Stability and Robustness of Homogeneous Differential Inclusions

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***Abstract*— The known results on asymptotic stability of homogeneous differential inclusions with negative homogeneity degrees and their accuracy in the presence of noises and delays are extended to arbitrary homogeneity degrees. Discretization issues are considered, which include explicit and implicit Euler integration schemes. Computer simulation illustrates the theoretical results.**

I. INTRODUCTION

Differential Inclusions (DIs) are often used for the adequate representation of uncertain dynamic systems. Also the control of uncertain systems is often discontinuous, i.e. is based on the Sliding-Mode (SM) Control (SMC) [24], [12], [23], [6]. The resulting SMC systems are most properly understood in the Filippov sense [14], i.e. are equivalent to some DIs. Thus, DIs and SMC methods are strongly related.

Sliding Modes (SMs) are used to suppress uncertainties by exactly keeping properly chosen outputs (sliding variables) at zero. Provided the system has a known well-defined relative degree, the problem is reduced to the finite-time (FT) stabilization of a less-order uncertain system [15]. Thus, the SMC problem turns to be a FT stabilization problem for a differential inclusion. The main idea is to design a control, producing a closed-loop FT stable DI of a negative homogeneity degree [8], [17].

Positive homogeneity degrees are employed when one wants to provide for the fixed-time convergence of the system trajectories [3], [11], [4], [22], [21]. Classical SMC systems [24], [12], [23] are often naturally described by discontinuous systems with the homogeneity degree 0.

The produced systems often feature such less desirable features as fast explosion of approximating discrete Euler schemes [18] in the case of positive homogeneity degrees, or extensive chattering in the operational mode in the case of discontinuous systems with non-positive homogeneity degrees. One of the methods to overcome the obstacle is to use implicit Euler schemes both in simulation and control [1], [2]. The questions on the convergence and accuracy of implicit Euler schemes applied to general DIs are still to be addressed.

Respectively, this paper briefly studies the coordinate homogeneity and stability features of general DIs. The asymptotic accuracy of disturbed asymptotically stable (AS) homogeneous DIs is calculated, and is shown to be directly determined by the coordinate weights and the system homogeneity degree. In particular, the stabilization accuracy is calculated in the presence of noises and variable delays.

The framework of the disturbed-Filippov-DIs' theory [14] allows to avoid difficult analysis of functional differential equations in infinite-dimension state spaces [17]. The idea is to use the homogeneity to extend the results obtained for small local disturbances.

The dicretization Euler methods are shown to be reduced to the study of DIs with variable *positive or negative* delays. Respectively new results on discretization are obtained as a consequence of the developed general theory.

Simulation results show the effectiveness of the proposed method and demonstrate its accuracy.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$. A function of a set is the set of function values on this set. The norm $||x||$ stays for the standard Euclidian norm of x, $B_\varepsilon = \{x \mid ||x|| \le \varepsilon\}$.

II. COORDINATE HOMOGENEITY OF DIFFERENTIAL INCLUSIONS

Recall that a solution of a DI

$$\dot{x} \in F(x),\ F(x) \subset \mathbb{R}^n, \tag{1}$$

is defined as any locally absolutely continuous function $x(t)$, satisfying the DI for almost all t. We call a DI (1) *Filippov DI*, if $F(x) \subset \mathbb{R}^n$ is non-empty, compact and convex for any x, and F is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$.

It is well-known that such DIs feature most standard properties, like existence and extendability of solutions, but not the uniqueness of solutions [14]. Asymptotically stable Filippov DIs have smooth Lyapunov functions [10].

A DI naturally appears when a dynamic system

$$\dot{x} = f(x),\ f : \mathbb{R}^n \to \mathbb{R}^n, \tag{2}$$

with a discontinuous vector field $f(x)$ is considered. The solutions of (2) are understood in the Filippov sense [14], i.e. as solutions of the corresponding Filippov DI.

Introduce the weights $m_1, m_2, \dots, m_n > 0$ of the coordinates $x_1, x_2, \dots, x_n$ in $\mathbb{R}^n$. Define the dilation [5]

$$d_\kappa : (x_1, x_2, ..., x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_n} x_n),$$

where $\kappa > 0$. Recall [5], [16] that a function $g : \mathbb{R}^n \to \mathbb{R}$ is said to have the homogeneity degree (weight) $q \in \mathbb{R}$,

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$\deg g = q$, if the identity $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^n$ and $\kappa > 0$.

Contrary to the case of a function, DI (1) implicitly involves time in the derivative. Consider the combined time-coordinate transformation

$$G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x), \quad \kappa > 0,$$

where p might naturally be considered as the weight of t. The DI $\dot{x} \in F(x)$ and the vector-set field $F(x)$ are called homogeneous of the homogeneity degree $q = -p$, if the DI is invariant with respect to the above transformation.

The respective homogeneity property can be rewritten as $\dot{x} \in F(x) \Leftrightarrow \frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x)$. Thus we come to the equivalent formal definition.

Definition 1: [17] A vector-set field $F(x) \subset \mathbb{R}^n$ (DI $\dot{x} \in F(x)$), $x \in \mathbb{R}^n$, is called *homogeneous of the degree* $q \in \mathbb{R}$, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$.

The differential equation (2), $\dot{x}_i = f_i(x)$, is a particular case of DI, when the set $F(x)$ contains only one vector $f(x)$. Then the above definition is reduced to the standard definition $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg f_i$ [5], [16]. Note that if f is discontinuous, (2) is equivalent to the corresponding homogeneous Filippov DI (1).

The homogeneity degree q can be positive, negative or zero. The identity $\kappa^{m_i} x_i = (\kappa^{1/\gamma})^{m_i \gamma} x_i$, $\gamma > 0$, shows that the weights $p = -q$, $m_1, ..., m_n$ are defined up to proportionality. In particular, the non-zero homogeneity degree q of a vector-set field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, ..., m_n$. As we will see in the sequel, the sign of the homogeneity degree determines many properties of DIs.

Any continuous positive-definite function of the homogeneity degree 1 is called a homogeneous norm. We denote it $||x||_h$. Note that the proportional change of all weights also changes the weight of each function. Respectively a function which is a homogeneous norm for the weight vector $m \in \mathbb{R}^n$ is not a homogeneous norm under another equivalent weight distribution γm, $\gamma \neq 1$, $\gamma > 0$. The following are examples of homogeneous norms:

$$||x||_{h\infty} = \max_i |x_i|^{\frac{1}{m_i}}, \; ||x||_{hd} = \left(\sum_i |x_i|^{\frac{d}{m_i}} \right)^{\frac{1}{d}}, \; d > 0. \tag{3}$$

The latter homogeneous norm is continuously differentiable for $x \neq 0$ if $d > \max m_i$. Note that all homogeneous norms are equivalent in the sense that for each two norms there exist $\gamma_1 \geq \gamma_2 > 0$, such that $\gamma_2 ||x||_{h1} \leq ||x||_{h2} \leq \gamma_1 ||x||_{h1}$.

III. Stability of homogeneous inclusions

All over the paper the differential inclusion has equilibrium at zero, and only the (strong) stability of this equilibrium is studied.

Recall that the inclusion (1) is called *asymptotically stable* (AS) if zero is a Lyapunov-stable equilibrium, each solution $x(t)$ is extended till infinity in time and $\lim_{t \to \infty} x(t) = 0$.

DI (1) is called exponentially stable if it is AS and there exist numbers $\gamma_1 > \gamma_2 > 0$ such that for any $0 < R_2 \leq R_1$ the time of any trajectory transient from the Euclidean sphere with radius R_1 to the sphere with the radius R_2 belongs to the segment $[\gamma_2, \gamma_1] \ln(R_1/R_2)$.

DI (1) is called *finite-time stable* (FTS) if it is asymptotically stable, and each solution converges to zero in finite time (FT).

DI (1) is called *fixed-time* (FxT) *attracted* (FxTA) [21], [22] to some vicinity Ω of zero, if each solution converges into Ω in FT, and the corresponding transient times posses a common finite upper bound.

DI (1) is called *fixed-time stable* (FxTS) [21], [22], if it is AS (or FTS) and FxT attracted (FxTA) to zero.

Basic conditions. All over the paper it is assumed that the DI (1) is an asymptotically stable homogeneous Filippov differential inclusion. In particular these conditions imply that $m_i - p \geq 0$ for all $i = 1, ..., n$ [19].

Definition 2 ([17]): A set $D \subset \mathbb{R}^n$ is called *dilation retractable* if $\forall \kappa \in [0, 1]$ $d_\kappa D \subset D$. A homogeneous DI (1) is called *contractive*, if there exist two nonempty compact sets D_1, D_2 and $T > 0$ satisfying the following conditions. The set D_1 is dilation retractable, D_2 lies in the interior of D_1, and each solution which starts in D_1 at time $t = 0$ is in D_2 at $t = T$.

Note that any ball $||x|| \leq R$ is dilation retractable. The following Theorem generalizes some results of [3], [7], [9], [17], [18], [20], [21] to DIs of arbitrary homogeneous degrees.

Theorem 1: A homogeneous Filippov DI (1) is AS iff it is contractive. Moreover:

1) If the homogeneity degree is negative, asymptotic stability is equivalent to FT stability.
2) If the homogeneity degree is zero, asymptotic stability is exponential.
3) If the homogeneity degree is positive, any open vicinity of zero attracts solutions in FxT. The convergence to zero is slower than exponential.

Here and further all proofs are placed in Section VI. The following theorem extends a corollary from [17] to any homogeneity degrees and establishes stability of local homogeneous approximations [3], [5].

Theorem 2: Let $\dot{x} \in \tilde{F}(x)$ be any other homogeneous Filippov DI with the same homogeneity dilation and degree. Then there exists such $\delta > 0$ that if the inclusion $\tilde{F}(x) \subset F(x) + B_\delta$ holds on the homogeneous sphere $||x||_h = 1$, it implies the asymptotic stability of the DI $\dot{x} \in \tilde{F}(x)$.

IV. Accuracy of disturbed homogeneous differential inclusions

Estimate the steady-state accuracy of the disturbed differential inclusion (1). The resulting DI is

$$\begin{array}{l} \dot{x}(t) \in F(x(t + [-\tau_-, \tau_+]) + \varepsilon[-1, 1]), \;\; x \in \mathbb{R}^n, \\ t \in [0, t_f], \; \varepsilon = (\varepsilon_1, ..., \varepsilon_n), \; \varepsilon_i \geq 0, \; \tau_-, \tau_+ \geq 0. \end{array} \tag{4}$$

Here ε_i is the magnitude of possible "measurement errors" for x_i, time delays vary in $[-\tau_-, \tau_+]$, i.e. *can be both positive* *and negative*. There is no connection between errors and delays of the same or different coordinates at the same or different times.

The presence of the time deviations requires initial and final conditions

$$\begin{array}{ll} x(t) = \xi_-(t), & t \in [-\tau_-, 0], \\ x(t) = \xi_+(t - t_f), & t \in [t_f, t_f + \tau_+]. \end{array} \tag{5}$$

The initial conditions should posses some natural homogeneity properties, which are automatically satisfied, provided ξ_-, ξ_+ satisfy the simple DI

$$\begin{array}{l} \dot{\hat{\xi}}_{\vartheta i} \in \varpi(\|\hat{\xi}_\vartheta\|_h + ||\varepsilon||_h + \tau_-^{1/p} + \tau_+^{1/p})^{m_i - p}[-1, 1], \\ i = 1, ..., n, \ \hat{\xi}_\vartheta(0) = 0, \ \vartheta \in \{+, -\}; \\ \xi_-(t) = x(0) + \hat{\xi}_-(t), \ t \in [-\tau_-, 0]; \\ \xi_+(t) = x(t_f) + \hat{\xi}_+(t), \ t \in [0, \tau_+]. \end{array} \tag{6}$$

Recall that $m_i \geq p$. It is formally defined here that $\forall c \geq 0 : c^0 \equiv 1$. If $p < 0$ the function ξ may be only defined over a subsegment of $[-\tau_-, \tau_+]$; $\varpi > 0$ is chosen so as to include the initial conditions of a considered concrete system.

We do not consider the questions of the solution existence. Obviously one can choose the functions ξ_-, ξ_+ and assign a distribution of delays in time and coordinates in such a way that there be no solutions. Nevertheless, in the considered case the set of solutions is not empty. In particular it trivially contains solutions of the original DI (1) (also see the next Section). The following theorems generalize [17] to any p.

Theorem 3: Let the homogeneity degree be negative, i.e. $\deg t = p > 0$, $\tau_M = \max(\tau_-, \tau_+)$. Then there exists such $\mu_0 > 0$ that for any $R \geq 0$ there exists $T_R \geq 0$ such that for any solution of (4), (5), (6) the inequalities $||x(0)||_h \leq R$ and $t_f > T_R$ imply that

$$||x(t)||_h \leq \mu_0 \max[||\varepsilon||_h, \tau_M^{1/p}]$$

is kept for any $t \geq T_R$.

Theorem 4: Let the homogeneity degree be zero, $p = 0$, $\tau_M = \max(\tau_-, \tau_+)$, and $||\varepsilon||_h > 0$. Then for some $\tau_M > 0$ there is $\mu_\tau > 0$ such that for any $R \geq 0$ and some $T_R \geq 0$ the inequalities $||x(0)||_h \leq R$ and $t_f > T_R$ imply that

$$||x(t)||_h \leq \mu_\tau ||\varepsilon||_h \tag{7}$$

is kept for any $t \geq T_R$. The transient time T_R tends to infinity as $||\varepsilon||_h \to 0$. In particular, if $\varepsilon = 0$ all indefinitely extendable solutions asymptotically converge to zero.

Theorem 5: Let the homogeneity degree be positive, i.e. $\deg t = p < 0$, $\tau_M = \max(\tau_-, \tau_+)$. Then there exist such μ_x, $\eta_{t,1}, \eta_{t,2}$, $\eta_R > 0$, $\eta_{t,2} \geq \eta_{t,1} > 0$ that for any $R \geq 0$

1) if $0 < ||\varepsilon||_h \leq \eta_{t,1} \tau_M^{1/p}$ and $R \leq \eta_R \tau_M^{1/p}$ then there is $T_R \geq 0$ such that the inequalities $||x(0)||_h \leq R$ and $t_f > T_R$ imply that

   $$||x(t)||_h \leq \mu_x ||\varepsilon||_h$$

   is kept for any $t \geq T_R$; $T_R \to \infty$ as $||\varepsilon||_h \to 0$;
2) if $||\varepsilon||_h \geq \eta_{t,2} \tau_M^{1/p}$ or $R > \eta_R \tau_M^{1/p}$ and $\tau_+ = 0$, $\tau_- > 0$ then some extendable-in-time solutions diverge to infinity.

V. Discretization issues

Simulation of homogeneous DIs is usually performed by the Euler method. Indeed, higher-order integration schemes are based on the Taylor expansion of solutions, which requires the right-hand side to be a sufficiently smooth vector field. This condition does never hold for negative homogeneous degrees, and obviously never holds when the DI contains more than one vector on the right-hand side.

One step of the (explicit) Euler scheme for DI (1) is described by the recursive equation

$$x(t_{k+1}) = x(t_k) + y_k h_k, \ y_k \in F(x(t_k)), \ t_{k+1} = t_k + h_k, \tag{8}$$

where $h_k > 0$ is the variable kth integration step. Arbitrary value $y_k \in F(x(t_k))$ is taken at each step.

The *implicit* Euler scheme has the form

$$x(t_{k+1}) = x(t_k) + y_k h_k, \ y_k \in F(x(t_{k+1})), \ t_{k+1} = t_k + h_k. \tag{9}$$

In order to realize one step (9) one needs to find both $x(t_{k+1})$ and $y_k \in F(x(t_{k+1}))$. It is not always simple, and sometimes even impossible. Nevertheless, the implicit scheme features better stability properties, which justifies its application.

The Euler solution is defined as a piece-wise linear function connecting the points $x(t_k)$. Thus one gets

$$\dot{x}(t) = y(t_k), \ y(t_k) \in F(x(t_k)), \ t \in [t_k, t_{k+1}) \tag{10}$$

for the explicit scheme, and

$$\dot{x}(t) = y(t_{k+1}), \ y(t_{k+1}) \in F(x(t_{k+1})), \ t \in [t_k, t_{k+1}) \tag{11}$$

for the implicit one.

Note that the special property of (10) and (11) is that both do not need initial/final functional conditions (5). Only the values $x(0)$ or $x(t_f)$ are respectively needed. The main difficulty of the implicit scheme implementation is that instead of $x(t_f)$ only $x(0)$ is usually available.

Further we respectively call solutions of (10) and (11) explicit/implicit Euler solutions. One can also consider (11) as an explicit Euler solution in the backward time, or the explicit Euler solution for the reversed system $\dot{x} \in -F(x)$.

Obviously any explicit Euler solution is easily extended in time till infinity. It means that no explicit Euler solution escapes to infinity in finite time, which is not very useful.

Obviously any implicit Euler solution is extendable in time till minus infinity. This means that implicit Euler solutions never demonstrate fixed-time attraction to a ball. In the particular case of system (2), i.e. $\dot{x} = f(x)$, with continuous f one has $f(0) = 0$, which means that non-zero implicit solution different from identical zero never vanishes. Respectively it never demonstrates finite-time convergence to zero.

Consider AS system (2) with continuous f. It can be shown that for non-zero homogeneity degree under some conditions extendable implicit solutions exist for any initial conditions and globally asymptotically converge to zero for any constant integration step h. The result appears in the paper [13] presented by the authors at this conference.

Let $\sup h_k \leq \tau$. Then due to the identity $t_k = t-(t-t_k)$ equation (10) implies

$$\dot{x}(t) \in F(x(t+\tau[-1,0])), \quad (12)$$

whereas $t_{k+1} = t + (t_{k+1} - t)$ and (11) imply

$$\dot{x}(t) \in F(x(t+\tau[0,1])). \quad (13)$$

Results of the previous section are applicable to these systems. In particular Theorem 4 implies that for the homogeneity degree 0 and sufficiently small τ all explicit and extendable implicit Euler solutions asymptotically converge to zero.

Theorem 6: Let the homogeneity degree be positive, i.e. $\deg t = p < 0$, $h_k = \text{const} = h$. Then for any $h > 0$ there exists such $R > 0$ that provided $||x(0)|| \geq R$ all explicit Euler solutions escape to infinity.

A simulation example for Theorem 6 is presented in [18]. It is also proved there that under the conditions of Theorem 6 some explicit Euler solutions with *variable* integration steps $h_k \leq \tau$ escape to infinity faster than any exponent.

VI. Proofs

Only the main points of the proofs are presented.

Proof of Theorem 1. Asymptotic stability of (1) implies existence of a smooth homogeneous Lyapunov function $V(x)$ [10], [7], $\deg V = d > \max(0,p)$. To prove the contractivity choose a ball $||x|| \leq R$ as the set D_1, and find $v_1 > v_2 > 0$ such that $\{V(x) \leq v_2\} \subset \{||x|| < R\}$ and $\{||x|| \leq R\} \subset \{V(x) \leq v_1\}$. Finally define $D_2 = \{V(x) \leq v_2\}$.

Now suppose the contractivity of (1) and prove its asymptotic stability. For some $\kappa_0 \in (0,1)$ get that $D_2 \subset d_{\kappa_0} D_1$. The time of transition from D_1 to $d_{\kappa_0} D_1$ does not exceed T. Introduce $\Omega_j = d^j_{\kappa_0} D_1$, were j can be any integer. Since D_1 contains a ball around 0, the sets Ω_j together cover the whole space $\mathbb{R}^n$. Respectively the time of transition from $\Omega_j = d^j_{\kappa_0} D_1$ to $\Omega_{j+1} = d^{j+1}_{\kappa_0} D_1$, $\Omega_{j+1} \subset \Omega_j$, is estimated by ${\kappa_0}^{jp} T$. Hence, the time of convergence from Ω_{j_0} to 0 is estimated by

$$T_{j_0} \leq T{\kappa_0}^{j_0 p}\left(1 + {\kappa_0}^p + {\kappa_0}^{2p} + ...\right).$$

This sum is finite if $p > 0$ and infinite if $p \leq 0$. Respectively there is finite-time convergence for $p > 0$ and only asymptotic convergence for $p \leq 0$.

Stability of the equilibrium 0 follows from the simple fact that the points of all trajectories starting in Ω_j at $t=0$ and finishing in Ω_{j+1} at $t = {\kappa_0}^{jp} T$ constitute a compact invariant set [14], which shrinks to 0 as $j \to \infty$. This proves the main statement and claim 1 of the theorem.

Prove claim 3. Let $p < 0$ and a closed ball B contain $\Omega_{j_0} = d^{j_0}_{\kappa_0} D_1$. Then for $j < j_0$ the time of convergence from Ω_j to B is estimated from above by

$$T_{j,j_0} \leq T{\kappa_0}^{j_0 p}\left(1 + {\kappa_0}^{-p} + {\kappa_0}^{-2p} + ... + {\kappa_0}^{-(j_0-j)p}\right).$$

It is a segment of converging series, therefore the transient time is uniformly bounded, which proves the FxT convergence to B.

According to [7] $\max_{y \in F(x)} \frac{\partial}{\partial x} V(x) y \leq -W(x)$, where W is a smooth positive-definite function, $\deg W = d - p > d > 0$. Thus $\deg V^{\frac{d-p}{d}} = \deg W$ and for some $0 < \lambda_1 \leq \lambda_2$ the DI $\dot{V} \in -[\lambda_1, \lambda_2] V^{\frac{d-p}{d}}$ is kept along any solution, which implies that the convergence is slower than exponential near 0.

It remains to prove claim 2. Since $p = 0$ get $\deg V = \deg W$, and $\dot{V} \in -[\lambda_1,\lambda_2]V$. The exponential stability is now obvious. $\square$

Proof of Theorem 2. Obviously the contractivity property is robust with respect to small homogeneous perturbations. $\square$

Further proofs use the following construction. Instead of the system (4) consider a system with a larger set of solutions

$$\begin{array}{l} \dot{x}(t) \in F(x(t+\tau_M[-1,1]) + d_{\varepsilon_M}(1,...,1)[-1,1]), \\ \varepsilon_M = ||\varepsilon||_{h\infty},\ \tau_M = \max(\tau_-,\tau_+),\ t \in [0,t_f]. \end{array} \quad (14)$$

The homogeneous norm $||\varepsilon||_{h\infty}$ is introduced in (3). Denote $\pi = (\tau_M, \varepsilon_M) \in \mathbb{R}^2$. The time-coordinate-parameter transformation

$$\begin{array}{l} \pi = (\tau_M, \varepsilon_M),\ \Delta_\kappa : \pi \mapsto (\kappa^p \tau_M, \kappa \varepsilon_M),\ \kappa > 0, \\ (t,x,\pi) \mapsto (\kappa^p t, d_\kappa x, \Delta_\kappa \pi) \end{array} \quad (15)$$

establishes a one-to-one correspondence between solutions of (14), (5) with parameters π and $\Delta_\kappa \pi$.

Proof of Theorem 3. Consider system (14). First show that there is a finite-time attractor. All solutions of (1) starting in a ball D_1 in some finite time T_0 gather in any fixed small ball D_2 to stay there. Due to the continuous dependence of the phase stream on the right-hand side [14] for sufficiently small ε_M, τ_M solutions of (14) gather in some larger ball $\hat{D}_2$, $\hat{D}_2 \subset D_1$.

Since $\hat{D}_2 \subset D_1$, solutions starting in $\hat{D}_2$ return to $\hat{D}_2$ in the time T_0. Thus, the set of all points of the solution segments over the time period $[0,T_0]$ which start in $\hat{D}_2$ constitute a compact [14] invariant set $\hat{D}$ for (14). For sufficiently small D_2, π also get $\hat{D} \subset$ interior D_1. Fix the corresponding parameters $\pi_0 = (\tau_{M0}, \varepsilon_{M0})$.

Show that $\hat{D}$ is a global FT attractor of (14), (5) . Choose a value $\kappa_0 \in (0,1)$ such that $\Omega_1 = d_{\kappa_0} D_1 \supset \hat{D}$. All solutions of (14) starting in D_1 enter $d_{\kappa_0} D_1$ in time T_0 to stay there forever. Build the sequence $\{\Omega_j\}$, $j = 0,-1,...$, $\Omega_j = d_{\kappa_0}\Omega_{j-1}$, $\kappa_0 \in (0,1)$, $\Omega_0 = D_1$. Transformation (15) with $\kappa = \kappa_0^{-j}$ transfers transient solutions of (14) between Ω_0 and Ω_1 into the solutions between Ω_j and Ω_{j+1} with the transient time $\kappa_0^{-jp} T_0$. These solutions include all solutions of the system with the fixed parameters π_0 since $\kappa_0^{-1} > 1$. Thus each solution in finite time enters Ω_0 and then $\hat{D}$ to stay there forever.

Note that this chain cannot be extended to $j > 1$. Indeed, transient solutions from Ω_1 to Ω_2 correspond to the DI (14),(5) with the parameters $\Delta_{\kappa_0}\pi_0$. Since $\kappa_0 < 1$ this solutions include only a part of the solutions of (14), which prevents the further convergence.

Let $\hat{D}$ lie in the set $||x||_h \leq \gamma$. Consider now DI (14), (5) with arbitrary disturbance parameters π. Transformation

(15) with $\kappa = \max[(\tau_M/\tau_{M0})^{1/p}, ||\varepsilon||_{h\infty}/\varepsilon_M]$ transfers solutions of (14) with parameters π_0 into solutions of (14) with parameters $\Delta_\kappa \pi_0$. Those solutions obviously include all solutions of (4) and converge to $d_\kappa \hat{D}$. The latter satisfies $||x||_h \le \kappa\gamma$. The equivalence of homogeneous norms finishes the proof. □

Proof of Theorem 4. In the case $p = 0$ transformation (15) does not change τ_M. Respectively all the reasoning of the previous proof is to be repeated with the same τ_M. After one gets the invariant attractor $\hat{D} \subset \{||x||_h \le \gamma\}$ for some $\pi_0 = (\tau_{M0}, \varepsilon_{M0})$, $\tau_M = \tau_{M0}$ is fixed. Applying transformation (15) with $\kappa = ||\varepsilon||_{h\infty}/\varepsilon_{M0}$ obtain the accuracy (7).

In the case $\varepsilon_M = 0$ the chain ... $\supset \Omega_j \supset \Omega_{j+1} \supset$ ... is extended to any $j > 0$. Respectively similarly to the proof of Theorem 1 one gets asymptotic convergence to zero. □

Proof of Theorem 5. In the case $p < 0$ transformation (15) increases τ_M and decreases ε_M for $\kappa < 1$. On the contrary, it decreases τ_M and increases ε_M for $\kappa > 1$. Choose some value $\pi_0 = (\tau_{M0}, \varepsilon_{M0})$ for which Ω_0 and Ω_1 can be built as in the proof of Theorem 3. The chain Ω_j cannot be now extended to $j > 1$, since ε_M decreases as the result excluding some system solutions. It also cannot be extended to $j < 0$, since τ_M decreases excluding solutions with larger absolute delay values.

Instead choose some $\pi_1 = (\tau_{M1}, \varepsilon_{M0})$ with $\tau_{M1} < \tau_{M0}$. The same sets Ω_0 and Ω_1 can be used for π_1. Now the chain can be extended to any $j < 0$ until the condition $\kappa_0^{jp}\tau_{M1} \le \tau_{M0}$ is violated. Thus for any τ_{M1} we get a bounded region of convergence to $\hat{D}$. If D_1 is a homogeneous ball $D_1 = \{||x||_h \le R_1\}$, also the region of attraction to $\hat{D}$ is a homogeneous ball determined by

$$||x||_h \le \kappa_0^{-j_M} R_1,\ j_M = \left[\frac{\ln(\tau_{M0}/\tau_{M1})}{p \ln \kappa_0}\right],$$

where $[\cdot]$ stays for the integer part. Therefore accuracy remains the same for any $\tau_{M1} < \tau_{M0}$, and is only determined by π_0, while the attraction region is determined by τ_{M1}. Similarly to the proof of Theorem 1 the convergence time is bounded from above by some T_1 and from below by some $T_0 > 0$.

Now take any parameter $\pi = (\tau_M, \varepsilon_M)$, and choose $\kappa = ||\varepsilon||_{h\infty}/\varepsilon_{M0}$. The convergence to the attractor is assured for

$$||x||_h \le \kappa_0^{-j_M} \tfrac{R_1}{\varepsilon_{M0}} ||\varepsilon||_{h\infty},\ j_M = \left[\frac{\ln((\frac{||\varepsilon||_{h\infty}}{\varepsilon_{M0}})^p \tau_{M0}/\tau_M)}{p \ln \kappa_0}\right]$$

while at least the accuracy $||x||_h \le \frac{R_1}{\varepsilon_{M0}} ||\varepsilon||_{h\infty}$ is provided. The convergence time belongs to $[T_0, T_1]\kappa^p$. The corresponding attraction region is empty if $\tau_M > (\frac{||\varepsilon||_{h\infty}}{\varepsilon_{M0}})^p \tau_{M0}$. This proves claim 1 of the theorem.

It is needed now to show that if ε_M is sufficiently large with respect to τ_M, or τ_M is sufficiently large with respect to $||x(0)||_h$, then some extendable in time solutions diverge to infinity.

Take explicit Euler solutions (8) with the constant integration step $h_k = \tau_M$. It corresponds to the case $\tau_+ = 0$. First take the case $\varepsilon_M = 0$, $h_k = \tau_M$, $t_{k+1} = t_k + \tau_M$.

Let $V(x)$ be the smooth Lyapunov function for (1), $\deg V = d > 0$. Let $x = d_\kappa z$, where $V(z) = 1$. Thus $V(x) = \kappa^d$. Recall that $-p = q > 0$, and $V(x) \to \infty$ when $||x|| \to \infty$. Also note that the set $F(S)$ is separated from zero over $S = \{x|\ V(x) = 1\}$.

Let $E_h(x) = x + f_x \tau_M$ be the Euler-step operator, $f_x \in F(x)$, $f_x = \kappa^q d_\kappa f_z$, $f_z \in F(z)$, $q = -p > 0$. Then one gets

$$\begin{aligned} V(E_h(x)) &= V(d_\kappa(z + f_z \kappa^q \tau_M)) \\ &= V(x) V(z + f_z V^{q/d}(x) \tau_M). \end{aligned}$$

For sufficiently large x one gets $V(x) \ge c_m ||x||^{d/\overline{m}}$ for some $c_m > 0$ and $\overline{m} = \max m_i$. Respectively for large x get

$$V(E_h(x)) \ge \tfrac{1}{2} c_m \tau_M^{d/\overline{m}} V^{1+q/\overline{m}}(x).$$

Hence for some $\rho_0 > 0$ the inequality $||x(0)|| \ge \rho_0$ implies that $V(x(t_k))$ monotonously tends to infinity. Since for large x also $V(x) \le c_M ||x||^{d/\underline{m}}$ for some $c_M > 0$ and $\underline{m} = \min m_i$, one gets $||x(t_k)|| \to \infty$.

Now take some ε, $||\varepsilon|| > \rho_0$, and consider the explicit Euler solution with the constant "measurement error"

$$\dot{x}(t) = y(t_k),\ y(t_k) \in F(x(t_k) + \varepsilon),\ t \in [t_k, t_{k+1}).$$

Obviously its solution starting at $t_0 = 0$ at $x(0) = 0$ escapes to infinity. Rewriting these relations with respect to homogeneity obtain claim 2. □

Proof of Theorem 6. The theorem proof is contained in the proof of the second statement of Theorem 5. □

VII. Simulation results

Consider an academic example of a disturbed AS homogeneous system of the homogeneity degree 0

$$\begin{aligned} &\dot{x}_1 = \hat{x}_2(t_k),\ t \in [t_k, t_{k+1}), \\ &\dot{x}_2 = \hat{x}_1(t_k) + \hat{x}_2(t_k) + u(\hat{x}_1(t_k), \hat{x}_2(t_k)), \\ &u(x_1, x_2) = \begin{cases} -15x_1 & \text{if } (x_1 + x_2)x_1 > 0, \\ +15x_1 & \text{if } (2x_1 + x_2)x_1 < 0, \\ 0 & \text{otherwise} \end{cases} \\ &\hat{x}_1(t) = x_1(t) + \eta_1(t),\ \ \hat{x}_2(t) = x_2(t) + \eta_2(t), \\ &\eta_1(t), \eta_2(t) \in [-\varepsilon, \varepsilon],\ t_{k+1} - t_k = h_k \in [0, \tau]. \end{aligned} \tag{16}$$

The homogeneous weights $\deg x_1 = \deg x_2 = 1$, $q = p = 0$ can be taken. Equation (16) describes an explicit Euler solution with noises not exceeding ε and the variable step h_k not exceeding τ. The solutions satisfy a Filippov differential inclusion [14]. The system is a particular case of (4) with $\tau_+ = 0$, $\tau_- = \tau$. According to Theorem 4 system (16) is AS for all sufficiently small τ and $\varepsilon = 0$. In the presence of noises the resulting accuracy should be proportional to ε.

Let $x(0) = (10, 20)$, the simulation time interval is $[0, 10]$. Simulation shows that the system is AS for any $\tau \le 0.1$. The transient trajectory of the original system (16) with $\varepsilon = \tau = 0$ is demonstrated in Fig. 1a. The visible sliding mode appearance means that the system is described by a Filippov DI. Convergence to the origin in the presence of the constant Euler step $h_k = 0.1$ is shown in Fig. 1b. The variable Euler integration step $h_k(t) = |\cos(19t)|$

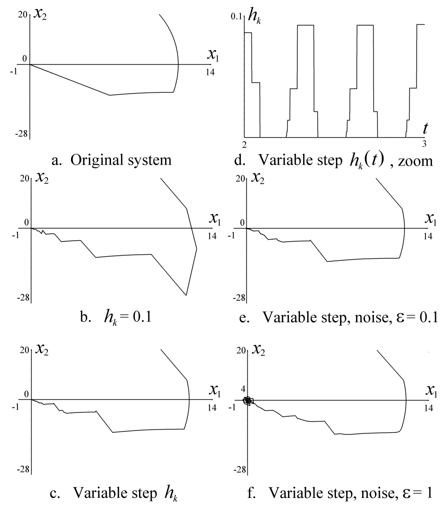

Fig. 1. Performance of system (16) with homogeneity degree 0.

yields the performance from Fig. 1c. The graph of the Euler integration/sampling step over the subinterval [2,3] is shown in Fig. 1d. In the absence of noise all trajectories exponentially converge to zero producing the accuracy of about 10^{-6} for both coordinates at $t = 10$.

Now introduce noisy sampling. The noises $\eta_1(t_k), \eta_2(t_k)$ are simulated as Gaussian noises with the dispersion 0.5ε, whicht roughly corresponds to the noise magnitude ε. The same variable step h_k is kept. The trajectories corresponding to $\varepsilon = 0.1$ and $\varepsilon = 1$ are presented in Fig. 1e and 1f respectively. Following are the resulting accuracies for different noise magnitudes:

$$|x_1| \leq 3.62 \cdot 10^{-4},\ |x_2| \leq 2.14 \cdot 10^{-3} \quad \text{for } \varepsilon = 0.001,$$
$$|x_1| \leq 3.78 \cdot 10^{-3},\ |x_2| \leq 2.20 \cdot 10^{-2} \quad \text{for } \varepsilon = 0.01,$$
$$|x_1| \leq 3.49 \cdot 10^{-2},\ |x_2| \leq 2.18 \cdot 10^{-1} \quad \text{for } \varepsilon = 0.1,$$
$$|x_1| \leq 3.65 \cdot 10^{-1},\ |x_2| \leq 2.17 \cdot 10^{0} \quad \text{for } \varepsilon = 1.$$

The claimed asymptotic accuracy is perfectly kept.

VIII. Conclusions

Stability and accuracy of disturbed AS homogeneous Filippov DIs is studied in the presence of sampling noises and variable delays. Similarly to the case of differential equations asymptotic stability features are determined by the system homogeneity degree. The steady-state accuracy is determined by the weights of the coordinates and the system degree.

Implicit and explicit Euler integration schemes are shown to be described by the original differential inclusion with respectively positive and negative time deviations (delays). Thus the general obtained results imply some new results on the convergence of implicit and explicit Euler schemes to the origin.

In particular, the explosion of explicit Euler schemes for systems with positive homogeneity degree is proved to take place for any sufficiently large initial conditions. Accuracy of both schemes are evaluated for variable integration steps in the presence of sampling noises. For example, it is shown that for the zero homogeneity degree the homogeneous radius of the steady-state attractor of a general nonlinear AS system with variable delays is proportional to the homogeneous norm of the noise magnitudes' vector. The results are confirmed by simulation of a disturbed system of nonlinear variable structure with the homogeneity degree 0.

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