# Low-chattering discretization of sliding mode control

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***Abstract*— A new chattering-mitigation method is proposed for discontinuous dynamics discretization. Its application to feedback and output-feedback homogeneous sliding-modes significantly diminishes the control chattering in the absence of noises, while preserving the system accuracy in their presence. Numeric experiments illustrate the approach efficacy.**

## I. INTRODUCTION

Sliding-mode (SM) control (SMC) [27], [30], [31] is known for its ability of effectively suppress uncertainties despite sampling noises and singular perturbations. The approach is based on enforcing a proper constraint $\sigma = 0$ on the system for some available output $\sigma$. The constraint is kept by discontinuous control preventing leaving the constraint under uncertainty conditions.

Unfortunately SMC often causes dangerous system vibrations (the chattering effect) due to the high-frequency control switching, discrete noisy sampling, and unaccounted for system dynamics [4], [6], [19]. The methods counteracting the chattering phenomena can be divided into three categories: SM regularization, dynamic extension (in particular, high-order SMC (HOSMC)) and SMC discretization.

SM regularization replaces the discontinuities with some continuous (often called "sigmoid") approximations [30] usually reducible to local singular perturbations. They compromise the SMC insensitivity to matched disturbances, and still feature chattering. The dynamic extension introduces additional integrators in the feedback hiding the discontinuities in the higher derivatives of the system states [16], [28], [29]. In particular, HOSMC [3], [17], [19] is capable of establishing and keeping constraints of any relative degrees [14]. Unfortunately it requires real-time estimation of additional derivatives of $\sigma$, making the system more susceptible to noises. Also the recent integral action method [25] suffers of similar informational issues.

By the discretization we generally understand any method involving replacing a continuous time-state dynamic system with a system having discrete components, provided the solutions of the discretized system converge to the solutions of the original system, as the maximal discretization step tends to zero. That definition is intentionally vague and covers sampled feedback systems and/or computer simulation of infinite- and finite-dimensional systems. The dependence on the sampling/discretization step is the main factor distinguishing this method from two other.

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The main discretization method for discontinuous systems has always been the classical possibly-modified Euler method, usually causing chattering. Implicit discretization methods [1], [7] developed to address this problem usually require additional knowledge on the system and are often computationally complicated for higher relative degrees.

Note that no feedback control can remove chattering due to the sampling noises [19]. Also "chattering uncertainties" generate the corresponding system chattering. In this paper we propose a new simple approach to the discretization of Filippov discontinuities [10]. Similarly to other methods it depends on the system structure and the chosen control, but this dependence is much weaker.

In our recent papers [11], [13] we developed low chattering discretizations of SM-based differentiators. In this paper we demonstrate a low-chattering discretization scheme for an arbitrary-order quasi-continuous homogeneous controller stabilizing disturbed integrator chains in finite time (FT). The new scheme is also applicable in the output-feedback format, in which case discrete differentiators [13] are used. Simulation shows the simplicity and the effectiveness of the method.

**Notation.** A binary operation $\diamond$ of two sets is defined as $A \diamond B = \{a \diamond b | \ a \in A, b \in B\}$. A function of a set is the set of function values on this set. The norm $||x||$ stays for the standard Euclidian norm of $x$, $B_\varepsilon = \{x \mid ||x|| \le \varepsilon\}$; $\vec{\gamma}_k = (\gamma_0, ..., \gamma_k)$ for any sequence $\gamma_i$; $\mathbb{R}_+ = [0, \infty)$; $\lfloor a \rceil^b = |a|^b \operatorname{sign} a$, $\lfloor a \rceil^0 = \operatorname{sign} a$; $\operatorname{sat} s = \max[-1, \min(1, s)]$.

## II. DISCRETIZATION OF DISCONTINUOUS DYNAMICS

Recall that solutions of any differential inclusion (DI)

$$\dot{x} \in F(x),\ x \in \mathbb{R}^n,\ F(x) \subset T_x\mathbb{R}^n, \tag{1}$$

are defined as locally absolutely continuous functions $x(t)$, satisfying the DI for almost all $t$. Here $T_x\mathbb{R}^n$ denotes the tangent space at the point $x \in \mathbb{R}^n$.

DI (1) is called *homogeneous* with the weights $\deg x_i = m_i > 0$, $i = 1, ..., n$, of the HD $q \in \mathbb{R}$, $\deg t = -q$, if it is invariant with respect to the transformation $x_i \mapsto \kappa^{m_i} x_i$, $t \mapsto \kappa^{-q} t$ for any $\kappa > 0$.

### A. Basics of the Filippov theory

We call DI (1) *Filippov DI*, if the vector-set field $F(x) \subset T_x\mathbb{R}^n$ is non-empty, compact and convex for any $x$, and $F$ is an upper-semicontinuous set function. The latter means that the maximal distance of the points of $F(x)$ from the set $F(y)$ tends to zero, as $x \to y$. Filippov DIs feature most standard properties (existence, extendability of solutions, etc.), but not the uniqueness of solutions [10].

The graph $\Gamma(F)$ of the DI (1) over the domain $G \subset \mathbb{R}^n$ is the set of pairs $\Gamma(F) = \{(x,\xi) \mid x \in G,\ \xi \in F(x)\}$. Although $\Gamma(F) \subset \mathbb{R}^n \times T\mathbb{R}^n$ for any fixed coordinates it is isomorphically embedded in $\mathbb{R}^{2n}$.

If $G$ is closed, and $F(x)$ is nonempty, compact and locally bounded for any $x \in G$, then $F$ is upper-semicontinuous in $G$ if and only if $\Gamma(F)$ is closed [10]. If $G$ is compact and $F$ is upper-semicontinuous, then also $\Gamma(F)$ is compact [10].

The solutions of the Filippov DI (1) defined over the segment $[a,b]$, $a \le 0 \le b$, with a fixed compact set $A \subset \mathbb{R}^n$ of initial conditions $x(0) \in A$ constitute a compact set in the $C$-metric, and their points constitute a compact set in $\mathbb{R}^n$.

Let $\phi : \mathbb{R}^{n_x} \to T\mathbb{R}^{n_x}$ be Lebesgue-measurable and locally essentially bounded. A differential equation (DE) $\dot{x} = \phi(x), x \in \mathbb{R}^{n_x}$, is understood *in the Filippov sense*, if it is replaced with the Filippov DI $\dot{x} \in K_F[\phi](x)$,

$$K_F[\phi](x) = \bigcap_{\mu_L N=0} \bigcap_{\delta>0} \overline{\text{co}}\ \phi((x+B_\delta)\backslash N). \tag{2}$$

Here $\overline{\text{co}}$ is the convex closure operation, $\mu_L$ is the Lebesgue measure, and (2) is the famous Filippov procedure [10]. In the non-autonomous case we formally add the DE $\dot{t} = 1$.

In the most usual case, when $\phi$ is almost everywhere continuous, $K_F[\phi](x)$ is just the convex closure of all possible limit values $\lim_{k\to\infty} \phi(y_k)$ obtained along continuity-points sequences $y_k$ converging to $x$.

**Approximation of solutions.** A locally absolutely continuous function $\xi : I \to G$ is further called a $\delta$-graph-approximating ($\delta$-GA) solution of the Filippov DI (1) defined in a closed domain $G \subset \mathbb{R}^n$, $\delta \ge 0$, $I \subset \mathbb{R}$, if for almost all $t \in I$ the distance from the pair $(\xi(t), \dot{\xi}(t))$ from $\Gamma(F)$ does not exceed $\delta$. Here $I$ can be any infinite or finite, closed, open, or one-side-open time interval.

Let the time interval $I$ be compact. Then for any $\varepsilon > 0$ there exists such $\delta > 0$ that every $\delta$-GA solution of the Filippov DI (1) defined in a closed domain $G \subset \mathbb{R}^n$ over $I$ is distanced in $C$-metric by not more than $\varepsilon$ from some solution of DI (1). Let $\delta_k \to 0$, then any sequence of $\delta_k$-GA solutions has a subsequence uniformly converging to a solution of (1) over $I$ [10].

### *B. Discretization of Filippov dynamic systems*

Let the controlled system $\dot{x} = X(t,x,u)$ have the output $\sigma \in \mathbb{R}^{n_s}$. Consider a closed-loop system

$$\begin{array}{l} \dot{x} = X(t,x,u),\ x \in \mathbb{R}^{n_x},\ y \in \mathbb{R}^{n_y},\ u \in \mathbb{R}^{n_u}, \\ \dot{y} = Y(t,y,\sigma(t,x)),\ u = U(t,y,\sigma(t,x)), \end{array} \tag{3}$$

with a general-form output feedback. Suppose that the right-hand sides be locally bounded and Lebesgue measurable. Let the system be understood in the Filippov sense, and the corresponding Filippov DI be $\frac{d}{dt}(t,x,y)^T \in F_{txy}(t,x,y)$.

Let $\vec{t}_d = \{t_j\} = t_0, t_1, ...$ be the sequence of sampling time instants, $t_j < t_{j+1}$, $t_j \to \infty$, or, alternatively, $t_j \in [t_a, t_b]$. Let $\sup_j(t_{j+1} - t_j) \le \tau$, and let it be called the density of $\vec{t}_d$. Assume that admissible sampling-time sequences exist for any density $\tau > 0$.

By a discretization of the closed-loop system (3) we understand any algorithm producing $\delta(\vec{t}_d)$-GA solutions of the corresponding Filippov DI, provided for any compact set of initial conditions $\delta(\vec{t}_d)$ uniformly tends to zero as the sampling density vanishes.

It follows from the above Filippov results that the discretized solutions uniformly converge to the Filippov solutions over any compact region of $(t,x,y)$.

There are two natural types of discretization: the discretization of the whole system corresponding to the computer simulation, and the feedback discretization leaving the continuous-time system dynamics intact. The latter models practical applications and results in hybrid systems.

Naturally one can include Lebesgue-measurable noises in the system. In order to keep the Lebesgue measurability of the right-hand side of (3), it may require some parts of the system being Borel-measurable.

In the following we demonstrate that a proper simple feedback discretization can significantly diminish the system chattering in the absence of noises, or when the noises are small (usually very small). Note once more that removing the chattering in the presence of sampling noises is in general impossible.

## III. Homogeneous SM control

The discretization approach proposed in Section II is most easily implemented locally in the time and the state. System homogeneity facilitates its global application. Even in that case effective discretization seemingly has to depend on the concrete system or controller. In the following we consider low chattering discretization of a family of homogeneous single-input single-output SM controllers [8].

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u,\ \ \sigma = \sigma(t,x), \tag{4}$$

where $x \in \mathbb{R}^n$, $a : \mathbb{R}^{n+1} \to \mathbb{R}^n$, $b : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^{n+1} \to \mathbb{R}$ are uncertain smooth functions, $u \in \mathbb{R}$ is the control. The output function $\sigma$ can be considered as a tracking deviation. For simplicity we assume that any solution of (4) is forward complete, i.e. indefinitely extended in time, provided the control remains bounded along the solution trajectory.

The system is assumed to have the known relative degree $r$ [14]. Respectively,

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{5}$$

holds, where $g \ne 0$ [14]. The functions $h(t,x)$ and $g(t,x)$ are unknown, smooth, bounded and satisfy the inequalities

$$|h(t,x)| \le C,\ 0 < K_m \le g(t,x) \le K_M. \tag{6}$$

Obviously (5), (6) imply

$$\sigma^{(r)} \in [-C,C] + [K_m, K_M]u. \tag{7}$$

Denote $\vec{\sigma}_k = (\sigma, \dot{\sigma}, \ldots, \sigma^{(k)}) \in \mathbb{R}^{k+1}$. Consider the discontinuous feedback SM control ("simple" SMC [8])

$$u = \alpha u_{*r}(\vec{\sigma}_{r-1}),$$
$$u_{*r}(\vec{\sigma}_{r-1}) = -\frac{\lfloor\sigma^{(r-1)}\rceil^{\frac{1}{1}} + \beta_{r-2}\lfloor\sigma^{(r-2)}\rceil^{\frac{1}{2}} + \ldots + \beta_0\lfloor\sigma\rceil^{\frac{1}{r}}}{|\sigma^{(r-1)}|^{\frac{1}{1}} + \beta_{r-2}|\sigma^{(r-2)}|^{\frac{1}{2}} + \ldots + \beta_0|\sigma|^{\frac{1}{r}}}. \tag{8}$$

It is known to establish and keep $\sigma \equiv 0$, provided parameters $\beta_0, ..., \beta_{r-2} > 0$ are properly chosen. Here $\alpha > 0$ is the control magnitude parameter. The value of $u_{*r}(0)$ does not matter, since it does not influence the Filippov procedure (2). Note that $u_{*0} = -\operatorname{sign}\sigma$ for $r = 1$.

Thus any solution of (4), (8), (6) satisfies the Filippov DI

$$\sigma^{(r)} \in [-C, C] + \alpha[K_m, K_M]K_F[u_{*r}](\vec{\sigma}_{r-1}). \tag{9}$$

The application of this control requires the real-time estimation of $r - 1$ derivatives of $\sigma$. Similarly, further any continuous-time feedback control in DIs is assumed to be replaced by its Filippov extension (2).

DI (9) is homogeneous with the weights $\deg\sigma^{(i)} = r - i$ and the homogeneity degree (HD) $-\deg t = -1$ [18]. Obviously $\deg\sigma^{(r)} = \deg u_{*r} = 0$. Let the sampling noise magnitude and the sampling step not exceed $\varepsilon_0 \geq 0$ and $\tau > 0$ respectively. Then due to the homogeneity the steady-state accuracy

$$|\sigma^{(i)}| \leq \mu_i \rho^{r-i}, \tag{10}$$

is established in FT for $\rho = \max[\tau, \varepsilon_0^{1/r}]$ and some constants $\mu_i > 0$ [18]. The formula remains correct also for the continuous-time exact sampling $\tau = \varepsilon_0 = 0$.

### A. SM-based differentiation and filtering

Let $\mathrm{Lip}_{n_d} L$ be the set of all scalar functions defined on $\mathbb{R}_+ = [0, \infty)$, whose $n_d$th derivative has the Lipschitz constant $L > 0$. Assume that the input signal $f(t)$, $t \geq 0$ has the form $f(t) = f_0(t) + \eta(t)$, where $f_0 \in \mathrm{Lip}_{n_d} L$ is unknown and $\eta(t)$ is a Lebesgue-measurable noise, $|\eta| \leq \varepsilon_0$. The numbers $L, n_d$ are assumed known, and the function $f(t)$ is available (sampled) in real time.

An $n_d$th-order differentiator is any algorithm producing functions $z_0, ..., z_{n_d} : \mathbb{R}_+ \to \mathbb{R}$ to be the real-time estimations of $f_0(t), \dot{f}_0(t), ..., f_0^{(n_d)}(t)$ respectively.

A differentiator exact on $f \in \mathrm{Lip}_{n_d} L$ is called **asymptotically optimal** [23], if for some $\mu_i > 0$, any $f_0 \in \mathrm{Lip}_{n_d}(L)$ and any bounded noise $\eta$, $\operatorname{ess\,sup}|\eta(t)| \leq \varepsilon_0$, it in FT establishes the estimation accuracy

$$|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L^{\frac{i}{n_d+1}} \varepsilon_0^{\frac{n_d+1-i}{n_d+1}}, \ i = 0, 1, ..., n_d. \tag{11}$$

That accuracy asymptotics is proved to be the best possible, moreover $\mu_i \geq 2^{\frac{i}{n_d+1}}$ always holds [23]. Arbitrary-order asymptotically-optimal differentiators have been first proposed in [17] and are based on SMs.

The asymptotically-optimal filtering differentiator [24], [22] of the differentiation order $n_d \geq 0$ and the filtering order $n_f \geq 0$,

$$\begin{aligned}
\dot{w}_1 &= -\tilde{\lambda}_{n_d+n_f} L^{\frac{1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+n_f}{n_d+n_f+1}} + w_2, \\
&\ldots \\
\dot{w}_{n_f-1} &= -\tilde{\lambda}_{n_d+2} L^{\frac{n_f-1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+2}{n_d+n_f+1}} + w_{n_f}, \\
\dot{w}_{n_f} &= -\tilde{\lambda}_{n_d+1} L^{\frac{n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+1}{n_d+n_f+1}} + w_{n_f+1}, \\
w_{n_f+1} &= z_0 - f(t),
\end{aligned} \tag{12}$$

$$\begin{aligned}
\dot{z}_0 &= -\tilde{\lambda}_{n_d} L^{\frac{n_f+1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d}{n_d+n_f+1}} + z_1, \\
&\ldots \\
\dot{z}_{n_d-1} &= -\tilde{\lambda}_1 L^{\frac{n_d+n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{n_d+n_f+1}} + z_{n_d}, \\
\dot{z}_{n_d} &= -\tilde{\lambda}_0 L \ \mathrm{sign}(w_1), \ |f_0^{(n_d+1)}| \leq L,
\end{aligned} \tag{13}$$

also filters out *unbounded* noises featuring a small local iterated integral of the order $n_f$ or less.

If $n_f = 0$ DEs (12) disappear and $w_1 = z_0 - f(t)$ is substituted in (13) producing the standard differentiator [17]. For example, $n_d = n_f = 0$ yields the 0-order differentiator $\dot{z}_0 = -\tilde{\lambda}_0 L \ \mathrm{sign}(z_0 - f(t))$, $|\dot{f}_0| \leq L$.

Introduce the short notation for (12), (13)

$$\dot{w} = \Omega_{n_d,n_f}(w, z_0 - f, L), \ \dot{z} = D_{n_d,n_f}(w_1, z, L), \tag{14}$$

for some proper $\tilde{\lambda} = (\tilde{\lambda}_0, ..., \tilde{\lambda}_{n_d+n_f})$ (see Appendix).

Let $\operatorname{ess\,sup}\eta(t) = \varepsilon_0$. It is proved in [17], [21] that in the presence of discrete measurements with the maximal sampling time interval $\tau > 0$ differentiator (14) in FT provides the steady-state accuracy

$$\begin{aligned}
&|z_i(t) - f_0^{(i)}(t)| \leq \mu_i L \rho^{n_d+1-i}, \ i = 0, 1, ..., n_d, \\
&|w_1(t)| \leq \mu_{w1} L \rho^{n_d+n_f+1}
\end{aligned} \tag{15}$$

for $\rho = \max[(\varepsilon_0/L)^{1/(n_d+1)}, \tau]$, and some $\mu_{w1} > 0$, $\mu_i > 0$ only depending on the choice of $\tilde{\lambda}$. Moreover, inequalities $|w_k(t)| \leq \mu_{wk} L \rho^{n_d+n_f+2-k}$ hold for all $k = 1, .., n_f$ and some $\mu_{wk} > 0$.

Formulas (15) are also formally applicable for continuous sampling which corresponds to $\tau = 0$ and $\rho = (\varepsilon_0/L)^{1/(n+1)}$. Thus, the filtering differentiator is asymptotically optimal. The differentiator also features strong noise-filtering capabilities [22], [13]. In particular, it directly extracts the equivalent control and its derivatives from the chattering SMC [24].

**Notation.** Denote by $\delta_j \phi = \phi(t_{j+1}) - \phi(t_j)$ the increment of any sampled vector signal $\phi(t_j)$.

**The discrete version of differentiator** (14) has the form

$$\begin{aligned}
\delta_j w &= \Omega_{n_d,n_f}(w(t_j), z_0(t_j) - f(t_j), L)\tau_j, \\
\delta_j z &= D_{n_d,n_f}(w_1(t_j), z(t_j), L)\tau_j + T_{n_d}(z(t_j), \tau_j),
\end{aligned} \tag{16}$$

where the Taylor-like term $T_{n_d} \in \mathbb{R}^{n_d+1}$ is defined as

$$\begin{aligned}
T_{n_d,0} &= \tfrac{1}{2!} z_2(t_j)\tau_j^2 + ... + \tfrac{1}{n_d!} z_{n_d}(t_j)\tau_j^{n_d}, \\
T_{n_d,1} &= \tfrac{1}{2!} z_3(t_j)\tau_j^2 + ... + \tfrac{1}{(n_d-1)!} z_{n_d}(t_j)\tau_j^{n_d-1}, \\
&\ldots \\
T_{n_d,n_d-2} &= \tfrac{1}{2!} z_{n_d}(t_j)\tau_j^2, \\
T_{n_d,n_d-1} &= 0, \ T_{n_d,n_d} = 0.
\end{aligned} \tag{17}$$

Terms $T_{n_d}$ provide for the homogeneity of the discrete error dynamics [2].

## B. Output feedback stabilization

First let only $\sigma$ be available by its noisy measurements $\sigma(t,x(t))+\eta(t)$. The corresponding FT stabilization is achieved for sufficiently large $\alpha>0$ for the system

$$\begin{array}{l} \sigma^{(r)}\in[-C,C]+\alpha[K_m,K_M]K_F[u_{*r}](z(t)),\\ \dot{w}=\Omega_{n_d,n_f}(w,z_0-\sigma-\eta(t),L),\\ \dot{z}=D_{r-1,n_f}(w_1,z,L),\\ L\geq C+K_M\alpha. \end{array} \tag{18}$$

The stabilization is exact for $\eta=0$. Note that in the case $\eta(t)=0$ system (18) is homogeneous with the HD $-1$ and the weights $\deg z_i=\deg\sigma^{(i)}=r-i$, $i=0,1,...,n_d$, $\deg w_k=r+n_f-k$, $k=1,2,...,n_f$. Its accuracy in the presence of noises is well-known [12], [15] and is described by (10) for $\rho=\max[(\varepsilon_0)^{1/(n_d+1)},\tau]$.

Let now $\sigma$ be available by its discrete noisy samples $\sigma(t_j)+\eta(t_j)$ for some sampling instants $t_0,t_1,..$, $\tau_j=t_{j+1}-t_j\leq\tau$, and the bounded noise $|\eta|\leq\varepsilon_0$.

Denote (16), (17) by $\delta_j(w,z)^T=\Delta_{n_d,n_f}(w,z,z_0-f,L,\tau_j)(t_j)$. Then the output-feedback closed-loop system gets the form

$$\begin{array}{l} \sigma^{(r)}=h(t,x)+\alpha g(t,x)u_{*r}(z(t_j)),\ t\in[t_j,t_{j+1}),\\ \delta_j(w,z)^T=\Delta_{r-1,n_f}(w,z,z_0-\sigma-\eta,L,\tau_j)(t_j),\\ L\geq C+K_M\alpha,\ |u_{*r}|\leq 1,\ 0<\tau_j=t_{j+1}-t_j\leq\tau. \end{array} \tag{19}$$

Then the above accuracy is established in FT for any initial conditions. Unbounded noises are considered in [22].

## IV. Low-chattering discretization

One cannot define the chattering of a separate signal [19]. Indeed, only the time scaling distinguishes between $\sin(10^6t)$ and $\sin(10^{-6}t)$. In this paper we intentionally restrict ourselves to the intuitive understanding of the phenomenon.

In the SM control $u$ in average approximates the equivalent control $u_{eq}=-h/g|_{\sigma\equiv0}$ [31], [24], thus one needs $u_{eq}$ itself not to chatter. In other words the ideal Filippov solution of (9) should not chatter. For the same reason, since measurement noises can mimic the chattering of $u_{eq}$, in general, one cannot remove the control chattering in the presence of noises [19]. We also do not consider the chattering due to parasitic dynamics [5].

*Thus our goal is to diminish the high-frequency significant-magnitude vibrations of the SMC* (19) *in the case of exact discrete measurements for small enough sampling step $\tau$ and relatively slowly changing $h,g$.*

All available problem solutions are obtained under the same assumptions and prove the system practical stability in the absence of noises. The widespread discontinuity regularization [30] is highly sensitive to noises [19]. Artificial increase of the relative degree [3], [9] raises the sensitivity to noises due to the required higher-order differentiation. Also continuous SM controllers with integral action [25] have differentiation issues and some chattering due to the non-Lipschitzian control. The implicit discretization schemes [1], [7] are computationally difficult and require the knowledge of the control coefficient $g$. Their performance in the presence of noises is not theoretically established.

*In the following we suggest a simple discretization of the output-feedback SMC* (19) *featuring significantly less chattering in the absence of noises and preserving the system performance in the presence of noises.* The case of the direct measurements of $\vec{\sigma}_{r-1}$ is obtained by trivially removing the observer from the feedback.

In the following *the upper bound $\tau$ of the sampling step is assumed available*.

**Low chattering differentiator discretization.** Let $k_L>0$ be the parameter of the low-chattering differentiator discretization chosen as in [13]. The following is the *low-chattering discrete filtering differentiator* [11]:

$$\begin{array}{l} \delta_j(w,z)^T=\Delta_{n_d,n_f}(w,z,z_0-f,\hat{L},\tau_j)(t_j),\\ \hat{L}(t_j)=L\,\mathrm{sat}\left(\frac{|w_1(t_j)|}{Lw_\tau}\right),\ w_\tau=k_L\tau^{n_d+n_f+1}. \end{array} \tag{20}$$

According to [13] it in FT provides for the accuracy of the form $|z_i-\sigma^{(i)}|\leq\mu_{di}L\tau^{r-i}$, $i=0,...,r-1$, in the absence of noises. The optional choice of parameters valid for any $n=n_d+n_f\leq12$, $n_d,n_f\geq0$, is proposed in the Appendix, Figs. 3,4 [13]. Recall that increasing $k_L$ preserves the validity of the parameters [13].

**Low chattering controller discretization.** The main idea is to complement the powers of coordinates $\sigma^{(i)}$ to 1 turning controller (8) into a linear one.

Each term $\lfloor\sigma^{(i)}\rceil^\gamma$ is replaced with the term $\mathrm{sat}^{1-\gamma}(|\sigma^{(i)}|/\zeta_{\tau i})\lfloor\sigma^{(i)}\rceil^\gamma$. The transformation is performed in infinitesimally thin layers $|\sigma^{(i)}|\leq\zeta_{\tau i}$ along the surfaces of discontinuity and/or non-smoothness, keeping the velocity vectors close to the graph of the Filippov inclusion (9). Since $\lim_{\tau\to0}\zeta_{\tau i}=0$ the distance of the vectors from the graph vanishes as $\tau\to0$, producing a discretization.

Choose some numbers $k_0,...,k_{r-1},k_h>0$, $k_h\in(0,1]$, to define the layer widths. Outputs $z_i$ of differentiator (20) are substituted for $\sigma^{(i)}$, $i=0,1,...,r-1$, producing the output feedback control

$$\begin{array}{l} u(t)=\alpha\,U_r(z(t_j)),\ t\in[t_j,t_{j+1}),\\ U_r(z)=-\,\mathrm{sat}\left[\frac{Q(z)}{k_hQ_\tau}\right]\frac{P(z)}{Q(z)},\\ Q(z)=|z_{r-1}|^{\frac{1}{1}}+\beta_{r-2}\,\mathrm{sat}(\frac{|z_{r-2}|}{\zeta_{\tau r-2}})^{\frac{1}{2}}|z_{r-2}|^{\frac{1}{2}}+...\\ \qquad\qquad+\beta_0\,\mathrm{sat}(\frac{|z_0|}{\zeta_{\tau0}})^{\frac{r-1}{r}}|z_0|^{\frac{1}{r}},\\ P(z)=z_{r-1}+\beta_{r-2}\,\mathrm{sat}(\frac{|z_{r-2}|}{\zeta_{\tau r-2}})^{\frac{1}{2}}\lfloor z_{r-2}\rceil^{\frac{1}{2}}+...\\ \qquad\qquad+\beta_0\,\mathrm{sat}(\frac{|z_0|}{\zeta_{\tau0}})^{\frac{r-1}{r}}\lfloor z_0\rceil^{\frac{1}{r}},\\ \zeta_{\tau i}=k_i^{r-i}\tau^{r-i},\ i=0,1,...,r-1,\\ Q_\tau=\zeta_{\tau r-1}^{\frac{1}{1}}+\beta_{r-2}\zeta_{\tau r-2}^{\frac{1}{2}}+...+\beta_0\zeta_{\tau0}^{\frac{1}{r}} \end{array} \tag{21}$$

In the case of the direct measurements of $\vec{\sigma}_{r-1}$, $\sigma^{(i)}$ are substituted back for $z_i$ in (21).

Simple calculation shows that

$$Q_\tau=q_\tau\tau,\ q_\tau=k_{r-1}+\beta_{r-2}k_{r-2}+...+\beta_0k_0.$$

It is easy to see that in the set $|z_i|\leq(k_i\tau)^{r-i}$, $Q(z)\leq k_hq_\tau\tau$ the control function $U_r$ from (21) gets the form

$$\begin{array}{l} U_r(z)=-(k_hq_\tau\tau)^{-1}[z_{r-1}+\hat{\beta}_{r-2}\tau^{-1}z_{r-2}\\ \qquad\qquad+...+\hat{\beta}_0\tau^{-(r-1)}z_0],\\ \hat{\beta}_i=\beta_ik_i^{-(r-1-i)},\ i=0,1....,r-2, \end{array} \tag{22}$$

which corresponds to the local output-feedback high-gain control with the small parameter $\tau$. Recall that it only takes place in its discrete form in the infinitesimally small vicinity of $\vec{\sigma}_{r-1} = z = 0$.

Obviously any choice of $\hat{\beta}_0, ..., \hat{\beta}_{r-2} > 0$ can be obtained by a proper choice of $k_0, ..., k_{r-1}$. Moreover, substituting $\kappa_0 k_i$ for $k_i$, $\kappa_0 > 0$, simultaneously divides the roots of the polynomial $s^{r-1} + \hat{\beta}_{r-2} s^{r-2} + ... + \hat{\beta}_0$ by the same number.

*Theorem 1:* Fix any $\hat{k}_{r-1} > 0$, $k_h \in (0, 1]$, and let $\hat{k}_0, ..., \hat{k}_{r-2} > 0$ be any sufficiently large positive numbers. Let the polynomial $s^{r-1} + \hat{\beta}_{r-2} s^{r-2} + ... + \hat{\beta}_0$ be Hurwitz. Then for any sufficiently large $\alpha, \kappa_* > 0$ the choice $k_i = \kappa_* \hat{k}_i$, $i = 0, 1, ..., r-1$, provides for the FT stabilization of the system (7), (20), (21) in the set $|\sigma^{(i)}| \leq k_i^{r-i} \tau^{r-i}$, $|z_i| \leq k_i^{r-i} \tau^{r-i}$ for any sufficiently small $\tau$.

*Theorem 2:* Under conditions of Theorem 1 the discretized system (7), (20), (21) is exponentially stable for $C = 0$, i.e. $\vec{\sigma}_{r-1}, z \to 0$ for any small enough $\tau$.

Too small values of $k_h$ require smaller $\tau$ to suppress chattering. The chattering is removed in the rare case when $u_{eq} = -h(t, x)/g(t, x) \equiv 0$ and the noises are absent. Otherwise, $u$ tracks $u_{eq}$, if it is sufficiently smooth and slow.

## V. Proof sketches

Consider the auxiliary set-valued control function

$$
\begin{aligned}
& u \in \alpha\, \hat{U}_r(z), \;\; \hat{U}_r(z) = -I_{\epsilon_1} \tfrac{\hat{P}(z)}{\hat{Q}(z)}, \; \epsilon_1, \epsilon \in (0, 1), \\
& J(\xi_1, \xi_2) \triangleq [\mathrm{sign}(\xi_1 - \xi_2), \mathrm{sign}(\xi_1 + \xi_2)] \max(|\xi_1|, |\xi_2|), \\
& J(\xi, \xi) = J(-\xi, \xi) \triangleq [-\xi, \xi], \; I_\xi \triangleq [1 - \xi, 1 + \xi], \\
& \hat{Q}(z) = I_\epsilon |z_{r-1}| + \beta_{r-2} I_\epsilon \max(|z_{r-2}|^{\frac{1}{2}}, \epsilon ||z||_h) \\
& \qquad\qquad + ... + \beta_0 I_\epsilon \max(|z_0|^{\frac{1}{r}}, \epsilon ||z||_h), \\
& \hat{P}(z) = I_\epsilon z_{r-1} + \beta_{r-2} I_\epsilon J(\lfloor z_{r-2} \rceil^{\frac{1}{2}}, \epsilon ||z||_h), \\
& \qquad\qquad + ... + \beta_0 I_\epsilon J(\lfloor z_0 \rceil^{\frac{1}{r}}, \epsilon ||z||_h), \\
& ||z||_h = \max_{j = 0, 1, ..., r-1} |z_j|^{\frac{1}{r-j}}.
\end{aligned}
\tag{23}
$$

It is easy to see [13] that, provided (18) is FT stable for $\eta = 0$, also the system

$$
\begin{aligned}
& \sigma^{(r)} \in [-C, C] + \alpha [K_m, K_M]\, \overline{\mathrm{co}}\, \hat{U}_r(z), \\
& \dot{w} = \Omega_{n_d, n_f}(w, z_0 - \sigma, L), \\
& \dot{z} = D_{r-1, n_f}(w_1, z, L)
\end{aligned}
\tag{24}
$$

is FT stable for sufficiently small $\epsilon, \epsilon_1 > 0$. Recall that after a FT transient $\vec{\sigma}_{r-1} \equiv z$ is kept. Obviously, $u_{*r}(z) \in \hat{U}_r(z)$.

Note that $U_r(z) \in \hat{U}_r(z)$ for $|z_i|/\zeta_{\tau i} \geq (1 - \epsilon)^{(r-i)/(r-i-1)}$, $i = 0, ..., r-2$, $|z_{r-1}|/\zeta_{\tau r-1} \geq 1 - \epsilon$, and $Q(z)/(k_h Q_\tau) \geq 1 - \epsilon_1$. Denote by $\Theta_z$ the compact set where this combined condition is violated and $\vec{\sigma}_{r-1} = z$.

Fix any $\zeta_{\tau i} > 0$. Let $\Theta_\zeta = \{\vec{\sigma}_{r-1} = z, |z_i| \leq \zeta_{\tau i}, \; i = 0, ..., r-1\}$, $\Theta_z \subset \Theta_\zeta$. Let $\zeta_{\tau 0}$ be sufficiently small. The limit system (21), $\tau \to 0$, is described by system (24) with a switching disturbance along the plane $z_0 = 0$ in the set $\Theta_z$. Due to the proximity of the graphs, irrespectively of the fixed values $\zeta_{\tau i}$, $i = 0, ..., r-1$, the system converges into a small vicinity of 0.

In that vicinity of 0 the control gets the form

$$
\begin{aligned}
u = -\tfrac{\alpha}{\max[k_h Q_\tau, Q(z)]} & [z_{r-1} + \hat{\beta}_{r-2} z_{r-2} \\
& + ... + \hat{\beta}_0 z_0], \\
\hat{\beta}_i = \beta_i (\tfrac{1}{\zeta_{\tau i}})^{r-1-i}&, \; i = 0, 1...., r-2,
\end{aligned}
\tag{25}
$$

The corresponding polynomial is Hurwitz, which implies that system (7), (25) is globally AS for $C = 0$ and sufficiently large $\alpha$ or small $k_h$. It is shown as in [20]. Otherwise it converges to a vicinity of zero proportional to $C$.

Now discretize the system as in (21) taking $\zeta_{\tau i} = (k_i \tau)^{r-i}$, $i = 0, ..., r-1$, for small enough $\tau$.

## VI. Simulation

Consider the kinematic model of vehicle motion [26]

$$
\begin{aligned}
& \dot{x} = V \cos(\varphi), \quad \dot{y} = V \sin(\varphi) \\
& \dot{\varphi} = \tfrac{V}{\Delta} \tan\theta, \quad\;\; \dot{\theta} = u,
\end{aligned}
\tag{26}
$$

where $x$ and $y$ are the Cartesian coordinates of the middle point of the rear axle (Fig. 1a), $\Delta$ is the distance between the two axles, $\varphi$ is the orientation angle, $V$ is the constant longitudinal velocity, $\theta$ is the steering angle (i.e. the actual real-life input), and $u = \dot{\theta}$ is the control input.

*The goal* is to track some smooth trajectory $y = g(x)$, whereas $g(x(t)), y(t)$ are available in real time. That is, the task is to make $s(x, y) = y - g(x)$ as small as possible. The function $s$ is measured with the sampling step $\tau$ and the noise $\eta$. Let $g(t) = 10 \sin(0.05 x(t)) + 5$. The Euler integration is performed with the integration step $10^{-4}$.

Control (20), (21) is applied with $L = 50$, $n_f = 2$, $k_L = 5$, $\alpha = 1$, $r = 3$, $\beta_1 = 2$, $\beta_0 = 1$, $k_0 = 10^{5/3}, k_1 = 4000^{1/2}, k_2 = 300, k_h = 0.3$. The control is kept at 0 till the time $t = 1$ to allow the differentiator convergence. The initial conditions are $(x(0), y(0), \varphi(0), \theta(0)) = (0, 15, 1, 0.5)$, $z(0) = 0, w(0) = 0$.

First take $\tau = 0.0001$ and apply control (19) without adaptation. The corresponding performance is shown in Fig. 1. The accuracy is described by the component-wise inequality $|\sigma| \leq 3.8 \cdot 10^{-8} m$, $|\dot{\sigma}| \leq 1.4 \cdot 10^{-4} m/s$, $|\ddot{\sigma}| \leq 0.022 m/s^2$.

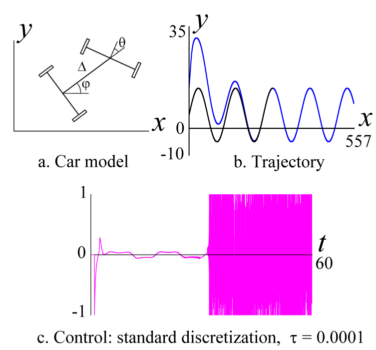


Fig. 1. a: The car model. b,c: Car trajectory and the control (steering angle derivative) with the standard discretization for the sampling step $\tau = 10^{-3}$.

Now apply the proposed discretization. The performance for $\tau = 0.02$ and $\tau = 0.0001$ is shown in Fig. 2. The corresponding accuracies are $|\sigma| \leq 0.11 m$, $|\dot{\sigma}| \leq 0.088 m/s$, $|\ddot{\sigma}| \leq 0.064 m/s^2$ and $|\sigma| \leq 1.4 \cdot 10^{-8} m$, $|\dot{\sigma}| \leq 1.3 \cdot 10^{-4} m/s$, $|\ddot{\sigma}| \leq 1.1 \cdot 10^{-5} m/s^2$ respectively.

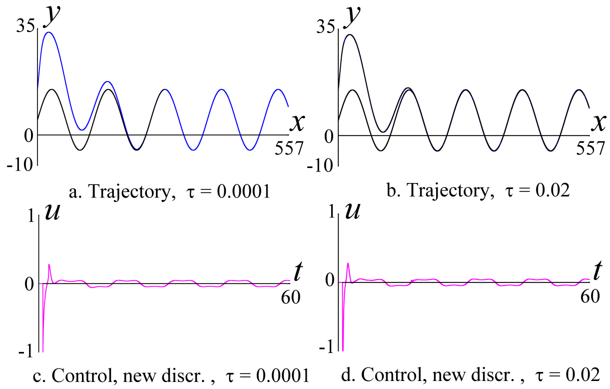


Fig. 2. New discretization of car SMC for the sampling steps $10^{-4}s$ (a, c), $0.02s$ (b, d). The graphs are almost indistinguishable in the current resolution.

## VII. Conclusions

The proposed new discretization scheme is computationally simple, significantly diminishes the SMC chattering and improves the system accuracy in the absence of noises, while preserving the standard accuracy in their presence.

## Appendix

**Choice of differentiator parameters**. The recommended parameters [13] of the filtering differentiators (12), (13) (Fig. 3) and their low-chattering discretization (20) (Fig. 4) for $n_d + n_f = 0, 1, ..., 12$ correspond to the indefinitely-extendable sequence of the recursive-form parameters $\vec{\lambda} = \{\lambda_0, \lambda_1, ..., \lambda_{n_d+n_f}, ...\}$ [17].

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.1 | | | | | | | | | | | | |
| 1 | 1.1 | 1.5 | | | | | | | | | | | |
| 2 | 1.1 | 2.12 | 2 | | | | | | | | | | |
| 3 | 1.1 | 3.06 | 4.16 | 3 | | | | | | | | | |
| 4 | 1.1 | 4.57 | 9.30 | 10.03 | 5 | | | | | | | | |
| 5 | 1.1 | 6.75 | 20.26 | 32.24 | 23.72 | 7 | | | | | | | |
| 6 | 1.1 | 9.91 | 43.65 | 101.96 | 110.08 | 47.69 | 10 | | | | | | |
| 7 | 1.1 | 14.13 | 88.78 | 295.74 | 455.40 | 281.37 | 84.14 | 12 | | | | | |
| 8 | 1.1 | 19.66 | 171.73 | 795.63 | 1703.9 | 1464.2 | 608.99 | 120.79 | 14 | | | | |
| 9 | 1.1 | 26.93 | 322.31 | 2045.8 | 6002.3 | 7066.2 | 4026.3 | 1094.1 | 173.72 | 17 | | | |
| 10 | 1.1 | 36.34 | 586.78 | 5025.4 | 19895 | 31601 | 24296 | 8908 | 1908.5 | 251.99 | 20 | | |
| 11 | 1.1 | 48.86 | 1061.1 | 12220 | 65053 | 138954 | 143658 | 70830 | 20406 | 3623.1 | 386.7 | 26 | |
| 12 | 1.1 | 65.22 | 1890.6 | 29064 | 206531 | 588869 | 812652 | 534837 | 205679 | 48747 | 6944.8 | 623.30 | 32 |

Fig. 3. Parameters $\tilde{\lambda}_0, \tilde{\lambda}_1, ..., \tilde{\lambda}_{n_d+n_f}$ of differentiator (12), (13) for $n_d + n_f = 0, 1, ..., 12$, $\vec{\lambda} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32, ...\}$

| $n$ | $n_f = 0$ | $n_f = 1$ | $n_f = 2$ | $n_f = 3$ | $n_f = 4, ..., n$ |
|---|---|---|---|---|---|
| 0 | 2 | | | | |
| 1 | 1 | 1 | | | |
| 2 | 3 | 3 | 3 | | |
| 3 | 10 | 6 | 6 | 6 | |
| 4 | 50 | 10 | 10 | 10 | 10 |
| 5 | 3000 | 400 | 400 | 400 | 400 |
| 6 | $1.5 \cdot 10^5$ | $2 \cdot 10^4$ | $10^4$ | $10^4$ | $10^4$ |
| 7 | $3 \cdot 10^7$ | $10^7$ | $10^7$ | $10^7$ | $10^7$ |
| 8 | $10^{10}$ | $1.5 \cdot 10^{10}$ | $1.5 \cdot 10^9$ | $1.5 \cdot 10^9$ | $1.5 \cdot 10^9$ |
| 9 | $10^{12}$ | $7 \cdot 10^{12}$ | $5 \cdot 10^{11}$ | $1.5 \cdot 10^{11}$ | $1.5 \cdot 10^{11}$ |
| 10 | $10^{14}$ | $10^{15}$ | $10^{13}$ | $3 \cdot 10^{12}$ | $3 \cdot 10^{12}$ |
| 11 | $8 \cdot 10^{15}$ | $10^{17}$ | $10^{15}$ | $8 \cdot 10^{13}$ | $2 \cdot 10^{14}$ |
| 12 | $3 \cdot 10^{18}$ | $8 \cdot 10^{18}$ | $10^{16}$ | $7 \cdot 10^{15}$ | $7 \cdot 10^{15}$ |

Fig. 4. Valid parameters $k_L$ of the discrete differentiator (20) corresponding to Fig. 3 and $\vec{\lambda}_{12} = (1.1, 1.5, 2, 3, 5, 7, 10, 12, 14, 17, 20, 26, 32)$, $n_d + n_f = n = 0, 1, ..., 12$

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