Chapter 1

On Discretization of High Order Sliding Modes

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Abstract

Output-feedback high-order sliding-mode (HOSM) controls include HOSM-based differentiators and, therefore, possess complicated discontinuous dynamics. Their practical application naturally involves discrete noisy output sampling and numeric integration of the internal variables. Resulting hybrid systems are shown to be stable, and the corresponding asymptotic sliding-mode accuracies are calculated in the presence of Euler integration and discrete sampling, whereas both might feature variable or constant time steps. Discrete differentiators are developed which restore the optimal accuracy of their continuous-time counterparts. Numeric criteria detect the end of the differentiator transient. Simulation confirms the presented results.

1.1 Introduction

According to the Sliding Mode Control (SMC) approach the dynamics uncertainty is to be removed keeping an appropriate constraint $\sigma = 0$. Due to the system uncertainty the control exactly keeping the constraint is unknown, and the control is switched providing for returning to the constraint each time the equality $\sigma = 0$ turns out to be violated. It results in high-frequency switching of the control, and the corresponding motion mode $\sigma \equiv 0$ features theoretically infinite control switching frequency and is called Sliding Mode (SM), whereas σ is called the sliding variable [52, 16, 49]. Further for simplicity σ is assumed to be a scalar function.

The advantages and disadvantages of the approach are obvious. On one hand the control is simple and effective. It is especially simple, if $\dot{\sigma}$ contains control (the relative-degree-1 case), and the control takes the form of a simple relay $u = -\alpha \operatorname{sign} \sigma$. The constraint $\sigma = 0$ is established in finite time, and the uncertainty is effectively diminished [52, 16, 49]. On the other hand the control is discontinuous on the constraint manifold, and the high-frequency switching can cause dangerous system vibrations (the chattering effect [52, 5, 20]).

High-Order Sliding Modes (HOSMs) [8, 27, 29, 41, 45, 50] were historically proposed to overcome the above chattering-effect problem and to solve the problem of establishing the constraint in finite time also in the case when the relative degree differs from 1.

Suppose that the equality $\sigma = 0$ is kept on the solutions of a closed-loop system. The sliding order r is the lowest integer r, such that the rth-order total-time derivative $\sigma^{(k)}$ is not a continuous function of the state variables and time [27, 29]. The corresponding motion $\sigma \equiv 0$ is called rth-order SM, and for brevity is called r-sliding mode (r-SM).

The chattering attenuation is obtained by artificially increasing the number k of the derivative $\sigma^{(k)}$ which contains the discontinuity. Consider $u^{(l)}$ as the virtual control, and suppose that $\sigma^{(r)}$ is the first total time derivative of σ to contain the control u, then $\sigma^{(r+l)}$ is to be the first to contain the virtual control $u^{(l)}$. Thus, one has to establish the $(r+l)$-SM $\sigma = 0$ in finite time by means of discontinuous $u^{(l)}$ [7, 27, 11]. It has been shown [33] that in this case the high-energy chattering is removed.

Note that the chattering reduction is not due to the continuity of the corresponding control $u(t)$, but due to simultaneously keeping continuous functions of system variables $\sigma, \dot{\sigma}, ..., \sigma^{(k+l-1)}$ at zero [33]. Homogeneous HOSMs also feature high accuracy in the presence of small switching imperfections and noises [27, 30].

Standard SMs [16, 52] are of the first order, i.e. already $\dot{\sigma}$ contains discontinuous control u, $r = 1$. Thus, the chattering attenuation is obtained using 2-SMs and discontinuous $\dot{u}$ [7, 8, 27].

Relative degree is defined as the lowest total derivative order of the output σ which explicitly contains control [24]. Families of universal controls are recursively constructed and solve the problem for any relative degree r [30, 49] of the output σ by means of discontinuous r-SM control. In particular, the finite-time stabilization of σ is possible by means of control, continuous everywhere except the manifold $\sigma = \dot{\sigma} = \cdots = \sigma^{(r-1)} = 0$ [31]. The controllers are complemented by the robust exact SM-based differentiators in finite time providing for the unavailable derivatives $\sigma, ..., \sigma^{(r-1)}$ [29].

SM control is proved to be insensitive to disturbances in the control channel (matched disturbances), robust with respect to sampling noises and small delays. Homogeneous SMs [30] are proved to be robust to small disturbances, including those which change the relative degree [39], and to the presence of fast stable sensors and actuators [21, 33].

It is natural to introduce appropriate nonlinear constraints, describing finite-time stable differential equations $\Sigma(t, \sigma, ..., \sigma^{(r-1)}) = 0$. Then keeping $\Sigma \equiv 0$ in 1-SM would lead to the establishment of the r-SM $\sigma \equiv 0$ in finite time. Unfortunately, this idea does only easily work with $r = 2$, when $\sigma = \dot{\sigma} + |\sigma|^{1/2} \operatorname{sign} \sigma$ can be taken. For higher r one typically encounters the problem of unbounded gradients of the respective functions Σ, which results in singular unbounded 1-SM controls [41]. The problem is usually solved in the framework of the homogeneity theory [9, 23, 30, 46].

The results described above constitute a solid foundation for extensive applications of SMC for solution of various control and observation problems under uncertainty conditions [2, 6, 8, 11, 14, 15, 17, 19, 22, 41, 45, 44, 50]. Lyapunov functions are found and used for HOSM controllers [42, 47, 48, 13].

The asymptotic accuracy of the r-SM was studied in [27]. It has been found there that the best possible accuracy of a SM in the presence of discrete switching is directly defined by its sliding order r. In particular, the best possible asymptotic accuracy with the sampling time interval $\tau > 0$ is $\sigma^{(j)} = O(\tau^{r-j})$, $j = 0, 1, \ldots, r-1$. And indeed the homogeneity technique [30] provides for that accuracy. Moreover, the accuracy is preserved, if the derivatives are estimated by homogeneous differentiators [29]. More exactly, the asymptotic accuracy $\sigma^{(j)} = O(\max(\tau^{r-j}, \varepsilon^{(r-j)/r}))$ is obtained for the sampling accuracy $\varepsilon > 0$ of the σ-measurements. Unfortunately, this result is restricted to the ideal case, when the system is described by the Filippov differential equations with zero-hold measurements.

The output-feedback HOSM control contains differentiators (i.e. observers) in the feedback. Also the chattering attenuation procedure described above inserts integrators in the feedback. Thus, the produced feedback often (actually almost always) has its own dynamics. In reality the control input is produced by a modern computer unit, integrators are replaced by some discrete integral approximation, most usually by one-step Euler approximation. The resulting closed loop system features complex interaction of the discrete controller with the continuous-time dynamic system, i.e. is a heterogeneous, hybrid system. The area was practically *terra incognita* 2-3 years ago. Not only the accuracy, even the stability of the hybrid system was not proved. The authors have got considerable advances in this field. This article presents an intermediate summary of the current state of the research.

1.2 Preliminaries: sliding order and SM accuracy

Realization of SM control inevitably includes discrete switching and noisy measurements. Here we estimate the worst and the best possible realization accuracy of HOSMs in the presence of noises and discrete switching, and show that possible practical SM accuracies are strictly determined by the numbers of the output derivatives in which the discontinuity appears for the first time.

1.2.1 Accuracy of SMs in the absence of noises

The following lemma actually describes SM accuracy in the presence of discrete control swithing. It extends a similar result of [27] and is similarly proved.

Lemma 1. *Let $\omega(t)$ be a scalar function having continuous derivative $\omega^{(l)}$ on the segment $[0, \tau]$, $\tau > 0$. Then for each natural number l there exist such $c_0, c_1, ..., c_{l-1} > 0$ and $d_1, d_2, ..., d_{l-1} > 0$ that for any $\delta > 0$*

1. *if $|\omega^{(l)}| \geq \delta$ holds on the segment, then*

$$\max|\omega| \geq c_0 \delta \tau^l, \max|\dot{\omega}| \geq c_1 \delta \tau^{l-1}, ..., \max|\omega^{(l-1)}| \geq c_{l-1} \delta \tau; \tag{1.1}$$

2. *if* $|\omega^{(l)}| \le \delta$ *and* $|\omega| \le d_0 \delta \tau^l$ *hold over the segment* $[0,\tau]$ *for some* d_0*, then*

$$\max|\dot{\omega}| \le (d_0 d_1 + 1)\delta\tau^{l-1}, ..., \max|\omega^{(l-1)}| \le (d_0 d_{l-1} + 1)\delta\tau. \quad (1.2)$$

Obviously the second statement of the Lemma provides for a rather crude estimation, since with $d_0 = 0$ all derivatives of ω should vanish. Lemma 1 shows that the accuracy of keeping an output $\sigma = 0$ is directly connected with the number of its continuous total time derivatives. This naturally implies the following definition [27], directly formulated for the vector output σ.

Definition 1. *Suppose the constraint* $\sigma(x) = 0$*,* $\sigma : \mathbb{R}^k \to \mathbb{R}^m$*, is identically kept on some solutions of a dynamic system* $\dot{x} = v(x)$*,* $x \in \mathbb{R}^k$*, understood in the Filippov sense,* $v(x)$ *is any Lebesgue-measurable locally bounded vector-function. Then the solutions keeping* $\sigma(x) = 0$ *are said to be in the* $(r_1, r_2, ..., r_m)$*th-order sliding mode, if*

1. *the total time derivatives* $\sigma_i^{(j)}(x)$ *are continuous functions of* x*,* $j \le r_i - 1$*;*
2. *the r-sliding set* $L_r = \left\{ x \,|\, \sigma_i^{(j)}(x) = 0,\ j \le r_i - 1,\ i = 1, ..., m \right\}$ *is not empty, and locally consists of Filippov solutions;*
3. $\sigma_1^{(r_1)}, \sigma_2^{(r_2)}, ..., \sigma_m^{(r_m)}$ *are discontinuous functions of* x *or do not exist.*

In the non-autonomous case the time t *is considered as an additional coordinate, and the equation* $\dot{t} = 1$ *is formally added.*

Recall that a scalar output $\sigma(t,x)$ of a smooth SISO system

$$\dot{x} = a(t,x) + b(t,x)u, \quad (1.3)$$

has a relative degree r, if the rth total time derivative of σ is the first to explicitly contain the control, and the corresponding control coefficient does not vanish. In the MIMO case the dimensions of u and σ are to be equal, for each component σ_i of the output a partial relative degree r_i is to exist with respect to some control component, and the matrix $\left(\frac{\partial \sigma_i^{(r_i)}}{\partial u_j} \right)$ is to be nonsingular [24].

It follows from the sliding-order definition that SM order is component-wise larger or equal than vector relative degree, if the latter exists. It can be higher, if, for example, the control itself features some discontinuous dynamics. An r-SM is called unstable, asymptotically stable, finite-time stable, etc., if the r-sliding manifold L_r features the same property.

Consider a MIMO dynamic system (1.3). Let the output $\sigma(t,x)$ and the input u be vectors, $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $u \in \mathbb{R}^l$, a, b be smooth. The system is assumed to have the partial relative degrees $r = (r_1, ..., r_m)$, $r_i > 0$, which means that the successive total time derivatives $\sigma_i^{(j)}$, $j = 0, 1, ..., r_i - 1$, $i = 1, ..., m$, do not contain controls, but controls appear in $\sigma_i^{(r_i)}$. Respectively, get a vector equation

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \quad (1.4)$$

where $\sigma^{(r)}$ denotes $(\sigma_1^{(r_1)},...,\sigma_m^{(r_m)})^T$. As a direct consequence of Lemma 1 obtain the following Theorem.

Theorem 1. *Let system* (1.3) *be smooth with partial relative degrees* $r = (r_1,...,r_m)$. *Then for some constants* $c_{r_i,0},...,c_{r_i,r_i-1}$, $d_{r_i,1},...,$ $d_{r_i,r_i-1} > 0$, $i = 1,...,m$ *the following is true for each component* σ_i. *Over any time interval of the length* τ *with continuous control* $u(t) \in \mathbb{R}^l$

1. *if* $|\sigma_i^{(r_i)}| \geq \delta_i$ *holds on the segment for some* $\delta_i > 0$, *then*

$$\max|\sigma_i| \geq c_{r_i,0}\delta_i\tau^{r_i}, \max|\dot{\sigma}_i| \geq c_{r_i,1}\delta_i\tau^{r_i-1},...,\max|\sigma^{(r_i-1)}| \geq c_{r_i,r_i-1}\delta_i\tau; \quad (1.5)$$

2. *if* $|\sigma_i^{(r_i)}| \leq \delta_i$ *and* $|\sigma_i| \leq d_{r_i,0}\delta_i\tau^r$ *hold over the segment for some* $d_{r_i,0}$ *and* $\delta_i > 0$, *then*

$$\max|\dot{\sigma}| \leq (d_{r_i,0}d_{r_i,1}+1)\delta_i\tau^{r_i-1},...,\max|\sigma^{(r_i-1)}| \leq (d_{r_i,0}d_{r_i,r_i-1}+1)\delta_i\tau. \quad (1.6)$$

In particular, in the case of the SISO SMC problem, it follows from the Theorem that no one can expect an accuracy better than $\sigma = O(\tau^r), \dot{\sigma} = O(\tau^{r-1}),...,\sigma^{(r-1)} = O(\tau)$ in the sliding mode $\sigma \approx 0$, if $\sigma^{(r)}$ is separated from zero between the switchings. On the other hand, if $\sigma^{(r)}$ exists and is bounded, then keeping $\sigma \approx 0$ implies that also $\sigma^{(j)} \approx 0$, $j = 1,\ldots,r-1$.

As follows from the second statement of the Theorem, the r-SM accuracy can be higher than $\sigma^{(j)} = O(\tau^{r-j})$, if $\sigma^{(r)}$ is kept close to zero. For example, the implicit Euler method [1] actually increases the order of the real (i.e. approximate) SM due to the on-line estimation of the equivalent control, which allows to decrease the discontinuous component of the control. Unfortunately such estimation requires some additional system knowledge.

1.2.2 Accuracy of sliding modes in the presence of noises

Once more consider the uncertain SMC problem (1.3), (1.4). Recall that it is possible to provide for the exact finite-time establishment of the r-SM $\sigma \equiv 0$ using only output measurements [29, 30].

Theorem 2. *Suppose that the control, based on the input measurements only, provides for the exact finite-time establishment of the r-SM* $\sigma \equiv 0$ *for any function h, satisfying* $\|h\| \leq C$. *Let* σ_i *be measured with a Lebesgue-measurable noise* $\eta_i(t)$ *of the maximal magnitude* $\varepsilon_i \geq 0$, $\eta_i(t) \leq \varepsilon_i$, *with unknown features,* $i = 1,\ldots,m$. *Then the worst-case SM accuracy cannot be better than*

$$\begin{aligned}
&|\sigma_i| \leq \varepsilon_i, |\dot{\sigma}| \leq \tilde{c}_{i,1}\varepsilon_i^{\frac{r_i-1}{r_i}},...,|\sigma_i^{(r_i-1)}| \leq \tilde{c}_{i,r_i-1}\varepsilon_i^{\frac{1}{r_i}},\\
&\tilde{c}_{i,j} = \left(\frac{C}{m^{\frac{1}{2}}}\right)^{\frac{j}{r_i}}, j = 1,\ldots,r_i, i = 1,\ldots,m.
\end{aligned} \quad (1.7)$$

Proof. Let the output satisfy the equation $\sigma^{(r)} = g(t,x)u$, i.e. (1.4) with $\|h\| \leq C$, $h \equiv 0$. Let now the measured signal $\hat{\sigma}_i$ be of the form $\hat{\sigma}_i(t,x) = \sigma_i(t,x) + \varepsilon_i \cos((m^{-1/2}C/\varepsilon)^{1/r_i}t)$, i.e. the noise be equal $\varepsilon \cos((m^{-1/2}C/\varepsilon)^{1/r_i}t)$. Then the noisy signal $\hat{\sigma}_i$ satisfies

$$\sigma_i^{(r_i)} = \left(\cos\left(\left(\frac{C}{\varepsilon m^{\frac{1}{2}}}\right)^{\frac{1}{r}} t\right)\right)^{(r)} + g(t,x)u, \left|\left(\cos\left(\left(\frac{C}{\varepsilon m^{\frac{1}{2}}}\right)^{\frac{1}{r}} t\right)\right)^{(r)}\right| \leq \frac{C}{m^{\frac{1}{2}}}.$$

Respectively, according to the assumptions, the control will successfully establish and keep $\hat{\sigma}_i \equiv 0$, which corresponds to $|\sigma_i^{(j)}| \leq \left(\frac{C}{\sqrt{m}}\right)^{\frac{j}{r}} \varepsilon^{\frac{r-j}{r}}$, $j = 0, \ldots, r-1$. □

Note that both Theorems 1, 2 are true for any τ and ε_i, neither τ nor ε_i need to be small.

Under the conditions of Theorem 2 let the output σ be measured with noises of the magnitudes $\varepsilon_i > 0$ at some discrete time instants, and let the control be updated at each sampling instant and remain constant between the sampling moments. Then the inequalities (1.5) hold independently of the noise presence over each sampling time interval of the length τ on which the inequality $|\sigma_i^{(r_i)}| > \delta_i > 0$ is held.

The situation is more complicated, if the maximal sampling step tends to zero. Some additional assumptions are needed to ensure that the corresponding solutions uniformly converge to solutions with continuous sampling. If such a convergence takes place, then, according to (1.7), the worst case SM accuracy is not better than $|\sigma_i^{(j)}| = O(\varepsilon_i^{\frac{r-j}{r}})$.

Example. The output σ of the SISO system (1.3), (1.19) of the relative degree r is traditionally nullified by keeping the constraint $\Sigma = \left(\frac{d}{dt} + \lambda\right)^{r-1} \sigma = 0$ in 1-SM $\Sigma \equiv 0$ [52, 16]. Let τ be the sampling step, then $\Sigma = O(\tau)$ is the only possible accuracy according to Theorem 1. The respective overall r-SM accuracy is $\sigma = O(\tau), \dot{\sigma} = O(\tau), ..., \sigma^{(r-1)} = O(\tau)$ [51]. It definitely satisfies (1.5), but is much worse than the best possible accuracy (1.6).

1.3 Accuracy of homogeneous differential inclusions

Recall that a solution of a DI $\dot{x} \in F(x)$, $F(x) \subset \mathbb{R}^n$, is defined as any absolutely continuous function $x(t)$, satisfying the DI for almost all t. We call a DI $\dot{x} \in F(x)$ *Filippov DI*, if $F(x) \subset \mathbb{R}^n$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

It is well-known that such DIs feature most standard features, i.e. existence and extendability of solutions, except the uniqueness of solutions [18]. Asymptotically stable Filippov DIs have smooth Lyapunov functions [12].

1.3.1 Weighted homogeneity of differential inclusions

Introduce the weights $m_1, m_2, \ldots, m_n > 0$ of the coordinates $x_1, x_2, \ldots, x_n$ in $\mathbb{R}^n$. Define the dilation

$$d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n),$$

where $\kappa > 0$. Recall [4], [25] that a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to have the homogeneity degree (weight) $q \in \mathbb{R}$, $\deg f = q$, if the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds for any x and $\kappa > 0$.

Definition 2 ([30]). *A vector-set field $F(x) \subset \mathbb{R}^n$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^n$, is called homogeneous of the degree $q \in \mathbb{R}$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$.*

Consider a differential equation $\dot{x} = f(x)$, $\dot{x}_i = f_i(x)$, as a particular case of DI, when the set $F(x)$ contains only one vector $f(x)$. Then the above definition is reduced to the standard definition $\deg \dot{x}_i = \deg f_i = m_i + q$ [4], [25]. Note that the non-zero homogeneity degree q of a vector-set field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, ..., m_n$.

Also note that the homogeneity of a vector-set field $F(x)$ can equivalently be defined as the invariance of the DI $\dot{x} \in F(x)$ with respect to the combined time-coordinate transformation

$$G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x), \;\; \kappa > 0,$$

where p, $p = -q$, might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as

$$\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x).$$

Theorem 3 ([30, 34, 43]). *Let a Filippov DI be homogeneous of a negative homogeneity degree. Then finite-time (FT) stability, asymptotic stability and contractivity features are equivalent. The maximal (minimal) stabilization time is a well-defined upper (lower) semi-continuous function of the initial conditions.*

Here the upper (lower) semi-continuity of a scalar function ϕ means that $\limsup_{x \to y} \phi(x) \leq \phi(y)$ ($\liminf_{x \to y} \phi(x) \geq \phi(y)$). The contractivity [30] is equivalent to the existence of $T > 0$, $R > r > 0$, such that all solutions starting in the ball $||x|| \leq R$ at the time 0 are in the smaller ball $||x|| \leq r$ at the time T. It can be also proved that FT stability of $\dot{x} \in F(x)$ implies the inequalities $\deg F = q < 0$, $\deg \dot{x}_i = \deg x_i + \deg F = m_i + q \geq 0$, $i = 1, \ldots, n$.

1.3.2 Accuracy of disturbed homogeneous differential inclusions

It is well-known that FT-stable homogeneous differential inclusions feature robustness with respect to various disturbances, delays and sampling errors [10, 30, 32, 39, 40, 43]. Estimate the steady-state accuracy of a disturbed differential inclusion

$$\dot{x} \in F(x, \gamma), \;\; x \in \mathbb{R}^n, \;\; \gamma \in \mathbb{R}^\nu, \tag{1.8}$$

where γ is the vector disturbance parameter. The set field $F(x,\gamma) \subset \mathbb{R}^n$ is a non-empty compact convex set-valued function, upper-semicontinuous at all points $(x,0)$, $x \in \mathbb{R}^n$.

Introduce the dilations

$$d_\kappa : (x_1,...,x_n) \mapsto (\kappa^{m_1} s_1,...,\kappa^{m_n} s_n),\ m_1,...,m_n > 0,$$
$$\Delta_\kappa : (\gamma_1,...,\gamma_\nu) \mapsto (\kappa^{\omega_1}\gamma_1,...,\kappa^{\omega_\mu}\gamma_\mu),\ \omega_1,...,\omega_\nu > 0.$$

Inclusion (1.8) is assumed homogeneous in both x and γ, while the undisturbed inclusion $\dot{x} \in F(x,0)$ is assumed FT stable with the homogeneity degree $q = -p$, $p > 0$. Hence, $m_i \geq p$. The homogeneity of (1.8) means that the transformation

$$(t,x,\gamma) \mapsto (\kappa^p t, d_\kappa x, \Delta_\kappa \gamma),\ \ \kappa > 0, \tag{1.9}$$

establishes a one-to-one correspondence between the solutions of the inclusion (1.8) with different parameters γ. In other words, $F(x,\gamma) = \kappa^p d_\kappa^{-1} F(d_\kappa x, \Delta_\kappa \gamma)$. In particular, the standard homogeneity $F(x,0) = \kappa^p d_\kappa^{-1} F(d_\kappa x, 0)$ is obtained for $\gamma = 0$.

In its turn $\gamma \in \Gamma(\rho,x) \subset \mathbb{R}^\nu$, where Γ is a homogeneous compact non-empty set-valued function with the magnitude parameter $\rho \geq 0$, i.e. $\forall \kappa,\rho > 0 \forall x \in \mathbb{R}^n$: $\Gamma(\kappa^{m_\rho}\rho, d_\kappa x) = \Delta_\kappa \Gamma(\rho,x)$, $m_\rho > 0$. The function Γ monotonously increases with respect to the parameter ρ, i.e. $0 \leq \rho \leq \hat{\rho}$ implies $\Gamma(\rho,x) \subset \Gamma(\hat{\rho},x)$. It is also assumed that $\Gamma(0,x) = \{0\} \subset \mathbb{R}^n$ and $\Gamma(\rho,x)$ is Hausdorff-continuous in ρ,x at the points $(0,x)$.

It is easy to see that the time-coordinate-parameter transformation

$$\tilde{G}_\kappa : (t,\rho,x) \mapsto (\kappa^p t, \kappa^{m_\rho}\rho, d_\kappa x) \tag{1.10}$$

establishes a one-to-one correspondence between the solutions of $\dot{x} \in F(x,\Gamma(\rho,x))$ with different values of ρ.

Obviously, due to the homogeneity of Γ and the compactness of the disk $||x|| \leq R$, for any $R > 0$ and any $\varepsilon > 0$ there exists $\rho > 0$, such that $||x|| \leq R$ implies that $\forall z \in \Gamma(\rho,x)$: $\|z\| < \varepsilon$. Also, with any fixed $\rho \geq 0$ the function Γ maps bounded sets to bounded sets.

Now, consider the general retarded differential inclusion

$$\dot{x} \in F(x(t-\tau[0,1]),\Gamma(\rho,x(t-\tau[0,1]))), \tag{1.11}$$

where $\tau \geq 0$ is the maximal possible time delay.

The presence of the delays in (1.11) requires some initial conditions

$$x(t) = \xi(t), \quad t \in [-\tau,0], \quad \xi \in \Xi(\tau,\rho,x_0). \tag{1.12}$$

The sets $\Xi(\tau,\rho,x_0)$ should posses some natural homogeneity properties, which are automatically satisfied, provided $\Xi = \tilde{\Xi}_\varpi(\tau,\rho,x_0)$, where $\tilde{\Xi}_\varpi(\tau,\rho,x_0)$ is comprised of the solutions of the simple Filippov differential inclusion

$$\begin{aligned} &\dot{\xi}_i \in \varpi\left(\|\xi\|_h + \rho^{1/m_\rho}\right)^{m_i - p}[-1,1], \\ &i = 1,...,n,\ \xi(0) = x_0,\ -\tau \leq t \leq 0. \end{aligned} \tag{1.13}$$

Recall that $m_i \geq p$. It is also formally assumed here that $\forall c \geq 0 : c^0 \equiv 1$. Inclusion (1.13) is homogeneous (i.e. invariant) with respect to the transformation

$(t,\tau,\rho,\xi) \mapsto (\kappa t, \kappa^p \tau, \kappa^{m_\rho}\rho, d_\kappa \xi)$. The parameter ϖ is chosen sufficiently large to include the initial conditions of a considered concrete system.

Obviously, regular solutions of $\dot{x} \in F(x,0)$ always satisfy (1.11), i.e., solutions of (1.11) always exist. Also, solutions of the inclusion with "discrete measurements" and uniformly-bounded "noises" always exist. They correspond to the solutions with the right-hand side of the inclusion frozen between the "sampling instants", $\dot{x}(t) = \dot{x}(t_k) \in F(x(t_k), \Gamma(\rho, x(t_k)))$, $t \in [t_k, t_{k+1}]$, with the time periods $t_{k+1} - t_k \leq \tau$. Both types of solutions are compatible with the above construction (1.13) of initial conditions.

Theorem 4 ([30, 39]). *After a finite-time transient all solutions of the disturbed differential inclusion* (1.11) *enter the region* $|x_i(t)| \leq \mu_i \delta^{w_i}$, $\delta = \max\{\rho^{1/m_\rho}, \tau^{1/p}\}$, *to stay there forever. The constants* $\mu_i > 0$ *do not depend on* $\rho \geq 0$.

1.3.3 Accuracy of finite-time stable homogeneous systems

This subsection demonstrates application of Theorem 4.

1.3.3.1 Accuracy of homogeneous finite-time stable systems

Consider a FT-stable Filippov homogeneous differential inclusion $\dot{x} \in F(x)$, $\deg t = 1$, $\deg x_i = m_i$, $i = 1, ..., n_x$. Let x be measured at discrete time instants t_k, $t_{k+1} - t_k \leq \tau$ with the measurement errors $|\eta_i(t_k)| \leq \varepsilon_i$. Then the solutions satisfy the differential inclusion $\dot{x} \in F(x(t - \tau[0,1]) + \varepsilon[-1,1])$. Respectively the accuracy $x_i = O(\rho^{m_j})$, $j = 1, ..., n_x$, is established in finite time, where $\rho = \max\{\tau, \max_i \varepsilon_i^{\frac{1}{m_i}}\}$. Note that that estimation corresponds to the simplest case, when the process is described by a differential inclusion between the sampling instants. No discrete dynamics is involved.

1.3.3.2 Accuracy of a hybrid system

Consider a FT-stable homogeneous differential inclusion

$$\dot{x} \in F(x,z), \tag{1.14}$$

$$\dot{z} \in \Psi(h(x),z). \tag{1.15}$$

Here (1.15) represents a possibly discontinuous "feedback" understood in the Filippov sense. The system is assumed to describe error dynamics of some controlled process. The uncertainties and time dependence are taken into account by some appropriate expansion of the sets on the right-hand side. Let the system homogeneity degree be -1, $\deg t = 1$, and $\deg x_i = m_i$, $i = 1, ..., n_x$, $\deg z_i = m_{zi}$, $i = 1, ..., n_z$. The output h is a homogeneous vector-function, $\deg h_i = m_{hi} > 0$, $i = 1, ..., n_h$. The vector variable $z \in \mathbb{R}^{n_z}$ is the internal controller state.

In "practice" the output $h(x)$ is sampled at the time instants t_k, $0 \leq t_{k+1} - t_k \leq \tau$, with the error $\eta \in \mathbb{R}^{n_h}$, $\eta_i \in \varepsilon_i[-1,1]$, $i = 1, ..., n_h$, and some "actuator" delay $\tilde{\tau}_k$. The dynamic feedback (1.15) becomes a discrete system defined at the discrete time instants $t_{k,j},\ j = 0,...,l_k,\ t_{k,0} = t_k + \tilde{\tau}_k,\ t_{k,l_k} = t_{k+1,0} = t_{k+1} + \tilde{\tau}_{k+1}$. Naturally $t_{k+1} + \tilde{\tau}_{k+1} > t_k + \tilde{\tau}_k$ is assumed to hold. It is also assumed that $0 < t_{k,j+1} - t_{k,j} = \underline{\tau}_{k,j} < \underline{\tau}(\tau)$, i.e. the maximal integration step $\underline{\tau}$ can be chosen in dependence on an upper bound τ of the sampling periods τ_k. Also $\tilde{\tau}_k \leq \tilde{\tau}$ holds.

The resulting feedback is approximated by the Euler integration, producing the hybrid system

$$\begin{array}{l} \dot{x} \in F(x, z(t_{k,j})), \\ z(t_{k,j+1}) \in z(t_{k,j}) + \Psi(h(x(t_k + \tilde{\tau}_k))) + \eta_i(t_k + \tilde{\tau}_k), z(t_{k,j}))\underline{\tau}_{k,j}. \\ j = 0,...,l_k - 1. \end{array} \tag{1.16}$$

Note that a random element is taken from the set Ψ at each time step k, j. The solution component $z(t_{k,j})$ of (1.16) can be equivalently described by solutions of equations with piece-wise-constant right-hand sides, which take the same values at the time instants $t_{k,j}$. Indeed, the intermediate values of z do not affect the process component x changing continuously in time. Thus, the components $x(t)$ of solutions of (1.16) coincide with the x components of some solutions of the differential inclusion

$$\begin{array}{l} \dot{x} \in F(x, z(t + (\tilde{\tau} + \tau)[-1, 0]))), \\ \dot{z}(t) \in \Psi(h(x(t + (\tilde{\tau} + \tau)[-1, 0])) + \varepsilon[-1, 1], z(t + \underline{\tau}[-1, 0])). \end{array} \tag{1.17}$$

In its turn these solutions satisfy

$$\begin{array}{l} \dot{x} \in F(x, z(t + \rho[-1, 0]))), \\ \dot{z}(t) \in \Psi(h(x(t + \rho[-1, 0])) + \vec{\rho}^{m_h}[-1, 1], z(t + \rho[-1, 0])). \end{array} \tag{1.18}$$

where $\rho = \max\{\tilde{\tau} + \tau, \max_i \varepsilon_i^{\frac{1}{m_{hi}}}\}$ and $\vec{\rho}^{m_h} = (\rho^{m_{h1}}, ..., \rho^{m_{hn_h}})^T$. Introduce also some appropriate initial values as it is done in section 1.3.2.

It follows now from Theorem 4 that the accuracy $x_j = O(\rho^{m_j}),\ j = 1,...,n_x$, is established in finite time.

1.4 Homogeneous continuous-time SM control

In this section we develop SMC which realizes the best possible asymptotic SM accuracy (1.6), (1.7) calculated in the previous section. All over this section the accuracies are calculated under the assumption that, whereas the sampling is discrete, the differential equations/inclusions still take place between the samplings.

1.4.1 Homogeneous SISO SM control

Consider the SISO SMC problem (1.3) with smooth functions a and b. The system is understood in the Filippov sense [18]. Let the smooth scalar output $\sigma(t,x)$ have the relative degree r, which means that

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{1.19}$$

where h, g are uncertain smooth functions, $g(t,x) \neq 0$. As usual [27, 29, 30] assume that h, g are bounded,

$$|h(t,x)| \leq C, \; 0 < K_m \leq g(t,x) \leq K_M, \tag{1.20}$$

Such bounds exist at least for any compact operational region. Any solution of (1.3) is assumed infinitely extendable in time, provided σ, its derivatives and u remain bounded along the solution.

The above are the only system conditions needed in Section 1.4.1. The system uncertainty is defined by the numbers r, C, K_m, K_M which are supposed to be known.

The uncertain dynamics (1.19) can be replaced by the concrete differential inclusion

$$\sigma^{(r)} \in [-C,C] + [K_m, K_M]u. \tag{1.21}$$

Most r-SM controllers are build as controllers for (1.21) making $\vec{\sigma} = (\sigma, \dot{\sigma}, ..., \sigma^{(r-1)})$ vanish in finite time.

In order to use the results from section 1.3.1 the closed-loop inclusion is to be homogeneous with negative homogeneity degree. Scaling the system homogeneity degree to -1, $\deg t = 1$, obtain that with $C > 0$ inevitably $\deg \sigma^{(r)} = 0$, on the other hand, $\deg \sigma^{(r)} = \deg \sigma^{(r-1)} - \deg t$. Thus, the only possible homogeneity weights are $\deg \sigma = r, \ldots, \deg \sigma^{(r-1)} = 1$. This homogeneity is called r-sliding homogeneity [30]. Respectively, the control

$$u = U_r(\vec{\sigma}) \tag{1.22}$$

is called r-sliding homogeneous, if $\deg u = 0$, i.e.

$$U_r(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}) \equiv U_r(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, ..., \kappa \sigma^{(r-1)}). \tag{1.23}$$

holds for any $\kappa > 0$, $\vec{\sigma} \in \mathbb{R}^r$. Since the control is required to be locally bounded [18], due to (1.23) it is also **globally bounded**. The right-hand side of the inclusion (1.21), (1.22) is assumed minimally enlarged at the points of the discontinuity of (1.22) to satisfy the Filippov conditions [18].

Replace $\sigma \in \mathbb{R}$ with $\omega \in \mathbb{R}$ in the following formulas, enabling the further usage of the controllers for different components of the vector output σ in the MIMO case. Let $\beta_{1,r}, \ldots, \beta_{r-1,r}$ be some predefined positive coefficients, and α be the chosen control magnitude. Then the simplest family of r-sliding homogeneous controllers of the form

$$u = -\alpha \Psi_{r-1,r}(\omega, \dot{\omega}, \ldots, \omega^{(r-1)}), \tag{1.24}$$

called embedded SM controllers [29], are provided by the following resursion. Let $d \geq r$, define

$$\begin{aligned}
&\varphi_{0,r} = \omega, N_{0,r} = |\omega|^{1/r}, \Psi_{0,r} = \operatorname{sign} \omega; \\
&\varphi_{j,r} = \omega^{(j)} + \beta_{j,r} N_{j-1,r}^{r-j} \Psi_{j-1,r}, \Psi_{j,r} = \operatorname{sign} \varphi_{j,r}, \\
&N_{j,r} = \left(|\omega|^{d/r} + |\dot{\omega}|^{d/(r-1)} + \ldots + |\omega^{(j-1)}|^{d/(r+1-j)} \right)^{1/d}.
\end{aligned} \tag{1.25}$$

The following are valid parametric sets $\{\beta_{1,r},\ldots,\beta_{r-1,r}\}, d_r$ for $r=2,\ldots,4$: $r=2$, $\{1\}$, $d_1=1$; $r=3$, $\{1,2\}$, $d_2=6$; $r=4$, $\{0.5,1,3\}$, $d_4=12$. It is further assumed that $\beta_{1,r},\ldots,\beta_{r-1,r}$ are always properly chosen, which means that the differential equations $\varphi_{r-1,r}=0$ are finite-time stable [38].

Provided the parameters $\beta_{j,r}$ are properly chosen [38] and α is sufficiently large, under the above assumptions (1.19), (1.20) the listed r-SM controllers solve the stated problem of finite-time establishing and keeping $\sigma\equiv 0$ [29] by means of uniformly bounded control and for any initial conditions.

Another well-known family of SM controllers, called quasi-continuous SM controllers [31], also features control continuous everywhere except the r-sliding set $\omega=\dot{\omega}=\ldots=\omega^{(r-1)}=0$. Such controllers feature considerably less chattering. Other constructions of similar homogeneous HOSM controllers and the choice of parameters are considered in [30, 38].

1.4.1.1 Differentiator.

Any r-sliding homogeneous controller can be combined with an $(r-1)$th-order differentiator [29] producing an output feedback controller. Its applicability in this case is possible, since $\sigma^{(r)}$ is bounded due to the boundedness of the feedback function $u=-\alpha\Psi_{r-1,r}(\vec{\sigma})$ in (1.21).

Let the input signal $f(t)$ consist of a bounded Lebesgue-measurable noise with unknown features, and an unknown basic signal $f_0(t)$, whose n_dth derivative has a known Lipschitz constant $L>0$. These are the only restrictions on the input of the differentiator. While the number L is to be known, one does not need to know the noise magnitude.

The outputs z_j of the following differentiator estimate the derivatives $f_0^{(j)}$, $j=0,\ldots,n_d$. The recursive form of the differentiator is

$$\begin{aligned}
&\dot{z}_0=-\lambda_{n_d}L^{\frac{1}{n_d+1}}|z_0-f(t)|^{\frac{n_d}{n_d+1}}\,\mathrm{sign}(z_0-f(t))+z_1,\\
&\dot{z}_1=-\lambda_{n_d-1}L^{\frac{1}{n_d}}|z_1-\dot{z}_0|^{\frac{n_d-1}{n_d}}\,\mathrm{sign}(z_1-\dot{z}_0)+z_2,\\
&\ldots\\
&\dot{z}_{n_d-1}=-\lambda_1L^{\frac{1}{2}}|z_{n_d-1}-\dot{z}_{n_d-2}|^{\frac{1}{2}}\,\mathrm{sign}(z_{n_d-1}-\dot{z}_{n_d-2})+z_{n_d},\\
&\dot{z}_{n_d}=-\lambda_0L\,\mathrm{sign}(z_{n_d}-\dot{z}_{n_d-1}).
\end{aligned}\tag{1.26}$$

Parameters λ_i of differentiator (1.26) are chosen in advance for each n_d. An infinite sequence of parameters λ_i can be built, valid for all natural n_d [29]. In particular, one can choose $\lambda_0=1.1$, $\lambda_1=1.5$, $\lambda_2=2$, $\lambda_3=3$, $\lambda_4=5$, $\lambda_5=8$ [31] or $\lambda_0=1.1$, $\lambda_1=1.5$, $\lambda_2=3$, $\lambda_3=5$, $\lambda_4=8$, $\lambda_5=12$, which is enough for $n_d\le 5$. In the absence of noises the differentiator provides for the exact estimations in finite time. Equations (1.26) can be rewritten in the usual non-recursive form

$$\begin{aligned}
\dot{z}_0&=-\tilde{\lambda}_{n_d}L^{\frac{1}{n_d+1}}|z_0-f(t)|^{\frac{n_d}{n_d+1}}\,\mathrm{sign}(z_0-f(t))+z_1,\\
\dot{z}_1&=-\tilde{\lambda}_{n_d-1}L^{\frac{2}{n_d+1}}|z_0-f(t)|^{\frac{n_d-1}{n_d+1}}\,\mathrm{sign}(z_0-f(t))+z_2,\\
&\ldots\\
\dot{z}_{n_d-1}&=-\tilde{\lambda}_1L^{\frac{n_d}{n_d+1}}|z_0-f(t)|^{\frac{1}{n_d+1}}\,\mathrm{sign}(z_0-f(t))+z_{n_d},\\
\dot{z}_{n_d}&=-\tilde{\lambda}_0L\,\mathrm{sign}(z_0-f(t)).
\end{aligned}\tag{1.27}$$

It is easy to see that $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_{n_d} = \lambda_{n_d}$, and $\tilde{\lambda}_j = \lambda_j \tilde{\lambda}_{j+1}^{j/(j+1)}$, $j = n_d - 1, n_d - 2, \ldots, 1$.
Notation. Assuming that the sequence λ_j, $j = 0, 1, \ldots$, is the same over the whole chapter, denote (1.27) by the equality $\dot{z} = D_{n_d}(z, f, L)$. Also for any $w \neq 0$ and $\gamma > 0$ denote $\lfloor w \rceil^{\gamma} = |w|^{\gamma} \operatorname{sign} w$; $\lfloor 0 \rceil^{\gamma} = 0$; $\lfloor w \rceil^0 = \operatorname{sign} w$.

Let the noise be absent. Subtracting $f^{(i+1)}(t)$ from the both sides of the equation for $\dot{z}_i$ of (1.27), denoting $\sigma_{d,i} = z_i - f^{(i)}$, $i = 0, \ldots, n_d$, and using $f^{(n_d+1)}(t) \in [-L, L]$ obtain the differentiator error dynamics

$$
\begin{array}{lcl}
\dot{\sigma}_{d,0} & = & -\tilde{\lambda}_{n_d} L^{\frac{1}{n_d+1}} \lfloor \sigma_{d,0} \rceil^{\frac{n_d}{n_d+1}} + \sigma_{d,1}, \\
\dot{\sigma}_{d,1} & = & -\tilde{\lambda}_{n_d-1} L^{\frac{2}{n_d+1}} \lfloor \sigma_{d,1} \rceil^{\frac{n_d-1}{n_d+1}} + \sigma_{d,2}, \\
 & & \ldots \\
\dot{\sigma}_{d,n_d-1} & = & -\tilde{\lambda}_1 L^{\frac{n_d}{n_d+1}} \lfloor \sigma_{d,n_d-1} \rceil^{\frac{1}{n_d+1}} + \sigma_{d,n_d}, \\
\dot{\sigma}_{d,n_d} & \in & -\tilde{\lambda}_0 L \operatorname{sign} \sigma_{d,n_d} + L[-1,1].
\end{array}
\tag{1.28}
$$

It is homogeneous with $\deg t = -1$, $\deg \sigma_{d,i} = n_d + 1 - i$ [29]. Thus, according to section 1.3.3.1 with sampling time periods not exceeding $\tau > 0$ and the maximal possible sampling error $\varepsilon \geq 0$ the differentiation accuracy $|z_j - f_0^{(j)}| \leq \mu_j L \rho^{n_d+1-j}$, $\rho = \max(\tau^{n_d+1-j}, (\varepsilon/L)^{(n_d+1-j)/(n_d+1)})$, is ensured, where the constant numbers $\mu_j > 0$ only depend on the parameters $\lambda_0, \ldots, \lambda_{n_d}$ of the differentiator. This accuracy is known to be asymptotically optimal in the presence of noises [26, 28], which means that only the coefficients μ_j can be improved.

1.4.1.2 Differentiator initialization.

Although one can take arbitrary initial values of differentiator for its feedback application, it may considerably destroy the initial system transient, since at the beginning the differentiator outputs will have no resemblance to the right derivatives. The overall performance can be drastically improved if the initial values of the differentiator are chosen right.

The most simple method is to take $z_0(t_0) = f(t_0)$ and $z_i(t_0) = 0$, $i = 1, \ldots, n_d$, where t_0 is the first sampling time. Then one just provides some reasonable time for the differentiator convergence prior to the control application.

Another method, *which we consider preferable if the noise magnitude is available*, is to choose some initial time increments of the length Δt, consisting of a number of real sampling intervals. The $n_d + 1$ sampling values of the input f are stored for n_d such successive time increments, and then the initial values of the differentiator are calculated by divided differences. During all this period the control is not applied, i.e. is kept at zero. Then the differentiator is practically already in the steady state from the very beginning. This initialization process is robust with respect to noises of the magnitude of the order Δt^{n_d+1}, i.e. Δt is to be chosen with respect to the maximal possible noise. A small additional time for the initial error elimination can still be considered.

One can also apply non-homogeneous differentiator modifications [3, 13, 35] with faster convergence. In that case the global system homogeneity is lost.

1.4.1.3 Differentiator with variable Lipschitz parameter L.

It is proved that if L continuously changes in time, then if at some moment differentiation errors are zero, they will stay at zero forever. Obviously such statement cannot withstand a practice exam.

Practically important result is that if L is differentiable, and $|\dot{L}|/L \leq M$ for some M, then for some $\delta > 0$ the differentiator converges provided the initial errors satisfy $|z_j - f_0^{(j)}| \leq \delta L$ [36]. The accuracy $|z_j - f_0^{(j)}| \leq \mu_j L \max(\tau^{n_d+1-j}, \varepsilon^{\frac{n_d+1-j}{n_d+1}})$ is kept provided the noise satisfies $|\eta(t)| \leq L(t)\varepsilon$ [36].

Globally convergent differentiator with fast convergence and $|\dot{L}|/L \leq M$ has been recently presented [35]. Note that its parameters depend on M.

1.4.1.4 Output feedback control.

Consider the system (1.3) of the relative degree r under the conditions (1.19), (1.20) with a bounded control (1.22). Incorporating the $(r-1)$th order differentiator into the feedback equations, obtain the SISO output-feedback r-sliding controller

$$u = U_r(z),\ \dot{z} = D_{r-1}(z, \sigma, L), \tag{1.29}$$

where $L \geq C + K_M \sup|U_r|$. Suppose that (1.21), (1.22) is finite-time stable. Then the output-feedback controller (1.29) ensures the finite-time establishment of the r-sliding mode $\vec{\sigma} = 0$. Moreover [30], if (1.22) is r-sliding homogeneous, the closed-loop inclusion (1.21), (1.29) is homogeneous with $\deg z_i = \deg \sigma^{(i)} = r - i$ and the system homogeneity degree -1. Respectively, due to section 1.3.3.1, if σ is sampled with the accuracy $\varepsilon \geq 0$ and the sampling intervals not exceeding $\tau > 0$, then the asymptotic SM accuracy $\sigma^{(j)} = O(\max(\tau^{r-j}, \varepsilon^{(r-j)/r}))$ is obtained.

1.4.2 Homogeneous MIMO SM control

Once more consider dynamic system (1.3),

$$\dot{x} = a(t,x) + b(t,x)u,\ \sigma = \sigma(t,x), \tag{1.30}$$

but let now σ and u be vectors, $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}^m$, $u \in \mathbb{R}^m$. The system is assumed to have the vector relative degree $r = (r_1, ..., r_m)$, $r_i > 0$. It means that the successive total time derivatives $\sigma_i^{(j)}$, $j = 0, 1, ..., r_i - 1$, $i = 1, ..., m$, do not contain controls, and can be used as a part of new coordinates [24]. Respectively, (1.19) turns to be a vector equation,

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{1.31}$$

where $\sigma^{(r)}$ denotes $(\sigma_1^{(r_1)}, ..., \sigma_m^{(r_m)})^T$, the functions h, and g are unknown and smooth. The function g is a non-singular matrix. It is often called high-frequency gain matrix.

Let g be represented in the form $g = K\bar{g}$, where $K > 0$ defines the "size" of the matrix g, and $\bar{g}$ corresponds to the matrix "direction". A nominal "direction" matrix $G(t,x)$ is assumed nonsingular and available in real time, so that

$$g(t,x) = K(t,x)(G(t,x) + \Delta g(t,x)),\ \left\|\Delta g G^{-1}\right\|_1 \leq p < 1. \tag{1.32}$$

Here Δg is the uncertain deviation of $\bar{g}$ from G, and the norm $\|\cdot\|_1$ of the matrix $A = (a_{ij})$ is defined as $\|A\|_1 = \max_i \sum_j |a_{ij}|$. The estimation G can be any Lebesgue-measurable function, p is a known constant. Mark that similar assumptions are adopted in [14].

Similarly to (1.20), assume that the uncertain vector function h and the scalar function K are bounded,

$$\|h(t,x)\| \leq C,\ 0 < K_m \leq K(t,x) \leq K_M, \tag{1.33}$$

where C, K_m, K_M are known constants.

It is also assumed that trajectories of (1.30) are infinitely extendible in time for any Lebesgue-measurable control with uniformly bounded $||gu||/H$.

The above are the only system conditions needed in Section 1.4.2. The system uncertainty is defined by r, C, K_m, K_M, p and $G(t,x)$ which are respectively assumed to be known or available.

Note that the avaialability of $G(t,x)$ in real time does not necessarily mean that $x(t)$ is available, and G is known analitically. For example, the aerodynamic characteristics of an aircraft are usually available as approximate table functions of the observable dynamic pressure and altitude.

Introduce a virtual control v,

$$u = G(t,x)^{-1} v. \tag{1.34}$$

Then dynamics (1.31) take the form

$$\sigma^{(r)} = h(t,x) + K(t,x)(I + \Delta g(t,x) G^{-1}(t,x))v, \tag{1.35}$$

where I is the unit matrix.

Introduce the notation $\vec{\sigma}_i = (\sigma_i, \ldots, \sigma_i^{(r_i-1)})$, $\vec{\sigma} = (\vec{\sigma}_1, \ldots, \vec{\sigma}_m)$. Choose the components of $v = (v_1, \ldots, v_m)^T$ in the form of the embedded r_i-sliding homogeneous controller (1.24), (1.25)

$$v_i = -\alpha \Psi_{r_i-1,r_i}(\vec{\sigma}_i),\ i = 0, 1, \ldots, m,\ u = G(t,x)^{-1} v, \tag{1.36}$$

where $\alpha > 0$. Now the closed-loop system satisfies the decoupled $(r_1, r_2, \ldots, r_m)$-sliding homogeneous inclusion

$$\sigma_i^{(r_i)} \in [-C, C] - \alpha [K_m(1-p), K_M(1+p)] \Psi_{r_i-1,r_i}(\vec{\sigma}_i),\ i = 1, \ldots, m, \tag{1.37}$$

with the weights $\deg \sigma_i^{(j)} = r_i - j$. According to section 1.4.1, (1.37) is finite-time stable with sufficiently large α.

Respectively the output-feedback control gets the form

$$\begin{aligned} v_i &= -\alpha \Psi_{r_i-1,r_i}(z_i),\ i = 0, 1, \ldots, m,\ u = G(t,x)^{-1} v, \\ \dot{z}_i &= D_{r_i-1}(L, \sigma_i, z_i),\ L \geq C + 2K_M \alpha. \end{aligned} \tag{1.38}$$

The closed-loop inclusion is still homogeneous with $\deg \sigma_i^{(j)} = \deg z_{i,j} = r_i - j$. The following theorem follows from the section 1.3.3.1.

Theorem 5. *Let the MIMO system* (1.30), (1.31) *satisfy conditions* (1.32), (1.33). *Then output-feedback control* (1.38) *provides for the finite-time establishment and keeping of the r-SM $\sigma = 0$. Let σ_i be measured with the sampling accuracy $\varepsilon_i \geq 0$, $i = 1, 2, \ldots, m$, and the sampling intervals not exceeding $\tau > 0$, then the asymptotic SM accuracy $\sigma_i^{(j)} = O(\max(\tau^{r_i - j}, \varepsilon_i^{(r_i - j)/r_i}))$ is obtained.*

As we have seen, the obtained SM asymptotics are the best possible. Note that one can here use quasi-continuous controllers [31, 38], but the corresponding technique is more complicated [37], though provides for superior performance.

1.5 Discretization of SM differentiators

In reality described differentiators are realized by means of computers. This turns a real-time differentiator into a discrete dynamic system. In this section we present the discretization methods for SM-based differentiators, analyze their accuracy and on-line detection of their convergence.

1.5.1 Discrete differentiators and their accuracy

Consider the differentiator (1.26) or (1.27), which is represented as $\dot{z} = D_{n_d}(f, z, L)$. Let the basic input $f_0(t)$ be sampled at the time instants t_k, $0 \leq t_{k+1} - t_k = \tau_k \leq \tau$, with the error $\eta \in \mathbb{R}$, $\eta \in \varepsilon[-1, 1]$, $f = f_0 + \eta$, $|f_0^{(n_d+1)}| \leq L$. Differentiator (1.27) is a discontinuous dynamic system. Therefore, its only reliable numeric integration is based on the Euler method. Also the discretization is naturally to be based on the Euler integration.

The simplest way is to perform one Euler integration step between each two successive measurements. The respective discretization is

$$z(t_{k+1}) = z(t_k) + D_{n_d}(f(t_k), z(t_k), L)\tau_k. \tag{1.39}$$

The accuracies obtained in section 1.4.1.1 correspond to the case when between the measurements the differentiator is described by differential equations. It corresponds to the infinite number of infinitesimally small Euler-integration steps between the measurements at t_k, t_{k+1}. In practice only finite number l_k of such steps is taken. Let the Euler steps take place at the discrete time instants $t_{k,j}$, $j = 0, \ldots, l_k$, $t_{k,0} = t_k$, $t_{k,l_k} = t_{k+1} = t_{k+1,0}$. Thus, all sampling instants are also the instants of the integration subdivision. It is also assumed that $0 < t_{k,j+1} - t_{k,j} = \underline{\tau}_{k,j} < \underline{\tau}(\tau)$, i.e. in general $\underline{\tau}$ may depend on τ.

It is natural to take the discretization

$$z(t_{k,j+1}) = z(t_{k,j}) + D_{n_d}(f(t_k), z(t_{k,j}), L)\underline{\tau}_{k,j},\ j = 0, \ldots, l_k - 1.$$

It is *not* the dicretization we use. The idea is never to use the differences of the input signal f and z_0 taken at *different* time instants. Thus, the proposed discretization is

$$z(t_{k,j+1}) = z(t_{k,j}) + \Delta_{n_d}(f(t_k), z(t_{k,j}), L)\underline{\tau}_{k,j},\ j = 0, \ldots, l_k - 1, \tag{1.40}$$

where the vector function $\Delta_{n_d}(f(t_k), z(t_{k,j}), L)$ has the components

$$\begin{array}{lcl}
\Delta_{n_d,0} & = & -\tilde{\lambda}_{n_d} L^{\frac{1}{n_d+1}} \lfloor z_0(t_k) - f(t_k) \rceil^{\frac{n_d}{n_d+1}} + z_1(t_{k,j}), \\
\Delta_{n_d,1} & = & -\tilde{\lambda}_{n_d-1} L^{\frac{2}{n_d+1}} \lfloor z_0(t_k) - f(t_k) \rceil^{\frac{n_d-1}{n_d+1}} + z_2(t_{k,j}), \\
 & & \dots \\
\Delta_{n_d,n_d-1} & = & -\tilde{\lambda}_1 L^{\frac{n_d}{n_d+1}} \lfloor z_0(t_k) - f(t_k) \rceil^{\frac{1}{n_d+1}} + z_{n_d}(t_{k,j}), \\
\Delta_{n_d,n_d} & = & -\tilde{\lambda}_0 L \,\mathrm{sign}(z_0(t_k) - f(t_k)).
\end{array} \quad (1.41)$$

One can expect that the resulting accuracy is worse than the standard differentiator accuracy from Section 1.4.1.1, but it is to be reclaimed for $\underline{\tau} \to 0$, $l_k \to \infty$.

Unfortunately the results of section 1.3.3.1 are not valid for such systems. One needs to use the technique described in section 1.3.3.2 to estimate the resulting accuracy. Following are the currently known results, some of which are declared here for the first time. In particular, the standard accuracy from section 1.4.1.1 is always preserved for the standard 1st-order differentiator, $n_d = 1$, [40].

1. Let the integration steps be equal, $t_{k,j+1} - t_{k,j} = \underline{\tau}_{k,j} = \underline{\tau}$. Let $\rho = \max[(\frac{\varepsilon}{L})^{\frac{1}{n_d+1}}, \tau]$. Also suppose that the derivatives $f_0^{(i)}$ of the orders $2, 3, ..., n_d + 1$, are bounded: $|f_0^{(i)}| \le D_i$, $D_{n_d+1} = L$. Then there exist such constants $\mu_i > 0$ that independently of the sampling intervals' choice the following inequalities hold after a finite time transient:

$$\begin{array}{l}
|z_0(t_{k,j}) - f_0(t_{k,j})| \le \mu_0 L \rho^{n_d+1}; \\
|z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})| \le \mu_i L \rho^{n_d+1-i} + j\underline{\tau} D_{i+1}, \; i = 1, 2, ..., n_d.
\end{array} \quad (1.42)$$

Note that this result is published in [40] for the case when the integration steps and the sampling intervals coincide, $l_k = 1$, $\underline{\tau} = \tau$. The proof is very similar to one presented in [40].

2. Let the maximal integration and sampling steps, $\underline{\tau}$ and τ, be small enough. Also suppose that the derivatives $f_0^{(i)}$ of the orders $2, 3, ..., n_d + 1$, are bounded: $|f_0^{(i)}| \le D_i$, $D_{n_d+1} = L$. Then there exist such constants $\mu_i > 0$ that independently of the sampling and integration intervals' choice the inequalities

$$|z_i(t_{k,j}) - f_0^{(i)}(t_{k,j})| \le \mu_i L \rho^{n_d+1-i}, \; i = 0, 1, ..., n_d, \quad (1.43)$$

hold after a finite time transient. Here $\rho = \max[(\varepsilon/L)^{1/(n_d+1)}, \underline{\tau}^{1/n_d}, \tau]$. Note that contrary to other cases here μ_i depend on $D_2/L, ..., D_{n_d}/L$. Obviously, the standard asymptotics of the section 1.4.1.1 is restored for $\underline{\tau} \le \tau^{n_d}$.

Also the latter result is published in [40] for the case when the integration steps and the sampling intervals coincide, $l_k = 1$, $\underline{\tau} = \tau$. The proof follows the general lines of section 1.3.3.2. One has to represent the errors of the discrete system (1.40) as solutions of some disturbed finite-time stable inclusion sampled at the times $t_{k,j}$.

As we see, in general the asymptotic accuracy of the continuous-time differentiator with discrete measurements is lost, when the differential equations are replaced with discrete Euler integration and the differention order exceeds 1. It is restored if the maximal integration step $\underline{\tau}$ and the maximal sampling interval τ satisfy the inequality $\underline{\tau} \le \tau^{n_d}$. That choice of the integration step still can be feasible for $n_d = 2$, but usually becomes impractical already for $n_d = 3$. Also the requirement for derivatives $f_0^{(i)}$, $i = 2, ..., n_d$, to be bounded is restrictive. The following discrete differentiator resolves all these issues.

3. *Homogeneous Discrete Differentiator.* Choose

$$\begin{aligned} \Delta_{n_d,i}(t_{k,j}) &= -\tilde{\lambda}_{n_d-i} L^{\frac{i+1}{n_d+1}} \lfloor z_0(t_k) - f(t_k) \rceil^{\frac{n_d-i}{n_d+1}} + z_{i+1}(t_{k,j}) \\ &\quad + \sum_{s=i+2}^{n_d-1} \frac{z_s(t_{k,j}) \tau_{k,j}^{s-1}}{(s-i)!}, i = 0, 1, ..., n_d - 1; \\ \Delta_{n_d,n_d}(t_{k,j+1}) &= -\tilde{\lambda}_0 L \,\mathrm{sign}(z_0(t_k) - f(t_k)). \end{aligned} \tag{1.44}$$

New terms appear in the second line of (1.44) and are only present if $n_d > 1$. Note that (1.44) can be also rewritten in the recursive form [35]. Let the maximal integration and sampling steps, $\underline{\tau} \leq \tau$, **be any positive numbers**. Then there exist such constants $\mu_i > 0$ that independently of the function f_0 and the choice of the sampling intervals and integration steps the inequalities (1.43) hold after a finite time transient for $\rho = \max[(\varepsilon/L)^{1/(n_d+1)}, \tau]$.

Obviously discrete differentiator (1.40), (1.44) completely reclaims the accuracy of its continuous-time analogue. This result has been published in [40] for the case when the integration steps and the sampling intervals coincide. It also seems that additional integration steps do not cause any noticeable accuracy improvement.

1.5.2 Convergence criteria

It is practically important to detect the moment when the differentiator starts to produce reliable derivative estimations. Choose some parameter $\rho > 0$. Taking into account the homogeneity features of the differentiator, we expect that if the noise $\eta(t)$, the maximal sampling interval τ, and the maximal integration step $\underline{\tau}$ satisfy

$$|\eta| \leq k_\eta L \rho^{n_d+1}, \ \tau \leq k_\tau \rho, \tag{1.45}$$

and, if integration steps are variable and differ from the sampling steps,

$$\underline{\tau} \leq \underline{k}_\tau \rho^{n_d}, \tag{1.46}$$

then, respectively to the chosen discretization scheme, the differentiation accuracy (1.42) or (1.43) is to be obtained in the steady state.

The only available real-time information consists here of the observed sampled differences $z_0(t_k) - f(t_k)$, which are corrupted by noise and taken at discrete times t_k. According to the results of the previous subsection we can expect the steady-state accuracy $|z_0(t_{k,j}) - f(t_{k,j})| \leq k_f L \rho^{n_d+1}$ to be kept.

The idea is that provided this accuracy is observed for sufficiently long time, one can assume that the transient is over. Due to the homogeneity reasoning that observation time is to be proportional to ρ. Thus one expects

$$|z_0(t_k) - f(t_k)| \leq k_f L \rho^{n_d+1}, \ t_k \in [t - k_t \rho, t] \tag{1.47}$$

to hold for some $k_t > 0$ starting from some moment t.

The general convergence criterion. For any sufficiently small $\rho > 0$

1. for any choice of positive parameters $\mu_0, ..., \mu_{n_d}$ there exists a set of positive parameters k_η, k_τ, k_f, k_t, and, maybe, $\underline{k}_\tau$,

2. for any choice of positive parameters k_η, k_τ, k_f, k_t, and, maybe, $\underline{k}_\tau$ there exists a set of positive parameters $\mu_0, ..., \mu_{n_d}$,

such that for any input $f(t)$, $|f_0^{(n_d+1)}| \leq L$ (plus boundedness of $|f_0^{(i)}|$, $i \geq 2$ for the discretization scheme (1.41)), provided (1.47) holds at the time moment $t = t_*$, it also holds for any $t \geq t_*$, and the accuracies (1.42) or (1.43) (respectively to the chosen discretization scheme) are kept starting from the time $t_* - k_t\rho$.

Let us concretize the above criterion to remove any doubts.

- Continuous-time criterion [2], differentiator (1.26). The numeric integration is excluded, (1.45) are the only restrictions. The accuracy (1.43) is observed *for any* $\rho > 0$ (not only small).

- Equal integration steps, scheme (1.40), (1.41). Only (1.45) is required. The observed accuracy is (1.42).

- Variable integration steps, scheme (1.40), (1.41). Both (1.45) and (1.46) are required. The observed accuracy is (1.43). Choice of k_η, k_τ, k_f, k_t, and $\underline{k}_\tau$ if sampling and integration steps do not coincide, depends on $D_2/L, ..., D_{n_d}/L$.

- Improved scheme (1.40), (1.44). Only (1.45) is required. The accuracy (1.43) is observed *for any* $\rho > 0$ (not only small).

All the above cases yield the same asymptotic accuracy in the case $n_d = 1$.

1.5.3 Discrete differentiator with variable Lipschitz parameter L.

Let L be a variable function of t, and $|\dot{L}|/L \leq M$ hold for some M, then all the above schemes (1.40), (1.41) or (1.44) make sense [36]. The only difference is that each time when the requirement of the boundedness of $f_0^{(j)}$ appears in the constant-L case, it is replaced by the boundedness of $f_0^{(j)}(t)/L(t)$. Let also $|\eta(t)| \leq L(t)\hat{\varepsilon}$ then the same accuracy (1.43) or (1.42) is obtained and kept for sufficiently small $\tau, \hat{\varepsilon}$, whereas $\hat{\varepsilon}$ is substituted for ε/L in the definitions of ρ. The globally convergent differentiator with variable L [35] features the same accuracy.

The above convergence criteria are literally true for such differentiators, but ρ and τ are required to be sufficiently small in all cases. Naturally the parameter L appearing in the accuracies (1.42) or (1.43) depends on time. The corresponding result has meantime been published for the improved discretization scheme (1.40), (1.44) with coinciding integration and sampling steps [35].

1.5.4 Simulation results

Due to the large scope of the presented results only a few illustrative examples are provided. Consider the input function

$$f_0 = \tfrac{1}{5}\sin(3t+0.3) - \tfrac{2}{5}\sin(2.2t+1.5) + \tfrac{1}{10}\sin(0.5t+4.7), \qquad (1.48)$$

which obviously has bounded derivatives. Assign $L = 4, 10, 26, 70$ and 200 for $n_d = 1, 2, 3, 4$ and 5 respectively. Choose the parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$ of differentiators. Recall that $\underline{\tau}$ and τ are respectively the maximal values of the integration steps $\underline{\tau}_{k,j}$ and the sampling steps τ_k; ε is the noise magnitude. Naturally $\underline{\tau}_{k,j} \le \tau_k$, $\underline{\tau} \le \tau$ hold. The variable values of τ_k and $\underline{\tau}_{k,j}$ are calculated on-line as random numbers uniformly distributed in the range from 0 to the corresponding upper bound.

One of the main presented results is that the theoretical asymptotically optimal differentiation accuracy $z_i - f_0^{(i)} = O(\rho^{n_d+1-i})$, $\rho = \max(\varepsilon^{\frac{1}{n_d+1}}, \tau)$ of the continuous-time differentiator is restored by the Euler-integration discrete differentiator with variable integration and sampling steps, provided $\underline{\tau}$ is of the order of τ^{n_d} or higher.

Let $n_d = 5$, $\varepsilon = 0$ for simplicity. For $\tau = 0.01$ the ideal accuracy reclamation would require taking $\underline{\tau}$ proportional to 10^{-12} which is practically impossible. Instead fix a reasonable value $\underline{\tau} = 0.0001$ and gradually increase τ starting from $\tau = \underline{\tau}$ calculating the corresponding accuracies $\sup|z_i - f_0^{(i)}| = ||z_i - f_0^{(i)}||_\infty$ on a sufficiently long steady-state time interval. One can expect that starting from some moment the accuracies obey the above standard asymptotics.

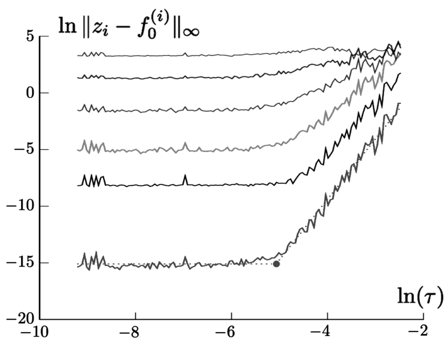

Figure 1.1: Logarithmic graphs of the 5th-order differentiator accuracies with $\underline{\tau} = 0.0001$. *The accuracies' lines correspond to the derivative orders 0, 1, 2, 3, 4, 5 from the bottom to the top. Integration and sampling steps are variable.*

It is seen from Fig. 1.1 that the standard asymptotics is restored for $\tau > \tau_c$, where the critical value $\tau_c \approx e^{-5} \approx 0.007$. It corresponds to $\underline{\tau} \le 6\cdot 10^6 \tau^5$. This relation is expected to hold for any τ. In particular, one needs to take $\underline{\tau} \le 0.0006$ for $\tau = 0.01$, which is still feasible, and $\underline{\tau} \le 6\cdot 10^{-9}$ for $\tau = 0.001$, which is already practically impossible.

It is also seen from the graphs that the accuracies are solely defined by $\underline{\tau}$ if $\tau < \tau_c$. In other words, if $\underline{\tau} \geq 6 \cdot 10^6 \tau^5$, too large integration steps influence the accuracy stronger than the sampling interval itself. According to this reasoning $\tau_c \approx 0.45 \underline{\tau}^{1/5}$. Note that τ_c depends not only on the parameters of the differentiator, but also on the upper bounds of the input derivatives starting from the order 2.

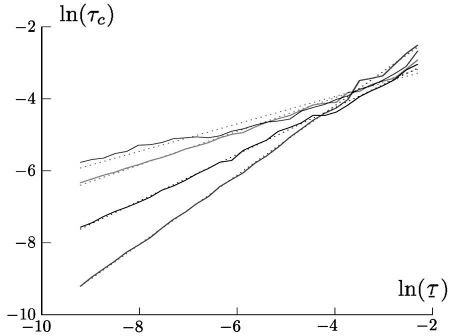

Figure 1.2: Logarithmic graphs of the critical sampling period τ_c vs $\underline{\tau}$ for $n_d =$ $1,2,3,4$ from the bottom to the top. Integration and sampling steps are variable.

In order to check the above reasoning the simulation was carried out for differentiators of the orders $1,2,3,4$ for the same input (1.48). The critical values τ_c are found as functions of $\underline{\tau}$. The hypothetical relation $\tau_c \approx \gamma_c \underline{\tau}^{1/n_d}$ is confirmed by the graphs in Fig. 1.2. The higher the differentiation order the more difficult and less reliable is the detection of τ_c. Thus, there is no graph corresponding to $n_d = 5$.

1.6 Discretization of SMs

Implementation of HOSM controllers usually requires the control values to be calculated and fed to the system at discrete time instants. In the case, when the controller does not involve its own dynamics, in particular, if all needed derivatives are directly sampled, the resulting system is adequately described by variable sampling noises and delays, and by the analysis from section 1.3.3.1 .

The situation changes when the output feedback is applied, which incorporates a dynamic observer. Let the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$. As we have seen (section 1.5), the differentiator accuracy deteriorates, if the differentiator is replaced by its Euler approximation. Therefore, one would expect that the accuracy of the output-feedback r-SM control also deteriorates. In fact, it is not the case due to the overall homogeneity of the closed-loop error dynamics.

1.6.1 SISO case

Consider the system (1.3) of the relative degree r under the conditions (1.19), (1.20) with a bounded control (1.22). Let the output-feedback control (1.29) be applied, and the differentiator be replaced by its simplest Euler discretization. Let the sampling take place at the time instants t_k, $0 < t_{k+1} - t_k = \tau_k \leq \tau$. The hybrid error dynamics satisfies the differential inclusion

$$\begin{aligned} &\sigma^{(r)} \in [-C, C] + [K_m, K_M] U_r(z(t_k)),\ t \in [t_k, t_{k+1}] \\ &z(t_{k+1}) = z(t_k) + \tau_k D_{r-1}(z(t_k), \sigma(t_k), L), \end{aligned} \tag{1.49}$$

where $L \geq C + K_M \sup|U_r|$. Let σ be sampled with the accuracy $\varepsilon \geq 0$. Applying the reasoning of section 1.3.3.1 obtain that after a finite-time transient the system features the asymptotic accuracies $\sigma^{(j)} = O(\max(\tau^{r-j}, \varepsilon^{(r-j)/r}))$.

Consider now the chattering attenuation procedure based on the artificial increase of the relative degree. Differentiating (1.19) obtain

$$\sigma^{(r+1)} = h_e(t,x,u) + g(t,x)\dot{u}. \tag{1.50}$$

Assume that the functions h_e and $h'_x b + g'_x bu$ are bounded in a vicinity of the $(r+1)$-sliding mode $\sigma \equiv 0$,

$$|h_e(t,x,u)| \leq C_e,\ |h'_x(t,x)b(t,x) + g'_x(t,x)b(t,x)u| \leq C_{1e}. \tag{1.51}$$

This assumption is natural, since u is close to $u_{eq} = -h/g$ in the vicinity of the $(r+1)$-sliding mode, and is therefore bounded at least locally. The applied feedback control gets the form

$$\dot{u}(t) = U_{r+1}(z(t_k)),\ \dot{z} = D_r(z, \sigma, L_e), \tag{1.52}$$

where $z \in \mathbb{R}^{r+1}$, $L_e \geq C_e + K_M \alpha$, $\alpha = \sup|U_{r+1}|$. According to section 1.4.1.4 the accuracy $\sigma^{(j)} = O(\max(\tau^{r+1-j}, \varepsilon^{(r+1-j)/(r+1)}))$ is to be obtained.

Its discretization produces

$$u(t_{k+1}) = u(t_k) + \tau_k U_{r+1}(z(t_k)),\ z(t_{k+1}) = z(t_k) + \tau_k D_r(z(t_k), \sigma(t_k), L_e).$$

The following theorem summarizes these two cases.

Theorem 6 ([37]). *In the SISO case discretization does not destroy the closed-system asymptotic accuracy under the standard conditions* (1.20). *Under additional conditions* (1.51) *the same is true for the chattering attenuation procedure if the maximal sampling interval τ is sufficiently small.*

Note that the first part of the theorem has actually been proved above. The second part is restricted to the case when only one integrator is inserted in the feedback and the relative degree increases from r to $r+1$. The result is probably true also for the introduction of any number of integrators, but the proof is still not available.

1.6.2 MIMO case

Like in the SISO case we differentiate (1.31) and obtain

$$\sigma^{(r+(1,\ldots,1))} = h_e(t,x,u) + g(t,x)\dot{u},$$

where $h_e(t,x,u) \in \mathbb{R}^m$. Similarly an additional assumption is needed. The functions h_e and $h'_x b + g'_x bu$ are bounded in a vicinity of the $(r + (1,...,1))$-sliding mode $\vec{\sigma} \equiv 0$,

$$||h_e(t,x,u)|| \leq C_e, \; ||h'_x(t,x)b(t,x) + g'_x(t,x)b(t,x)u|| \leq C_{1e}. \tag{1.53}$$

The theorem exactly analogous to Theorem 6 is also true in the MIMO case due to the effective decoupling (1.35) of the system. Mark that that the proposed homogeneous SM control (1.34), (1.36) or (1.38) is applied here.

1.6.3 Example

Consider a simple second-order system

$$\begin{aligned} \dot{x}_0 &= \sin t + x_1, \\ \dot{x}_1 &= \cos x_0 + u, \quad \sigma = x_0, \end{aligned}$$

to be stabilized by continuous SM control. The solution is to apply the dynamic control $\dot{u} = -\alpha\Psi_{2,3}(z)$, $\dot{z} = D_2(z,\sigma,L)$. Its discretization yields

$$\begin{aligned} &\dot{x}_0 = \sin t + x_1, \\ &\dot{x}_1 = \cos x_0 + u(t_k), \quad \sigma = x_0, \\ &u(t_{k+1}) = u(t_k) - \tau_k \alpha \Psi_{2,3}(z_0(t_k), z_1(t_k), z_2(t_k)), \\ &z(t_{k+1}) = z(t_k) + \tau_k D_2(z(t_k), \sigma(t_k), L), \end{aligned} \tag{1.54}$$

where the standard embedded 3-SM controller (1.25) is taken,

$$\Psi_{2,3}(z) = \text{sign}\,(z_2 + 2(|z_1|^3 + |z_0|^2)^{1/6}\,\text{sign}(z_1 + |z_0|^{2/3}\,\text{sign}\, z_0)).$$

The continuous-time part of system (1.54) was integrated by the Euler method with the integration step 10^{-4} and initial values $x_0(0) = 10$, $x_1(0) = 5$. The discrete-time subsystem in (1.54) has the parameters $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 2$, $L = 16$, $\alpha = 8$ and $z_0(0) = z_1(0) = z_2(0) = 0$.

The random positive sampling steps τ_k are uniformly distributed in the segment $[10^{-4}, \tau]$. The stabilization of σ, $\dot{\sigma}$, $\ddot{\sigma}$ with $0 < \tau_k \leq \tau = 0.01$ is demonstrated in Figs. 1.3a, b, and c respectively. Now let the maximal sampling step τ take values $0.01, 0.02, ..., 0.1$. Logarithmic plots of $\max_{[30,40]} |\sigma^{(i)}|$, $i = 0,1,2$, together with the corresponding best-fitting lines $2.9 \ln \tau + 8.1$, $1.9 \ln \tau + 5.6$ and $0.9 \ln \tau + 3.9$ are shown in Fig. 1.3d. According to Theorem 6, the worst-case accuracy orders correspond to the slope values 3, 2, 1, respectively. Thus, the simulation results are in good compliance with the theory.

1.7 Conclusions

The accuracy of disturbed homogeneous finite-time stable differential inclusions is analyzed and the results are applied to the accuracy analysis of homogeneous sliding mode control systems. The current knowledge on the accuracy and performance of dynamic systems closed by discretized dynamic homogeneous sliding-mode controllers is presented.

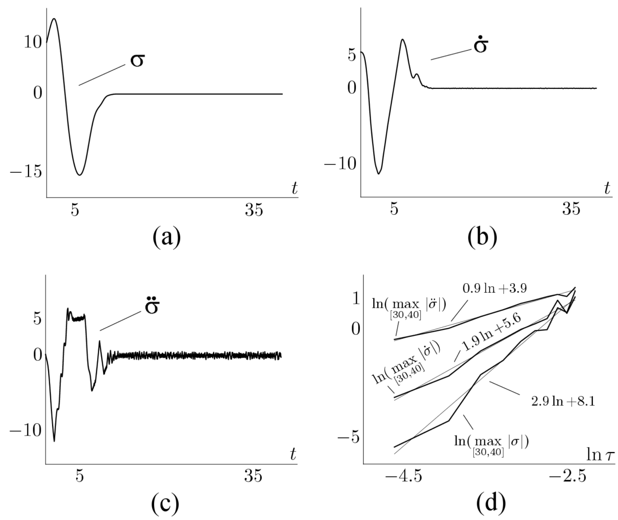

Figure 1.3: (a), (b), (c): Stabilization of $\sigma, \dot{\sigma}, \ddot{\sigma}$ *with* $\tau = 0.01$*; (d): asymptotics of* $\sigma, \dot{\sigma}, \ddot{\sigma}$ *for the maximal sampling steps* $\tau = 0.01, 0.02, ..., 0.1$.

Different discretization schemes of homogeneous sliding-mode-based differentiators are considered, and their accuracy is analyzed. For the first time the internal numeric Euler integration is considered between the sampling instants, and the corresponding effect on the accuracy is studied. Differentiator convergence criteria are presented. All the results are extended to the case of the variable Lipschitz parameter.

Discretization of output-feedback homogeneous sliding-mode controllers is shown not to destroy the overall system accuracy in the presence of noises and discrete sampling. The same is true if one discrete Euler integrator is introduced in the feedback in order to effectively attenuate system chattering by the artificial increase of the relative degree. The results are true for both single-input single-output and multi-input multi-output cases.

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