_j) + \underline{\xi}_{1,0}(t_j)]/L + \\
& [\bar{\xi}_{n_f}^{(n_f-1)}(t_j) + ... + \dot{\bar{\xi}}_2(t_j) + \bar{\xi}_1(t_j)]/L, \\
\sigma_i(t_j) = \quad & [z_i(t_j) - f_0^{(i)}(t_j)]/L.
\end{aligned}
$$

Now, taking into account that $|\bar{\xi}_l^{(k)}(t_{j+1}) - \bar{\xi}_l^{(k)}(t_j)|/L \leq \sup |\bar{\xi}_l^{(k+1)}|\tau/L \leq \gamma_1\rho^{n+l-k+1}$ for some $\gamma_1 > 0$, $k = 0, ..., l$, $l = 1, ..., n_f$, obtain piece-wise linear solutions of the retarded DI

$$
\begin{aligned}
\dot{\omega} \in &\quad \Omega_{n,n_f}(\omega + \vec{\rho}, \sigma_0 + \rho^{n+1}[-1,1], 1, \vec{\lambda})|_{t=t_j} + \vec{\rho}_1, \\
\dot{\Sigma} \in &\quad [D_{n,n_f}(\omega_1 + \rho^{n+n_f+1}[-1,1], \Sigma, 1, \vec{\lambda}) + h]|_{t=t_j} \\
&\qquad\qquad\qquad + T_n(\Sigma, \rho)|_{t=t_j}\rho^{-1} + \hat{R}; \\
\vec{\rho} = &\quad \gamma_2[-1,1](\rho^{n+n_f+1}, \rho^{n+n_f}, ..., \rho^{n+2})^T, \\
\vec{\rho}_1 = &\quad \gamma_2[-1,1](\rho^{n+n_f}, \rho^{n+n_f-1}, ..., \rho^{n+1})^T, \ t \in [t_j, t_{j+1}), \\
h = &\quad [-1,1](0, ..., 0, 1)^T, \ \hat{R} = [-1,1](\tfrac{\tau^n}{(n+1)!}, ..., \tfrac{\tau}{2!}, 0, 0)^T.
\end{aligned}
$$

It only differs from (.13) by the presence of some $\gamma_2 > 1$ in $\vec{\rho}$ and the additional external disturbance term $\vec{\rho}_1$. Now the theorem once more follows from [46]. Note that the disturbance is of a bit more general form than in Section 2.1. □

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