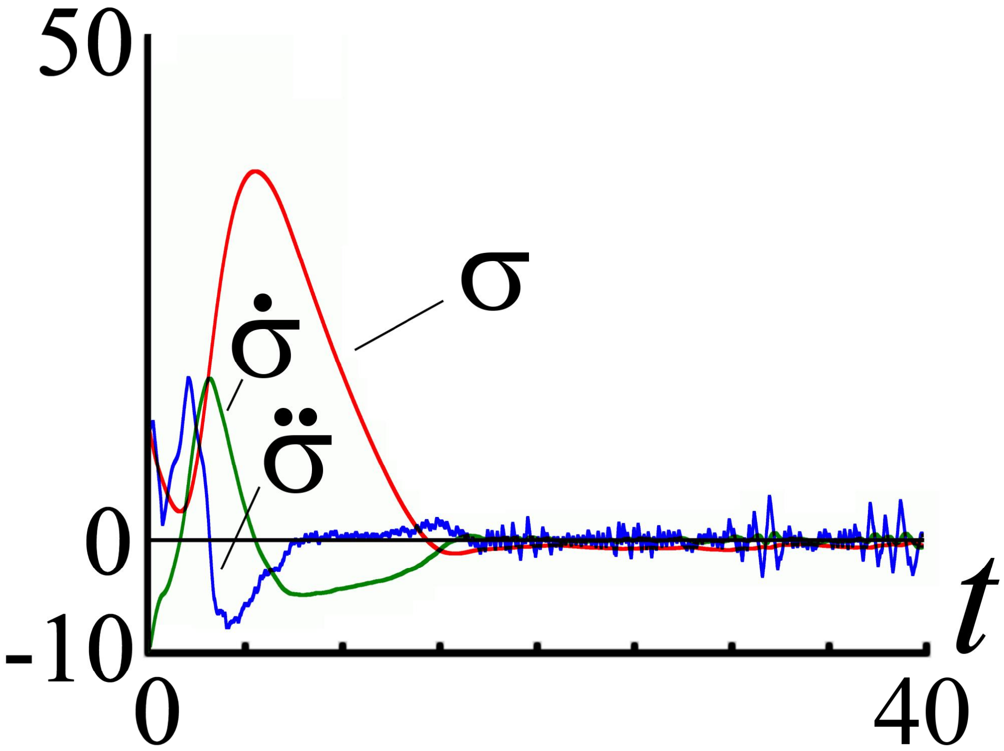
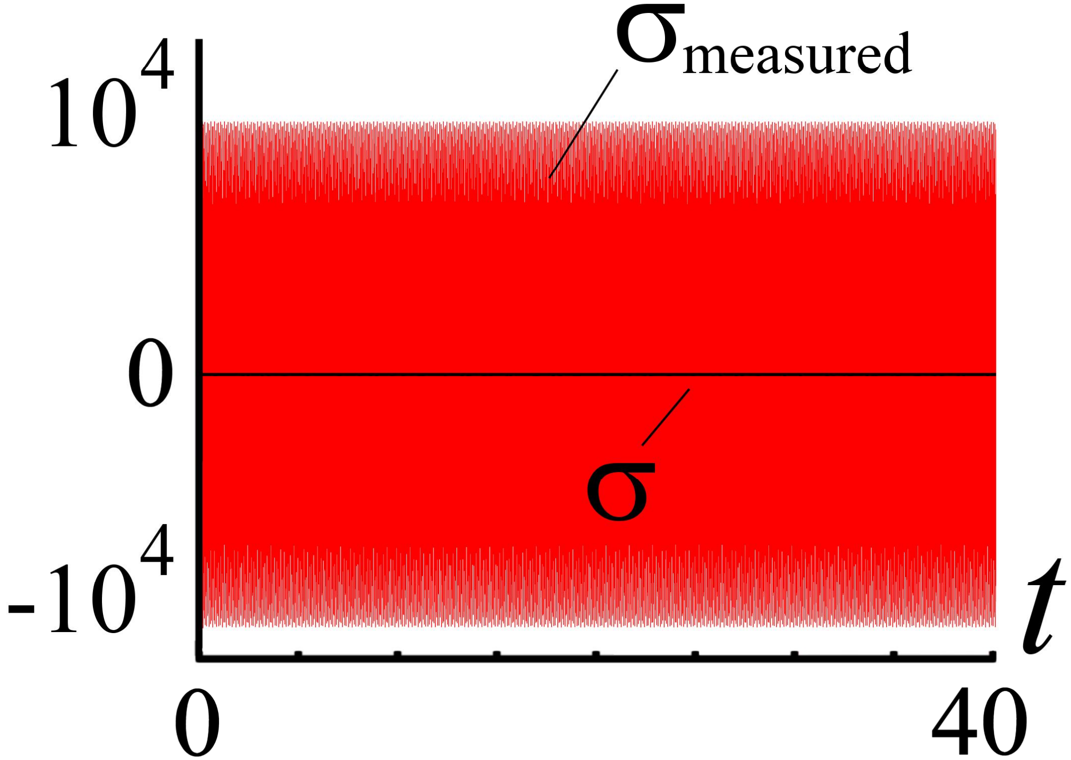
Homogeneous control and observation of uncertain SISO systems

Dalian, July 27, 2020 - Mexico, January 13, 2021

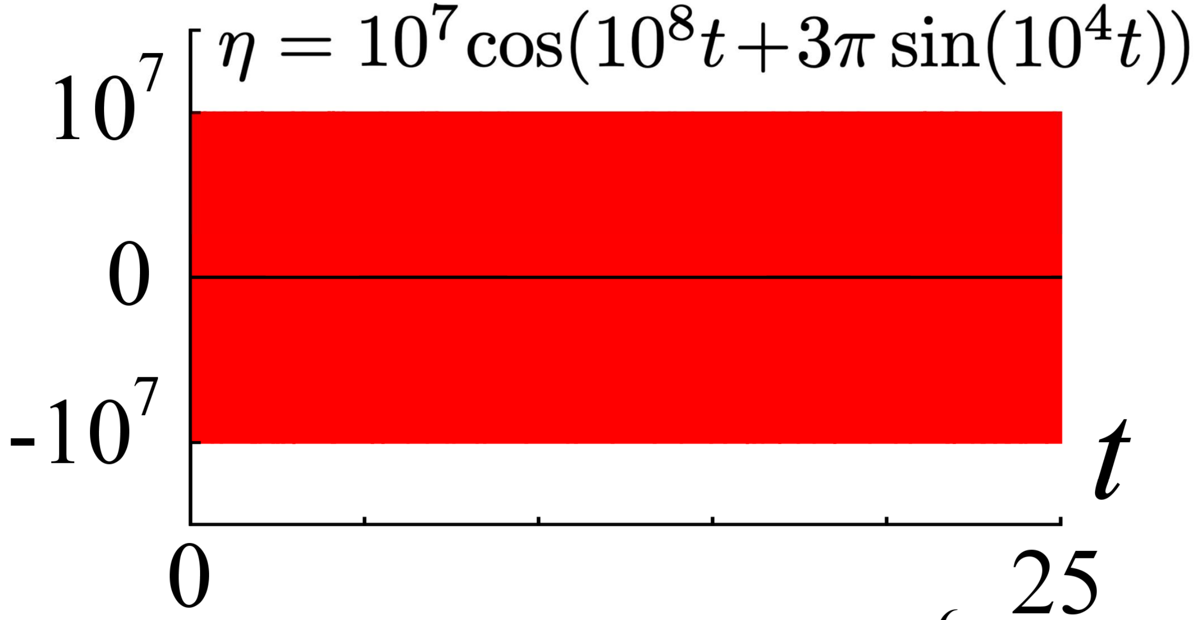
Avi Hanan, Adam Jbara, Arie Levant

Tel-Aviv University, Israel,
http://www.tau.ac.il/~levant/

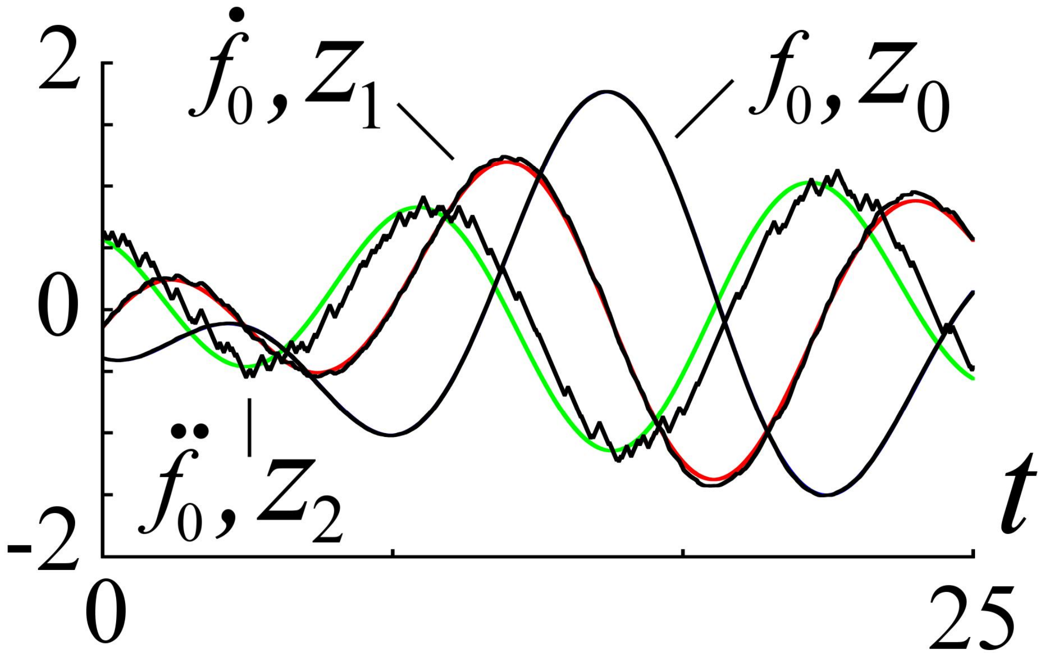




Numeric homogeneous differentiation of the 2nd order



Noisy input: sampling step $\tau = 10^{-6}$,
$f(t) = f_0(t) + \eta(t),\ f_0(t) = 0.8\cos t - \sin(0.6t)$



Output stabilization problem of the relative degree *n*.

$$\frac{d^n}{dt^n}\sigma(t,x(t)) = h(t,x) + g(t,x)u, \quad g(t,x) > 0$$

The method

Uncertainty $\Rightarrow$ differential inclusion (DI)

$$\frac{d^n}{dt^n}\sigma \in H(\vec{\sigma}_{n-1}) + G(\vec{\sigma}_{n-1})u$$

Notation: $\vec{\sigma}_k = (\sigma, \dot{\sigma}, \ldots, \sigma^{(k)})$

Plan:

1. **Control** $u = U(\vec{\sigma}_{n-1})$
2. **Differentiator**
3. **Method: homogeneity theory**

Preliminary conclusions

- SISO homogeneous differential inclusions (disturbed integrator chains) of any homogeneous degree (HD) are robustly globally asymptotically stabilized by a standardized homogeneous output feedback.
- Non-Lyapunov control design is easy for any HD.
- The feedback includes standardized filtering observers. HGOs, SM-based differentiators are particular (non-filtering) cases for zero and negative HDs.
- The robustness is obtained with respect to bounded and some kinds of **unbounded** sampling noises.

References (July 2020):

A. Bacciotti and L. Rosier. Liapunov Functions and Stability in Control Theory. Springer Verlag, London, 2005.

E. Bernuau, D. Efimov, W. Perruquetti, and A. Polyakov. On an extension of homogeneity notion for differential inclusions. In Proc. of 12th European Control Conference ECC, 17-19 July, 2013, Zurich, pages 2204–2209, 2013.

E. Cruz-Zavala and J.A. Moreno. Lyapunov approach to higher-order sliding mode design. In Recent Trends in Sliding Mode Control, pages 3–28. Institution of Engineering and Technology IET, 2016.

E. Cruz-Zavala and J.A. Moreno. Lyapunov functions for continuous and discontinuous differentiators. IFAC-PapersOnLine, 49(18):660–665, 2016.

E. Cruz-Zavala and J.A. Moreno. Levant's arbitrary order exact differentiator: a Lyapunov approach. IEEE Transactions on Automatic Control, 64(7):3034–39, 2019.

C. Qian. A homogeneous domination approach for global output feedback stabilization of a class of nonlinear systems. In Proceedings of the IEEE American Control Conference 2005, pages 4708–4715, 2005.

W. Perruquetti, T. Floquet, and E. Moulay. Finite-time observers: application to secure communication., IEEE TAC, 53(1):356–360, 2008.

L.K. Vasiljevic and H.K. Khalil. Error bounds in differentiation of noisy signals by high-gain observers. Systems & Control Letters, 57(10):856–862, 2008.

J.-P. Barbot, A. Levant, M. Livne, and D. Lunz. Discrete differentiators based on sliding modes. Automatica, 112: available online, 2020.

A. Hanan, A. Jbara, A. Levant. Homogeneous Output-Feedback Control, In Proc. of IFAC World Congress, July 13-17, Berlin, 2020.

A. Levant. Higher order sliding modes, differentiation and output-feedback control. International J. Control, 76(9/10):924–941, 2003

A. Levant. Homogeneity approach to high-order sliding mode design. Automatica, 41(5):823–830, 2005

A. Levant. Non-Lyapunov homogeneous SISO control design. In 56th Annual IEEE Conference on Decision and Control (CDC) 2017, 6652–6657, 2017.

A. Levant, D. Efimov, A. Polyakov, and W. Perruquetti. Stability and robustness of homogeneous differential inclusions. In Proc. of the 55th IEEE Conference on Decision and Control, Las-Vegas, December 12-14, 2016.

A. Levant and M. Livne. Robust exact filtering differentiators. European Journal of Control, available online, 2019.

A. Levant, M. Livne, and X. Yu. Sliding-mode-based differentiation and its application. IFAC-PapersOnLine, 50(1):1699–1704, 2017.

A. Levant and X. Yu. Sliding-mode-based differentiation and filtering. IEEE Transactions on Automatic Control, 63(9):3061–3067, 2018.

Homogeneity Basics

Weighted Homogeneity

Homogeneity of a function f: $\mathbf{R}^n \to \mathbf{R}$

Dilation: $x \in \mathbf{R}^n$, $\deg x_i = m_i > 0$

$$d_\kappa\colon (x_1, x_2, \ldots, x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, \ldots, \kappa^{m_n} x_n)$$

$$\deg f = q \;\Leftrightarrow\; \forall x\ \forall \kappa > 0 \quad f(d_\kappa x) = \kappa^q f(x)$$

A function f is called a **homogeneous norm** if it is positive definite and $\deg f = 1$.

$$\text{Example: } \|x\|_{hd} = \left[|x_1|^{\frac{d}{m_1}} + \ldots + |x_n|^{\frac{d}{m_n}} \right]^{\frac{1}{d}}, \; d > 0.$$

The homogeneity degree is defined up to proportionality

$$s > 0, \kappa := \kappa_1^{s}$$

$$\kappa^{q} = \kappa_1^{qs}, \quad \kappa^{m_i} = \kappa_1^{sm_i},$$

New weights are qs, $m_i s$

Homogeneity of differential inclusions

Levant 2005

The homogeneity degree $q = - p$ (deg $t = p$) iff

$$\forall x \ \forall \kappa > 0 \qquad (t, x) \mapsto (\kappa^p t, d_\kappa x)$$

preserves the differential inclusion (equation).

$$\frac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x) \Leftrightarrow \dot{x} \in F(x) \subset T_x \mathbb{R}^n$$

$$\Rightarrow F(x) = \kappa^p d_\kappa^{-1} F(d_\kappa x)$$

$$\dot{x}_i = f_i(x) \Rightarrow \deg \dot{x}_i = \deg x_i - \deg t = \deg f_i(x) = m_i - p$$

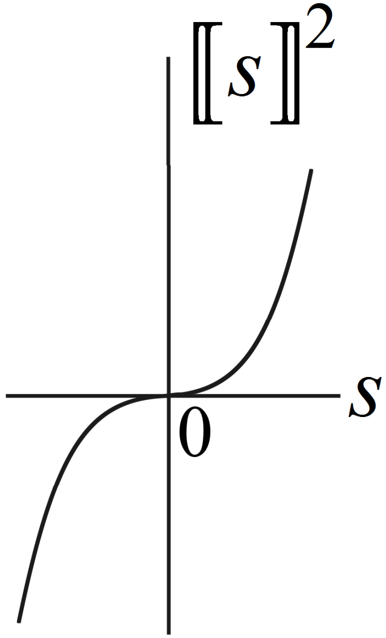
Homogeneous arithmetic

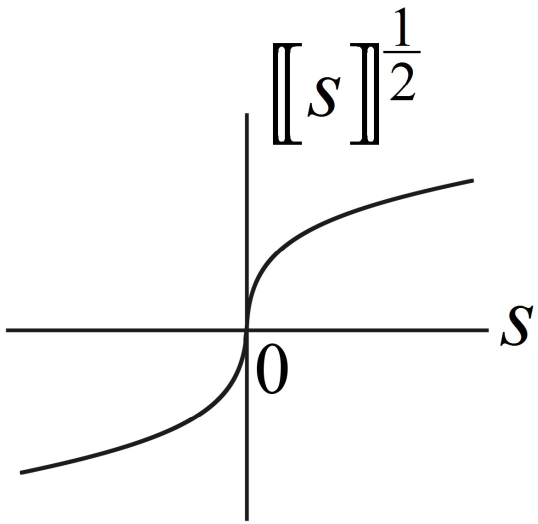
1. $\deg(A+B) = \deg A = \deg B$
2. $\deg(AB) = \deg A + \deg B$
3. $\deg(A\,/\,B) = \deg A - \deg B$
4. $\deg(A^{\beta}) = \beta \deg A$
5. $\deg \frac{\partial}{\partial x_i} A = \deg A - \deg x_i$
6. $\deg \dot{A} = \deg(\frac{\partial}{\partial x} A\, \dot{x}) = \deg A - \deg t$

Special power functions

$$\lfloor s \rceil^{\gamma} = [\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s$$

$$[\![s]\!]^{0} = \operatorname{sign} s$$





Asymptotic Stability (AS) implies

1. Finite-Time (FT) Stability for $q < 0$

Example:

$$\dot{\sigma} \in \tan(\tfrac{1}{3}\operatorname{sign}\sigma)\,|\sigma|^{\frac{2}{3}} - [2,3]\llbracket\sigma\rrbracket^{\frac{2}{3}},\ n = 1,\ q = -\tfrac{1}{3}$$

2. Exponential stability for $q = 0$

Example: $\dot{\sigma} \in [-1,1]\,|\sigma| - [2,3]\sigma,\ n = 1,\ q = 0$

3. Fixed Time (FxT) convergence from infinity to any ball for $q > 0$

Example: $\dot{\sigma} \in \cos(\operatorname{sign}\sigma)\,|\sigma|^{\frac{4}{3}} - [2,3]\llbracket\sigma\rrbracket^{\frac{4}{3}},\ n = 1,\ q = \tfrac{1}{3}$

Back to stabilization

The homogeneity dominance/extension

Uncertainty $\Rightarrow$ homogeneous differential inclusion (DI)

$$\frac{d^n}{dt^n}\sigma \in H(\vec{\sigma}_{n-1}) + G(\vec{\sigma}_{n-1})u(\vec{\sigma}_{n-1})$$

Homogeneity weights:

$$\deg \sigma^{(i)} = 1 + iq, \quad i = 0,\ldots,n, \quad 1 + nq \geq 0, \quad \deg t = -q$$

$$\text{Rule: } \deg \tfrac{d}{dt} f = \deg \dot{f} = \deg f - \deg t = \deg f + q$$

Homogeneous norm: $\| \vec{\sigma}_k \|_{h\infty} = \max(|\sigma|, |\dot{\sigma}|^{\frac{1}{1+q}}, \ldots, |\sigma^{(k)}|^{\frac{1}{1+kq}})$

Homogeneous stabilization problem

System output satisfies the DI

$$\sigma^{(n)} \in [-C, C] \| \vec{\sigma}_{n-1} \|_h^{1+nq} + [K_m, K_M] u,$$

$$C \geq 0,\ 0 < K_m \leq K_M,$$

Only σ is available, possibly with noise

Solution:

stabilizer+ filtering observer/differentiator

If $q = -1/n$ obtain n-SMC problem

$$\sigma^{(n)} \in [-C, C] + [K_m, K_M] u,$$

then filtering hybrid differentiator is used

Control Design

Homogeneous Control Templates

Theorem (Levant 2017):

AS DE: $\sigma^{(r-1)} + \varphi_{r-1}(\vec{\sigma}_{r-2}) = 0, \quad \varphi_{r-1} \in C$

Sign equivalency: QC = quasi continuous, continuous everywhere accept 0

$$\phi_r : \mathbb{R}^r \to \mathbb{R},\ \phi_r \in \mathrm{QC}\ ,\ \deg \phi_r = \deg \sigma^{(r)}$$

$$\vec{\sigma}_{r-1} \neq 0 \Rightarrow \operatorname{sign} \phi_r(\vec{\sigma}_{r-1}) = \operatorname{sign}[\sigma^{(r-1)} + \varphi_{r-1}(\vec{\sigma}_{r-2})]$$

$|\phi_r(\vec{\sigma}_{r-1})| + |\sigma^{(r-1)} + \varphi_{r-1}(\vec{\sigma}_{r-2})| \neq 0$

Then for large enough $\alpha, \beta > 0$ get AS DE and DI

$$\sigma^{(r)} + \beta \phi_r(\vec{\sigma}_{r-1}) = 0$$

$$\sigma^{(r)} \in [-C, C] \| \vec{\sigma}_{r-1} \|_h^{\deg \sigma^{(r)}} - [K_m, K_M] \alpha \phi_r(\vec{\sigma}_{r-1})$$

Example: *r*-SMC design

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]u,$$

General option: $\deg \sigma = 1, \quad \deg t = -q$

$$\deg \sigma^{(i)} = 1 + iq, \quad i = 0, \ldots, r, \quad 1 + rq = 0, \quad \deg t = -q = \frac{1}{r+1}$$

Traditional weights:

$$\deg \sigma = r, \quad \deg \dot{\sigma} = r - 1, \quad \ldots, \quad \deg \sigma^{(r-1)} = 1$$

Sign equivalency: $\quad \forall \gamma > 0: \quad [\![A]\!]^{\gamma} + [\![B]\!]^{\gamma} \sim A + B$

Example: 3-SMC design

$$\ddot{\sigma} \in [-C, C] + [K_m, K_M]u,$$

$$\deg \sigma = 3, \;\; \deg \dot{\sigma} = 2, \;\; \deg \ddot{\sigma} = 1$$

$r = 1$: $\dot{\sigma} + \beta_0 [\![\sigma]\!]^{2/3} = 0$ any $\beta_0 > 0$ fits

$r = 2$: $\ddot{\sigma} + \beta_1 \tan \frac{\dot{\sigma} + \beta_0 [\![\sigma]\!]^{2/3}}{|\dot{\sigma}| + \beta_0 |\sigma|^{2/3}} \max[|\dot{\sigma}|^{1/2}, |\sigma|^{1/3}] = 0$

$$u_{3-SM} = -\alpha \frac{[\![\ddot{\sigma}]\!]^{\frac{1}{2}} + \tilde{\beta}_1 \left[\!\!\left[\tan \frac{\dot{\sigma} + \beta_0 [\![\sigma]\!]^{2/3}}{|\dot{\sigma}| + \beta_0 |\sigma|^{2/3}} \max[|\dot{\sigma}|^{1/2}, |\sigma|^{1/3}] \right]\!\!\right]^{\frac{1}{2}}}{|\ddot{\sigma}|^{\frac{1}{2}} + |\dot{\sigma}|^{\frac{1}{4}} + |\sigma|^{\frac{1}{6}}}$$

$\tan s$ can be replaced with $\arctan s + 0.5 \sin s$

General form for *r* = 3:

$$\deg\sigma = 1,\ \ \deg\dot\sigma = 1+q,\ \ \deg\ddot\sigma = 1+2q,\ \ q \ge -\tfrac{1}{3}$$

$$u_{3,q} = -\alpha \frac{\lfloor \ddot\sigma \rceil^{\frac{1}{2}} + \tilde\beta_1 \left\lfloor \tan\frac{\dot\sigma + \beta_0 \lfloor\sigma\rceil^{1+q}}{|\dot\sigma| + \beta_0 |\sigma|^{1+q}} \max[\,|\dot\sigma|^{\frac{1+2q}{1+q}}, |\sigma|^{1+2q}\,] \right\rceil^{\frac{1}{2}}}{|\ddot\sigma|^{\frac{1}{2}} + |\dot\sigma|^{\frac{1+2q}{2(1+q)}} + |\sigma|^{\frac{1+2q}{2}}}$$

Any $\beta_0 > 0$, then $\tilde\beta_1$, α are to be found by simulation one by one for each q

Note: $u_{3-SM} = u_{3,-1/3}$!

(r+1)-SMC Design (Levant, Dorel –CDC2008)

$$\sigma^{(r)} = h(t,x) + g(t,x)u$$

$$u = \alpha U_r(\sigma, \dot{\sigma}, \ldots, \sigma^{(r-1)})$$

Discontinuous control causes chattering

Nowadays (Moreno, Fridman, et al): Continuous SMC

$$g \in [K_m, K_M], \quad |\tfrac{d}{dt}\tfrac{h}{g}| \leq \Delta$$

Simple HOSM chattering attenuation

$$\dot{x} = a(t,x) + b(t,x)u, \quad \dot{u} = \alpha U_{r+1}(\sigma, \dot{\sigma}, ..., \sigma^{(r)})$$

$$\sigma^{(r+1)} = h_1(t,x,u) + g(t,x)\,\dot{u},$$

$$h_1 = h'_t + h'_x a + (h'_x b + g'_t + g'_x a)u + g'_x b u^2.$$

Required assumptions:

$$|h'_t + h'_x a| \le c_a,\ |h'_x b + g'_t + g'_x a| \le c_b,\ |g'_x b| \le c_d$$

$\dot{u}$ prevails for sure only for $\sigma \approx \dot{\sigma} \approx ... \approx \sigma^{(r)} \approx 0$

since $u \approx u_{eq} = - h(t,x)/g(t,x)$ is kept

$\Rightarrow$ local convergence only

Alternative way (2008): Integral (r+1)-SMC

Levant, Dorel CDC2008, IEEETAC 2007 (see the Levant's homepage)

Choosing $s(t)$ with desired trajectory: $\sigma(t, x(t)) = s(t)$,

Initial conditions at $t = t_0$:

$$s(t_0)=\sigma(t_0),\ \dot{s}(t_0)=\dot{\sigma}(t_0),\ \ldots,\ s^{(r)}(t_0)=\sigma^{(r)}(t_0)$$

$$s(t) = 0 \text{ with } t \geq t_f. \quad | s^{(r+1)}(t)| \leq const$$

SOLUTION: $S(t, x) = \sigma(t, x) - s(t) \equiv 0$,

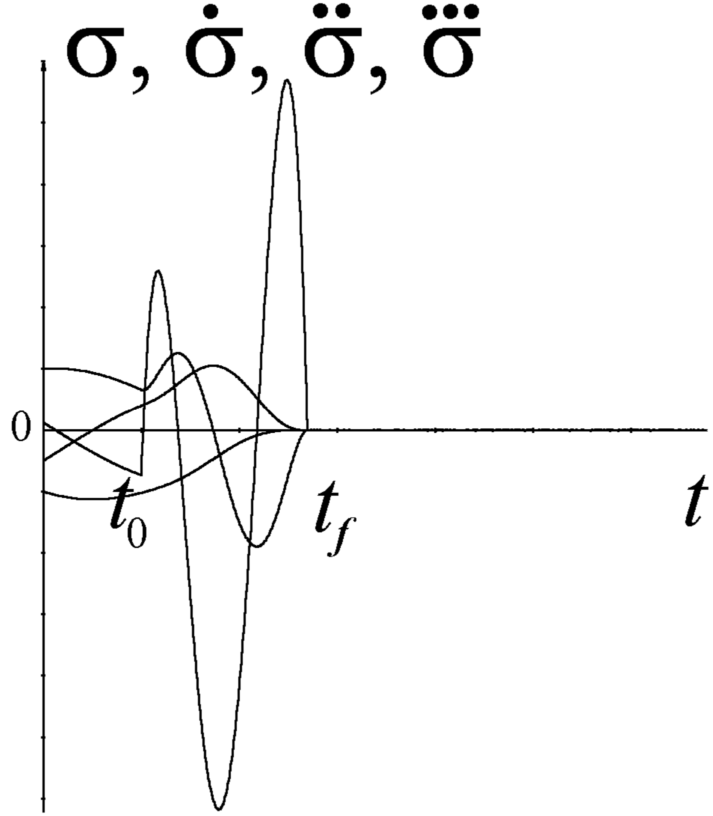
The sliding mode starts from the very beginning.

Differentiator is directly applicable

Options for the trajectory choice:

1. Optimal trajectory (i.e. minimal transient time under a restriction on $|\,s^{(r+1)}(t)|$)
2. Choice of the transient time as a function of initial conditions, and another optimization (optionally).

Levant, Alelishvilli, IEEETAC, 2007



New (2008) simple solution idea

Auxiliary dynamic system:

$$s^{(r+1)} = v,\ |v| \le \alpha_0$$

$$S = \sigma - s,$$

$$s(t_0) = \sigma(t_0),\ \ldots,\ s^{(r)}(t_0) = \sigma^{(r)}(t_0)$$

$$\dot{u} = -\alpha U_{r+1}(S, \dot{S}, \ldots, S^{(r)}),$$

α is sufficiently large $\Rightarrow s \equiv \sigma, \ldots, s^{(r)} \equiv \sigma^{(r)}$

v directly controls $\sigma, \dot{\sigma}, \ldots, \sigma^{(r)}$

v is the actual new bounded control

Integral HOSM chattering attenuation

Thus the auxiliary control v directly dictates the chosen transient trajectory to $\sigma \equiv 0$. The original system does not affect it.

Theorem.
Semi-global finite-time convergence to the $(r+1)$-sliding mode $\sigma \equiv 0$.
Output-feedback control is readily produced.

Simulation (CDC 2008)

$\ddot{x} = -\sin x + R(t)\,u,\quad R(t) \in [0.5, 1.5],$

$R = 2 + \sin 4t,\qquad x_c = 0.5 \sin 0.5t + 0.5 \cos t$

$\sigma = x - x_c\,,\ r = 2$

$x(0) = -6,\ \dot{x}(0) = -15,\ u(0) = 3.$

$\dot{u} = -40\,[S_2 + 2(|S_1| + |S_0|^{2/3})^{-1/2}(S_1 + |S_0|^{2/3}\text{sign } S_0)]/[|S_2| + 2(|S_1| + |S_0|^{2/3})^{1/2}];$

$$\sigma = x - x_c,\ S_0 = z_0 - s,\ S_1 = z_1 - \dot{s},\ S_2 = z_2 - \ddot{s};$$

$\dddot{s} = -\lambda^3 20[\ddot{s} + 2\lambda^{3/2}(|\dot{s}| + \lambda|s|^{2/3})^{-1/2}(\dot{s} + \lambda|s|^{2/3}\text{sign } s)]/$

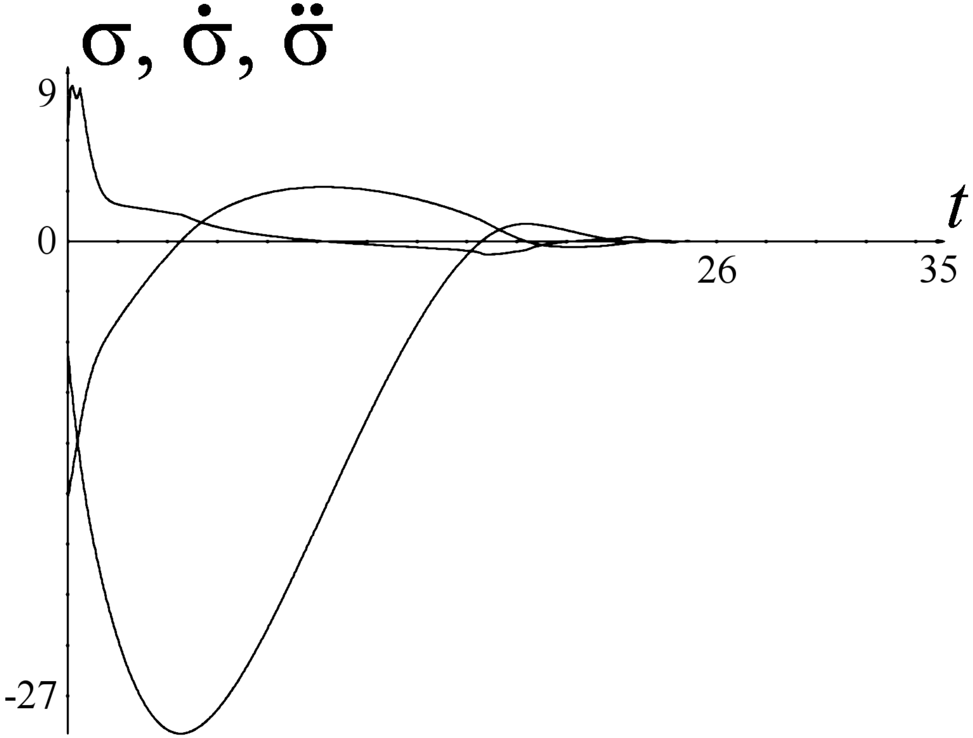
$[|\ddot{s}| + 2\lambda^{3/2}(|\dot{s}| + \lambda|s|^{2/3})^{1/2}],$

$s(t_0) = z_0(t_0),\ \dot{s}(t_0) = z_1(t_0),\ \ddot{s}(t_0) = z_2(t_0);$

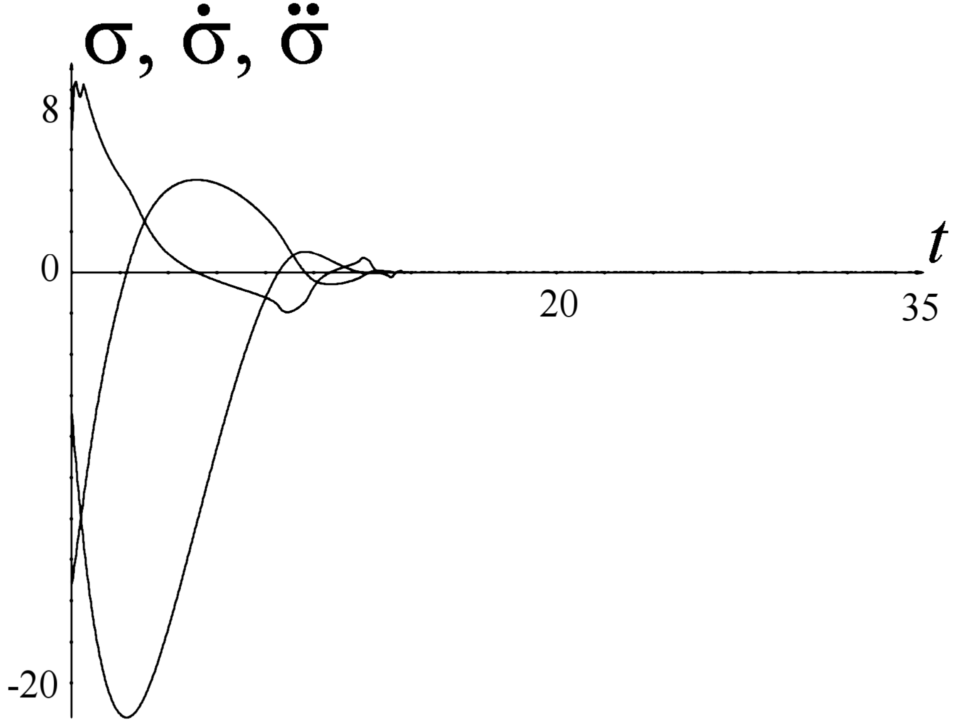
$\dot{z}_0 = v_0,\quad v_0 = -18.5664\,|z_0 - \sigma|^{2/3}\,\text{sign}(z_0 - \sigma) + z_1,$

$\dot{z}_1 = v_1,\quad v_1 = -42.4264\,|z_1 - v_0|^{1/2}\,\text{sign}(z_1 - v_0) + z_2,$

$\dot{z}_2 = -880\,\text{sign}(z_2 - v_1).\qquad L = 800,\ z_0(0) = \sigma(0),\ z_1(0) = z_2(0) = 0$



$\lambda = 0.8, \varepsilon = 0$



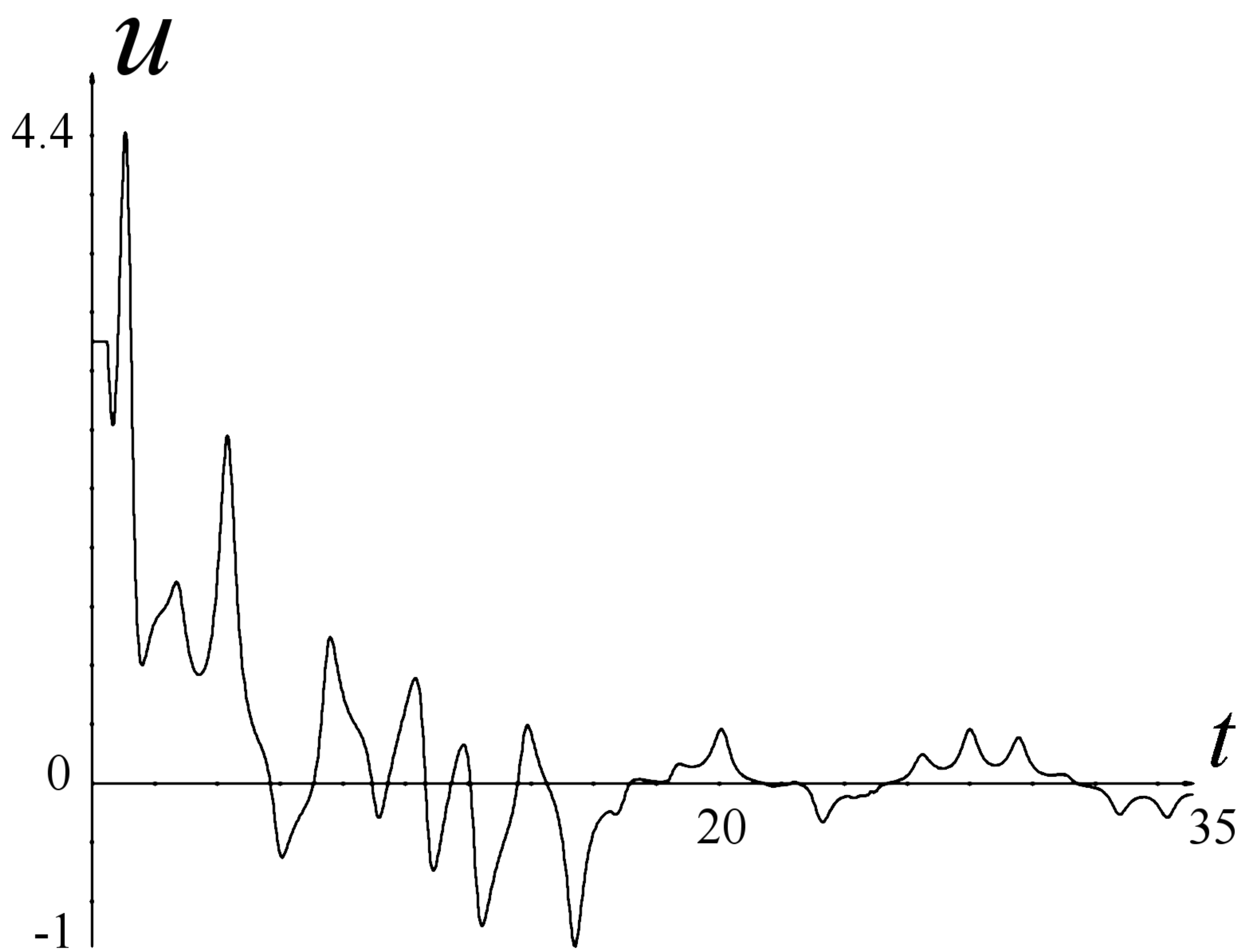
$\lambda = 0.5, \varepsilon = 0$

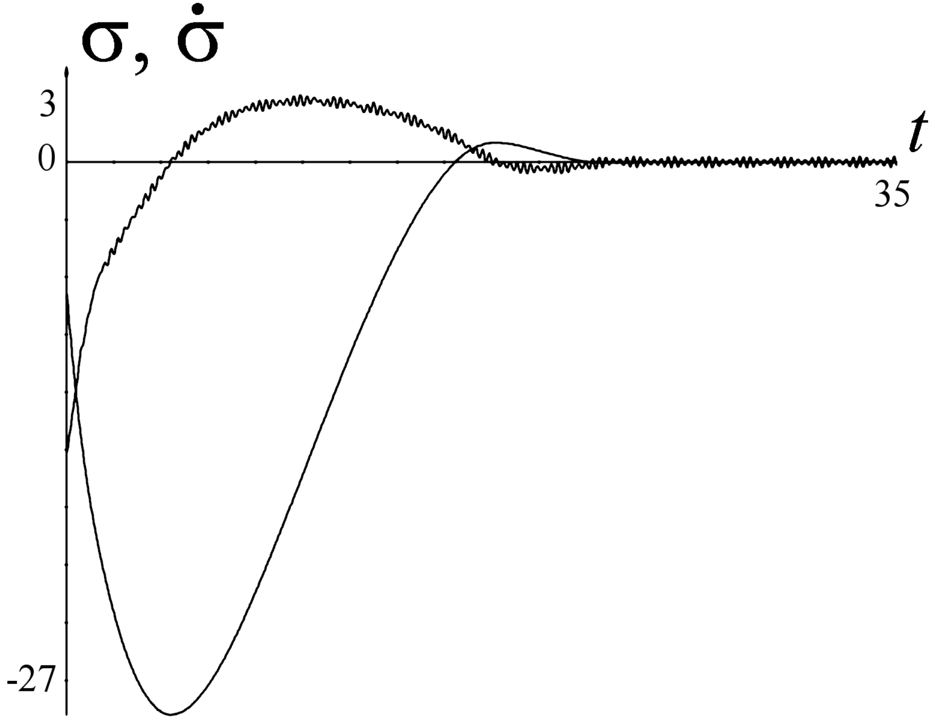
Control starts from $t_0 = 0.5$ (for the differentiator convergence).

ε is the noise magnitude.

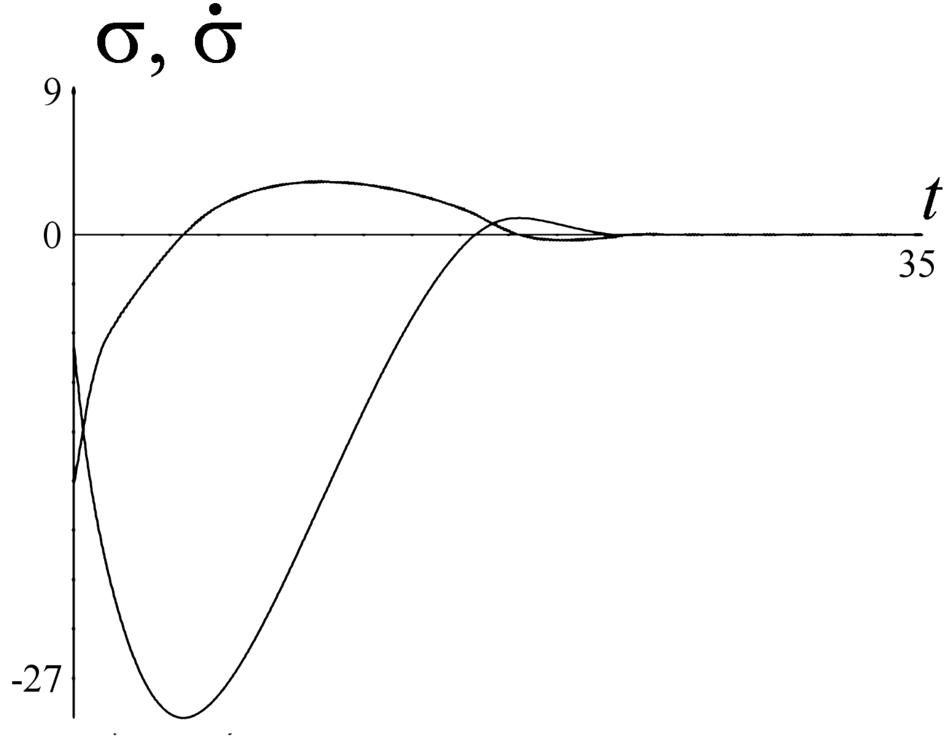
$|\sigma| \leq 1.4 \cdot 10^{-10}$, $|\dot{\sigma}| \leq 6.0 \cdot 10^{-7}$, $|\ddot{\sigma}| \leq 0.007$ are obtained with $\tau = 10^{-5}$

The resulting Lipschitzian control for $\lambda = 0.5$





$\lambda = 0.5$, Noise: $\varepsilon = 0.01$



$\lambda = 0.5$, Noise: $\varepsilon = 0.001$

σ and $\dot{\sigma}$ (coordinates of the original system) demonstrate infinitesimal chattering.

Observation Differentiation

"Standard" n_dth-order differentiator

(non-recursive form)

Levant 2003, Lyapunov function: Cruz-Zavala, Moreno 2017

$$\dot z_0 = -\tilde\lambda_{n_d} L^{\frac{1}{n_d+1}} \left[\!\left[z_0 - f(t) \right]\!\right]^{\frac{n_d}{n_d+1}} + z_1,$$

$$\dot z_1 = -\tilde\lambda_{n_d-1} L^{\frac{2}{n_d+1}} \left[\!\left[z_0 - f(t) \right]\!\right]^{\frac{n_d-1}{n_d+1}} + z_2,$$

$$\ldots \qquad\qquad |f_0^{(n_d+1)}| \le \mathrm{L}$$

$$\dot z_{n_d-1} = -\tilde\lambda_1 L^{\frac{n_d}{n_d+1}} \left[\!\left[z_0 - f(t) \right]\!\right]^{\frac{1}{n_d+1}} + z_{n_d},$$

$$\dot z_{n_d} = -\tilde\lambda_0 L \ \mathrm{sign}(z_0 - f(t)), \quad z_i - f_0^{(i)} \to 0.$$

n_dth-order homogeneous differentiator of the filtering order n_f, $-\frac{1}{n_d+1} \le q < \frac{1}{n_f}$

$$
n_f \left[
\begin{aligned}
\dot{w}_1 &= -\tilde{\lambda}_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1-(n_f-1)q}{1-n_f q}} + w_2, \\
&\ \ \dots \\
\dot{w}_{n_f} &= -\tilde{\lambda}_{n_d+1} L^{\frac{n_f |q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1}{1-n_f q}} + w_{n_f+1}, \ w_{n_f+1} = z_0 - f(t),
\end{aligned}
\right.
$$

$$
n_d \left[
\begin{aligned}
\dot{z}_0 &= -\tilde{\lambda}_{n_d} L^{\frac{(n_f+1)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+q}{1-n_f q}} + z_1, \\
&\ \ \dots \\
\dot{z}_{n_d-1} &= -\tilde{\lambda}_1 L^{\frac{(n_d+n_f)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+n_d q}{1-n_f q}} + z_{n_d}, \\
\dot{z}_{n_d} &= -\tilde{\lambda}_0 L^{\frac{(n_d+n_f+1)|q|}{1-n_f q}} \lfloor w_1 \rceil^{\frac{1+(n_d+1)q}{1-n_f q}}.
\end{aligned}
\right.
$$

Differentiator input: $f(t) = f_0(t) + \eta(t)$

Differentiator: $f(\cdot) \mapsto (z_0(\cdot), \ldots, z_{n_d}(\cdot)),\ z_i \approx f_0^{(i)}$

Exact convergence condition for $\eta = 0$:

$$\zeta_i = z_i - f_0^{(i)},\quad i = 0, 1, \ldots, n_d,\ \gamma_L > 0,\ L \geq 1$$

$$| f_0^{(n_d+1)}(t) | \leq \gamma_L \max_{i=0,\ldots,n_d} [L^{\frac{(n_d+1-i)|q|}{1+iq}} | \zeta_i(t) |^{\frac{1+(n_d+1)q}{1+iq}}]$$

$$q = -1/(n_d+1) \Rightarrow | f_0^{(n_d+1)}(t) | \leq \gamma_L L,\quad L > 0$$

$$\gamma_L = 0 \Rightarrow f_0 \text{ is a polynomial, } f_0 = c_0 + c_1 t + \ldots + c_{n_d} t^{n_d}\ .$$

$$q = 0 \Rightarrow | f_0^{(n_d+1)}(t) | \leq \gamma_L \, \| (\zeta_0, \ldots, \zeta_{n_d}) \|$$

Recursive form, $1/n_f \geq q \geq -1/(n_d + 1)$

$$
n_f \left[
\begin{aligned}
\dot{w}_1 &= -\lambda_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \left[\!\left[w_1 \right]\!\right]^{\frac{1-(n_f-1)q}{1-n_f q}} + w_2, \\
\dot{w}_2 &= -\lambda_{n_d+n_f-1} L^{\frac{|q|}{1-(n_f-1)q}} \left[\!\left[w_2 - \dot{w}_1 \right]\!\right]^{\frac{1-(n_f-2)q}{1-(n_f-1)q}} + w_3, \\
&\ldots \\
\dot{w}_{n_f} &= -\lambda_{n_d+1} L^{\frac{|q|}{1-q}} \left[\!\left[w_{n_f} - \dot{w}_{n_f-1} \right]\!\right]^{\frac{1}{1-q}} + w_{n_f+1}, \; w_{n_f+1} = z_0 - f(t),
\end{aligned}
\right.
$$

$$
n_d \left[
\begin{aligned}
\dot{z}_0 &= -\lambda_{n_d} L^{\frac{|q|}{1}} \left[\!\left[w_{n_f+1} - \dot{w}_{n_f} \right]\!\right]^{\frac{1+q}{1}} + z_1, \\
&\ldots \\
\dot{z}_{n_d-1} &= -\lambda_1 L^{\frac{|q|}{1+(n_d-1)q}} \left[\!\left[z_{n_d-1} - \dot{z}_{n_d-2} \right]\!\right]^{\frac{1+n_d q}{1+(n_d-1)q}} + z_{n_d}, \\
\dot{z}_{n_d} &= -\lambda_0 L^{\frac{|q|}{1+n_d q}} \left[\!\left[z_{n_d} - \dot{z}_{n_d-1} \right]\!\right]^{\frac{1+(n_d+1)q}{1+n_d q}}.
\end{aligned}
\right.
$$

Recursive coefficients

$$\tilde{\lambda}_k = \lambda_k, \text{ where } k = n_d + n_f;$$

$$\tilde{\lambda}_{k-i} = \lambda_{k-i} \cdot \tilde{\lambda}_{k-i+1}^{\frac{1-(n_f-i)q}{1-(n_f-i+1)q}}, \; i = 1, 2, \ldots, n_f,$$

$$\tilde{\lambda}_{k-i} = \lambda_{k-i} \cdot \tilde{\lambda}_{n_d-i+1}^{\frac{1+(n_d-i+1)q}{1+(n_d-i)q}}, \; i = n_d, n_d+1, \ldots, k.$$

Theorem. $\exists \vec{\lambda} = \{\lambda_0, \lambda_1, \ldots, \lambda_{n_d+n_f}\}$ for each $\gamma_L > 0$, $L \geq 1$.

To find $\vec{\lambda}$ perform proportional change of the homogeneity degrees:

$$\deg t = \pm 1, 0, \quad \deg \dot{z}_{n_d} = \deg z_{n_d} - \deg t = d \geq 0$$

Recursive form, deg *t* = 1, -1, *d* ≥ 0; deg *t* = 0, *d* = ∞

$$
n_f \left[
\begin{aligned}
\dot{w}_1 &= -\lambda_{n_d+n_f} L^{\frac{1}{d-(n_d+n_f+1)s_q}} \left[\!\left[w_1 \right]\!\right]^{\frac{d-(n_d+n_f)s_q}{d-(n_d+n_f+1)s_q}} + w_2, \\
&\dots \\
\dot{w}_{n_f} &= -\lambda_{n_d+1} L^{\frac{1}{d-(n_d+2)s_q}} \left[\!\left[w_{n_f} - \dot{w}_{n_f-1} \right]\!\right]^{\frac{d-(n_d+1)s_q}{d-(n_d+2)s_q}} + w_{n_f+1}, \\
w_{n_f+1} &= z_0 - f(t), \qquad \dot{w}_{n_f} = 0 \text{ for } n_f = 0,
\end{aligned}
\right.
$$

$$
n_d \left[
\begin{aligned}
\dot{z}_0 &= -\lambda_{n_d} L^{\frac{1}{d-(n_d+1)s_q}} \left[\!\left[w_{n_f+1} - \dot{w}_{n_f} \right]\!\right]^{\frac{d-n_d s_q}{d-(n_d+1)s_q}} + z_1, \\
\dot{z}_1 &= -\lambda_{n_d} L^{\frac{1}{d-n_d s_q}} \left[\!\left[z_1 - \dot{z}_0 \right]\!\right]^{\frac{d-(n_d-1)s_q}{d-n_d s_q}} + z_2, \\
&\dots \\
\dot{z}_{n_d} &= -\lambda_0 L^{\frac{1}{d-s_q}} \left[\!\left[z_{n_d} - \dot{z}_{n_d-1} \right]\!\right]^{\frac{d}{d-s_q}}, \; s_q = \operatorname{sign} q
\end{aligned}
\right.
$$

Universal recursive coefficients

$$d = \begin{cases} 1/|q| + (n_d + 1)\,\mathrm{sign}\, q, & q \neq 0, \\ \infty, & q = 0, \end{cases} \quad L \geq 1,$$

Theorem. $\forall q \geq -1, \lambda_0 > \gamma_L, d \geq 0,\ \ s_q = \mathrm{sign}\, q$

$\forall k = n_d + n_f \geq 0, d > (k+1)s_q\ \exists \vec{\lambda} = \{\lambda_0, \lambda_1, \ldots, \lambda_k, \ldots\}$

- $q \leq 0$ then $\vec{\lambda}$ is infinite
- $q > 0$ then $\vec{\lambda}$ is finite

$d = 0$, then $q = -1/(n_d + 1)$ and for $\gamma_L = 1$

$\vec{\lambda} = \{1.1, 1.5, 2, 3, 5, 7, 9, 12, 14, 17, 20, 26, 32, \ldots\}$, $k \leq 12$

SMC case, $q = -1/(n_d + 1)$

The parameters $\tilde{\lambda}_0, \ldots, \tilde{\lambda}_{n_d+n_f}$ for $n_d + n_f = 0, \ldots, 12$ can be taken from the table

0	1.1												
1	1.1	1.5											
2	1.1	2.12	2										
3	1.1	3.06	4.16	3									
4	1.1	4.57	9.30	10.03	5								
5	1.1	6.75	20.26	32.24	23.72	7							
6	1.1	9.91	43.65	101.96	110.08	47.69	10						
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12					
8	1.1	19.66	171.73	795.63	1703.9	1464.2	608.99	120.79	14				
9	1.1	26.93	322.31	2045.8	6002.3	7066.2	4026.3	1094.1	173.72	17			
10	1.1	36.34	586.78	5025.4	19895	31601	24296	8908	1908.5	251.99	20		
11	1.1	48.86	1061.1	12220	65053	138954	143658	70830	20406	3623.1	386.7	26	
12	1.1	65.22	1890.6	29064	206531	588869	812652	534837	205679	48747	6944.8	623.30	32

Filtering High-Gain Observer, $q = 0$

$$
n_f \left[\begin{array}{ll}
\dot{w}_1 & = -\tilde{\lambda}_{n_d+n_f} w_1 + w_2, \\
\dot{w}_2 & = -\tilde{\lambda}_{n_d+n_f-1} w_1 + w_3, \\
 & \dots \\
\dot{w}_{n_f} & = -\tilde{\lambda}_{n_d+1} w_1 + w_{n_f+1}, \; w_{n_f+1} = z_0 - f(t),
\end{array} \right.
$$

$$
n_d \left[\begin{array}{ll}
\dot{z}_0 & = -\tilde{\lambda}_{n_d} w_1 + z_1, \\
 & \dots \\
\dot{z}_{n_d-1} & = -\tilde{\lambda}_1 w_1 + z_{n_d}, \\
\dot{z}_{n_d} & = -\tilde{\lambda}_0 w_1.
\end{array} \right.
$$

$$\tilde{\lambda}_k = \tilde{\lambda}_{k0} \hat{\varepsilon}^{-(n_d+n_f+1-k)}, \quad 0 < \hat{\varepsilon} << 1, k = 0,1,\dots,n_d + n_f$$

Hurwitz: $$p_0(s) = s^{n_d+n_f+1} + \tilde{\lambda}_{n_d+n_f,0} s^{n_d+n_f} + \dots + \tilde{\lambda}_{0,0}$$

Bihomogeneous differentiator, $0 > q \geq -1/(n_d + 1)$

$$n_f \begin{cases} \dot{w}_1 = -\lambda_{n_d+n_f} L^{\frac{|q|}{1-n_f q}} \left[\!\left[w_1 \right]\!\right]^{\frac{1-(n_f-1)q}{1-n_f q}} - \mu_{n_d+n_f} M w_1 + w_2, \\ \dot{w}_2 = -\lambda_{n_d+n_f-1} L^{\frac{|q|}{1-(n_f-1)q}} \left[\!\left[w_2 - \dot{w}_1 \right]\!\right]^{\frac{1-(n_f-2)q}{1-(n_f-1)q}} - \mu_{n_d+n_f-1} M (w_2 - \dot{w}_1) + w_3, \\ \dots \\ \dot{w}_{n_f} = -\lambda_{n_d+1} L^{\frac{|q|}{1-q}} \left[\!\left[w_{n_f} - \dot{w}_{n_f-1} \right]\!\right]^{\frac{1}{1-q}} - \mu_{n_d+1} M (w_{n_f} - \dot{w}_{n_f-1}) + w_{n_f+1}, \; w_{n_f+1} = z_0 - f(t), \end{cases}$$

$$n_d \begin{cases} \dot{z}_0 = -\lambda_{n_d} L^{\frac{|q|}{1}} \left[\!\left[w_{n_f+1} - \dot{w}_{n_f} \right]\!\right]^{\frac{1+q}{1}} - \mu_{n_d} M (w_{n_f+1} - \dot{w}_{n_f}) + z_1, \\ \dots \\ \dot{z}_{n_d-1} = -\lambda_1 L^{\frac{|q|}{1+(n_d-1)q}} \left[\!\left[z_{n_d-1} - \dot{z}_{n_d-2} \right]\!\right]^{\frac{1+n_d q}{1+(n_d-1)q}} - \mu_1 M (z_{n_d-1} - \dot{z}_{n_d-2}) + z_{n_d}, \\ \dot{z}_{n_d} = -\lambda_0 L^{\frac{|q|}{1+n_d q}} \left[\!\left[z_{n_d} - \dot{z}_{n_d-1} \right]\!\right]^{\frac{1+(n_d+1)q}{1+n_d q}} - \mu_0 M (z_{n_d} - \dot{z}_{n_d-1}), \quad \left| \frac{\dot{L}}{L} \right| \leq M \end{cases}$$

Bihomogeneous parameters for $q = -1/(n_d + 1)$

$$\vec{\lambda} = \{1.1, 1.5, 2, 3, 5, 7, 10, 12, ...\}$$

$$\vec{\mu} = \{2, 3, 4, 7, 9, 13, 19, 23, ...\}$$

Notation for the differentiators

$$\dot{w} = \Omega_{n_d, n_f, q}(w, z_0 - f, L, M, \vec{\lambda}_{n_d + n_f}, \vec{\mu}_{n_d + n_f}),$$

$$\dot{z} = D_{n_d, n_f, q}(w_1, z, L, M, \vec{\lambda}_{n_d + n_f}, \vec{\mu}_{n_d + n_f}),$$

Main Stabilization Result

Let $u = U(\vec{\sigma}_{n-1})$, $\deg U(\vec{\sigma}_{n-1}) = 1 + nq \geq 0$, stabilize the DI, then

$$\sigma^{(n)} \in [-C, C] \| \vec{\sigma}_{n-1} \|_h^{1+nq} + [K_m, K_M] U(z),$$

$$\dot{w} = \Omega_{n-1,n_f,q}(w, z_0 - \sigma, L, M, \vec{\lambda}_{n+n_f-1}, \vec{\mu}_{n+n_f-1}),$$

$$\dot{z} = D_{n-1,n_f,q}(w_1, z, L, M, \vec{\lambda}_{n+n_f-1}, \vec{\mu}_{n+n_f-1}),$$

is AS for $L > 0$ large enough and any $M \geq 0$.

For $q \neq -1/(n+1)$ only the case $M = 0$ is considered.

The accuracy

σ is measured as $\sigma + \eta(t)$, where the noise can be represented as $\eta(t) = \eta_0(t) + \eta_1(t) + ... + \eta_{n_f}(t)$,

$\xi_k^{(k)}(t) = \eta_k(t), |\xi_k| \leq \varepsilon_k, k = 0,1,...n_f$ ALSO LOCALLY!

Theorem. The established accuracy is

$$|\sigma_i| \leq \gamma_i \rho^{1+iq}, i = 0,1,...,n,$$

$$\rho = \max\left[\varepsilon_0^{\frac{1}{1}}, \varepsilon_1^{\frac{1}{1-q}}, ..., \varepsilon_{n_f}^{\frac{1}{1-n_f q}}\right]$$

Discrete sampling, $\varepsilon_1 = ... = \varepsilon_{n_f} = 0 \Rightarrow \rho = \max(\tau^{-1/q}, \varepsilon_0)$,

general case as in Levant, Livne 2020.

Discrete Filtering Differentiator

Let $\tau_j > 0$ be the sampling step. Then the discretization is

$$w(t_{j+1}) = w(t_j) + \tau_j \Omega_{n_d, n_f, q}(w(t_j), z_0(t_j) - f(t_j), L, M, \vec{\lambda}_{n_d+n_f}, \vec{\mu}_{n_d+n_f}),$$

$$z(t_{j+1}) = z(t_j) + \tau_j D_{n_d, n_f, q}(w_1(t_j), z(t_j), L, M, \vec{\lambda}_{n_d+n_f}, \vec{\mu}_{n_d+n_f})$$
$$+ T_{n_d}(z(t_j), \tau_j),$$

$$T_{n_d,0} = \tfrac{1}{2!} z_2(t_j)\tau_j^2 + \ldots + \tfrac{1}{n_d!} z_{n_d}(t_j)\tau_j^{n_d},$$

$$T_{n_d,1} = \tfrac{1}{2!} z_3(t_j)\tau_j^2 + \ldots + \tfrac{1}{(n_d-1)!} z_{n_d}(t_j)\tau_j^{n_d-1},$$

...

$T_{n_d,n_d-2} = \frac{1}{2!} z_{n_d}(t_j)\tau_j^2,$ IN FEEDBACK $T_{n_d} = 0$ can be taken

$$T_{n_d,n_d-1} = 0, T_{n_d,n_d} = 0.$$

SIMULATION

Numeric Differentiation

$$f(t) = f_0(t) + \eta(t),\ f_0(t) = 0.8\cos t - \sin(0.6t),\ n_d = 2, |\dddot{f}_0| \le 1,\ L = 1$$

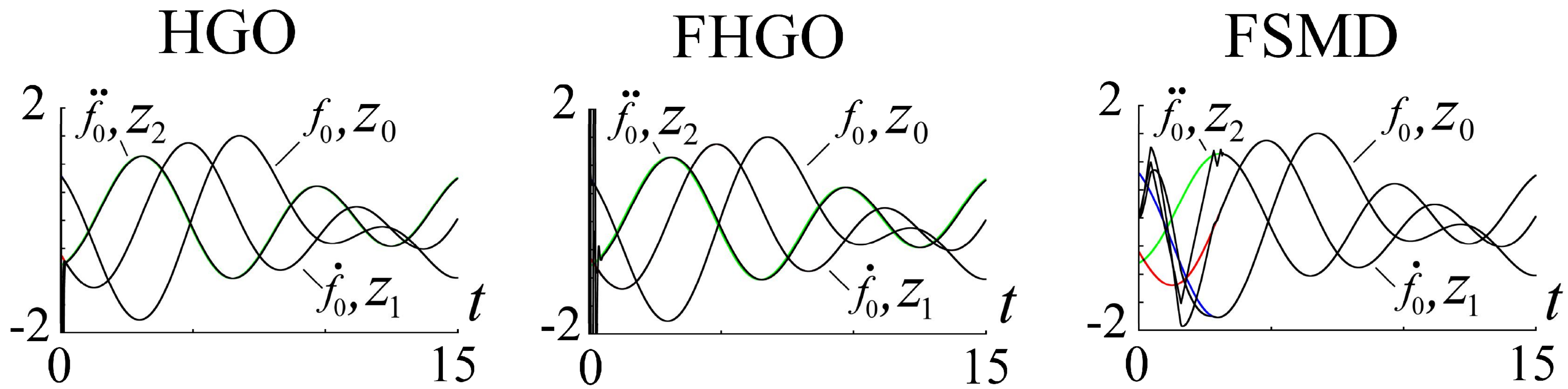
$$\hat{\varepsilon} = 0.01\ ,\quad \text{HGO: p(s)=(s+}\hat{\varepsilon}^{-1})^3,\ n_f = 0;\ \text{FHGO: p(s)=(s+}\hat{\varepsilon}^{-1})^{10},\ n_f = 8$$

Accuracies for $\eta = 0$:

$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (8.1\cdot 10^{-7}, 2.8\cdot 10^{-4}, 2.4\cdot 10^{-2}),\ n_f = 0,\ q = 0;$$

$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (1.3\cdot 10^{-4}, 4.4\cdot 10^{-3}, 8.9\cdot 10^{-2}),\ n_f = 8,\ q = 0;$$

$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (6.5\cdot 10^{-12}, 7.5\cdot 10^{-5}, 2.9\cdot 10^{-4}),\ n_f = 8,\ q = -\tfrac{1}{3}.$$



$$n_d = 2,\ n_f = 8,\ L = 1,\ q = -1/3$$

Output-Feedback Stabilization

$$\ddot{\sigma} = \cos(12t)\,|\sigma|^{1+3q} + |\ddot{\sigma}|^{\frac{1+3q}{1+2q}}\,\mathrm{sign}(\dot{\sigma}) + [2+\cos(t-\dot{\sigma})]u$$

Correspondingly

$$\ddot{\sigma} \in [-2,2]\,\|\vec{\sigma}_2\|_h^{1+3q} + [1,3]u,$$

$$\|\vec{\sigma}_2\|_h = |\sigma| + |\dot{\sigma}|^{\frac{1}{1+q}} + |\ddot{\sigma}|^{\frac{1}{1+2q}}\,.$$

$$u = -5\,\|(z_0, z_1, z_2)\|_h^{\frac{1}{2}+3q}\left[\,[\![z_2]\!]^{\frac{1}{2(1+2q)}} + 2[\![z_1]\!]^{\frac{1}{2(1+q)}} + [\![z_0]\!]^{\frac{1}{2}}\,\right]$$

Here z_i are the differentiator outputs, $q = 0.1, -0.1, -\frac{1}{3}$

Differentiator/observer parameters:

Integration step $\tau = 10^{-6}$

$$q = 0.1, n_d = 2, n_f = 3, L = 6 \cdot 10^{15}, \vec{\lambda}_5 = \{1, 2, 3, 5, 8.5, 900\}$$

$$q = -0.1,\ n_d = 2,\ n_f = 5,\ \tau = 10^{-6}$$

$$L = 10^{12}, \vec{\lambda}_7 = \{1.1, 1.4, 2.4, 5, 6, 12, 25, 35\}$$

$q = -\frac{1}{3}, n_d = 2, n_f = 5,$ SMC, Hybrid diff., $\tau = 10^{-5}$

$$L = 35\max(2 - 0.1t, 1) + 0.35\cos(1111t), M = 0.2,$$

$$\vec{\lambda}_7 = \{1.1, 1.5, 2, 3, 5, 7, 10, 12\}, \vec{\mu}_7 = \{2, 3, 4, 7, 9, 13, 19, 23\}$$

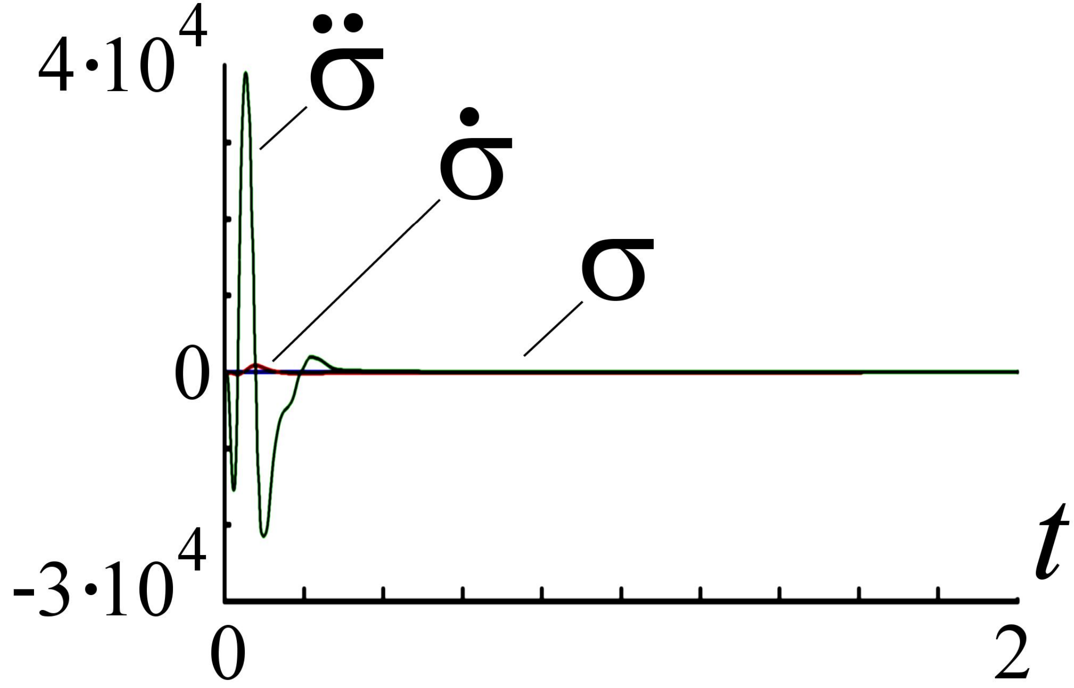
$q = 0.1$, no noise

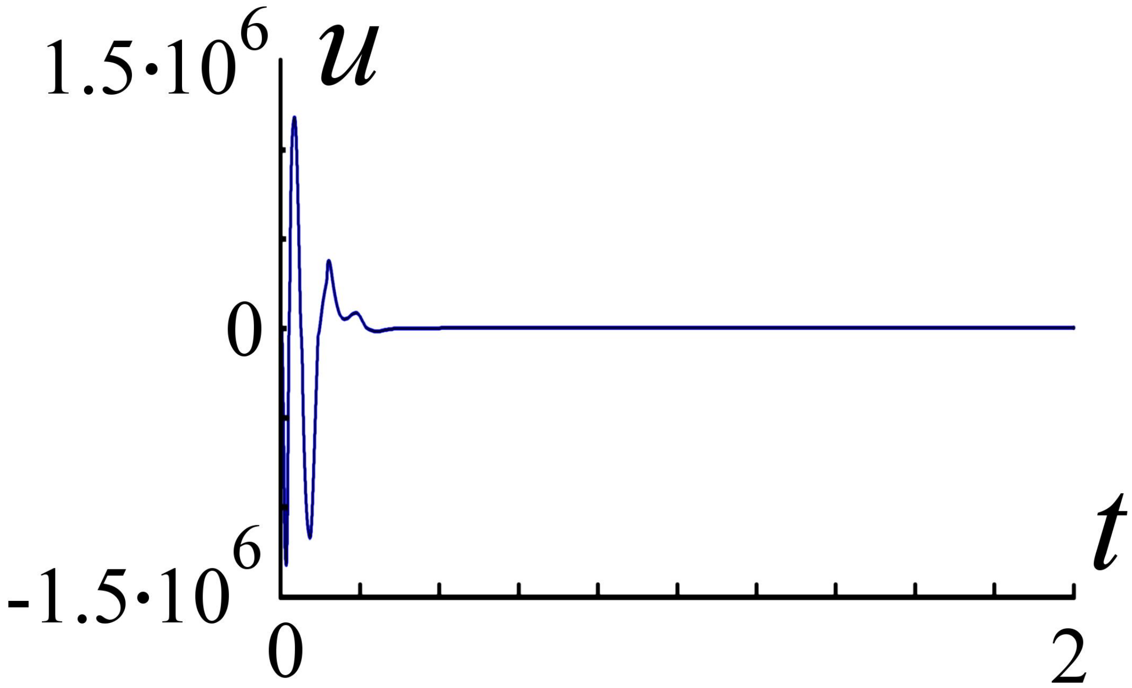
$\vec{\sigma}_2(0) = (10, -10, 10)$

For $t \geq 30$:

$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq$

$(3.5 \cdot 10^{-6}, 1.5 \cdot 10^{-5}, 8 \cdot 10^{-4})$





The noise, $q = \pm 0.1$

$$\vec{\sigma}_2(0) = (1000, -1000, 1000),\ \tau = 10^{-6}$$

$$\hat{\sigma} = \sigma + \eta,\ \eta(t) = \eta_1(t) + \eta_2(t) + \eta_3(t)$$

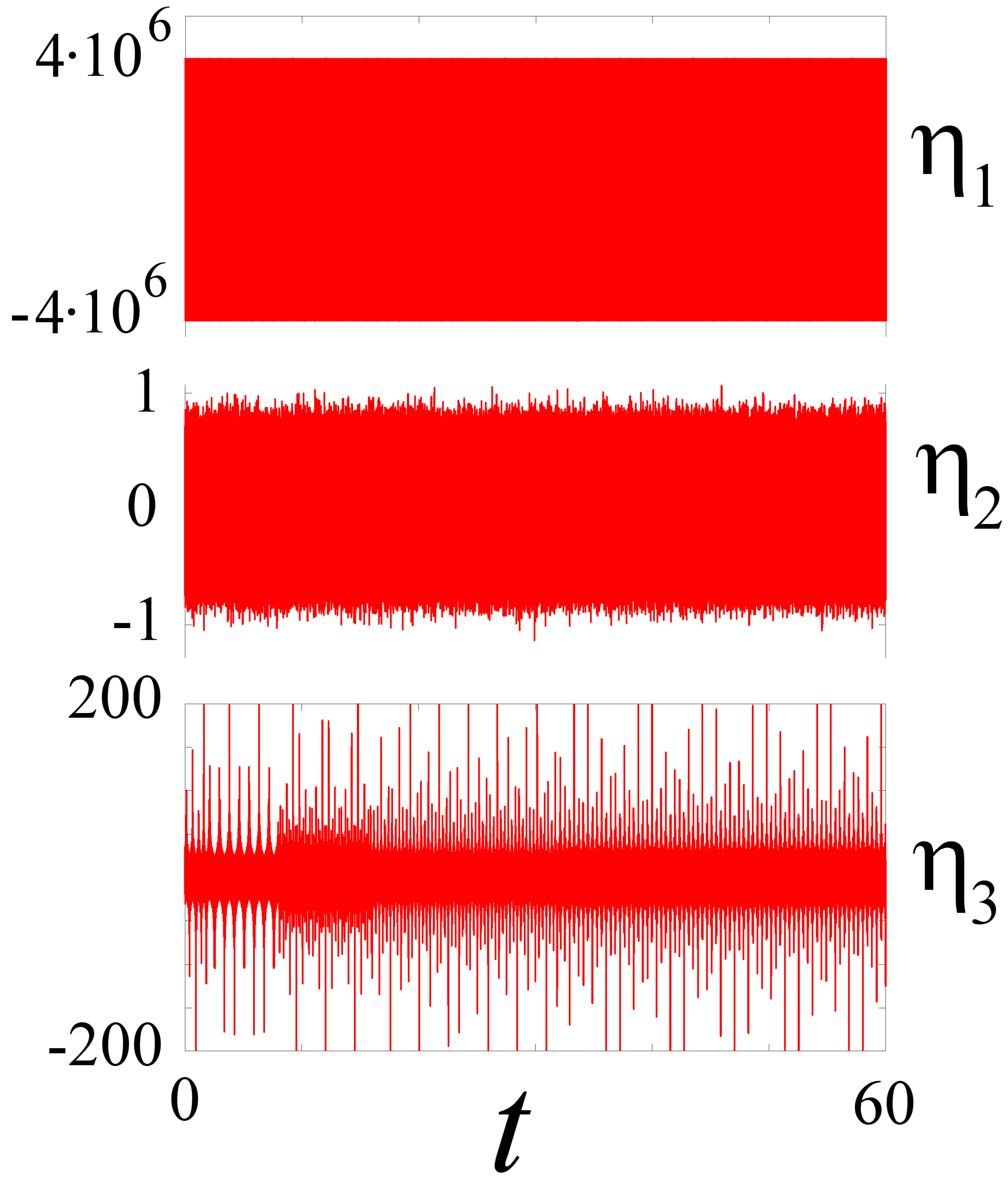
$$\eta_1 = 10^6 \cos(10^5 t) - 2 \cdot 10^6 \cos(5 \cdot 10^5 t) + 10^6 \sin(7 \cdot 10^5 t)$$

$$\eta_2 \in N(0, 0.2^2)$$

$$\eta_3 = 0.2 \frac{\sin(7 \cdot 10^5 t)}{|\cos(7 \cdot 10^5 t)|^{1/2}}$$

$q = \pm 0.1$

Noise

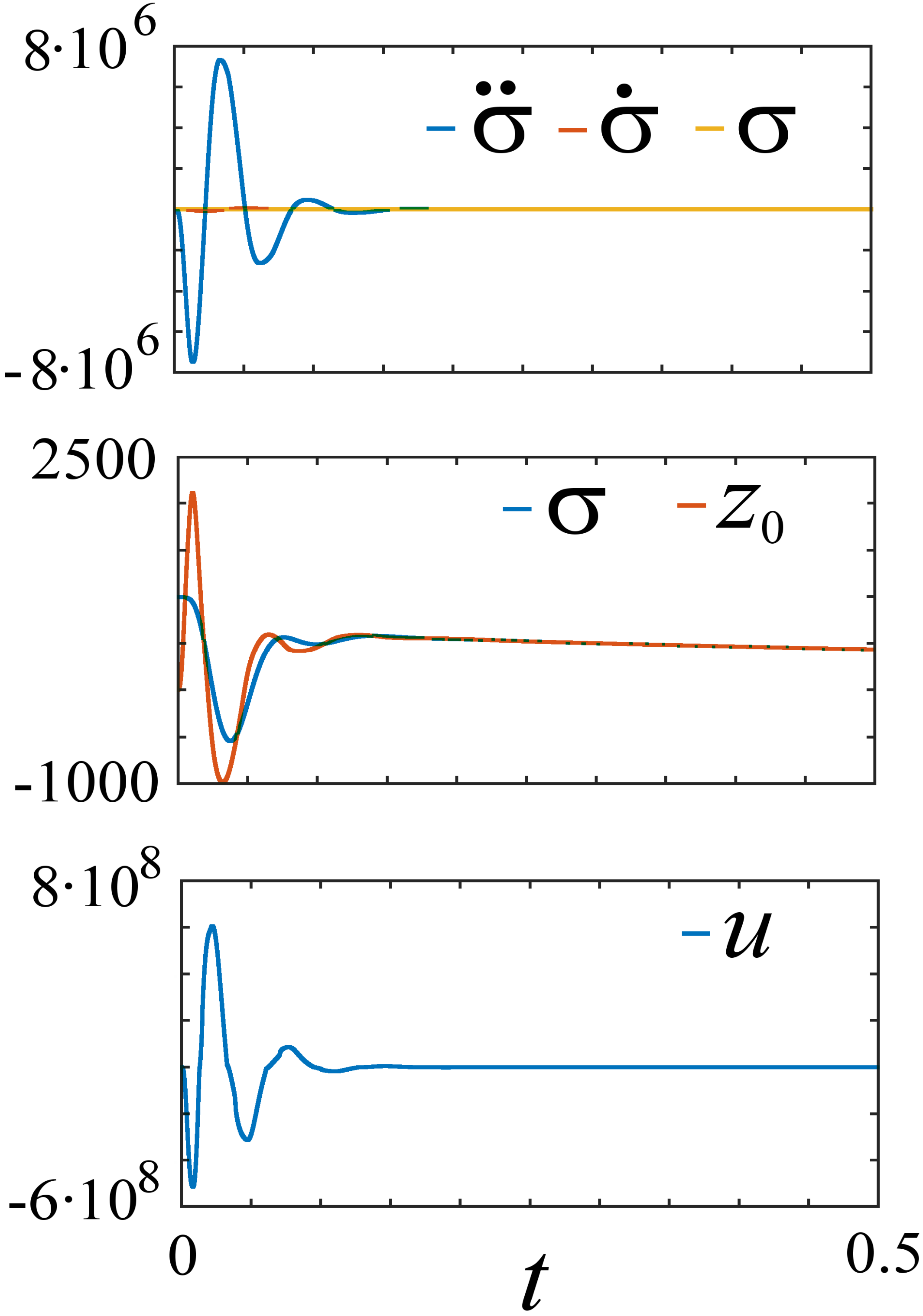


$$q = 0.1,$$

$$\vec{\sigma}_2(0) = (10^3, -10^3, 10^3)$$

For $t \geq 50$:

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.25, 3, 80)$$



q = -0.1,

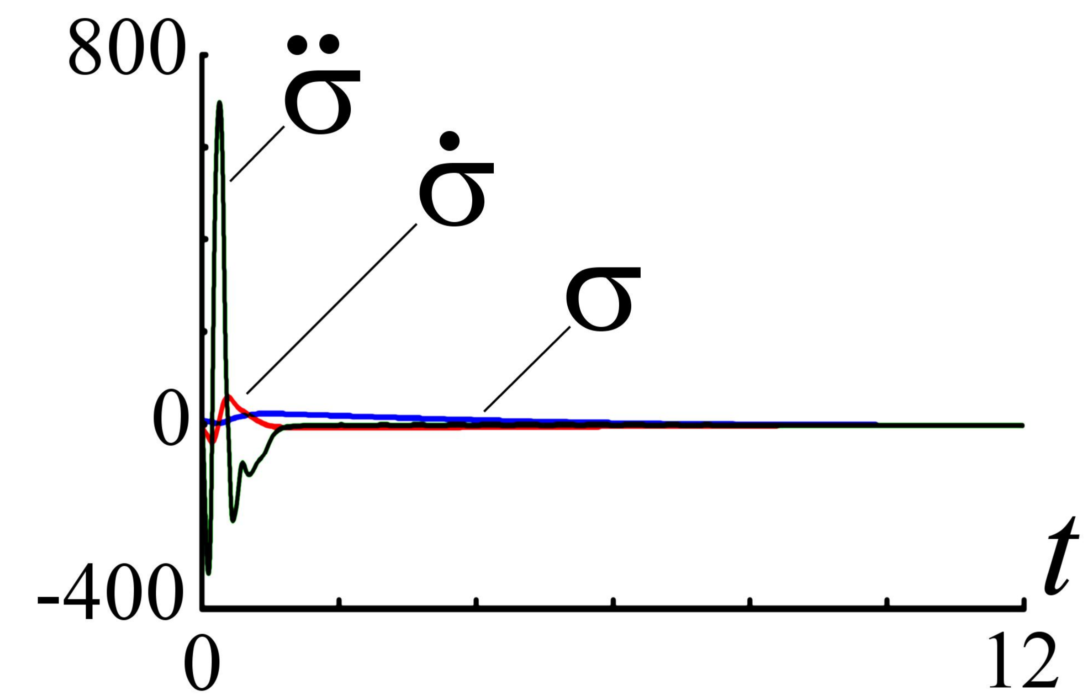
no noise

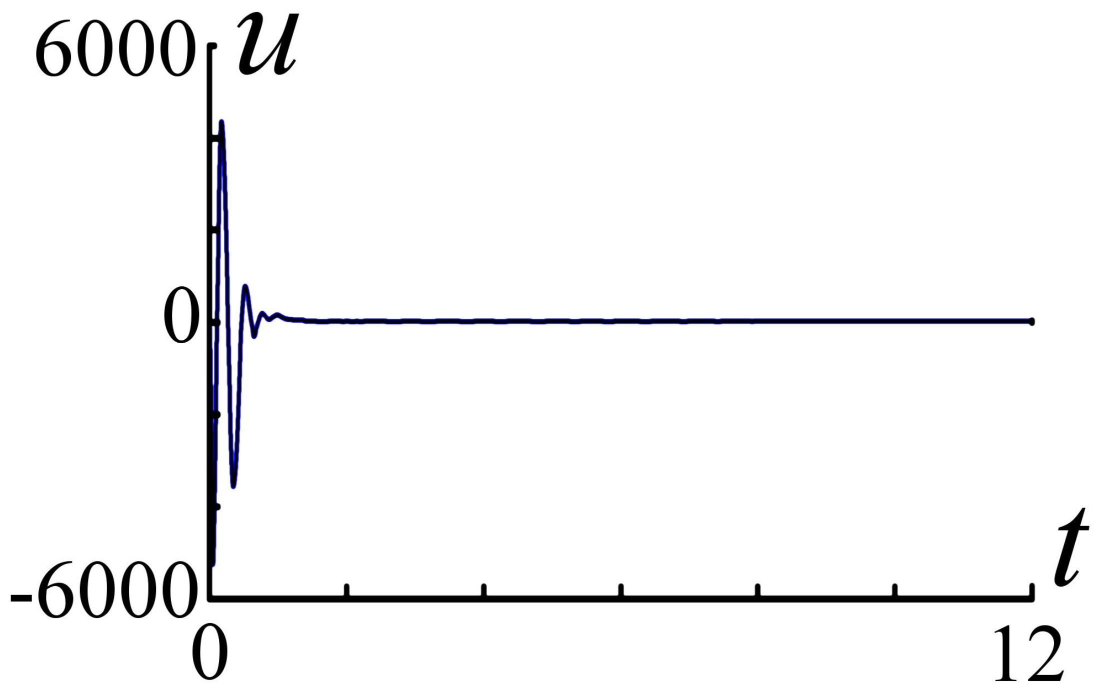
$\vec{\sigma}_2(0) = (10, -10, 10)$

For $t \geq 10$:

$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq$

$(3.5 \cdot 10^{-34}, 1.5 \cdot 10^{-30}, 8 \cdot 10^{-26})$



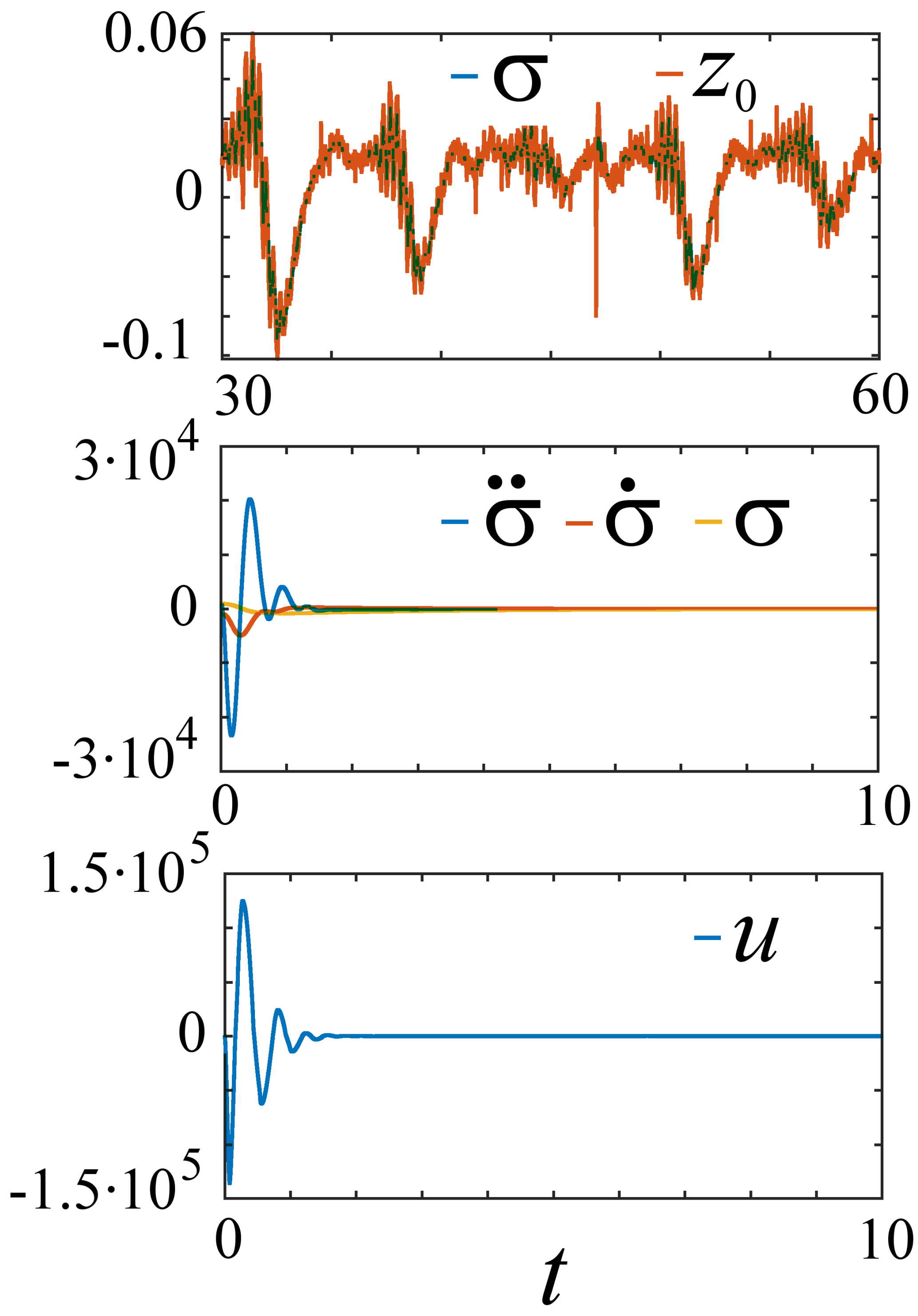


q =-0.1,

$\vec{\sigma}_2(0) = (10^3, -10^3, 10^3)$

For $t \geq 40$:

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.1, 0.4, 10)$$



$q = \text{-}1/3$, SMC,

no noise, $\tau = 10^{-5}$

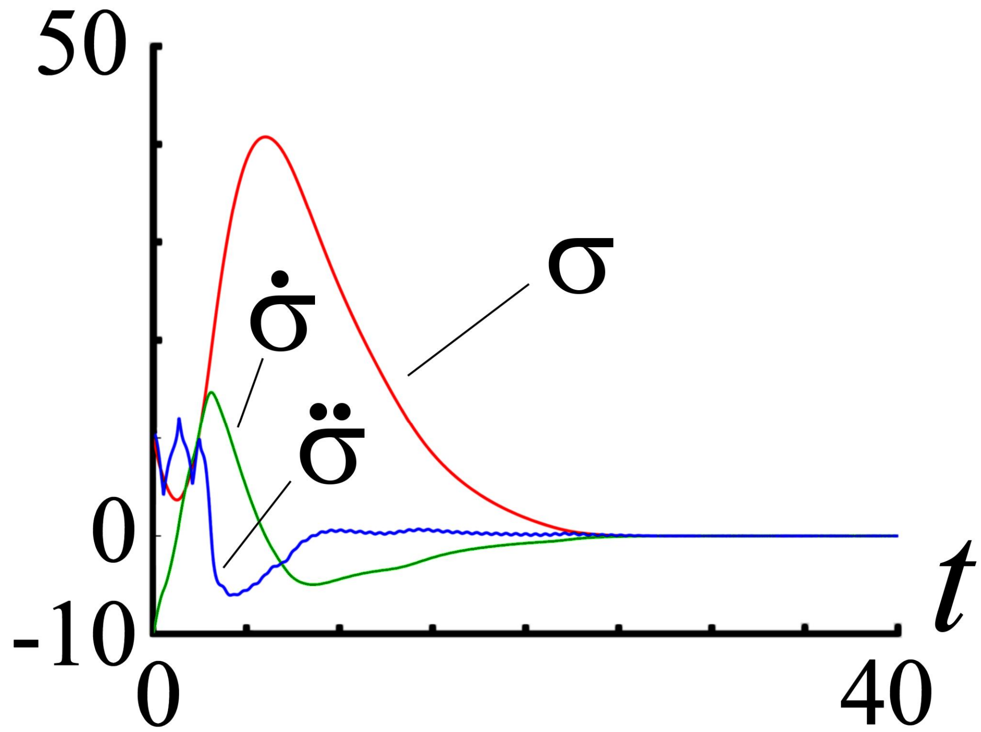
$z(0) = (-100, 100, 100)$,

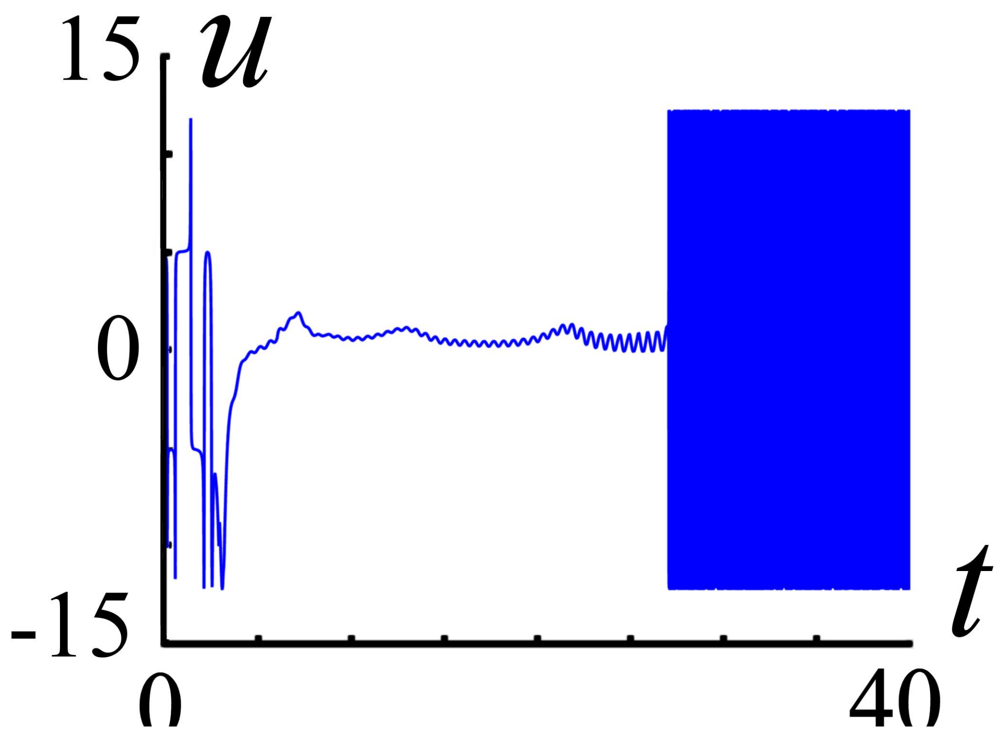
$\vec{\sigma}_2(0) = (10, -10, 10)$

For $t \geq 30$:

$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq$

$(4 \cdot 10^{-8}, 2.2 \cdot 10^{-5}, 2.7 \cdot 10^{-2})$





SMC: noise and differentiator

$$\hat{\sigma} = \sigma + \eta,\ \eta(t) = \eta_1(t) + \eta_2(t) + \eta_3(t),\ \ \tau = 10^{-5}$$

$$\eta_1 = 10^4 \cos(10^7 t)$$

$$\eta_2 \in N(0, 0.2^2)$$

$$\eta_3 = 0.1 \frac{\sin(2 \cdot 10^6 t)}{|\cos(2 \cdot 10^6 t)|^{1/2}}$$

$$L = 35 \max(2 - 0.1t, 1) + 0.35 \cos(1111t), M = 0.2$$

$$\vec{\sigma}_2(0) = (10, -10, 10),\ \ z(0) = (-100, 100, 100)$$

q = -1/3, SMC,

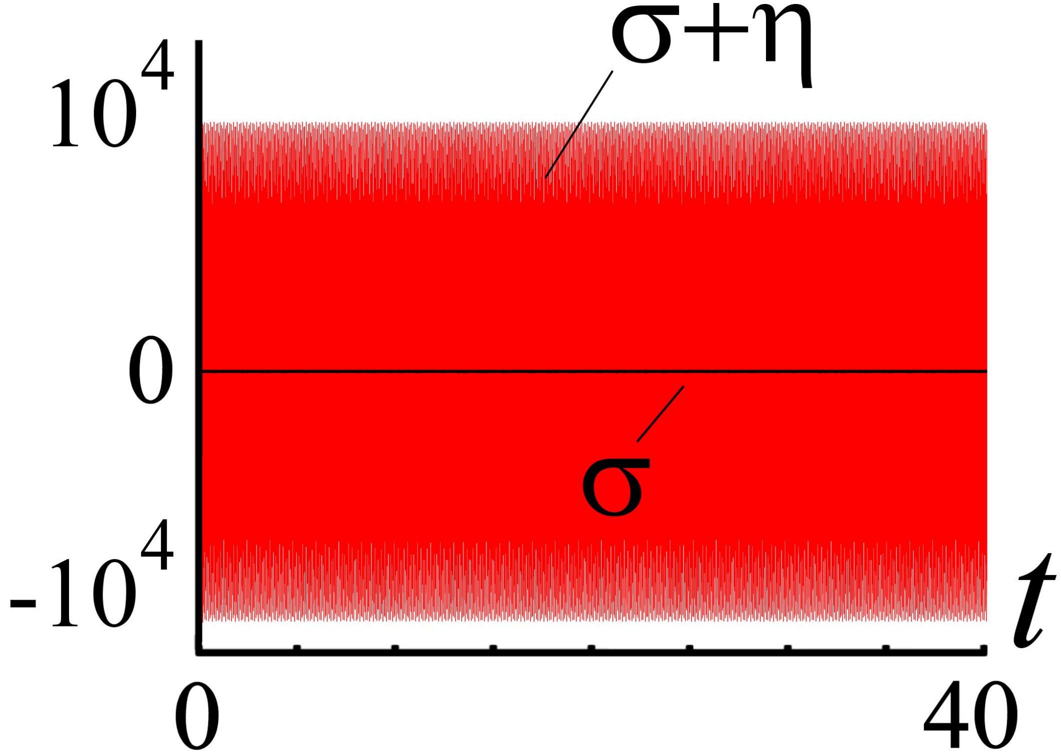
noise, $\tau = 10^{-5}$

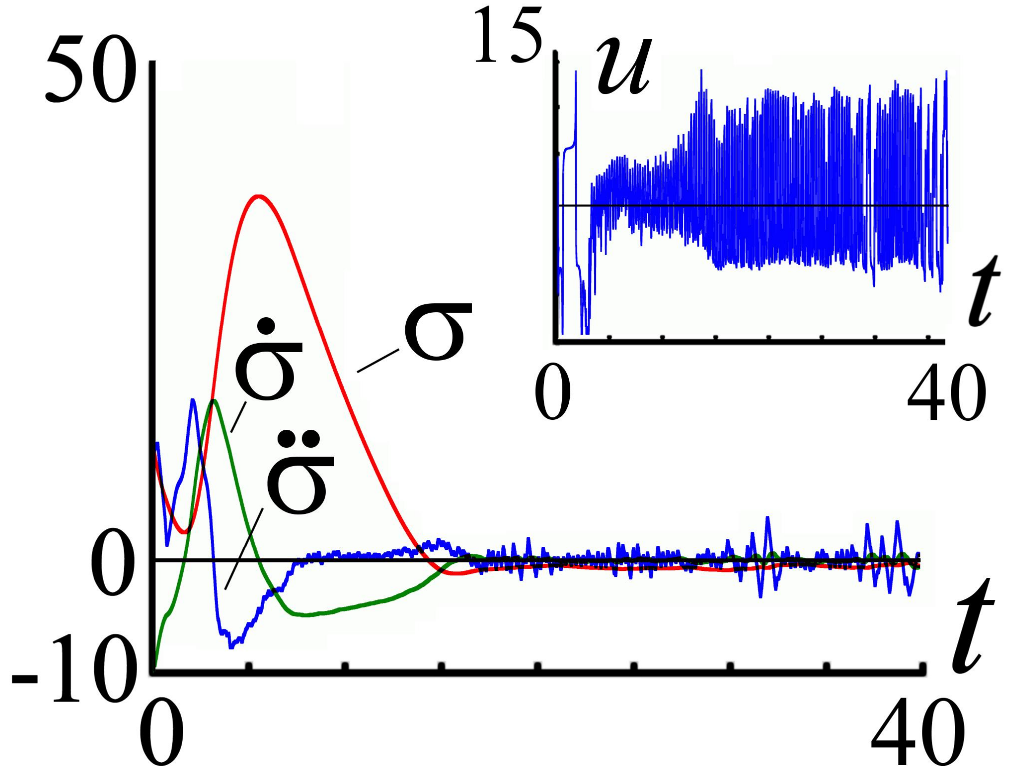
$z(0) = (-100, 100, 100)$,

$\vec{\sigma}_2(0) = (10, -10, 10)$

For $t \geq 30$:

$$(|\sigma|, |\dot{\sigma}|, |\ddot{\sigma}|) \leq (0.89, 0.4, 2.9)$$





Thank you very much!

Health to everybody

Appendices

Asymptotically optimal differentiation

$$f(t) = f_0(t) + \eta(t), \; |\eta| \le \varepsilon, \; |f_0^{(n_d+1)}(t)| \le L$$

The worst-case error is not better (Levant, Yu, 2017) than

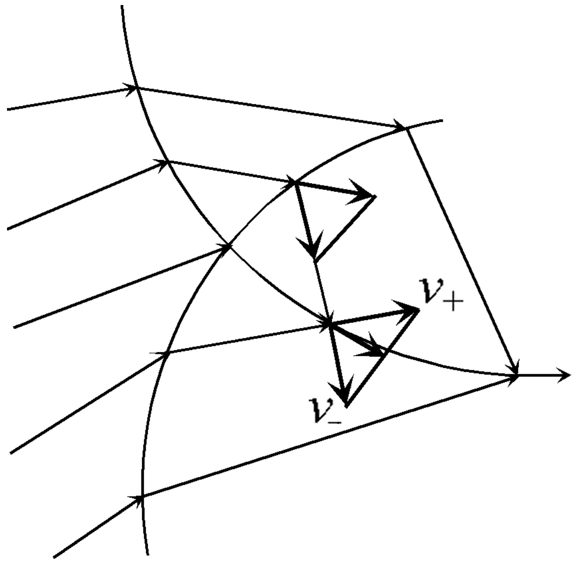
$$\inf_{f_0} \sup_t |z_i(t) - f_0^{(i)}(t)| \ge 2^{\frac{i}{n_d+1}} L^{\frac{i}{n_d+1}} \varepsilon^{\frac{n_d+1-i}{n_d+1}}.$$

(the Kolmogorov asymptotics)

A differentiator is called **asymptotically optimal**,

if for any $f(t) = f_0(t) + \eta(t), \; |\eta| \le \varepsilon,$

$$|z_i(t) - f_0^{(i)}(t)| \le \gamma_i L^{\frac{i}{n_d+1}} \varepsilon^{\frac{n_d+1-i}{n_d+1}} = \gamma_i L \left(\frac{\varepsilon}{L}\right)^{\frac{n_d+1-i}{n_d+1}}$$



Filippov Procedure

$$\dot{x} = f(x) \;\Leftrightarrow\; \dot{x} \in K_F[f](x)$$

$x(t)$ is an absolutely continuous function

$$K_F[f](x) = \bigcap_{\varepsilon>0}\bigcap_{\mu N=0} \text{convex\_closure}\, f(O_\varepsilon(x)\setminus N)$$

Non-autonomous case: $\dot{t} = 1$ is added.

$$\sigma^{(n)} \in [-C, C]\,\|\vec{\sigma}_{n-1}\|_h^{1+nq} + [K_m, K_M]\,u(\vec{\sigma}_{n-1})$$

is understood as

$$\sigma^{(n)} \in [-C, C]\,\|\vec{\sigma}_{n-1}\|_h^{1+nq} + [K_m, K_M]\,K_F[u](\vec{\sigma}_{n-1})$$