Low-chattering discretization of homogeneous differentiators

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***Abstract*—The proposed simple modification of discrete homogeneous sliding-mode-based differentiators improves the accuracy and significantly lowers the output chattering in the absence of noises, while preserving the differentiation accuracy in their presence. The approach is extended to general discretely sampled homogeneous discontinuous systems. Numeric experiments confirm the theoretical results.**

***Index Terms*—Sliding-mode control, nonlinear filtering, estimation, discrete event systems, uncertain systems.**

I. Introduction

Sliding-mode (SM) control (SMC) [17], [45], [47] is based on exactly keeping proper functions (SM variables) at zero. It is known for its finite-time (FT) convergence and high accuracy in spite of system uncertainties.

SM-based n_dth-order differentiators in FT produce exact derivative estimations $z_i = f_0^{(i)}(t)$, $i = 0, 1, ..., n_d$, of the signal $f_0(t)$, provided $|f_0^{(n_d+1)}| \leq L_0$ holds for some known $L_0 > 0$ [2], [15], [16], [18], [30]. The asymptotics of errors $z_i - f_0^{(i)}$ in the presence of bounded sampling noises are proved to be the best possible [39]. The recent filtering differentiators [40], [38] also suppress unbounded noises having a small local kth-order iterated integral, $k \leq n_f$. The number n_f is called the filtering order.

SM algorithms are often undermined by the chattering effect due to additional dynamics, noises or discretization [11], [17], [32], [45], [46], [47]. The SM regularization [21], [46] compromises the SMC accuracy and its insensitivity to matched disturbancies. Increasing the relative degree moves the discontinuities to the higher derivatives of sliding variables [6], [7], [20], [29], [30], [42] and often utilizes the homogeneity approach [8], [31], but the additional dynamics may rise the sensitivity to noises [11]. Implicit discretization schemes [1], [24] are effective, but often difficult to implement.

In this paper we propose a simple low-chattering discretization of homogeneous SM-based differentiators and generalize it to a general homogeneous SM-discretization approach.

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Practical modern filters are discrete-time computation schemes with a maximal sampling period τ. *Such a scheme is further considered a filter discretization, if in the noise-free case its outputs converge to the outputs of their continuous-time counterparts as* $\tau \to 0$. In our case the convergence is uniform over any compact region of errors and time.

Unfortunately, the outputs of the standard discrete SM-based differentiators [39], [38], [40] feature strong chattering for low sampling rates and large L_0.

Recent sophisticated low-chattering discrete schemes [27], [26], [43], [44], and implicit SM discretizations [12], [13], [14] are proposed to counteract these phenomena. These schemes usually require complicated calculations sometimes including real-time numeric solution of nonlinear algebraic equations at each sampling step. Their analysis in the presence of sampling noises is difficult. In numeric experiments these schemes typically perform worse or similarly to the standard schemes [38], [39], if significant noises are combined with overestimated L_0.

The new discretization scheme is only activated when some homogeneous inequality on errors and τ holds. It guaranties at least the original accuracy [38] in the presence of noises. In the absence of noises it reduces outputs' chattering and removes any influence of overestimated L_0.

The approach is extended to a general chattering-mitigation method of SM discretization which guaranties the preservation of the homogeneity-based accuracy.

Our conference paper [23] restricts the approach to differentiators with the low-chattering discretization of the zero homogeneity degree and contains no proofs. Here we provide proofs, new extensive simulation results and all needed parameters up to $n_d + n_f = 12$.

Notation. A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b |\ a \in A, b \in B\}$, $a \diamond B = \{a\} \diamond B$. A function of a set is the set of function values on this set. The norm $||x||$ stays for the standard Euclidian norm of x, $B_\varepsilon = \{x \mid ||x|| \leq \varepsilon\}$; $||x||_h$ is a homogeneous norm, $B_{h,\varepsilon} = \{x \mid ||x||_h \leq \varepsilon\}$.

For any sampled sequence $\phi(t_j)$ denote $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$, also $\mathrm{sat}\, a = \max(-1, \min(1, a))$; $\vec{\gamma}_k = (\gamma_0, ..., \gamma_k)$ for any sequence γ_i, $\vec{\gamma}_{i,j} = (\gamma_i, ..., \gamma_j)$, $i \leq j$; $\mathbb{R}_+ = [0, \infty)$; $\lfloor a \rceil^b = |a|^b \,\mathrm{sign}\, a$, $\lfloor a \rceil^0 = \mathrm{sign}\, a$.

II. Weighted homogeneity basics

Recall that a solution of a differential inclusion (DI)

$$\dot{x} \in F(x),\ x \in \mathbb{R}^{n_x}, F(x) \subset T_x\mathbb{R}^{n_x}. \tag{1}$$

is any locally absolutely continuous function $x(t)$, satisfying DI (1) for almost all t. Here $T_x\mathbb{R}^{n_x}$ denotes the tangent space to $\mathbb{R}^{n_x}$ at the point x.

DI (1) is further called *Filippov DI*, if the vector set $F(x)$ is non-empty, compact and convex for any x, and F is upper-semicontinuous [19], [31], i.e. the maximal distance of the vectors of $F(x)$ from the vector set $F(y)$ vanishes as $x \to y$.

Let $\phi : \mathbb{R}^{n_x} \to T\mathbb{R}^{n_x}$ be Lebesgue-measurable and locally essentially bounded. A differential equation (DE) $\dot{x} = \phi(x), x \in \mathbb{R}^{n_x}$, is understood *in the Filippov sense*, if it is replaced with the Filippov DI $\dot{x} \in K_F[\phi](x)$,

$$K_F[\phi](x) = \bigcap_{\mu_L N=0} \bigcap_{\delta>0} \overline{\mathrm{co}}\ \phi((x + B_\delta)\backslash N). \tag{2}$$

Here $\overline{\mathrm{co}}$ is the convex closure operation, μ_L is the Lebesgue measure, and (2) is the famous Filippov procedure [19]. In the non-autonomous case we formally add the DE $\dot{t} = 1$.

All DEs are further understood in the Filippov sense.

Weighted-homogeneity notions. The homogeneous weights $m_1, ..., m_{n_x} > 0$ of the coordinates $x_1, ..., x_{n_x}$ in $\mathbb{R}^{n_x}$ are some fixed numbers $\deg x_i = m_i > 0$. The homogeneous dilation [4] is defined for any $\kappa \geq 0$ as the transformation

$$d_\kappa : (x_1, x_2, ..., x_{n_x}) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, ..., \kappa^{m_{n_x}} x_{n_x}).$$

A function $g : \mathbb{R}^{n_x} \to \mathbb{R}^{m_x}$ is said to have the homogeneity degree (HD) (weight) $q \in \mathbb{R}$, $\deg g = q$, if $g(x) = \kappa^{-q} g(d_\kappa x)$ holds for any $x \in \mathbb{R}^{n_x}$ and any $\kappa > 0$.

The DI $\dot{x} \in F(x), x \in \mathbb{R}^{n_x}$, and the vector-set field $F(x) \subset T_x\mathbb{R}^{n_x}$ are called homogeneous of the HD q, if the identity $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$ holds for any x and $\kappa > 0$ [31]. It implies the invariance of the DI with respect to the time-coordinate transformation $(t, x) \mapsto (\kappa^{-q} t, d_\kappa x),\ \ \kappa > 0$ [31]. The number $-q \in \mathbb{R}$ can be interpreted as the weight of the time t, $\deg t = -q$.

In the case of DEs $\dot{x}_i = \phi_i(x)$, $i = 1, ..., n_x$, $F(x) = \{\phi(x)\}$, and the DE homogeneity is reduced to $\deg \dot{x}_i = \deg x_i - \deg t = m_i + q = \deg \phi_i$ [4]. Filippov's procedure (2) preserves the homogeneity of the DE $\dot{x} = \phi(x)$.

A set $S \subset \mathbb{R}^{n+1}$ is called homogeneous if $d_\kappa S = S$.

The HDs q, $m_1, ..., m_{n_x}$ are defined up to proportionality.

Any continuous positive-definite function $||x||_h$ of the HD 1 is called a *homogeneous norm*. The quotient of any two homogeneous norms is uniformly bounded and separated from zero for $x \neq 0$.

In particular, denote $||x||_{h\infty} = \max_i |x_i|^{1/m_i}$. This norm induces norms in the lower-dimensional coordinate subspaces.

If $q > 0$ then asymptotic stability (AS, also meaning "asymptotically stable") of the DI implies fixed-time (FxT) convergence to any ball around 0, AS is exponential for $q = 0$, and AS implies FT stability if $q < 0$ [31], [36], [35].

Disturbed system accuracy (adapted from [36]). Let (1) be a homogeneous AS Filippov DI of the HD $q < 0$. Consider

$$\begin{aligned} &\dot{x} \in F_d(t, x(\cdot), \rho),\ t, \rho \geq 0,\ \deg \rho = 1, \\ &F_d = F(x(t - [0, \rho^{-q}]) + B_{h,\phi(x(t),\rho)}) + \Psi(x(t), \rho), \\ &\Psi = (\psi_1(x, \rho), ..., \psi_{n_x}(x, \rho))^T [-1, 1],\ i = 1, ..., n_x, \\ &\phi, \psi_i : \mathbb{R}^{n_x+1} \to \mathbb{R}_+,\ \deg \phi = 1,\ \deg \psi_i = m_i + q, \end{aligned} \tag{3}$$

where ϕ, Ψ are continuous, ρ measures the disturbance intensity, $\phi(x, 0) \equiv 0, \Psi_i(x, 0) \equiv 0$, and ϕ, ψ_i are monotonously increasing in ρ. Then for some $\mu > 0$ and any $\rho > 0$ all its extendable-in-time solutions in FT establish the inequality $||x||_h \leq \mu\rho$. The required initial-condition assumptions [36] automatically hold in the widespread case of sampled systems, when only $x(t_0), t_0 = 0$, influences the solution $x(t), t \geq 0$.

III. Introduction to homogeneous differentiation

By $\mathrm{Lip}_{n_d} L_0$ we denote the set of all scalar functions defined on $\mathbb{R}_+ = [0, \infty)$, whose n_dth derivative has the Lipschitz constant $L_0 \geq 0$.

Let the input signal $f(t) = f_0(t) + \eta(t)$, $t \geq 0$, be available (sampled) in real time. The Lebesgue-measurable noise $\eta(t)$ and $f_0 \in \mathrm{Lip}_{n_d} L_0$ are unknown, $L_0 > 0, n_d \geq 0$ are known.

It is proved [39] that $\forall \varepsilon_* > 0\ \exists t_0 > 0$ (also $\forall t_0 > 0\ \exists \varepsilon_* > 0$) such that for any ε_0, $0 < \varepsilon_0 \leq \varepsilon_*$, and any $f_0, f_1 \in \mathrm{Lip}_n L_0$ the inequality $\sup_{t \geq 0} |f_1(t) - f_0(t)| \leq \varepsilon_0$ implies

$$\sup_{t \geq t_0} |f_1^{(i)}(t) - f_0^{(i)}(t)| \leq K_{i,n_d} (2L_0)^{\frac{i}{n_d+1}} \varepsilon_0^{\frac{n_d+1-i}{n_d+1}} \tag{4}$$

for $i = 0, 1, ..., n_d$. Here K_{i,n_d} are the Kolmogorov constants [28], and (4) turn into equalities on certain functions [39].

It is known that $K_{i,n_d} \in [1, \pi/2]$, in particular, $K_{1,1} = \sqrt{2}$. Taking $\eta = f_1 - f_0$ obtain an unremovable restriction on the best possible accuracy of noisy differentiation.

Let an n_dth-order differentiator produce locally absolutely-continuous estimations $z_0, ..., z_{n_d} : \mathbb{R}_+ \to \mathbb{R}$ of the derivatives

$f_0(t), \dot f_0(t), ..., f_0^{(n_d)}(t)$, and let it be exact on any inputs from $\mathrm{Lip}_{n_d} L_0$ after some FT transient.

A differentiator is called *asymptotically optimal* [39], if for some $\mu_i > 0$, any $f_0 \in \mathrm{Lip}_{n_d}(L_0)$ and any bounded noise η, $\operatorname{ess\,sup} |\eta(t)| \le \varepsilon_0$, it in FT provides the estimation accuracy

$$|z_i(t) - f_0^{(i)}(t)| \le \mu_i L_0^{\frac{i}{n_d+1}} \varepsilon_0^{\frac{n_d+1-i}{n_d+1}},\ i = 0, 1, ..., n_d. \quad (5)$$

Obviously, $\mu_i \ge 2^{\frac{i}{n_d+1}} K_{i,n_d} \ge 2^{\frac{i}{n_d+1}}$ holds.

Asymptotically-optimal differentiators introduced in [30] were recently extended to the filtering SM-based differentiators [40], [38]. The filtering differentiator of the differentiation order $n_d \ge 0$ and the filtering order $n_f \ge 0$ has the form

$$\begin{aligned}
\dot w_1 &= -\tilde\lambda_{n_d+n_f} L^{\frac{1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+n_f}{n_d+n_f+1}} + w_2,\\
&\dots\\
\dot w_{n_f-1} &= -\tilde\lambda_{n_d+2} L^{\frac{n_f-1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+2}{n_d+n_f+1}} + w_{n_f},\\
\dot w_{n_f} &= -\tilde\lambda_{n_d+1} L^{\frac{n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+1}{n_d+n_f+1}} + w_{n_f+1},\\
w_{n_f+1} &= z_0 - f(t),
\end{aligned} \quad (6)$$

$$\begin{aligned}
\dot z_0 &= -\tilde\lambda_{n_d} L^{\frac{n_f+1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d}{n_d+n_f+1}} + z_1,\\
&\dots\\
\dot z_{n_d-1} &= -\tilde\lambda_1 L^{\frac{n_d+n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{n_d+n_f+1}} + z_{n_d},\\
\dot z_{n_d} &= -\tilde\lambda_0 L \operatorname{sign}(w_1).
\end{aligned} \quad (7)$$

$$L = L_0 > 0. \quad (8)$$

In the sequel L becomes variable. If $n_f = 0$ DEs (6) reduce to $w_1 = z_0 - f(t)$ producing the standard differentiator [30].

Let $\eta = 0$. Denote $n = n_d + n_f$, $\zeta_i = w_{i+1}/L_0$ for $i = 0, ..., n_f - 1$, and $\zeta_{i+n_f} = (z_i - f_0^{(i)})/L_0$ for $i = 0, ..., n_d$. Now subtracting $f_0^{(k+1)}$ from the both sides of the equation for z_k in (7), $k = 0, ..., n_d$, dividing (6) and (7) by L_0, and taking into account $f_0^{(n_d+1)}/L_0 \in [-1, 1]$ obtain the FT stable Filippov DI

$$\begin{aligned}
\dot\zeta_0 &= -\tilde\lambda_n \lfloor \zeta_0 \rceil^{\frac{n}{n+1}} + \zeta_1,\\
&\dots\\
\dot\zeta_{n-1} &= -\tilde\lambda_1 \lfloor \zeta_0 \rceil^{\frac{1}{n+1}} + \zeta_n,\\
\dot\zeta_n &\in -\tilde\lambda_0 \operatorname{sign}(\zeta_0) + [-1, 1],
\end{aligned} \quad (9)$$

where $K_F[\operatorname{sign}(\cdot)](0) = [-1, 1]$ is substituted for $\operatorname{sign} 0$ when $\zeta_0 = 0$ [19].

The error dynamics (9) are homogeneous with $\deg \zeta_i = n + 1 - i$, $\deg t = 1$ [31] and coincide with the error dynamics of the standard differentiator [30] corresponding to $n_d = n$, $n_f = 0$. Hence, the standard features [30], [5] are directly extended to the filtering differentiators. In particular, parameters $\tilde\lambda_i$, $i = 0, ..., n$, are the same for all n_d, n_f keeping $n_d + n_f = n$.

Filtering explanation by example. Let $\eta(t) = \eta_0(t) + \eta_1(t)$. Then $n_d = 0, n_f = 1$ yields the 0th-order differentiator

$$\begin{aligned}
\dot w_1 &= -\tilde\lambda_1 L_0^{\frac{1}{2}} \lfloor w_1 \rceil^{\frac{1}{2}} + z_0 - f_0 - \eta_0 - \eta_1,\\
\dot z_0 &= -\tilde\lambda_0 L_0 \operatorname{sign}(z_0 - f_0 - \eta_0 - \eta_1),
\end{aligned} \quad (10)$$

where $|\dot f_0| \le L_0$. Let η_0 and $\xi = \int_0^t \eta_1(s) ds$ be small; $\zeta_0 = (w_1 + \xi)/L_0$, $\zeta_1 = (z_0 - f_0)/L_0$. Now (10) is rewritten as

$$\begin{aligned}
\dot\zeta_0 &= -\tilde\lambda_1 \lfloor \zeta_0 - \xi/L_0 \rceil^{\frac{1}{2}} + \zeta_1 - \eta_0/L_0,\\
\dot\zeta_1 &\in -\tilde\lambda_0 \operatorname{sign}(\zeta_0 - \xi/L_0) + [-1, 1],
\end{aligned} \quad (11)$$

which corresponds to the FT stable dynamics (9) for $n = 0 + 1 = 1$ with small disturbances $\xi/L_0, \eta_0/L_0$. □

Differentiator (6), (7) is identically rewritten in the recursive form [30], [38] with the parameters $\lambda_0, \lambda_1, ..., \lambda_{n_d+n_f} > 0$, whereas the relations $\tilde\lambda_0 = \lambda_0$, $\tilde\lambda_n = \lambda_n$, and $\tilde\lambda_j = \lambda_j \tilde\lambda_{j+1}^{j/(j+1)}$, $j = n-1, n-2, \dots, 1$, $n = n_d + n_f$, hold.

An infinite sequence of parameters $\vec\lambda = \{\lambda_0, \lambda_1, ...\}$ can be built starting with any $\lambda_0 > 1$ [25], [30]. The sequence $\vec\lambda_7$ [37], [39] is extended here to $\vec\lambda_{12}$, $\vec\lambda = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32, ...\}$. Corresponding parameters $\vec{\tilde\lambda}_{n_d+n_f}$, are presented in Fig. 1 [23].

0	1.1												
1	1.1	1.5											
2	1.1	2.12	2										
3	1.1	3.06	4.16	3									
4	1.1	4.57	9.30	10.03	5								
5	1.1	6.75	20.26	32.24	23.72	7							
6	1.1	9.91	43.65	101.96	110.08	47.69	10						
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12					
8	1.1	19.66	171.73	795.63	1703.9	1464.2	608.99	120.79	14				
9	1.1	26.93	322.31	2045.8	6002.3	7066.2	4026.3	1094.1	173.72	17			
10	1.1	36.34	586.78	5025.4	19895	31601	24296	8908	1908.5	251.99	20		
11	1.1	48.86	1061.1	12220	65053	138954	143658	70830	20406	3623.1	386.7	26	
12	1.1	65.22	1890.6	29064	206531	588869	812652	534837	205679	48747	6944.8	623.30	32

Fig. 1. Parameters $\tilde\lambda_0, \tilde\lambda_1, ..., \tilde\lambda_{n_d+n_f}$ of differentiator (6), (7) for $n_d + n_f = 0, 1, ..., 12$, $\vec\lambda = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32, ...\}$

Parameters $\vec{\tilde\lambda}_{n_d+n_f}$ and $\vec\lambda = \lambda_0, \lambda_1, ...$ are further always assumed properly chosen. Introduce short notation for (6), (7),

$$\dot w = \Omega_{n_d,n_f}(w, z_0 - f, L, \vec\lambda),\ \dot z = D_{n_d,n_f}(w_1, z, L, \vec\lambda). \quad (12)$$

Let $\operatorname{ess\,sup} \eta(t) = \varepsilon_0$. In the presence of discrete measurements with the maximal sampling time interval $\tau > 0$ differentiator (12), (8) in FT provides the steady-state accuracy

$$\begin{aligned}
&|z_i(t) - f_0^{(i)}(t)| \le \mu_i L_0 \rho^{n_d+1-i},\ i = 0, 1, ..., n_d,\\
&|w_1(t)| \le \mu_{w,1} L_0 \rho^{n_d+n_f+1} \text{ if } n_f > 0
\end{aligned} \quad (13)$$

for $\rho = \max[(\varepsilon_0/L_0)^{1/(n_d+1)}, \tau]$, and some $\mu_{w,1}, \mu_i > 0$ only depending on parameters $\vec{\tilde\lambda}_n$ [30], [36]. Inequalities $|w_k(t)| \le \mu_{w,k} L_0 \rho^{n_d+n_f+2-k}$ hold for $k = 1, .., n_f$ and some $\mu_{w,k} > 0$.

Formulas (13) also remain true for continuous sampling formally corresponding to $\tau = 0$ and $\rho = (\varepsilon_0/L_0)^{1/(n_d+1)}$. It renders the filtering differentiator asymptotically optimal. The same accuracy asymptotics (with different constants $\mu_i, \mu_{w,k}$)

is maintained by properly discretized differentiators [22], [36].

A. Continuous-time filtering

A (noise) function $\nu(t)$, $\nu : [0, \infty) \to \mathbb{R}$, is called a *signal of the (global) filtering order* $k \geq 0$ [38], if ν is a locally integrable Lebesgue-measurable function, and there exists a globally bounded locally-absolutely-continuous solution $\xi(t)$ of the equation $\xi^{(k)} = \nu$. Any number exceeding $\operatorname{ess\,sup} |\xi(t)|$ is called the *kth-order integral magnitude of* ν.

Assumption 1. *Assume that the noise $\eta(t)$ incorporated in the input signal $f(t) = f_0(t) + \eta(t)$, is representable in the form $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$, where each η_k, $k = 0, ..., n_f$, is a signal of the global filtering order k and the kth-order integral magnitude $\varepsilon_k \geq 0$. Thus,* $\operatorname{ess\,sup} |\eta_0| \leq \varepsilon_0$, *whereas η_k, $k = 1, ..., n_f$, are potentially unbounded.*

It is proved in [38] that under Assumption 1 differentiator (12), (8) in FT provides accuracy (13) for

$$\rho = \max[(\tfrac{\varepsilon_0}{L_0})^{\frac{1}{n_d+1}}, (\tfrac{\varepsilon_1}{L_0})^{\frac{1}{n_d+2}}, ..., (\tfrac{\varepsilon_{n_f}}{L_0})^{\frac{1}{n_d+n_f+1}}], \quad (14)$$

where $\{\mu_i\}$, $\mu_{w,1}$ only depend on the choice of $\vec{\lambda}$.

The accuracy estimation (13), (14) holds for any possible expansion $\eta = \eta_0 + ... + \eta_{n_f}$ (Assumption 1). Thus, the actual accuracy always corresponds to *the best (mostly unknown) noise expansion*. Moreover, experiments show (Section VI) that the filtering differentiator is capable of suppressing noises of a wider class.

Example 1. The noise $\eta = A \frac{d^k}{dt^k} \lfloor \cos \omega_* t \rceil^{k-0.5}$, $A > 0$, $k \geq 1$, is unbounded, has the global filtering order k and the integral magnitude A.

Example 2. The noise $\eta = A \cos(\omega_* t)$, $A > 0$, features any filtering order $k \geq 0$ and the integral magnitude $2A/\omega_*^k$. Consider the trivial expansion $\eta = \eta_{n_f}$. It follows from (13), (14) that $\rho = O(A^{\frac{1}{n_d+n_f+1}} \omega_*^{\frac{-n_f}{n_d+n_f+1}})$. Hence, the higher n_f the better the accuracy, provided $\omega_* > 1$. Moreover, the influence of the magnitude A diminishes for large n_f.

Thus, increasing the filtering order n_f widens the class of filterable noises and correspondingly improves the accuracy asymptotics (13), (14) (Examples 1, 2).

Locally filterable signals are introduced and proved to be sums of three signals of the same or lower global filtering orders [25], [38].

B. Discrete filtering

Any modern filter is discrete and handles a discretely sampled input $f(t_j)$. Assumption 1 is sensitive to sampling rate: for example, high-frequency signals can be sampled as a constant. Thus, *the sampling instants sequence $t_0, t_1, ...$, $t_0 = 0$, is assumed to satisfy $t_{j+1} - t_j = \tau_j \leq \tau$ and exist for any possibly unknown maximal sampling interval $\tau > 0$.*

The discrete version of differentiator (12) has the form

$$\begin{aligned} &\delta_j w = \Omega_{n_d,n_f}(w(t_j), z_0(t_j) - f(t_j), L, \vec{\lambda})\tau_j, \\ &\delta_j z = D_{n_d,n_f}(w_1(t_j), z(t_j), L, \vec{\lambda})\tau_j + T_{n_d}(z(t_j), \tau_j), \end{aligned} \quad (15)$$

where $T_{n_d} \in \mathbb{R}^{n_d+1}$ has the Taylor-like components

$$\begin{aligned} &T_{n_d,0} = \tfrac{1}{2!} z_2(t_j)\tau_j^2 + ... + \tfrac{1}{n_d!} z_{n_d}(t_j)\tau_j^{n_d}, \\ &T_{n_d,1} = \tfrac{1}{2!} z_3(t_j)\tau_j^2 + ... + \tfrac{1}{(n_d-1)!} z_{n_d}(t_j)\tau_j^{n_d-1}, \\ &\quad ... \\ &T_{n_d,n_d-2} = \tfrac{1}{2!} z_{n_d}(t_j)\tau_j^2, \; T_{n_d,n_d-1} = T_{n_d,n_d} = 0. \end{aligned} \quad (16)$$

The vector term T_{n_d} provides for the homogeneity of the discrete error dynamics [5]. Recall that meantime $L = L_0$.

A discretely sampled signal $\nu : \mathbb{R}_+ \to \mathbb{R}$ is said to be a *signal of the (global) sampling filtering order* $k \geq 0$ and *the (global) kth order integral sampling magnitude* $a \geq 0$ if for each admissible sequence t_j there exists a discrete vector signal $\xi(t_j) = (\xi_0(t_j), ..., \xi_k(t_j))^T \in \mathbb{R}^{k+1}$, $j = 0, 1, ...$, which satisfies the relations $\delta_j \xi_i = \xi_{i+1}(t_j)\tau_j$, $i = 0, 1, ..., k-1$, $\xi_k(t_j) = \nu(t_j)$, and the bound $|\xi_0(t_j)| \leq a$.

Assumption 2. *The discrete noise $\eta(t_j)$ is representable as $\eta = \eta_0 + ... + \eta_{n_f}$, where the signals $\eta_l(t_j)$ are of the sampling filtering order l and integral magnitude ε_l, $l = 0, ..., n_f$.*

It is proved [38] that under Assumption 2 differentiator (12), (8) provides the accuracy (13) for

$$\rho = \max[\tau, \max_{0 \leq l \leq n_f} (\tfrac{\varepsilon_l}{L_0})^{\frac{1}{n_d+l+1}}]. \quad (17)$$

In general one needs very small sampling steps to reveal the small average value of the noise. The following *alternative* assumption is natural in processing Fourier series and audio filtering.

Assumption 3. *Each noise η_l is absolutely continuous with $|\dot{\eta}_l| \leq L_{\eta l}$, $L_{\eta l} > 0$, $l = 1, ..., n_f$.*

Formally define $L_{\eta 0} = 1$, then [38] under Assumptions 1, 3 the resulting accuracy (13) corresponds to

$$\rho = \max[\tau, \max_{0 \leq l \leq n_f} \max_{0 \leq k \leq l} (\tfrac{L_{\eta l}}{L_0} (\tfrac{\varepsilon_l}{L_{\eta l}})^{\frac{k+1}{l+1}})^{\frac{1}{n_d+k+1}}]. \quad (18)$$

Assumptions 2 and 3 can also be combined [38]. Also here signals of *local sampling filtering orders* are defined and shown to be representable as combinations of three globally filterable sampled signals [25], [38].

Example 3. Independent equally-distributed *random* sampling noises $\nu(t_j)$ of the zero mean value in practice constitute a noise of the local discrete filtering order 1 [34]. The SMC is representable as the sum of the equivalent control and a chattering noise of the discrete filtering order 1 and a small integral magnitude. Filtering differentiators with $n_f \geq 1$ practically completely remove such noises, provided the sampling rate is sufficiently high [34], [40].

Harmonic noises (Example 2) satisfy Assumption 3 and have any global or local sampling filtering order. Unbounded noises of Example 1 are of the same local sampling order k, provided they are saturated at some (even large) value.

Thus, discrete filtering differentiators are practically exact for infinitesimal τ in the absence of noises, and asymptotically optimal in the presence of bounded noises. They suppress some very large noises, in particular singled out by Assumptions 1, and 2 or 3.

IV. New discrete differentiation scheme

Two drawbacks of differentiators (15), (8) are their slow convergence from large initial errors and outputs chattering. The first one is, for example, treated in [2], [37], [41].

Outputs z_i, $i = 0, ..., n_d$, chatter due to the discontinuity in (7), large maximal sampling period τ, too large L_0, and sampling noises. A tight estimation L_0 of $\sup |f_0^{(n_d+1)}(t)|$ is rare, also τ is often hardware determined.

Our goal is to propose a new ***simple*** *scheme which would preserve accuracy asymptotics* (13)*,* (17) *or* (13)*,* (18) *of SM-based differentiators for $f_0 \in \mathrm{Lip}_{n_d} L_0$, while improving the performance in the absence of noises for $f = f_0 \in \mathrm{Lip}_{n_d} L_f$ for* ***any*** *$L_f < L_0$ in spite of its unavailability.*

A. New discretization scheme

According to the accuracy (13), (17) the values of w_1 and $w_{n_f+1} = z_0 - f$ indicate the convergence of scheme (15).

Let an upper bound τ of the sampling step be available. Choose some $q \geq -1/(n+1)$, $k_\tau > 0$, $n = n_d + n_f$. Denote $\mathrm{sat}\, s = \max[-1, \min(1, s)]$. *The proposed discretization* is

$$\begin{aligned} &L(t_j) = L_0 \,\mathrm{sat} \left(\tfrac{|w_1(t_j)|}{L_0 w_\tau}\right)^{1+(n_d+n_f+1)q}, \\ &q \geq -\tfrac{1}{n_d+n_f+1},\ w_\tau = k_\tau \tau^{n_d+n_f+1},\ k_\tau > 0. \end{aligned} \tag{19}$$

As $\tau \to 0$, new dynamics (19) *are only applied over infinitely thin state-space layer*. Hence (19) is just a special discretization applied at the discontinuity surface $w_1 = 0$. For the brevity we further call it *discrete L-adaptation*.

Theorem 1. ***A:*** *Discrete filter* (15)*,* (16)*,* (19) *in FT provides for the steady-state accuracy* (13)*,* (17) *or* (13)*,* (18) *with coefficients independent of q. In the absence of noises the corresponding piece-wise-linear-in-time Euler solutions $w(t), z(t)$ uniformly converge to solutions of* (6)*,* (7)*,* (8)*.*
B: *Let $k_\tau > 0$ be sufficiently large, then for sufficiently small $\varepsilon_\tau > 0$, any $\tau > 0$ and any noise $\eta(t)$, $|\eta| \leq \varepsilon_\tau L_0 \tau^{n_d+1}$, each solution of* (15)*,* (19) *in FT stabilizes in the set $|w_1| \leq L_0 w_\tau$. The choice of k_τ only depends on the choice of $\vec{\lambda}_n$.*

All proofs are placed in the Appendix. Due to Theorem 1B any sufficiently large value of $k_\tau > 0$ fits all q for any fixed n_d, n_f. Numerically checked values of $k_\tau(n_d, n_f)$, $n = n_d + n_f \leq 12$, are listed in Section VI, Tab. I.

B. Homogeneity features of the proposed adaptation

Starting from some moment $|w_1| \leq L_0 w_\tau$ is kept. Substituting (19) under that condition obtain

$$\begin{aligned} &L(t_j)^{\frac{i+1}{n+1}} \lfloor w_1 \rceil^{\frac{n-i}{n+1}} = \varepsilon_w^{-(i+1)} \lfloor w_1 \rceil^{1+(i+1)q}, \\ &\varepsilon_w = L_0^q k_\tau^{q+\frac{1}{n+1}} \tau^{1+(n+1)q},\ i = 0, 1, ..., n. \end{aligned} \tag{20}$$

Therefore, the corresponding local dynamics are

$$\begin{aligned} &\delta_j w_1 = \tau_j [-\tilde{\lambda}_n \varepsilon_w^{-1} \lfloor w_1 \rceil^{1+q} + w_2]|_{t_j}, \\ &\delta_j w_2 = \tau_j [-\tilde{\lambda}_{n-1} \varepsilon_w^{-2} \lfloor w_1 \rceil^{1+2q} + w_3]|_{t_j}, \\ &\quad ... \\ &\delta_j w_{n_f} = \tau_j [-\tilde{\lambda}_{n_d+1} \varepsilon_w^{-n_f} \lfloor w_1 \rceil^{1+n_f q} + w_{n_f+1}]|_{t_j}, \\ &w_{n_f+1} = z_0 - f,\ n = n_d + n_f, \end{aligned} \tag{21}$$

$$\begin{aligned} &\delta_j z_0 = \tau_j [-\tilde{\lambda}_n \, \varepsilon_w^{-(n_f+1)} \lfloor w_1 \rceil^{1+(n_f+1)q} + z_1]|_{t_j} + T_{n_d,0}, \\ &\quad ... \\ &\delta_j z_{n_d-1} = \tau_j [-\tilde{\lambda}_1 \, \varepsilon_w^{-n} \lfloor w_1 \rceil^{1+nq} + z_{n_d}]|_{t_j} + T_{n_d,n_d-1}, \\ &\delta_j z_{n_d} = \tau_j [-\tilde{\lambda}_0 \, \varepsilon_w^{-(n+1)} \lfloor w_1 \rceil^{1+(n+1)q}]|_{t_j} + T_{n_d,n_d}. \\ &T_{n_d,n_d-1} = T_{n_d,n_d} = 0, \end{aligned} \tag{22}$$

where T_{n_d} is introduced in (16). In its turn scheme (20), (21), (22) is the homogeneous discretization [25] of the filter

$$\begin{aligned} &\dot{w}_1 = -\tilde{\lambda}_n \varepsilon_w^{-1} \lfloor w_1 \rceil^{1+q} + w_2, \\ &\quad ... \\ &\dot{w}_{n_f} = -\tilde{\lambda}_{n_d+1} \varepsilon_w^{-n_f} \lfloor w_1 \rceil^{1+n_f q} + w_{n_f+1}, \\ &w_{n_f+1} = z_0 - f,\ n = n_d + n_f, \end{aligned} \tag{23}$$

$$\begin{aligned} &\dot{z}_0 = -\tilde{\lambda}_n \, \varepsilon_w^{-(n_f+1)} \lfloor w_1 \rceil^{1+(n_f+1)q} + z_1, \\ &\quad ... \\ &\dot{z}_{n_d-1} = -\tilde{\lambda}_1 \, \varepsilon_w^{-n} \lfloor w_1 \rceil^{1+nq} + z_{n_d}, \\ &\dot{z}_{n_d} = -\tilde{\lambda}_0 \, \varepsilon_w^{-(n+1)} \lfloor w_1 \rceil^{1+(n+1)q}. \end{aligned} \tag{24}$$

Parameter ε_w is small for small τ and $q > -1/(n+1)$. Filter (23), (24) is naturally to be called *homogeneous filtering high-*

gain observer (FHGO) [25]. It turns into the classical high-gain observer (HGO) [3] for $q = 0$ and $n_f = 0$.

If $f = 0$ system (23), (24) becomes homogeneous of the HD q with $\deg w_1 = 1$, $\deg w_2 = 1+q$, ..., $\deg z_{n_d} = 1+nq$, $\deg \dot{z}_{n_d} = 1 + (n+1)q \geq 0$. It is AS for a proper choice of the coefficients, and can be used for the differentiation of the input f provided $\varepsilon_w << 1$ [25].

The case $q = 0$ implies that $\varepsilon_w = k_\tau^{1/(n+1)}\tau$ and both continuous-time and discrete dynamics, (23), (24) and (21), (22), *do not depend on* L_0. The roots of the corresponding characteristic polynomial $p_\tau(s) = s^{n+1} + \varepsilon_w^{-1}\tilde{\lambda}_n s^n + ... + \varepsilon_w^{-(n+1)}\tilde{\lambda}_0$ are proportional to ε_w^{-1} [3].

The so-called continuous differentiator [15], [41] is obtained for $n_f = 0$, $-1/(n_d+1) < q < 0$ and finite ε_w. Small positive values of q were considered in [2] for $n_f = 0$.

In the case $q = q_n = -1/(n+1)$ get $\varepsilon_w = L_0^{-1/(n+1)}$, and the original filtering differentiator is restored, which does not contain high gains.

Lemma 1. *An infinite sequence $\lambda_0, \lambda_1, ...$, can be built for the HD $q_n = -\frac{1}{n+1}$, so that in the absence of noises continuous-time filters* (23), (24) *be asymptotically stable for any f_0, $f_0^{(n_d+1)}(t) \equiv 0$, any $n_d, n_f \geq 0$, and $q \in \{q_n, 0\}$. For that end for each $n \geq 1$, $n = n_d + n_f$, one simply takes λ_{n+1} large enough, provided $\lambda_0, ..., \lambda_n > 0$ are already properly chosen. Any $\lambda_0 > 1$ is valid for $n = 0$. The sequence segments $\vec{\lambda}_n$, $\tilde{\lambda}_0, ..., \tilde{\lambda}_n$ stay valid for sufficiently small variations of q.*

Suppose that the noise is bounded, $|\eta| \leq \varepsilon_0$. Introduce auxiliary parameters $\hat{\varepsilon}_0 = \varepsilon_w^{n_f}\varepsilon_0$, and $\hat{\tau} = \varepsilon_w^{-1}\tau = L_0^{-q}k_\tau^{-q-\frac{1}{n+1}}\tau^{-(n+1)q}$ to be used in the following Theorem.

Theorem 2. *Following Theorem 1B let k_τ be large enough, $n = n_d + n_f$, $\tau > 0$ and coefficients $\tilde{\lambda}_0, ..., \tilde{\lambda}_n$ be valid for $q = -(n+1)^{-1}, 0$ and any close values of q (Lemma 1).*

- *Let $|f_0^{(n_d+1)}(t)| \leq L_f$ for some L_f, $0 < L_f \leq L_0$. Then, for τ small enough if $q < 0$, any τ if $q = 0$, and τ large enough if $q > 0$, there exists such $\varepsilon_0 > 0$ that for any noise $\eta(t)$, $|\eta| \leq \varepsilon_0$, after a FT transient the accuracy*

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \leq \mu_i \varepsilon_w^{-n_f - i}\rho^{1+(n_f+i)q}, \\ &|w_k(t)| \leq \mu_{wk}\varepsilon_w^{-k+1}\rho^{1+(k-1)q}, \\ &i = 0, 1, ..., n_d, \ k = 1, 2, ..., n_f, \end{aligned} \tag{25}$$

is established for some $\mu_i, \mu_{wk} > 0$ only depending on the choice of $\lambda_0, ..., \lambda_n$, k_τ and q for

$$\rho = \begin{cases} \max[\hat{\varepsilon}_0^{\frac{1}{1+n_f q}}, (L_f \varepsilon_w^{n+1})^{\frac{1}{1+(n+1)q}}], \ q \geq 0; \\ \max[\hat{\tau}^{-\frac{1}{q}}, \hat{\varepsilon}_0^{\frac{1}{1+n_f q}}, (L_f \varepsilon_w^{n+1})^{\frac{1}{1+(n+1)q}}], \ q < 0. \end{cases}$$

The discrete filter (15), (19) *is AS if $q \geq 0$, $\varepsilon_0 = 0$ and $L_f = 0$ or $\sup\lim_{t\to\infty} f_0^{(n_d+1)}(t) = 0$.*

- *In the case $q = 0$ the corresponding steady-state dynamics of the scheme* (15), (19) ***do not depend on*** L_0. *The accuracy formula* (25) *takes the simpler form*

$$\begin{aligned} &|z_i(t) - f_0^{(i)}(t)| \leq \mu_i \max[\rho_\tau^{-i}\varepsilon_0, L_f\rho_\tau^{n_d+1-i}], \\ &|w_k(t)| \leq \mu_{wk}\max[\rho_\tau^{n_f+1-k}\varepsilon_0, L_f\rho_\tau^{n+2-k}]. \\ &\rho_\tau = k_\tau^{\frac{1}{n+1}}\tau = w_\tau^{\frac{1}{n+1}}. \end{aligned} \tag{26}$$

Remark 1. *The number L_f is not assumed available. We keep the presence of the constant k_τ in* (26)*, since it tends to be large for large n (Section VI, Tab. I).*

If $q < 0, L_f = \varepsilon_0 = 0$ the local discrete filter (21), (22) *in FT converges into a ball [35]. I.e. the chattering due to large τ still exists, and it weakens as q approaches* 0.

Remark 2. *Theorem 1 assures that the standard accuracy asymptotics are never lost. The presented accuracies* (25) *and* (26) *improve for $L_f < L_0$, but only hold for noises which do not stir the system from the layer $|w_1| \leq L_0 k_\tau \tau^{n+1}$. We have no good estimation for the case, when the steady-state system spends only part of the time in the layer.*

*Also note that even large **filterable** noises still keep the layer, if $\vec{\varepsilon}_{n_f}$ from Assumption 2 are small enough (Example 4 of Section VI). The corresponding technical analysis is omitted.*

Parameters $\tilde{\lambda}_i$ listed in Section III, Fig. 1, satisfy the requirements of Theorem 2 for $q = 0$, and any q close to 0 (Lemma 1, also the Bhat-Bernstein principle [2], [10], [41]).

V. General lower-chattering discretization

Let the homogeneous AS Filippov DI (1) of the HD $q < 0$ be presentable in the form $\dot{x} \in F(x) = F_\xi(x, \operatorname{sign} x_1)$ for $x_1 \neq 0$, and $F_\xi(x, [-1, 1]) \subset F(x)$ at the hyperplane $x_1 = 0$. Here $F_\xi(x, \theta) \subset T_x\mathbb{R}^{n_x}$ depends on the parameter $\theta \in \mathbb{R}$, $|\theta| \leq 1$, is non-empty, compact, upper-semicontinuous in x, θ at $x_1 = 0$, and homogeneous in x of the HD q for each θ.

Introduce the noisy-sampling parameter ρ and a discretization of (1) satisfying the inclusion

$$\begin{aligned} &\delta_j x \in \tau_j[F_\xi(\hat{x}(t_j), \operatorname{sign}\hat{x}_1(t_j)) + \Psi(\hat{x}(t_j), \tau_j^{-1/q})], \\ &x \in \mathbb{R}^{n_x}, \ \rho = \max[\varepsilon, \tau^{-1/q}]. \end{aligned} \tag{27}$$

Here $0 < \tau_j = \delta_j t \leq \tau$, τ is the maximal sampling period, $\hat{x} \in x + B_{h,\varepsilon}$ is the available "noisy" value of x, $\Psi(\cdot, \cdot)$ is a vector-set field described in (3) and below (Section II), and $\varepsilon \geq 0$ is the unknown noise intensity. According to Section II after a FT transient solutions of (27) keep $||x||_h \leq \mu\rho$ for some $\mu > 0$ [36].

Choose any $k_\tau > 0$. Then by the *discrete* Θ-*adaptatation* of (27) we understand a modified discretization of the form

$$\delta_j x \in \tau_j [F_\xi(\hat{x}(t_j), \Theta(t_j)) + \Psi(\hat{x}(t_j), \tau_j^{-1/q})], \qquad (28)$$

where the variable $\Theta(t_j)$ satisfies the conditions $|\Theta(t_j)| \leq 1$, and $\Theta(t_j) = \operatorname{sign} \hat{x}_1(t_j)$ whenever $|\hat{x}_1(t_j)| \geq k_\tau \tau^{-m_1/q}$.

Replacing $\delta_j x/\tau_j$ with piece-wise constant $\dot{x}$ obtain piece-wise linear Euler solutions $x_\tau(t)$ corresponding to (28).

Theorem 3. *Euler solutions $x_\tau(t)$ uniformly over any compact region of the state (x, t) converge to solutions of* (1) *as $\rho \to 0$. For any $k_\tau > 0$ there is such $\mu_\xi > 0$ that for any $\varepsilon \geq 0, \tau > 0$ solutions of* (28) *in FT provide for $||x(t_j)||_{h,\infty} \leq \mu_\xi \rho$.*

Hence, one can test any discrete Θ-adaptatation, $|\Theta(t_j)| \leq 1$, in attempt of diminishing the chattering of scheme (27) at the surface $x_1 = 0$ without comromising the system accuracy. The approach is applicable in simulation and control and is extendable to higher dimensions of x_1, θ.

VI. Numeric experiments

Assigning a valid value to k_τ is extremely simple, though not completely mathematically rigorous. One takes the input $f(t) = 10^{-18}$ canceling the digital-noise influence, sets $L_0 = 1$, $\tau = 1$, and then runs the discrete differentiator (15), (16) over the time interval $t \in [0, 40000]$ for $w(0) = 0$, $z(0) = 0$. Additional initial values $z(0) = (\pm 1, ..., \pm 1)^T$ are needed for $n \leq 5$. Then the observed $\max |w_1|$ is moderately increased and assigned to k_τ. For example, $k_\tau = 1.3 \max |w_1|$ may be used, but we have preferred round values. Recall that $w_1 = z_0 - f$ for $n_f = 0$.

Produced values of k_τ are listed in Tab. I, whereas parameters $\tilde{\lambda}_i$, $i = 0, ..., n$, appear in Fig. 1. They have been checked to ensure claims of Theorem 1B for any q and of Theorem 2 for $q = 0$. Recall that increasing k_τ is always allowed.

TABLE I
PARAMETERS k_τ OF THE DISCRETE DIFFERENTIATOR (15), (19) FOR $\vec{\lambda}_{12} = 1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32$, $n = 0, 1, ..., 12$

n	$n_f = 0$	$n_f = 1$	$n_f = 2$	$n_f = 3$	$n_f = 4, \dots, n$
0	2				
1	1	1			
2	3	3	3		
3	10	6	6	6	
4	50	10	10	10	10
5	3000	400	400	400	400
6	$1.5 \cdot 10^5$	$2 \cdot 10^4$	10^4	10^4	10^4
7	$3 \cdot 10^7$	10^7	$3 \cdot 10^6$	$3 \cdot 10^6$	$3 \cdot 10^6$
8	10^{10}	$1.5 \cdot 10^{10}$	$1.5 \cdot 10^9$	$1.5 \cdot 10^9$	$1.5 \cdot 10^9$
9	10^{12}	$7 \cdot 10^{12}$	$5 \cdot 10^{11}$	$1.5 \cdot 10^{11}$	$1.5 \cdot 10^{11}$
10	10^{14}	10^{15}	10^{13}	$3 \cdot 10^{12}$	$3 \cdot 10^{12}$
11	$8 \cdot 10^{15}$	10^{17}	10^{15}	$8 \cdot 10^{13}$	$2 \cdot 10^{14}$
12	$3 \cdot 10^{18}$	$8 \cdot 10^{18}$	10^{16}	$7 \cdot 10^{15}$	$7 \cdot 10^{15}$

In the sequel parameters $\tilde{\lambda}_i, k_\tau$ are taken from Fig. 1 and Table I. We also use component-wise inequalities and the notation $|\vec{\sigma}_{n_d}| = (|z_0 - f_0|, ..., |z_{n_d} - f_0^{(n_d)}|)$, $|\vec{w}_{1,n_f}| = (|w_1|, ..., |w_{n_f}|)$ (which differs from the notation in proofs).

First consider the input signal

$$f(t) = f_0(t) + \eta(t), \; f_0(t) = \cos(t) - \ln(t + 1). \qquad (29)$$

Obviously $\sup \lim_{t\to\infty} |f_0^{(n_d+1)}| = 1$ for $n_d = 0, 1, ...,$ i.e. $L_0 = 1$ is applicable for any n_d. In particular $|f_0^{(4)}| < 1.01$ and $|f_0^{(6)}| < 1.03$ hold for $t \geq 1$ and $t \geq 1.1$ respectively. The input is sampled with the constant time step $\tau_j = \tau$ and the accuracy is calculated over the interval $t \in [15, 30]$. Initial values $z(0) = 0, w(0) = 0$ are taken in all runs.

For τ small enough, any "standard" discrete filtering differentiator (DFD) (15) with $n_d = 3$, $n_f \geq 0$, $L_0 \geq 1$ "almost" exactly differentiates input (29) in the absence of noises, i.e. for $\eta(t) \equiv 0$. Still, though the accuracy asymptotics (13), (17) are the same, the coefficients μ_i of (13) are larger for larger n_f lowering the accuracy for the same τ. Also the convergence is slower for higher n_f. The same is true for the DFD with discrete L-adaptation (ADFD) (15), (19). Higher filtering orders n_f still ensure much better accuracy in the presence of large noises.

1. ADFD preserves the accuracy of DFD. Compare the ADFD (15), (19) with $n_d = 3$, $n_f = 7$, $k_\tau = 3 \cdot 10^{12}$, $q = 0$, to the DFD (15) with $n_d = 3$, $n_f = 7$ for $L_0 = 1$ and $L_0 = 1000$. First let $\tau = 10^{-5}$, $\eta = 0$. Recall that the DFD coincides with the ADFD for $q = -1/(3 + 7 + 1) = -1/11$. The corresponding accuracies are as follows:

$$\begin{aligned}
&(|\vec{w}_{1,7}|, |\vec{\sigma}_3|) \leq \\
&\quad (2 \cdot 10^{-43}, 3 \cdot 10^{-38}, 3 \cdot 10^{-33}, 1 \cdot 10^{-28}, \\
&\quad 4 \cdot 10^{-24}, 7 \cdot 10^{-20}, 7 \cdot 10^{-16}, \\
&\quad 3 \cdot 10^{-12}, 7 \cdot 10^{-9}, 7 \cdot 10^{-6}, 4 \cdot 10^{-3}), \; L_0 = 1, q = -\tfrac{1}{11}; \\
&(|\vec{w}_{1,7}|, |\vec{\sigma}_3|) \leq \\
&\quad (3 \cdot 10^{-43}, 4 \cdot 10^{-38}, 4 \cdot 10^{-33}, 2 \cdot 10^{-28}, \\
&\quad 7 \cdot 10^{-24}, 1 \cdot 10^{-19}, 1 \cdot 10^{-15}, \\
&\quad 6 \cdot 10^{-12}, 1 \cdot 10^{-8}, 1 \cdot 10^{-5}, 4 \cdot 10^{-3}), \; L_0 = 1, 10^3, q = 0;
\end{aligned}$$

Both ADFD and DFD visually have exactly the same performance for $L_0 = 1$ (Fig. 2). The adaptation keeps $L(t) < 1$ corresponding to $|w_1| \leq L_0 w_\tau$ for $L_0 = 1, 1000$. Accuracies for $L_0 = 1, 1000$ coincide in 4 digits for $q = 0$.

Also for the enormous practically random noise $\eta = 10^5 \cos(10^8 t + \pi \sin(10^4 t))$ ADFD and DFD demonstrate the same performance and accuracy $|\vec{\sigma}_3| \leq (0.0009, 0.01, 0.1, 0.4)$ for $L_0 = 1$ (Fig. 2). Moreover, $L(t_j) \equiv L_0$ holds.

2. Chattering reduction. Let $n_d = 3$, $n_f = 2$, $\tau = 0.05$, $\eta = 0$. The steady-state accuracies by the DFDs are provided by the component-wise inequalities $|\vec{\sigma}_3| \leq (0.006, 0.06, 0.3, 0.8)$ for $L_0 = 1$ and $|\vec{\sigma}_3| \leq (3, 15, 40, 60)$ for $L_0 = 1000$ (Fig. 3). Outputs' chattering is considerable.

In comparison, the ADFD with $q = 0$ provides for the same accuracy $|\vec{\sigma}_3| \leq (0.007, 0.06, 0.3, 0.8)$ (practically as DFD with $L_0 = 1$) for both $L_0 = 1$ and $L_0 = 1000$ (accuracies coincide!) and removes the output chattering (Fig. 3).

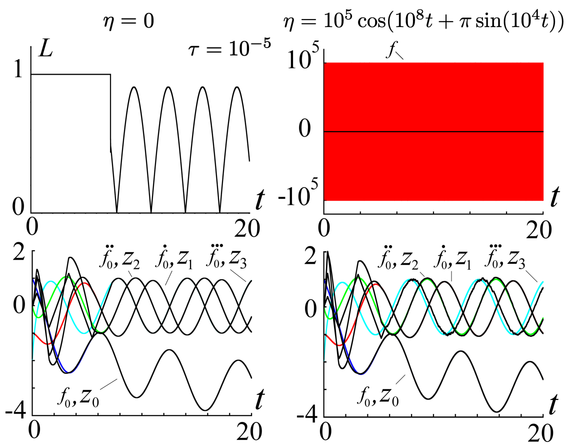

Fig. 2. The ADFD (15), (19), $L_0 = 1$, $n_d = 3$, $n_f = 7$, $\tau = 10^{-5}$, $q = 0$, $k_\tau = 3 \cdot 10^{12}$, preserves the accuracy and performance of DFD (12) for the input (29) in the absence of noises and also in the presence of the noise $\eta = 10^5 \cos(10^8 t + \pi \sin(10^4 t))$. Note that this noise is practically random.

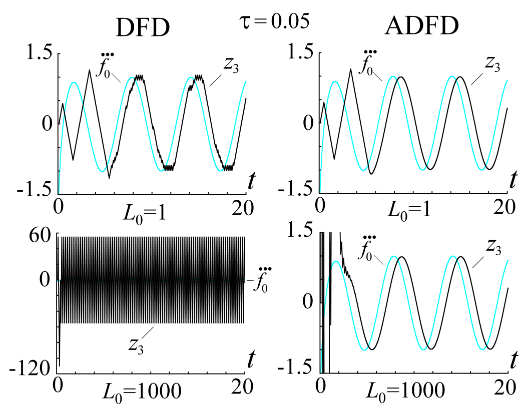

Fig. 3. The chattering of the ADFD (15), (19), $n_d = 3$, $n_f = 2$, $\tau = 0.05$, $q = 0$, $k_\tau = 400$, on the right in comparison to DFD (12) on the left for the input (29) and $L_0 = 1$, $L_0 = 1000$ in the absence of noise, $\eta = 0$. The graph of z_3 for ADFD, $L_0 = 1000$, is cut from above and below.

The following experiments are not practice oriented. They are to demonstrate the differentiation problematics and the correctness of the presented theoretical results.

3. Digital noise effect. The following experiment studies the influence of computer round-up errors as the small-noise case of Theorem 1. The double precision accuracy, usually provided by the modern software, corresponds to 15 meaningful decimal digits. Thus, any number $a \in \mathbb{R}$ in its computer representation is indistinguishable from $(1 \pm 5 \cdot 10^{-16})a$.

That seemingly small noise turns out to be significant for the high-order differentiation whenever exactness is pursued. Errors of algebraic finite-differences-based estimations tend

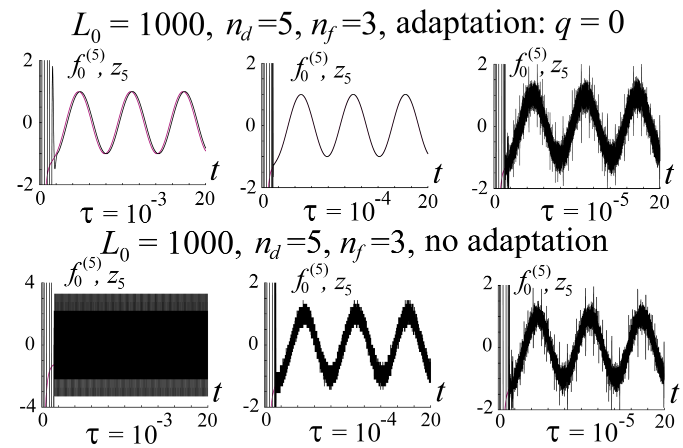

Fig. 4. Effect of the round-up computer noise on high-order differentiation. 5th-order derivative estimation of the input $f_0(t) = \cos t - \ln(1+t)$ by DFD (12) $n_d = 5$, $n_f = 3$, $L_0 = 1000$ (below), and ADFD (15), (19) for $q = 0$, $k_\tau = 1.5 \cdot 10^9$ (above) in the absence of sampling noise.

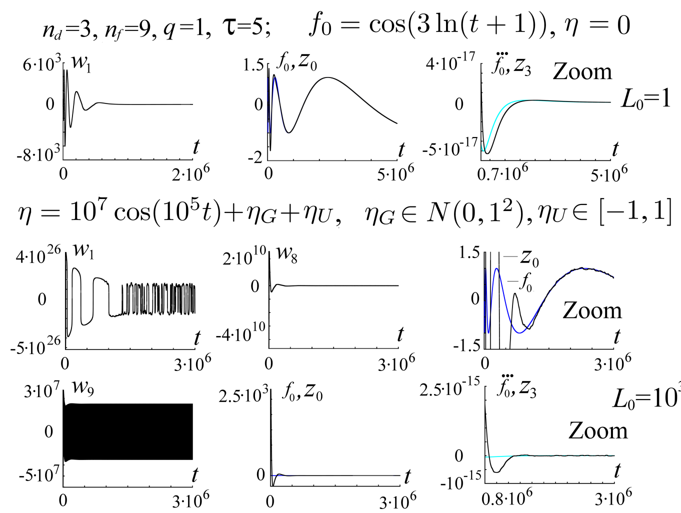

Fig. 5. Asymptotic convergence of ADFD (15), (19), $n_d = 3$, $n_f = 9$, $L_0 = 1$, $q = 1$, $k_\tau = 7 \cdot 10^{15}$, $\tau = 5$ for the input $f_0(t) = \cos(3\ln(1+t))$ in the absence of noise (above), and its performance for very large noise and $L = 1000$ (below). The noise is seen to be absorbed by $\vec{w}_{1,9}$.

to infinity as $\tau \to 0$, whereas HGO's errors tend to infinity as the high gain grows [5]. Homogeneous differentiators (6), (7) cease improving their accuracy for small-enough sampling periods [5].

Differentiation exactness on inputs $f_0 \in \mathrm{Lip}_{n_d} L_0$ invokes Kolmogorov asymptotics (5) as τ approaches 0. Asymptotics (5) dictate significant chattering for any fixed ε_0, $|\eta| \leq \varepsilon_0$, and large L_0. That outputs' chattering is actually due to the nearly-exact differentiation of some noisy signals $f = f_0 + \eta \in \mathrm{Lip}_{n_d} L_0$ (Fig. 4, $\tau = 10^{-5}$).

The proposed discretization (19) introduces the local HGO (23), (24) nonlinear for $q \neq 0$. It is easily seen from (26) that such HGO amplifies the noise as $\tau \to 0$, since the corresponding high gain grows. As the result the invariancy of the adaptation layer $|w_1| \leq k_\tau L_0 \tau^{n+1}$ is lost, and the ADFD is to feature the typical performance and accuracy of both discrete and continuous-time differentiators (12) and (15) respectively (Theorem 1, Fig. 4, $\tau = 10^{-5}$).

Indeed, consider the input (29), DFD (15), $n_d = 5, n_f = 3$, $L_0 = 1000$, and the ADFD (15), (19) for $q = 0$, $k_\tau = 1.5{\cdot}10^9$. The accuracies provided by the ADFD and DFD for $\tau = 10^{-5}$ completely coincide and are $|\vec{\sigma}_5| \leq (2 \cdot 10^{-14}, 7 \cdot 10^{-11}, 7 \cdot 10^{-8}, 4 \cdot 10^{-5}, 10^{-2}, 1.6)$ (Fig. 4, $\tau = 10^{-5}$).

On the other hand no discrete differentiation scheme is exact on $\mathrm{Lip}_{n_d} L_0$ for a **fixed** $\tau > 0$. This removes the "Kolmogorov-accuracy trap" and allows chattering attenuation for all signals $f_0 \in \mathrm{Lip}_{n_d} L_f$, $L_f << L_0$, and $\varepsilon_0 > 0$ small enough, considerably improving the Kolmogorov accuracy (5) in spite of $L_0 = 1000$. Indeed, the accuracies corresponding to $\tau = 10^{-3}$ are $|\vec{\sigma}_5| \leq (7 \cdot 10^{-10}, 2 \cdot 10^{-7}, 2 \cdot 10^{-5}, 8 \cdot 10^{-4}, 10^{-2}, 0.2)$ and $|\vec{\sigma}_5| \leq (3 \cdot 10^{-7}, 3 \cdot 10^{-5}, 2 \cdot 10^{-3}, 5 \cdot 10^{-2}, 0.6, 4.3)$ for the ADFD and DFD respectively (Fig. 4).

4. Asymptotic convergence of ADFD. Let the ADFD with $n_d = 3$, $n_f = 9$, $q = 1$, $k_\tau = 7 \cdot 10^{15}$, $\tau = 5$ handle the input

$$\begin{aligned} f(t) &= f_0(t) + \eta(t) = \cos(3\ln(t+1)) + \eta(t), \\ \eta(t) &= 10^7 \cos(10^5 t) + \eta_G(t) + \eta_U(t), \end{aligned} \tag{30}$$

where η_1 is a large high-frequency harmonic signal, $\eta_G \in N(0,1)$ is the Gaussian signal of the standard deviation 1, and η_U is uniformly distributed in $[-1, 1]$. The frequency 10^5 of the harmonic component is *extremely high* for the chosen sampling step $\tau = 5$, which also practically makes it random.

Obviously $f_0^{(4)}$ tends to zero. Hence Theorem 2 implies the *asymptotic convergence* of $z_i(t_j)$ to $f_0^{(i)}(t)$ for $i = 0, 1, 2, 3$, $t \in [t_j, t_{j+1}]$, in the absence of noises (Fig. 5). The convergence is very slow due to the low "high-gain" value ε_w^{-1} of the local filter (23), (24) (see (20)). The accuracy $|\vec{\sigma}_3| \leq (1 \cdot 10^{-3}, 4 \cdot 10^{-8}, 8 \cdot 10^{-13}, 7 \cdot 10^{-18})$ holds for t over the interval $[2 \cdot 10^6, 3 \cdot 10^6]$ for $L_0 = 1$, $\eta = 0$.

In the presence of noise (30) the accuracy changes to $|\vec{\sigma}_3| \leq (4 \cdot 10^{-2}, 7 \cdot 10^{-7}, 7 \cdot 10^{-12}, 3 \cdot 10^{-17})$ for $L_0 = 1000$ (Fig. 5). The filter is almost insensitive to that noise and very large L_0. It is seen from the graphs of w_1, w_8, w_9 in Fig. 5 that the noise is absorbed by the filtering variables w.

VII. CONCLUSIONS

A homogeneity-based chattering-alleviation discretization technique is proposed which independently of the chosen discontinuity discretization prevents the degradation of the homogeneous-system accuracy. The method is applied to the chattering attenuation of SM-based filtering differentiators.

The proposed new low-chattering discrete scheme (15), (19) for n_dth-order homogeneous SM-based differentiators is simple, significantly diminishes the output chattering even for low sampling rates and improves the accuracy for input signals f_0 having lower $(n_d{+}1)$th derivative bound $L_f = \sup |f_0^{n_d+1}|$. Increasing the differentiator parameter $L_0 \geq L_f$ does not affect the steady-state dynamics and accuracy in the absence of noises. Both the optimal accuracy asymptotics and the filtering capabilities of the standard continuous-time/discrete SM-based differentiation schemes stay intact.

APPENDIX

Proof of Theorem 3. The proof comprises two steps. First, a special continuous-time homogeneous approximation is built which includes replacing the layer $||x_1|| \leq w_\tau = k_\tau \tau^{-m_1/q} = k_\tau \tau^{m_1/|q|}$ with a homogeneous set. Second, the discrete error dynamics are considered as a perturbation of this extension.

Step 1. Let $S \subset \mathbb{R}^{n+1}$ be some set. Introduce the functions

$$\begin{aligned} \theta_S(\omega,\sigma) &= \begin{cases} \{\operatorname{sign} x_1\} \text{ for } x \notin S, \\ [-1,1] \quad \text{for } x \in S \text{ or } x_1 = 0, \end{cases} \\ \overline{F}_\xi(x,\theta) &= \begin{cases} \cap_{\delta>0} \overline{\mathrm{co}}\, F_\xi(x + B_\delta, \theta) \text{ for } x_1 \neq 0, \\ F(x) \text{ for } x_1 = 0. \end{cases} \end{aligned} \tag{31}$$

Note that $\dot{x} \in \overline{F}_\xi(x, \theta_S)$ is a homogeneous Filippov DI for any homogeneous set S. In particular, (1) is rewritten as $\dot{x} \in F(x) = \cap_{\delta>0} \overline{\mathrm{co}}\, F(x + B_\delta) = \overline{F}_\xi(x, \theta_\emptyset)$.

Denote $B_{h,\delta} = \{||x||_{h\infty} \leq \delta\}$ and

$$\begin{aligned} S_\gamma &= \{x \in \mathbb{R}^{n_x} |\ |x_1| \leq \gamma ||\vec{x}_{2,n_x}||_{h\infty}\}, \\ \check{S}_\gamma &= S_\gamma \cup B_{h,\gamma^{1/m_1}}, \\ \hat{S}_\gamma &= \{x \in \mathbb{R}^{n_x} |\ |x_1| \leq \gamma\}, \end{aligned}$$

where $\gamma \geq 0$. Then the DI $\dot{x} \in \overline{F}_\xi(x, \theta_{S_\gamma})$ is FT stable for sufficiently small $\gamma \geq 0$, since it is a small homogeneous perturbation of DI (1) [31], [35], [36].

Obviously solutions of $\dot{x} \in \overline{F}_\xi(x, \theta_{\check{S}_\gamma})$ in FT stabilize in some compact level set $\hat{B}_h$, $B_{h,2\gamma^{1/m_1}} \subset \hat{B}_h$, of a radially-unbounded homogeneous Lyapunov function existing for the AS DI $\dot{x} \in \overline{F}_\xi(x, \theta_{S_\gamma})$ [4], [9]. Hence, solutions of

$$\dot{x} \in \overline{F}_\xi(x, \theta_{\hat{S}_\gamma}) \tag{32}$$

in FT gather in $\hat{B}_h$ to stay there, since $\hat{S}_\gamma \subset \check{S}_\gamma$.

Step 2. Let τ be small enough to satisfy $w_\tau \le \gamma$. Then solutions of the discrete adaptation (28) satisfy the inclusion

$$\delta_j x \in \tau_j[\overline{F}_\xi(x(t_j) + B_{h,\varepsilon}, \theta_{\hat{S}_\gamma}) + \Psi(x(t_j) + B_{h,\varepsilon}, \tau_j^{-\frac{1}{q}})] \quad (33)$$

Thus, as $\rho \to 0$, for any compact set of initial conditions they uniformly approach solutions of (32) over each compact time interval [19]. Similarly to [36] one shows that for any sufficiently small $\rho_* > 0$ and the maximal sampling-time interval $\tau_* = \rho_*^{-q}$ all solutions of (33) FT stabilize in $d_2 \hat{B}_h$.

Due to $\hat{S}_{w_\tau} \subset \check{S}_{w_\tau}$ solutions of (28) also satisfy

$$\delta_j x \in \tau_j[\overline{F}_\xi(x(t_j) + B_{h,\varepsilon}, \theta_{\check{S}_{w_\tau}}) + \Psi(x(t_j) + B_{h,\varepsilon}, \tau_j^{-\frac{1}{q}})] \quad (34)$$

Discrete dynamics (34) are homogeneous for the weights $\deg \tau = \deg \tau_j = \deg t = -q$, $\deg \varepsilon = \deg \rho = 1$.

It follows now from the homogeneity of the discrete dynamics (34) that its solutions stabilize in $d_{2\kappa} \hat{B}_h$ for the new maximal sampling interval $\tau = \kappa^{-q} \tau_*$, $\rho = \kappa \rho_*$, $\kappa > 0$.

Now the accuracy of the homogeneous dynamics (34) follows from results [38] presented in Section II . □

Proof of Theorem 1. Let $|f_0^{(n_d+1)}| \le L_f \le L_0$, then

$$\begin{aligned}\delta_j f_0^{(i)} \in\ & \tfrac{1}{1!} f_0^{(i+1)}(t_j)\tau_j + \tfrac{1}{2!} f_0^{(i+2)}(t_j)\tau_j^2 + ... \\ & + \tfrac{1}{(n_d-i)!} f_0^{(n_d)}(t_j)\tau_j^{n_d-i} + \tfrac{[-1,1]}{(n_d-i+1)!} L_f \tau_j^{n_d-i+1}, \\ & 0 \le L_f \le L_0. \end{aligned} \quad (35)$$

Let $\omega_l = w_l/L_0$, $l = 1, ..., n_f$, $\sigma_i = (z_i - f_0^{(i)})/L_0$, $i = 0, ..., n_d$, $\omega_{n_f+1} = \sigma_0 + \eta/L_0$. Subtracting (35) from the equation for z_i in (15), $i = 0, ..., n_d$, taking $L_f = L_0$ and dividing by L_0 obtain

$$\begin{aligned}&\delta_j \omega \in \Omega_{n_d,n_f}(\omega, \omega_{n_f+1}, L(t_j)/L_0, \vec{\lambda})|_{t_j} \tau_j, \\ &\delta_j \sigma \in D_{n_d,n_f}(\omega_1, \sigma, L(t_j)/L_0, \vec{\lambda})|_{t_j} \tau_j + \\ &\qquad T_{n_d}(\sigma(t_j), \tau_j) + h(t_j, \tau_j)[-1, 1], \\ &h(t_j, \tau_j) = (0, ..., 0, \tfrac{\tau_j^{n_d+1}}{(n_d+1)!}, ..., \tfrac{\tau_j^2}{2!}, \tfrac{\tau_j}{1})^T.\end{aligned} \quad (36)$$

Theorem 3 finishes the proof of the part A for the case $|\eta| \le \varepsilon_0$, $\rho = \max[(\varepsilon_0/L)^{1/(n_d+1)}, \tau]$. The filterable noises are handled in exactly the same way as in [38].

Recall that $\deg t = 1$, $\deg \sigma_i = n_d+1-i$, $\deg \omega_j = n+2-j$, $i = 0, ..., n_d$, $j = 1, ..., n_f$. According to Theorem 3 solutions of the error system (36), (19) in FT establish the inequalities $|\omega_j| \le \mu_{w,j} \rho^{n+2-j}$, $|\sigma_i| \le \mu_i \rho^{n_d+1-i}$.

Choose any $R > 0$. Let $||(\omega, \sigma)||_{h,\infty} \le R$ hold after some transient for sufficiently small $\rho > 0$. Consider the continuous-time scalar dynamics of ω_1 in the case $n_f > 0$,

$$\dot{\omega}_1 \in -\tilde{\lambda}_n \lfloor \omega_1 + [-\rho, \rho] \rceil^{\frac{n}{n+1}} + [-R^n, R^n].$$

Obviously there exists such $\gamma > 0$ that in the absence of noise ω_1 stabilizes in the layer $|\omega_1| \le \gamma/2$. Then ω_1 stabilizes in the layer $|\omega_1| \le \gamma$ for some small enough $\tau = \tau_0$, $\varepsilon_\tau > 0$ and discrete dynamics (36). The case $n_f = 0$ is similarly considered. Fix the corresponding value $k_\tau = \gamma \tau_0^{-(n+1)}$.

The remark that if the set $|\omega_1| \le k_\tau \tau^{n+1}$ is FT attracting for $\tau = \tau_0$ and small enough ε_τ, then the homogeneity of the discrete system (36) renders it true for any $\tau > 0$, finishes the proof. □

Proof of Lemma 1. Without loss of generality consider the case $n_f = 0$ featuring the same error dynamics (9).

Prove the lemma by induction. It is trivial for $n = 0$ and any $\tilde{\lambda}_0 = \lambda_0 > 1$. Denote the coefficients of AS filter having HD $q_n = -1/(n+1)$ by $\tilde{\lambda}_{n,k}$, $k = 0, ..., n$. Let the filters with these coefficients be AS for both HDs $q_n, 0$.

Note that $(1 - q_n)^{-1} = 1 + q_{n+1}$, and it is proved in [25] that the error dynamics of the $(n + 1)$th-order differentiator with HD q_n can be rewritten in the form

$$\begin{array}{lcl}\dot{\sigma}_0 & = & -\lambda_{n+1} \lfloor \sigma_0 \rceil^{\frac{1}{1-q_n}} + \sigma_1,\ q_n = -\frac{1}{n+1}, \\ \dot{\sigma}_1 & = & -\tilde{\lambda}_{n,n} \lfloor \sigma_1 - \dot{\sigma}_0 \rceil^{1+q_n} + \sigma_2, \\ & \dots & \\ \dot{\sigma}_{n-1} & = & -\tilde{\lambda}_{n,1} \lfloor \sigma_1 - \dot{\sigma}_0 \rceil^{1+nq_n} + \sigma_n, \\ \dot{\sigma}_n & \in & -\tilde{\lambda}_{n,0} \lfloor \sigma_1 - \dot{\sigma}_0 \rceil^{1+(n+1)q_n} + [-\frac{L_f}{L_0}, \frac{L_f}{L_0}],\end{array} \quad (37)$$

which is AS for any sufficiently large $\lambda_{n+1} = \tilde{\lambda}_{n+1,n+1}$. Recall that $f_0 = 0, L_f = 0$. Show that additionally increasing λ_{n+1} ensures the AS of the linear filter with the same coefficients.

According to the induction assumption the characteristic polynomial of the nth-order linear filter $p_{n+1}(s) = s^{n+1} + \tilde{\lambda}_{n,n} s^n + ... + \tilde{\lambda}_{n,0}$ is Hurwitz. Show that also p_{n+2} is Hurwitz for sufficiently large λ_{n+1}.

It is obvious for $n = 0$, since coefficients of p_2 are positive.

Let $n > 0$ which implies $1 - q_n > 0$. Then $\tilde{\lambda}_{n+1,k} = \tilde{\lambda}_{n,k} \lambda_{n+1}^{1+(n+1-k)q_n} = \tilde{\lambda}_{n,k} \lambda_{n+1}^{1-q_n+(n+2-k)q_n}$ holds for $k = 0, ..., n + 1$. It is easily checked that $\hat{p}_{n+2}(s) = s^{n+2} + \lambda_{n+1}^{1-q_n} p_{n+1}(s)$ is Hurwitz for λ_{n+1} large enough. For example, one can use the template theorem [33] for HD 0. Finally, multiplying the coefficients of monomials s^k from $\hat{p}_{n+2}$ with $\lambda_{n+1}^{(n+2-k)q_n}$ only proportionally changes the roots of the polynomial. □

Proof of Theorem 2. Subtract the Taylor expansion (35) from the both sides of the equation for z_i of (22), and perform the time-coordinate transformation $\hat{\tau}_j = \varepsilon_w^{-1}\tau_j,\ \hat{\tau} = \varepsilon_w^{-1}\tau = L_0^{-q}k_\tau^{-q-\frac{1}{n+1}}\tau^{-(n+1)q},\ \hat{\omega}_i = \varepsilon_w^{i-1}w_i,\ i=1,...,n_f,\ \hat{t} = \varepsilon_w^{-1}t,\ \hat{\sigma}_i = \varepsilon_w^{n_f+i}(z_i - f_0^{(i)}),\ i=0,...,n_d,\ n = n_d+n_f.$

Note that $\varepsilon_w^{n_f+i}\sigma_{i+k}\tau^k = \hat{\sigma}_{i+k}\hat{\tau}^k$. Thus, due to Theorem 1 from some moment errors satisfy the inclusion

$$\begin{aligned}
&\delta_j\hat{\omega}_1 = [-\tilde{\lambda}_n\lfloor\hat{\omega}_1\rceil^{1+q} + \hat{\omega}_2]|_{\hat{t}_j}\hat{\tau}_j,\\
&\quad ...\\
&\delta_j\hat{\omega}_{n_f} = [-\tilde{\lambda}_{n_d+1}\lfloor\hat{\omega}_1\rceil^{1+n_f q} + \hat{\omega}_{n_f+1}]|_{\hat{t}_j}\hat{\tau}_j,\\
&\hat{\omega}_{n_f+1} \in \hat{\sigma}_0 + \hat{\varepsilon}_0[-1,1],
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\delta_j\hat{\sigma}_0 \in [-\tilde{\lambda}_{n_d}\lfloor\hat{\omega}_1\rceil^{1+(n_f+1)q} + \hat{\sigma}_1]|_{\hat{t}_j}\hat{\tau}_j +\\
&\quad [\tfrac{\hat{\tau}_j^2}{2!}\hat{\sigma}_2 + ... + \tfrac{\hat{\tau}_j^{n_d}}{n_d!}\hat{\sigma}_{n_d}]|_{\hat{t}_j} + \varepsilon_w^{n+1}L_f\tfrac{\hat{\tau}_j^{n_d+1}}{(n_d+1)!}[-1,1],\\
&\quad\quad ...\\
&\delta_j\hat{\sigma}_{n_d-1} \in [-\tilde{\lambda}_1\lfloor\hat{\omega}_1\rceil^{1+nq} + \hat{\sigma}_{n_d}]|_{\hat{t}_j}\hat{\tau}_j +\\
&\quad\quad\quad\quad \varepsilon_w^{n+1}L_f\tfrac{\hat{\tau}_j^2}{2!}[-1,1],\\
&\delta_j\hat{\sigma}_{n_d} \in [-\tilde{\lambda}_0\lfloor\hat{\omega}_1\rceil^{1+(n+1)q} + \varepsilon_w^{n+1}L_f[-1,1]]|_{\hat{t}_j}\hat{\tau}_j,
\end{aligned} \tag{39}$$

where $\hat{\varepsilon}_0 = \varepsilon_w^{n_f}\varepsilon_0$.

Discrete solutions of the DI (38), (39) can be presented as the node points of the peace-wise linear (Euler) solutions of the retarded disturbed homogeneous DI

$$\begin{aligned}
&\dot{\hat{\omega}}_1 \in [-\tilde{\lambda}_n\lfloor\hat{\omega}_1\rceil^{1+q} + \hat{\omega}_2]|_{\hat{t}-\hat{\tau}[0,1]},\\
&\quad ...\\
&\dot{\hat{\omega}}_{n_f} \in [-\tilde{\lambda}_{n_d+1}\lfloor\hat{\omega}_1\rceil^{1+n_f q} + \hat{\omega}_{n_f+1}]|_{\hat{t}-\hat{\tau}[0,1]},\\
&\hat{\omega}_{n_f+1}(\hat{t}-\hat{\tau}[0,1]) \in \hat{\sigma}_0(\hat{t}-\hat{\tau}[0,1]) + \hat{\varepsilon}_0[-1,1],
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\dot{\hat{\sigma}}_0 \in [-\tilde{\lambda}_{n_d}\lfloor\hat{\omega}_1\rceil^{1+(n_f+1)q} + \hat{\sigma}_1]|_{\hat{t}-\hat{\tau}[0,1]} +\\
&\quad [\tfrac{\hat{\tau}_j^1}{2!}\hat{\sigma}_2 + ... + \tfrac{\hat{\tau}_j^{n_d-1}}{n_d!}\hat{\sigma}_{n_d}]|_{\hat{t}-\hat{\tau}[0,1]} +\\
&\quad\quad\quad\quad \varepsilon_w^{n+1}L_f\tfrac{\hat{\tau}^{n_d}}{(n_d+1)!}[-1,1],\\
&\quad ...\\
&\dot{\hat{\sigma}}_{n_d-1} \in [-\tilde{\lambda}_1\lfloor\hat{\omega}_1\rceil^{1+nq} + \hat{\sigma}_{n_d}]|_{\hat{t}-\hat{\tau}[0,1]} +\\
&\quad\quad\quad\quad \varepsilon_w^{n+1}L_f\tfrac{\hat{\tau}}{2!}[-1,1],\\
&\dot{\hat{\sigma}}_{n_d} \in -\tilde{\lambda}_0\lfloor\hat{\omega}_1\rceil^{1+(n+1)q}|_{\hat{t}-\hat{\tau}[0,1]} + \varepsilon_w^{n+1}L_f[-1,1],
\end{aligned} \tag{41}$$

Define the weights $\deg\hat{\omega}_i = 1 + (i-1)q,\ i = 1,..,n_f$, $\deg\hat{\sigma}_i = 1 + (n_f+i)q, i = 0,..,n_d$, $\deg t = \deg\tau = \deg\tau_j = -q$. Introduce the disturbance parameter ρ as in (25) [35], [36]. Then $\deg\rho = 1$ and the steady state accuracy (25),

$$\begin{aligned}
&|w_i| \le \mu_{wi}\varepsilon_w^{-(i-1)}\rho^{1+(i-1)q},\ i = 1,2,...,n_f,\\
&|z_i - f_0^{(i)}| \le \mu_i\varepsilon_w^{-(n_f+i)}\rho^{1+(n_f+i)q},\ i = 0,1,...,n_d,
\end{aligned}$$

is established for sufficiently small $\hat{\tau}$ and $\hat{\varepsilon}_0$ [35], [36]. The errors asymptotically converge to zero for $q \ge 0,\ L_f = 0$.

In the case $q = 0$ get $\hat{\tau} = k_\tau^{-1/(n+1)},\ \varepsilon_w = k_\tau^{-1/(n+1)}\tau$, $\hat{\varepsilon}_0 = k_\tau^{-n_f/(n+1)}\tau^{n_f}\varepsilon_0$, and formula (26). □

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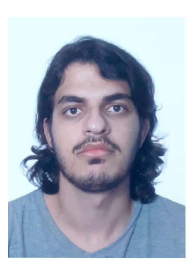

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