Accuracy of Homogeneous Sliding Modes in the Presence of Fast Actuators

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***Abstract*—It is shown that the higher the order r of the homogeneous sliding mode is, the less sensitive it is to the presence of unaccounted-for fast stable actuators. With μ being an infinitesimal actuator time constant, the sliding variable magnitude is proved to be proportional to μ^r. The system chattering is shown to be not amplified in the presence of actuators.**

***Index Terms*—Chattering, high-order sliding mode, singular perturbation, sliding-mode control.**

I. INTRODUCTION

Sliding mode control is accurate and insensitive to disturbances. Unfortunately standard sliding modes feature the so-called chattering effect due to high control switching frequency (e.g. [1], [5], [16]), and require the relative degree 1 of the sliding variable (e.g. [15]).

High order sliding modes (HOSMs) (e.g. see [2], [7]) were created to remove these restrictions by hiding the switching in higher derivatives of the sliding variable. Their application needs the robustness to possible small imperfections to be shown. Till now the robustness of homogeneous sliding modes was proved with respect to switching imperfections, small delays and noises [8].

In reality control affects a plant by means of an actuator. A proper mathematical model of the actuator is often uncertain, and, as a result, it is not accounted for at the control-design stage. The purpose of the actuator is to properly transmit the input, and it performs well, when the input changes smoothly and slowly. For this end the actuator is to be fast, exact and stable. Unfortunately, high-frequency discontinuous inputs cause uncontrolled vibrations of the actuator and of the closed HOSM system (see [3] and references therein).

Most of known HOSM controllers are homogeneous [8]–[10], [14]. The aim of this note is to show the robustness of homogeneous HOSMs with respect to the presence of unaccounted-for fast stable actuators. The corresponding asymptotic sliding accuracy is estimated with respect to the small actuator time constant and is proved to depend on the sliding-mode order only. It is shown that the higher the order is, the less sensitive is the HOSM accuracy to the presence of fast stable actuators. The system chattering is shown to be not intensified by such actuators. Introduction of an integrator at the actuator output and the corresponding increase of the order of the HOSM controller are proved to remove dangerous high-energy chattering phenomena.

II. THE PROBLEM STATEMENT

Let a smooth dynamic system with a smooth output function σ be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense [4]. Then, provided that successive total time derivatives $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}$ are continuous functions of the closed-system state-space variables, and the set $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$ is a non-empty integral set, the motion on the set is called r-sliding (rth order sliding) mode [7], [8]. Sliding modes used in the most variable structure systems are of the first order (σ is continuous, and $\dot{\sigma}$ is discontinuous; see [15], for example).

Let the dynamic system and the output (sliding variable) σ have the form

$$\dot{x} = a(t,x) + b(t,x)v, \quad \sigma = \sigma(t,x) \tag{1}$$

where $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, $\sigma \in \mathbf{R}$ and $r-1$ its total time derivatives are measured or estimated in real time, $v \in \mathbf{R}$ is the input, n is uncertain. Provided an r-sliding mode $\sigma = 0$ is established in (1), asymptotics of σ is to be estimated in the presence of unaccounted-for fast stable actuators. The local consideration is natural here, though also the global problem statement [7], [13] is possible.

Assumption 1°: Smooth uncertain functions a, b and σ are defined in some open region $\Omega \subset \mathbf{R}^{n+1}$. It is supposed that all solutions starting from an open region $\Omega_x \subset \mathbf{R}^n$ at $t = t_a$ can be extended in time up to $t = t_b > t_a$ without leaving the region Ω, provided the input v is a Lebesgue-measurable function of time, $|v| \le v_{\mathrm{M}}$. The constant $v_{\mathrm{M}} > 0$ is introduced in Assumption 4°.

Existence of such t_b is trivial for any $v_{\mathrm{M}} > 0$ and bounded Ω_x [4] (Ch. 2, Section 7).

Assumption 2°: The relative degree r of the system is assumed to be constant and known. That means that for the first time the input variable v explicitly appears in the rth total time derivative of σ [6]. It can be checked [6] that

$$\sigma^{(r)} = h(t,x) + g(t,x)v \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{v=0}$, $g(t,x) = (\partial/\partial v)\sigma^{(r)}$ are some unknown smooth functions. The set Ω_x is supposed to contain r-sliding points at the time $t = t_a$, i.e. points satisfying $\sigma = \dot{\sigma} = \ldots = \sigma^{(r-1)} = 0$.

Assumption 3°: It is supposed that

$$0 < K_{\mathrm{m}} \le \frac{\partial}{\partial v}\sigma^{(r)} \le K_{\mathrm{M}}, \quad \left|\sigma^{(r)}\right|_{v=0} | \le C \tag{3}$$

hold in Ω for some $K_{\mathrm{m}}, K_{\mathrm{M}}, C > 0$. Conditions (3) are formulated in terms of input-output relations.

The actuator model is described by the equations

$$\mu\dot{z} = f(z,u), \quad v = v(z) \tag{4}$$

where $z \in \mathbf{R}^m$, $u \in \mathbf{R}$ is the control and the input of the actuator, output $v(z)$ is continuous and $f(z,u)$ is a locally bounded Borel-measurable function, the time constant $\mu > 0$ is a small parameter. Recall that all differential equations are understood in the Filippov sense [4].

The control u is determined by a feedback of the form

$$u = U\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \tag{5}$$

where U is a function continuous almost everywhere, and bounded by some constant u_{M}, $u_{\mathrm{M}} > 0$, in its absolute value.

Being applied directly to (1), i.e. with

$$v = u \tag{6}$$

it is supposed to locally establish the r-sliding mode $\sigma \equiv 0$ (see also Assumption 6° below). In order to apply (5) one needs to measure or estimate $r-1$ derivatives of σ.

Assumption 4°: Initial values of actuator (4) belong to a compact region Ω_{z0}. The actuator features Bounded-Input-Bounded-State (BIBS)

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property with some value of μ. As $|u| \leq u_{\mathrm{M}}$, this provides for infinite extension in time of any solution of (4) and z belonging to another compact region Ω_z independent of μ. Indeed, μ can be excluded by the time transformation $\tau = t/\mu$. This assumption causes also the actuator output v to be bounded in its absolute value by some constant v_{M}, $v_{\mathrm{M}} > u_{\mathrm{M}} > 0$.

Assumption 5°: The dynamic output-feedback (5) is supposed to be r-sliding homogeneous [8], which means that the identity

$$U\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \equiv U\left(\kappa^r \sigma, \kappa^{r-1}\dot{\sigma}, \ldots, \kappa\sigma^{(r-1)}\right) \quad (7)$$

is kept for any $\kappa > 0$. It is also assumed that the control function U is locally Lipschitzian everywhere except a finite number of smooth manifolds comprising closed set Γ in the space with coordinates $\Sigma = (\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$. Note that due to the homogeneity property (7) set Γ contains the origin $\Sigma = 0$, where the function U is inevitably discontinuous [8], [9].

Most of known HOSM controllers [2], [7]–[10], [13] satisfy Assumption 5°. As follows from (2), (3):

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]v. \quad (8)$$

This inclusion does not "remember" anything on system (1) except the constants r, C, K_{m}, K_{M}.

Assumption 6°: It is assumed that with control (5) applied directly to inclusion (8), a finite-time stable inclusion (5), (6), (8) is created. This is the standard way to implement HOSM controllers [7], [8].

The differential inclusions are understood here in the Filippov sense. That means [8] that the right-hand-side vector set of (5), (6), (8) is enlarged at the discontinuity points of U in order to satisfy certain convexity and semicontinuity conditions [4].

Assumption 7°: The actuator is assumed exact in the following sense. With $\mu = 1$ and any constant value of u, $|u| \leq u_M$, the output v uniformly tends to u. In other words, for any $\delta > 0$ there exists $T > 0$ such that with any u, $u = \mathrm{const}$, $|u| \leq u_M$, $z(0) \in \Omega_z$, the inequality $|v - u| \leq \delta$ is kept after the transient time T. It is also required that the function $f(z, u)$ in (4) be uniformly continuous in u, which means that $\|f(z, u) - f(z, u + \Delta u)\|$ tends to 0 with $\Delta u \to 0$ uniformly in $z \in \Omega_z$, $|u| \leq u_M$.

While Assumptions 1°–7° can be considered natural, the next Assumption is to be separately checked for each controller (5).

Assumption 8°: It is supposed that the change of (5), (6) at the set Γ to

$$v \in \begin{cases} \{U(\Sigma)\}, & \Sigma \notin \Gamma \\ [-v_M, v_M], & \Sigma \in \Gamma \end{cases} \quad (9)$$

does not destroy the finite-time convergence, i.e. (8), (9) is also finite-time stable.

Remark 1: Any stable linear actuator with the transfer function $P(\mu p)/Q(\mu p)$ satisfies Assumptions 4°, 7°, if $\deg Q > \deg P$, Q is a Hurwitz polynomial, $P(0)/Q(0) = 1$.

Remark 2: A slightly generalized Assumption 7° can be considered, when the actuator instead of tracking the input u tracks ku, where $k > 0$ is an uncertain constant. Most HOSM controllers easily compensate for such actuator error, provided their magnitudes are sufficiently large.

Remark 3: Assumption 8 is significant, since solutions of differential inclusions in general depend on the right-hand-side values taken on sets of zero measure. Solutions of (8), (9) contain all solutions of (5), (6), (8), for $v_{\mathrm{M}} > u_{\mathrm{M}}$.

III. Main Results

The asymptotic sliding accuracy is calculated in the following main Lemma.

Lemma 1: Under assumptions 1°–6° suppose that for some $\mu = \mu_0$ there exist a ball B centered at $\Sigma = 0$ and a bounded invariant set Θ, which in finite time attracts all trajectories of the inclusion (4), (5), (8) starting within $B \times \Omega_{z0}$. Then there exist a time moment $t_1 \in (t_a, t_b)$, $a_0, a_1, \ldots, a_{r-1} > 0$, and a vicinity Q of the r-sliding set in Ω_x at $t = t_a$, such that with sufficiently small $\mu > 0$ the inequalities $|\sigma| < a_0\mu^r$, $|\dot{\sigma}| < a_1\mu^{r-1}, \ldots, |\sigma^{(r-1)}| < a_{r-1}\mu$ are kept with $t \geq t_1$ for any trajectory of (1), (4), (5) starting within Q at $t = t_a$.

Additional assumptions 7°, 8° provide for the existence of the invariant set needed in Lemma 1. The proofs of Lemma 1 and the following Theorem are presented in the Appendix.

Theorem 1: Let assumptions 1°–8° hold. Then the conditions of Lemma 1 hold and the corresponding asymptotic sliding accuracy is obtained.

A global theorem can be formulated in the global case [13] with $t_b = \infty$. Assumption 8° is to be checked for each controller. Fortunately it holds for most known HOSM controllers. In particular, consider homogeneous HOSM controllers [7]–[10], [14] of the form

$$u = -\alpha\Psi_r\left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right). \quad (10)$$

Note that enlarging α increases the class (3) of systems, to which the controller is applicable, whereas the internal parameters of the function Ψ_r are tuned to provide for the needed convergence rate [14]. Controller (5) is called quasi-continuous [8], [9], [14], if it is continuous everywhere except the r-sliding set $\Sigma = 0$. Since Assumption 8° is easily checked [12] for the nested controllers [7] and is trivial for all quasi-continuous controllers [8]–[10], [13], the following Lemma holds.

Lemma 2: Families of sliding-mode controllers defined in [7]–[9], [14] satisfy Assumption 8°.

All 2-sliding controllers from [10], which do not require switching on the axis $\dot{\sigma} = 0$, can also be shown to satisfy Assumption 8°. The popular sub-optimal 2-sliding controller [2] does satisfy Assumption 8°, though it does not have the form (5), and Theorem 1 is true also for it. The twisting controller [10] does not formally satisfy Assumption 8°, but this can be corrected changing the switching logic, while preserving the same trajectories.

As follows from Assumptions 1°, 2°, the rth derivative of the output σ is uniformly bounded by $C + K_M v_M$. Thus, an $(r-1)$th-order exact robust homogeneous differentiator [7] with finite-time convergence can be applied here, producing exact estimations of $\dot{\sigma}, \ldots, \sigma^{(r-1)}$ and preserving the asymptotic accuracy of Lemma 1 and Theorem 1. It can be shown that the resulting system is also robust with respect to small measurement noises.

Chattering Estimation: Consider an absolutely continuous signal $\xi(t, \varepsilon) \in \mathbf{R}, t \in [0, \mathrm{T}], \varepsilon \in \mathbf{R}^l$, where ε measures some imperfections and tends to zero. Define the nominal signal as the limit signal $\bar{\xi}(t) = \lim_{\varepsilon \to 0} \xi(t, \varepsilon)$, $t \in [0, \mathrm{T}]$. The (mathematical) chattering of the signal $\xi(t, \varepsilon)$ on [0, T] is defined as $\mathrm{chat}(\xi, \bar{\xi}; 0, T) = \int_0^T |\dot{\xi}(t, \varepsilon) - \dot{\bar{\xi}}(t)| dt$, and can be understood as virtual "heat release" [11], [12]. The chattering functional is not defined in the case when the limit absolutely continuous signal $\bar{\xi}(t)$ does not exist. General definitions and details can be found in [11], [12].

The chattering of the system is called infinitesimal if the chattering of each its coordinate tends to zero with $\varepsilon \to 0$; it is called bounded if it is not infinitesimal, and chattering of all coordinates is bounded when $\varepsilon \to 0$; and it is called unbounded if it is neither infinitesimal nor bounded [12] (respectively soft, hard and destructive chattering in [11]). Control is not considered as a coordinate. This classification does not depend on the choice of coordinates and the time scale. Infinitesimal chattering is the least possible, bounded and unbounded chattering phenomena are potentially devastating. Sliding-mode control systems generally feature bounded chattering, which leads to dangerous large "heat release" on prolonged time intervals. Some subsystems can be excluded from the chattering evaluation, if they correspond to mathematical models of devices insensitive to chattering [12]. It is the case of computer-based units and of some actuators and sensors.

According to [6], system (1) can be rewritten in some new local coordinates θ, Σ as

$$\sigma^{(r)} = h(t,\theta,\Sigma) + g(t,\theta,\Sigma)v, \quad \dot{\theta} = \Theta(t,\theta,\Sigma), \quad \theta \in \mathbf{R}^{n-r}.$$

As follows from Theorem 1, $\sigma, \dot{\sigma}, \ldots, \sigma^{(r-2)}, \theta$ feature infinitesimal chattering, and only $\sigma^{(r-1)}$ can reveal bounded chattering due to the boundedness of the actuator output v. Thus the actuator does not amplify the chattering and does not change the introduced chattering classification [12].

Suppose that (1) was obtained as a result of the standard chattering-attenuation procedure [2], [7], [10], i.e. $v = \dot{x}_n$, where x_n is the original control, and $r-1$ is the original relative degree. Respectively $x_1, \ldots, x_{n-1}$ are the coordinates of the original $(n-1)$-dimensional control system, which are to be checked for chattering. Therefore, the following Theorem is obtained, which can be shown true also in the presence of sampling noises and a differentiator [7] in the feedback.

Theorem 2: Let $\mu \to 0$ and the actuator variables z be excluded from the chattering analysis. Then the system features not more than bounded chattering. If the chattering-attenuation procedure was applied, the original $(n-1)$-dimensional system features only infinitesimal chattering.

IV. Simulation

The already traditional example of the kinematic car model

$$\dot{x} = V\cos\varphi, \quad \dot{y} = V\sin\varphi, \quad \dot{\varphi} = \frac{V}{l}\tan\theta, \quad \dot{\theta} = v \qquad (11)$$

is chosen. Here x and y are Cartesian coordinates of the rear-axle middle point, φ is the orientation angle, v is the longitudinal velocity, l is the length between the two axles and θ is the steering angle (Fig. 1(a)), v is the actuator output. The task is to steer the car from a given initial position to the trajectory $y = g(x)$, while $g(x)$ and y are assumed to be measured in real time. Let $V = const = 10$ m/s, $l = 5$ m, $g(x) = 10\sin(0.05x) + 5$, $x = y = \varphi = \theta = 0$ at $t = 0$.

Define $\sigma = y - g(x)$. The relative degree of the system is 3, and the less-popular quasi-continuous 3-sliding controller can be applied here, which was suggested in [8]. The resulting output-feedback controller is defined as

$$\begin{aligned} N_3 &= \left(|w_0|^2 + |w_1|^3 + |w_2|^6\right)^{1/6}, \\ \mathrm{sat}(p, 0.2) &= \min\left[1, \max(-1, 5p)\right], \\ u &= -0.5 \\ &\quad \cdot \mathrm{sat}\left\{\left[w_2 + 2\left(|w_1|^3 + |w_0|^2\right)^{1/6}\right.\right. \\ &\quad \left.\times \mathrm{sat}\left(\left(w_1 + |w_0|^{2/3}\mathrm{sign}\sigma\right)/N_3, 0.2\right)\right] \\ &\quad /N_3, 0.2\} \end{aligned}$$

where w_i are the real-time estimations of the derivatives $\sigma^{(i)}$, $i = 0$, 1, 2, obtained by the differentiator

$$\begin{aligned} \dot{w}_0 &= \xi_0, \quad \xi_0 = -9|w_0 - \sigma|^{2/3}\mathrm{sign}(w_0 - \sigma) + w_1, \\ \dot{w}_1 &= \xi_1, \quad \xi_1 = -15|w_1 - \xi_0|^{1/2}\mathrm{sign}(w_1 - \xi_0) + w_2, \\ \dot{w}_2 &= -110\mathrm{sign}(w_2 - \xi_1). \end{aligned}$$

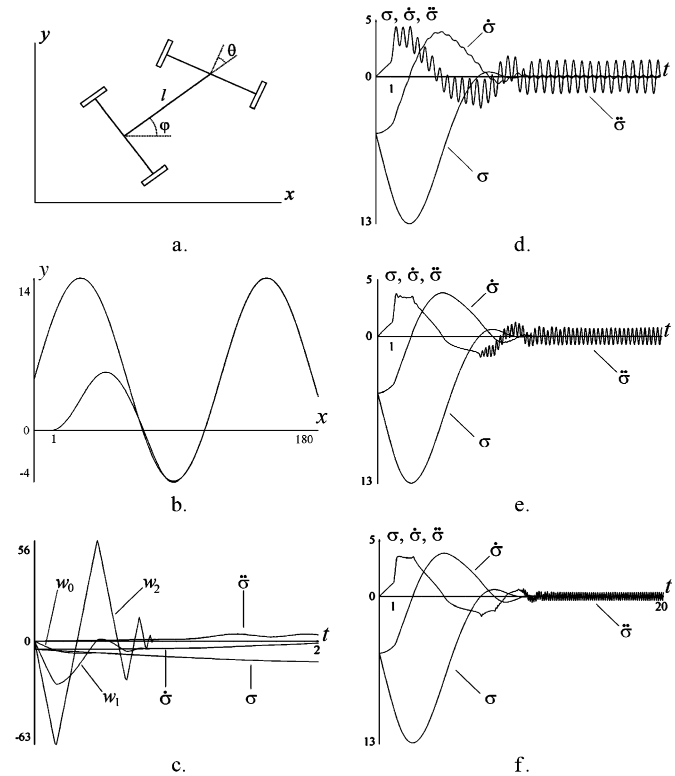

Fig. 1. Car model (a), trajectory tracking (b) and differentiator convergence (c) with $\boldsymbol{\mu} = \mathbf{0.08}$; comparison of 3-sliding deviations with $\boldsymbol{\mu} = \mathbf{0.08}$, 0.04, 0.02 (d, e, f).

TABLE I
Tracking Accuracies With Different Actuator Time Constants

μ	Sup $\|\sigma\|$	Sup $\|\dot{\sigma}\|$	Sup $\|\ddot{\sigma}\|$
0.01	0.0000765	0.00294	0.189
0.02	0.000644	0.0102	0.374
0.04	0.00529	0.0408	0.746
0.08	0.0433	0.182	1.50

The initial conditions of the differentiator are $w_0(0) = \sigma(0)$, $w_1(0) = w_2(0) = 0$. The control is applied only starting from $t = 1$ in order to provide some time for the differentiator convergence.

The actuator with the transfer function $(\mu s + 1)/(\mu^3 s^3 + 2\,\mu^2 s^2 + 2\,\mu s + 1)$ is realized in the form $\mu\dot{z}_1 = z_2$, $\mu\dot{z}_2 = z_3$, $\mu\dot{z}_3 = -z_1 - 2z_2 - 2z_3 + u$, $v = z_1 + z_2$ with zero initial conditions.

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics) with the integration step $\tau = 10^{-4}$. Tracking accuracies are listed in Table I. It is seen that the accuracies of σ, $\dot{\sigma}$, $\ddot{\sigma}$ are proportional to μ^3, μ^2, and μ respectively (Fig. 1(d), (e), (f), Table I). Other linear and non-linear actuators were also checked providing for similar simulation results.

System (11) can be considered as a system obtained from the "original" system x, y, φ by means of the chattering attenuation procedure. The relative degree of the system x, y, φ with respect to the "original control" $\tan\theta$ equals 2, and, therefore, its chattering is determined by the chattering of σ, $\dot{\sigma}$ (see Section III, [12]). The chattering functionals of σ and $\dot{\sigma}$ are calculated as integrals of $|\dot{\sigma}|$ and $|\ddot{\sigma}|$ respectively, and are infinitesimal in accordance with Theorem 2. The dangerous chattering has been moved to θ.

It is seen from Fig. 1(c) that the differentiator convergence takes about 0.9 s. The system performs remarkably well with a rather large

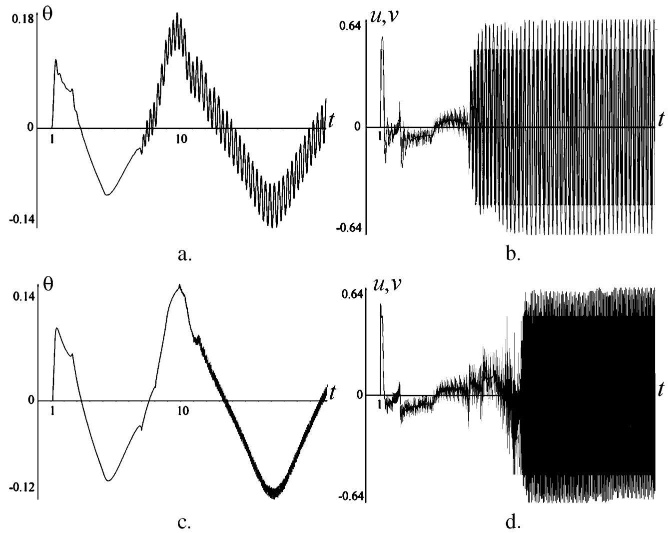

Fig. 2. Comparison of the steering angle (a, c), the control $\boldsymbol{u}$ and the actuator output $\boldsymbol{v}$ (b, d) corresponding to $\boldsymbol{\mu} = \mathbf{0.04}$ and to $\boldsymbol{\mu} = \mathbf{0.01}$.

actuator time constant $\mu = 0.08$. Indeed, the tracking deviation is only 4 cm. (Fig. 1(b), Table I). The steering angles with $\mu = 0.01$ and $\mu = 0.04$ are demonstrated in Fig. 2(a), (c). The corresponding actuator performance is shown in Fig. 2(b), (d).

In spite of the infinitesimal vibration magnitude of the steering angle θ with $\mu \to 0$ its vibration energy remains significant, and it features the potentially-dangerous bounded chattering [12]. One needs to additionally increase the relative degree [7] in order to get the infinitesimal chattering of θ.

V. Conclusion

The main conclusion is that stable fast actuators do not really destroy the performance of homogeneous high-order sliding-mode controllers. The resulting asymptotic sliding accuracy does not depend on the relative degree of the actuator and is only determined by the sliding order. The only exclusion is a rare case, when an asymptotically stable sliding mode $\sigma \equiv 0$ arises with the sliding order being equal to the sum of the system and actuator relative degrees. In such a case the residual chattering gradually disappears, and the asymptotic-accuracy coefficients of Lemma 1 can be taken arbitrarily small. Probably, it is only possible, when both relative degrees equal one.

The presence of unaccounted-for fast stable actuators does not amplify the chattering and does not influence the chattering classification [11], [12].

One can consider application of a smoothing filter at the input of an actuator device, which does not accept discontinuous inputs. If the time constant of the additional artificial actuator is sufficiently small, the asymptotic sliding accuracy of Theorem 1 is preserved, and the resulting combined actuator will still provide for good performance due to the high sliding order.

The most widely used application of high-order sliding modes is based on the artificial increase of the relative degree, when the control derivative is treated as a new control. For this end an integrator can be introduced at the output of the actuator in order to remove the most-dangerous high-energy chattering of the controlled system (Theorem 2). The HOSM controller is to be replaced according to the new relative degree. In this case also the asymptotic accuracy is improved due to the increased relative degree. Introduction of an integrator at the actuator input is also possible and is considered in [12]. Robustness is proved, but the asymptotic accuracy is not estimated in that case.

Appendix
Proofs of Lemma 1 and Theorem 1

Proof of Lemma 1: Due to the homogeneity property (7) with $\kappa > 0$ the transformation

$$\begin{aligned} &G_\kappa : (t, \Sigma, z, \mu) \mapsto (\kappa t, d_\kappa \Sigma, z, \kappa\mu), \\ &d_\kappa : \left(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)}\right) \mapsto \left(\kappa^r \sigma, \kappa^{(r-1)} \dot{\sigma}, \ldots, \kappa \sigma^{(r-1)}\right) \end{aligned} \quad (12)$$

transfers the trajectories of the inclusion (4), (5), (8) into the trajectories of the same inclusion but with the actuator parameter changed from μ to $\kappa\mu$. Choose any t_2, $0 < t_2 < t_b - t_a$. Let $\Theta \subset \Theta_\Sigma \times \Theta_z$, where $\Theta_\Sigma \subset \mathbf{R}^r$, $\Theta_z \subset \Omega_z$ are some bounded regions. With λ small enough get that with $\mu = \lambda\mu_0$ the trajectories starting from $d_\lambda B \times \Omega_{z0}$ converge to the invariant set $\Theta_1 \subset d_\lambda \Theta_\Sigma \times \Theta_z$ in the time t_2.

Let the set $d_\lambda \Theta_\Sigma \times \Theta_z$ satisfy the inequalities $|\sigma^{(i)}| \leq \gamma$, $i = 0, \ldots, r-1$, and $\|z\| \leq \gamma$, then for an arbitrary parameter μ and $\kappa = \mu/(\lambda\mu_0)$ obtain that (12) transfers $\Theta_1 \subset d_\lambda \Theta_\Sigma \times \Theta_z$ into $\Theta_2 \subset d_{\kappa\lambda} \Theta \times \Theta_z$ being the invariant set of (4), (5), (8). The set $d\Theta$ satisfies the inequalities $|\sigma| < a_0 \mu^r$, $|\dot{\sigma}| < a_1 \mu^{r-1}, \ldots, |\sigma^{(r-1)}| < a_{r-1}\mu$ with $a_i = \gamma(\lambda\mu_0)^{i-r}$. The new convergence time does not exceed $\mu t_2/(\lambda\mu_0)$.

Define $Q \subset \Omega_x$ as the subset of points with Σ belonging to $d_\lambda B$ at $t = t_a$, and let $t_1 = t_a + t_2$. ■

Lemma 3: Under Assumptions 4°, 7° let the input $u(t)$ of the actuator (4) be a Lipschitzian function of time $u(t)$ with some fixed Lipschitz constant. Then for any $\delta, \varepsilon > 0$ with sufficiently small μ the inequality $|v - u| \leq \varepsilon$ is established in the time δ and is kept afterwards.

Proof: Let the Lipschitz constant of $u(t)$ be $L > 0$. Consider the time transformation $t = \mu\tau$. Then (4) takes the form $\dot{z} = f(z, u_1(\tau))$, $v = v(z)$, $u_1(\tau) = u(\mu\tau)$. The function $u_1(\tau)$ is also Lipschitzian, but with the Lipschitz constant μL, i.e. is "almost" constant with $\mu \ll 1$. The further proof is based on the continuous dependence of the solutions of (4) on the right-hand side and Assumption 7°. ■

Proof of Theorem 1: Define the homogeneous vicinity $h_\delta(\Gamma)$ of the control singularity set Γ as the set comprised of the orbits of the group of dilations d_κ passing through the δ-vicinity of the intersection of Γ and the unit sphere. Let $H_\delta(\Gamma)$ be the union of $h_\delta(\Gamma)$ and the δ-vicinity of the origin $\Sigma = 0$. Outside of $H_{\delta/2}(\Gamma)$ the control $U(\Sigma)$ is a locally Lipschitzian function (Assumption 5°). Moreover, due to the homogeneity (7) the Lipschitz constant uniformly tends to zero along any orbit $d_\kappa \Sigma$, when $\kappa \to \infty$. Thus, outside of $H_{\delta/2}(\Gamma)$ the control is *globally* Lipschitzian. According to Lemma 3, with small μ, after an arbitrarily-short transient, $|u - v| \leq \gamma$ is kept outside of $H_\delta(\Gamma)$, $\gamma \ll 1$. Since $|v| \leq v_M$, the trajectories of (4), (5), (8) satisfy (8) with

$$v \in \begin{cases} U(\Sigma) + [-\gamma, \gamma], & \Sigma \notin H_\delta(\Gamma) \\ [-v_M, v_M], & \Sigma \in H_\delta(\Gamma). \end{cases}$$

The rest of the proof is based now on the continuous dependence of the solutions on the right-hand side with $\delta \to 0$, Assumption 8° and Lemma 1. ■.

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