Filtering Differentiation

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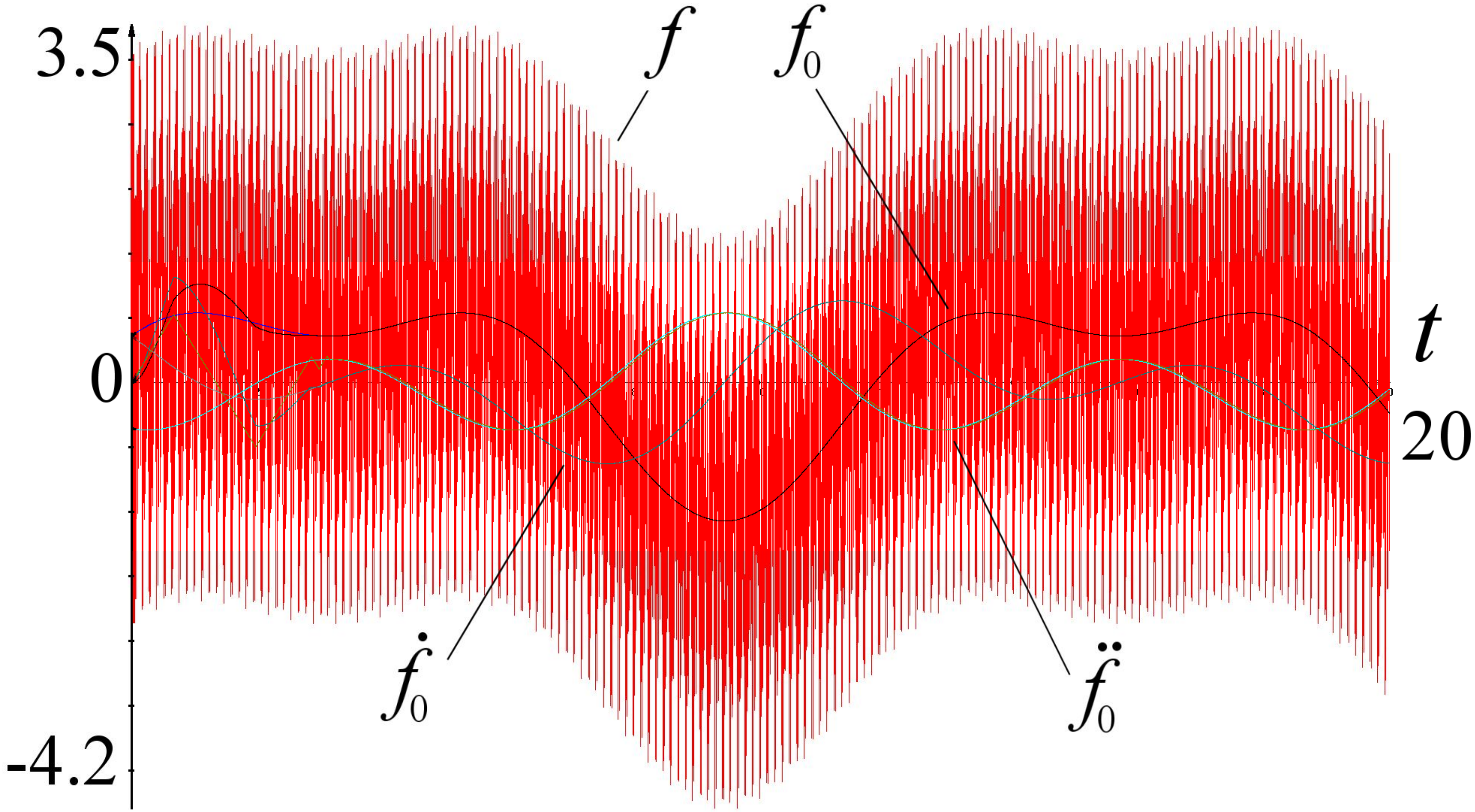
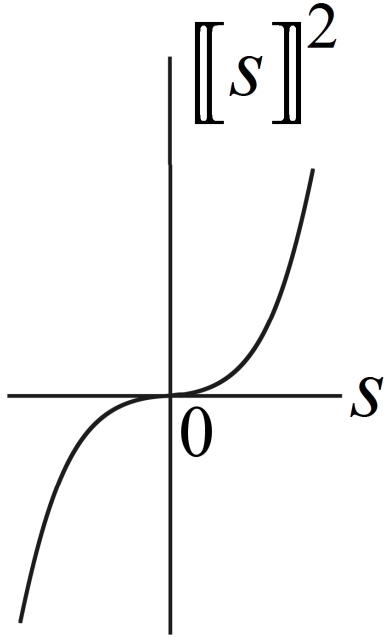


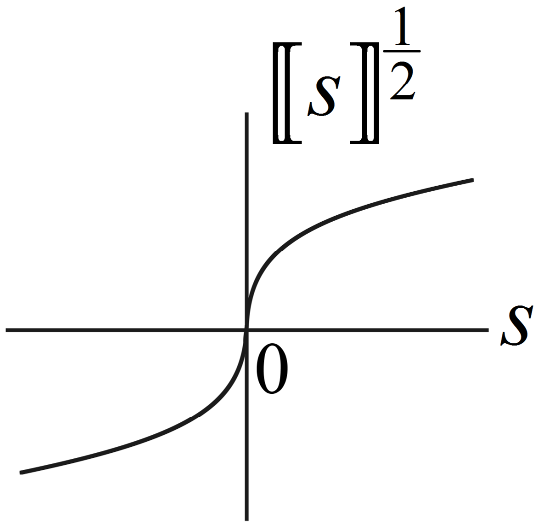
Special power functions

(standard notation)

$$\lfloor s \rceil^{\gamma} = [\![s]\!]^{\gamma} \triangleq |s|^{\gamma} \operatorname{sign} s$$

$$[\![s]\!]^{0} = \operatorname{sign} s$$





n_dth-order filtering differentiator of the filtering order n_f

(Levant, Yu 2018, Levant, Livne 2019)

$$
n_f \left[
\begin{aligned}
\dot{w}_1 &= -\lambda_{n_d+n_f} L^{\frac{1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+n_f}{n_d+n_f+1}} + w_1, \\
&\dots \\
\dot{w}_{n_f} &= -\lambda_{n_d+1} L^{\frac{n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d+1}{n_d+n_f+1}} + z_0 - f(t),
\end{aligned}
\right.
$$

$$
n_d \left[
\begin{aligned}
\dot{z}_0 &= -\lambda_{n_d} L^{\frac{n_f+1}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{n_d}{n_d+n_f+1}} + z_1, \\
&\dots \\
\dot{z}_{n_d-1} &= -\lambda_1 L^{\frac{n_d+n_f}{n_d+n_f+1}} \lfloor w_1 \rceil^{\frac{1}{n_d+n_f+1}} + z_{n_d}, \\
\dot{z}_{n_d} &= -\lambda_0 L \,\mathrm{sign}\, w_1.
\end{aligned}
\right.
$$

The parameters can be taken from the table

n	λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	λ_9	λ_{10}	λ_{11}	λ_{12}
0	1.1												
1	1.1	1.5											
2	1.1	2.12	2										
3	1.1	3.06	4.16	3									
4	1.1	4.57	9.30	10.03	5								
5	1.1	6.75	20.26	32.24	23.72	7							
6	1.1	9.91	43.65	101.96	110.08	47.69	10						
7	1.1	14.13	88.78	295.74	455.40	281.37	84.14	12					
8	1.1	19.66	171.73	795.63	1703.9	1464.2	608.99	120.79	14				
9	1.1	26.93	322.31	2045.8	6002.3	7066.2	4026.3	1094.1	173.72	17			
10	1.1	36.34	586.78	5025.4	19895	31601	24296	8908	1908.5	251.99	20		
11	1.1	48.86	1061.1	12220	65053	138954	143658	70830	20406	3623.1	386.7	26	
12	1.1	65.22	1890.6	29064	206531	588869	812652	534837	205679	48747	6944.8	623.30	32

which is enough for $n = n_d + n_f \leq 12.$

Denote the above differentiator as

$$\dot{w} = \Omega_{n_d, n_f}(w, z_0 - f, L), \; \dot{z} = D_{n_d, n_f}(w_1, z, L)$$

Examples

$n_d = 0,\ n_f = 0$: $\dot{z}_0 = -1.1L\ \mathrm{sign}(z_0 - f(t)),\ |\dot{f}_0| \le L.$

$n_d = 1,\ n_f = 0$:

$$\dot{z}_0 = -1.5L^{\frac{1}{2}}\ [\![z_0 - f(t)]\!]^{\frac{1}{2}} + z_1,$$
$$\dot{z}_1 = -1.1L\ \ \mathrm{sign}\,(z_0 - f(t)),\ \ |\ddot{f}_0| \le L,$$

$n_d = 0,\ n_f = 2$:

$$\dot{w}_1 = -2L^{\frac{1}{3}}\ [\![w_1]\!]^{\frac{2}{3}} + w_2,$$
$$\dot{w}_2 = -2.12L^{\frac{2}{3}}\ \ [\![w_1]\!]^{\frac{1}{3}} + z_0 - f(t),$$
$$\dot{z}_0 = -1.1L\ \ \mathrm{sign}\, w_1,\ |\dot{f}_0| \le L,$$

$$n_d = 2,\ n_f = 2$$

$$\dot{w}_1 = -5L^{\frac{1}{5}}\ [\![w_1]\!]^{\frac{4}{5}} + w_2,$$

$$\dot{w}_2 = -10.03L^{\frac{2}{5}}\ [\![w_1]\!]^{\frac{3}{5}} + z_0 - f(t),$$

$$\dot{z}_0 = -9.30L^{\frac{3}{5}}\ [\![w_1]\!]^{\frac{2}{5}} + z_1,$$

$$\dot{z}_1 = -4.57L^{\frac{4}{5}}\ [\![w_1]\!]^{\frac{1}{5}} + z_2,$$

$$\dot{z}_2 = -1.1L\ \ \operatorname{sign} w_1,\ |\dddot{f}_0| \le L.$$

Filtering Differentiator Accuracy

$$f(t) = f_0(t) + \eta(t), \ \ f_0 \in \mathrm{Lip}_{\mathbb{R}_+}(n, L), \ \ |f_0^{(n_d+1)}(t)| \le L$$

$$\eta(t) = \eta_0(t) + \eta_1(t) + \ldots + \eta_{n_f}(t)$$

$$\xi_k^{(k)}(t) = \eta_k(t), |\xi_k| \le \varepsilon_k, \ k = 0, 1, \ldots n_f$$

Theorem.

$$|z_i(t) - f_0^{(i)}(t)| \le \mu_i L \rho^{n_d+1-i}, \ i = 0, 1, \ldots, n_d,$$

$$\rho = \max\left[\left(\frac{\varepsilon_0}{L}\right)^{\frac{1}{n_d+1}}, \left(\frac{\varepsilon_1}{L}\right)^{\frac{1}{n_d+2}}, \ldots, \left(\frac{\varepsilon_{n_f}}{L}\right)^{\frac{1}{n_d+n_f+1}}\right]$$

$$|w_1(t)| \le \mu_{w1} L \rho^{n_d+n_f+1}$$

Discrete sampling+proper discretization

Let τ_j be the sampling step. Then the discretization is

$$w(t_{j+1}) = w(t_j) + \Omega_{n_d,n_f}(w(t_j), z_0(t_j) - f(t_j), L)\tau_j,$$

$$z(t_{j+1}) = z(t_j) + D_{n_d,n_f}(w_1(t_j), z(t_j), L)\tau_j + T_{n_d}(z(t_j), \tau_j),$$

where

$$T_{n_d,0} = \tfrac{1}{2!} z_2(t_j)\tau_j^2 + \ldots + \tfrac{1}{n!} z_n(t_j)\tau_j^n,$$

$$T_{n_d,1} = \tfrac{1}{2!} z_3(t_j)\tau_j^2 + \ldots + \frac{1}{(n-1)!} z_n(t_j)\tau_j^{n-1},$$

...

$$T_{n_d,n_d-2} = \tfrac{1}{2!} z_n(t_j)\tau_j^2,$$

$$T_{n_d,n_d-1} = 0,\ T_{n_d,n_d} = 0.$$

Notation

The increment operator:

$$\delta_j f = f(t_{j+1}) - f(t_j)$$

And once more:

$$[\![s]\!]^{\gamma} = | s |^{\gamma} \operatorname{sign} s$$

$$[\![s]\!]^{0} = \operatorname{sign} s$$

Discrete differentiator: $n_d = 5,\ n_f = 2$

$$\delta_j w_1 = [-12\, L^{\frac{1}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{7}{8}} + w_2(t_j)]\tau_j,$$

$$\delta_j w_2 = [-84.14\, L^{\frac{2}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{6}{8}} + z_0(t_j) - f(t_j)]\tau_j,$$

$$\delta_j z_0 = [-281.37\, L^{\frac{3}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{5}{8}} + z_1(t_j)]\tau_j + z_2(t_j)\frac{\tau_j^2}{2} + z_3(t_j)\frac{\tau_j^3}{6} + z_4(t_j)\frac{\tau_j^4}{24} + z_5(t_j)\frac{\tau_j^5}{120},$$

$$\delta_j z_1 = [-455.40\, L^{\frac{4}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{4}{8}} z_2(t_j)]\tau_j + z_3(t_j)\frac{\tau_j^2}{2} + z_4(t_j)\frac{\tau_j^3}{6} + z_5(t_j)\frac{\tau_j^4}{24},$$

$$\delta_j z_2 = [-295.74\, L^{\frac{5}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{3}{8}} + z_3(t_j)]\tau_j + z_4(t_j)\frac{\tau_j^2}{2} + z_5(t_j)\frac{\tau_j^3}{6},$$

$$\delta_j z_3 = [-88.78\, L^{\frac{6}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{2}{8}} + z_4(t_j)]\tau_j + z_5(t_j)\frac{\tau_j^2}{2},$$

$$\delta_j z_4 = [-14.13\, L^{\frac{7}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{1}{8}} + z_5(t_j)]\tau_j,$$

$$\delta_j z_5 = -1.1\, L\ \operatorname{sign}(w_1(t_j))\ \tau_j$$

Discrete differentiator: $n_d = 2,\ n_f = 5$

$$\delta_j w_1 = [-12\, L^{\frac{1}{8}} [\![w_1(t_j)]\!]^{\frac{7}{8}} + w_2(t_j)]\tau_j,$$

$$\delta_j w_2 = [-84.14\, L^{\frac{2}{8}} [\![w_1(t_j)]\!]^{\frac{6}{8}} + w_3(t_j)]\tau_j,$$

$$\delta_j w_3 = [-281.37\, L^{\frac{3}{8}} [\![w_1(t_j)]\!]^{\frac{5}{8}} + w_4(t_j)]\tau_j,$$

$$\delta_j w_4 = [-455.40\, L^{\frac{4}{8}} [\![w_1(t_j)]\!]^{\frac{4}{8}} + w_5(t_j)]\tau_j,$$

$$\delta_j w_5 = [-295.74\, L^{\frac{5}{8}} [\![w_1(t_j)]\!]^{\frac{3}{8}} + z_0(t_j) - f(t_j)]\tau_j,$$

$$\delta_j z_0 = [-88.78\, L^{\frac{6}{8}} [\![w_1(t_j)]\!]^{\frac{2}{8}} + z_1(t_j)]\tau_j + z_2(t_j)\frac{\tau_j^2}{2},$$

$$\delta_j z_1 = [-14.13\, L^{\frac{7}{8}} [\![w_1(t_j)]\!]^{\frac{1}{8}} + z_2(t_j)]\tau_j,$$

$$\delta_j z_2 = -1.1\, L\, \mathrm{sign}(w_1(t_j))\ \tau_j$$

Pure discrete filter: $n_d = 0,\ n_f = 7$

$$\delta_j w_1 = [-12\, L^{\frac{1}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{7}{8}} + w_2(t_j)]\tau_j,$$

$$\delta_j w_2 = [-84.14\, L^{\frac{2}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{6}{8}} + w_3(t_j)]\tau_j,$$

$$\delta_j w_3 = [-281.37\, L^{\frac{3}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{5}{8}} + w_4(t_j)]\tau_j,$$

$$\delta_j w_4 = [-455.40\, L^{\frac{4}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{4}{8}} + w_5(t_j)]\tau_j,$$

$$\delta_j w_5 = [-295.74\, L^{\frac{5}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{3}{8}} + w_6(t_j)]\tau_j,$$

$$\delta_j w_6 = [-88.78\, L^{\frac{6}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{2}{8}} + w_7(t_j)]\tau_j,$$

$$\delta_j w_7 = [-14.13\, L^{\frac{7}{8}} \left[\!\left[w_1(t_j) \right]\!\right]^{\frac{1}{8}} + z_0(t_j) - f(t_j)]\tau_j,$$

$$\delta_j z_0 = -1.1\, L\, \mathrm{sign}(w_1(t_j))\ \tau_j$$

SIMULATION

Numeric differentiation is difficult

Input:

$$f(t) = 0.8\cos t - \sin(0.2t) + \nu(t)$$

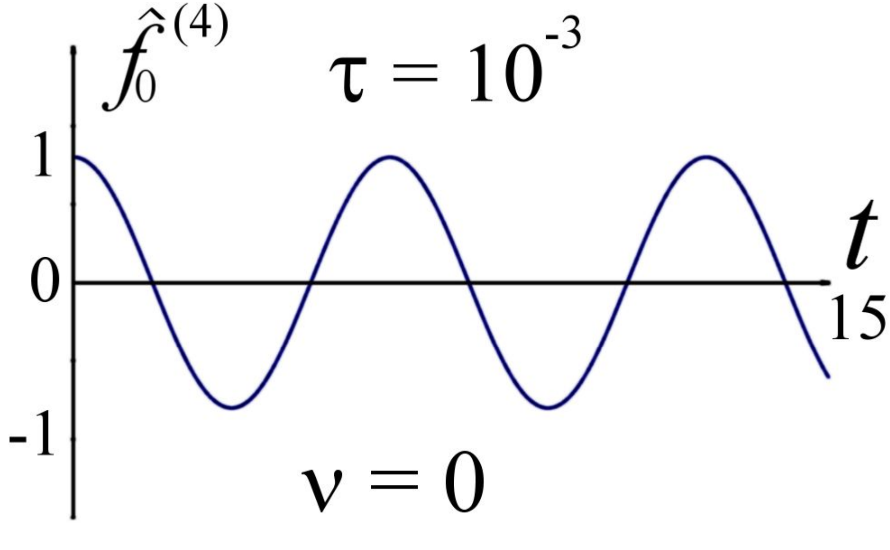
HGO:

high gain observer by Khalil (1990s),

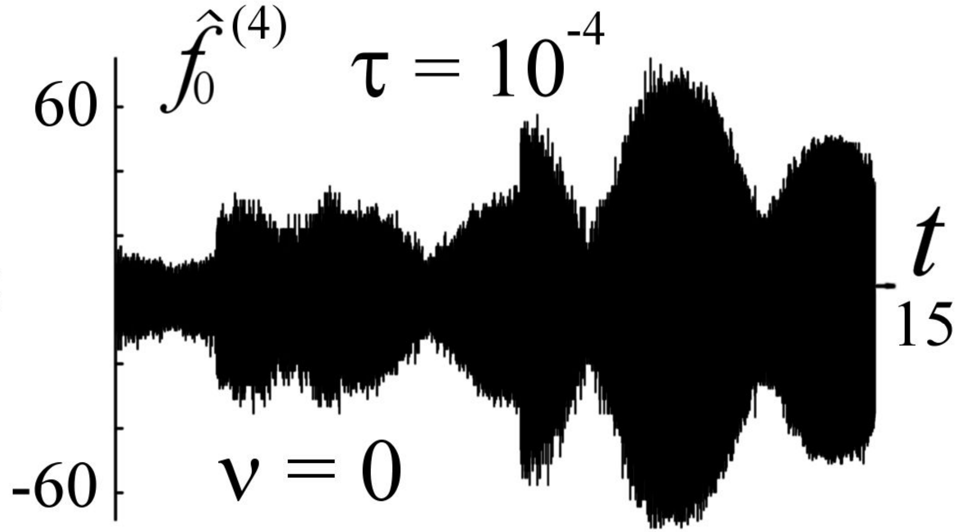
multiple eigenvalues -1000, $\tau = 10^{-6}$

Divided differences:

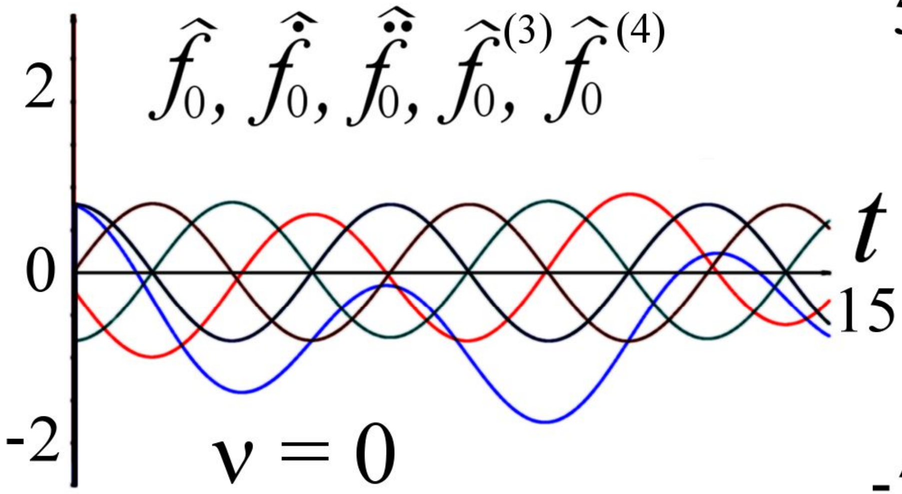
the standard MatLab differentiator



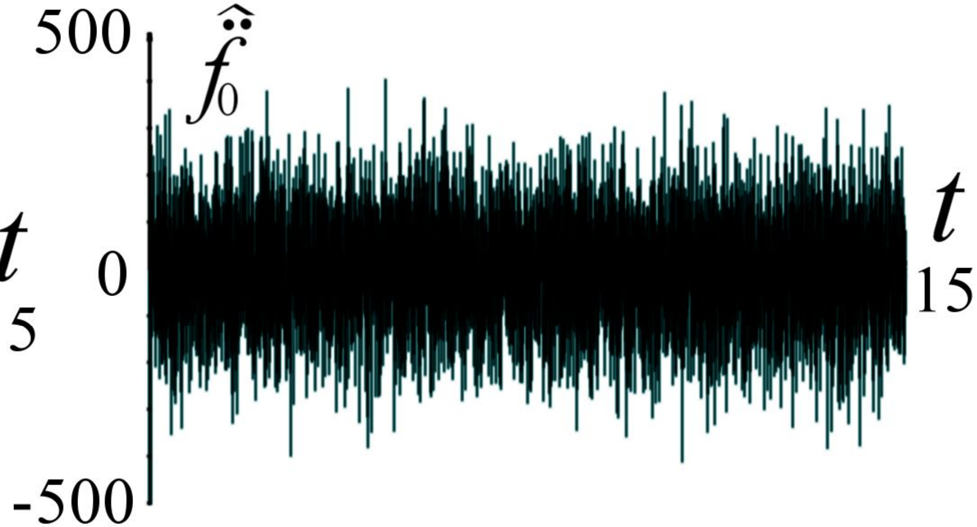
a. Divided differences



b. Divided differences



c. HGO: no noise



d. HGO: $\nu \in N(0, 0.001^2)$

Input and large noise

$$f_0(t) = 0.6\cos(t) - 0.9\sin(0.7t),$$

$$f(t) = f_0(t) + \eta_1(t) + \eta_2(t) + \eta_3(t),$$

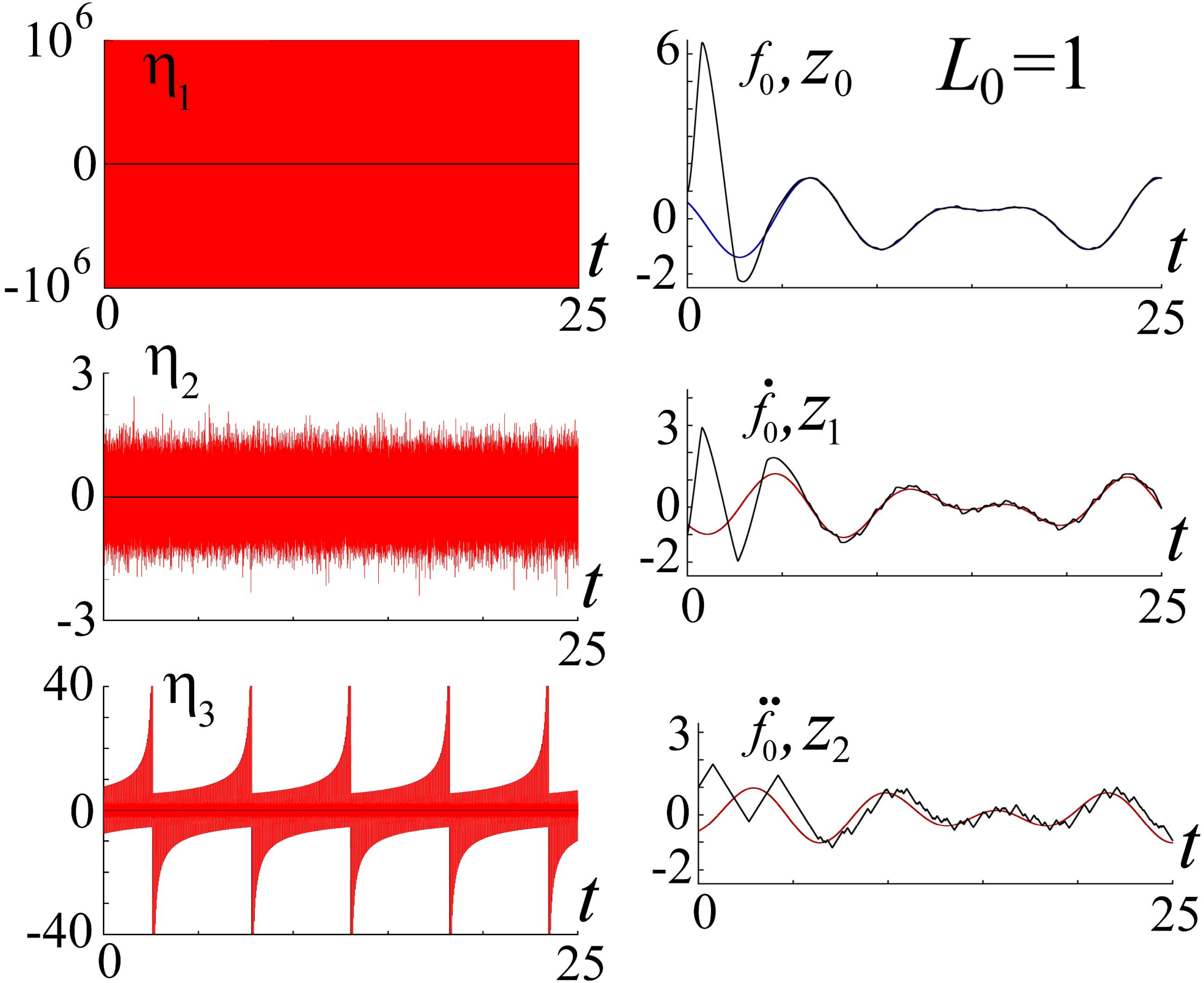
$$\eta_1(t) = 10^6\cos(10^8 t),\ \eta_2(t) \in N(0, 0.5^2),$$

$$\eta_3(t) = -5\cdot 10^{-5}\,\tfrac{d}{dt}\left[\!\left[\cos(20000t)\right]\!\right]^{\frac{1}{2}}$$

$$= 0.5\sin(20000t)\left|\cos(20000t)\right|^{-\frac{1}{2}},$$

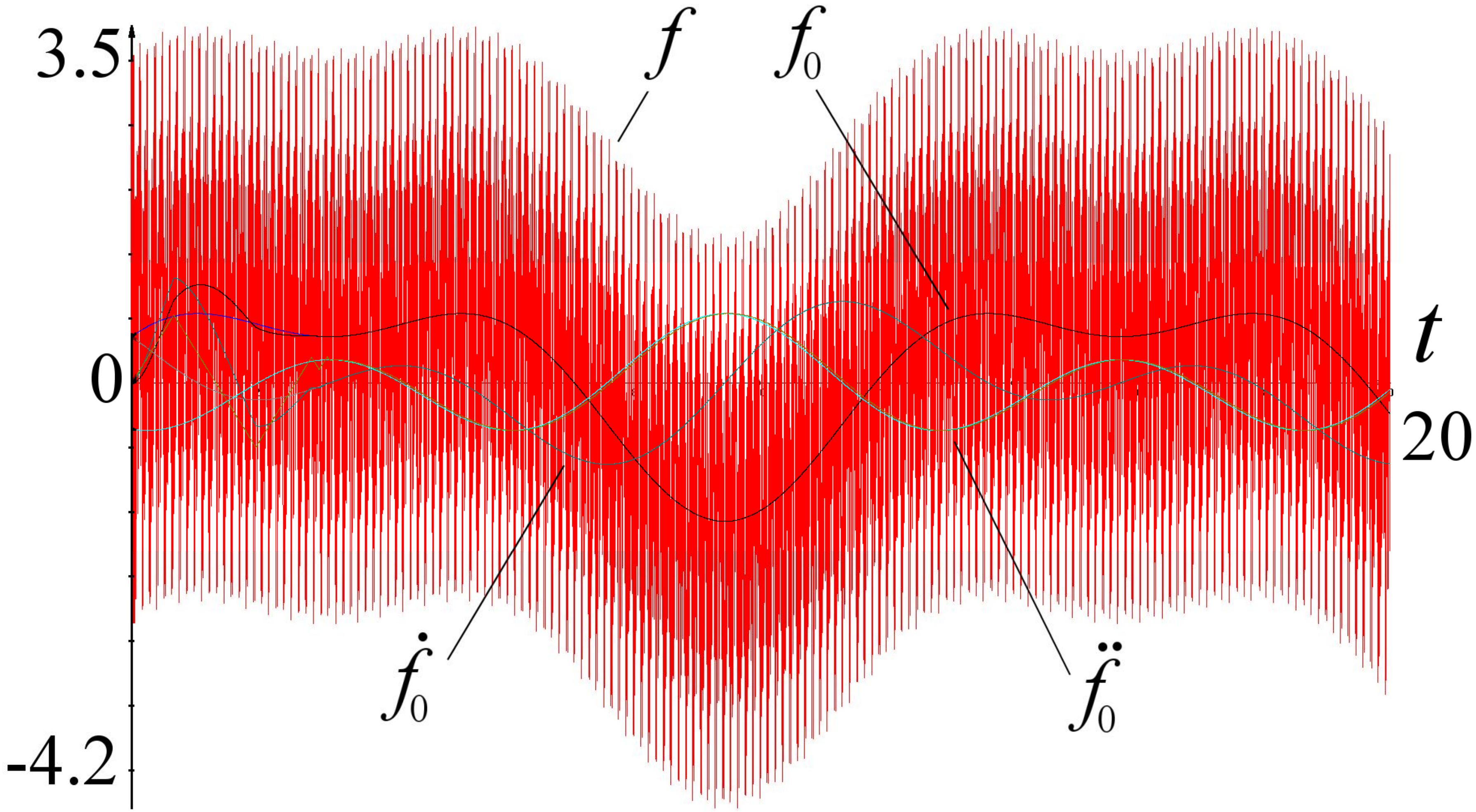
$$\tau = 10^{-5},\quad L = 1,\ n_d = 2,\ n_f = 3$$

Differentiating $f_0 + \eta_1 + \eta_2 + \eta_3, \tau = 10^{-5}$



Filtering differentiation: $n_d = 2, n_f = 5, \tau = 10^{-4}$

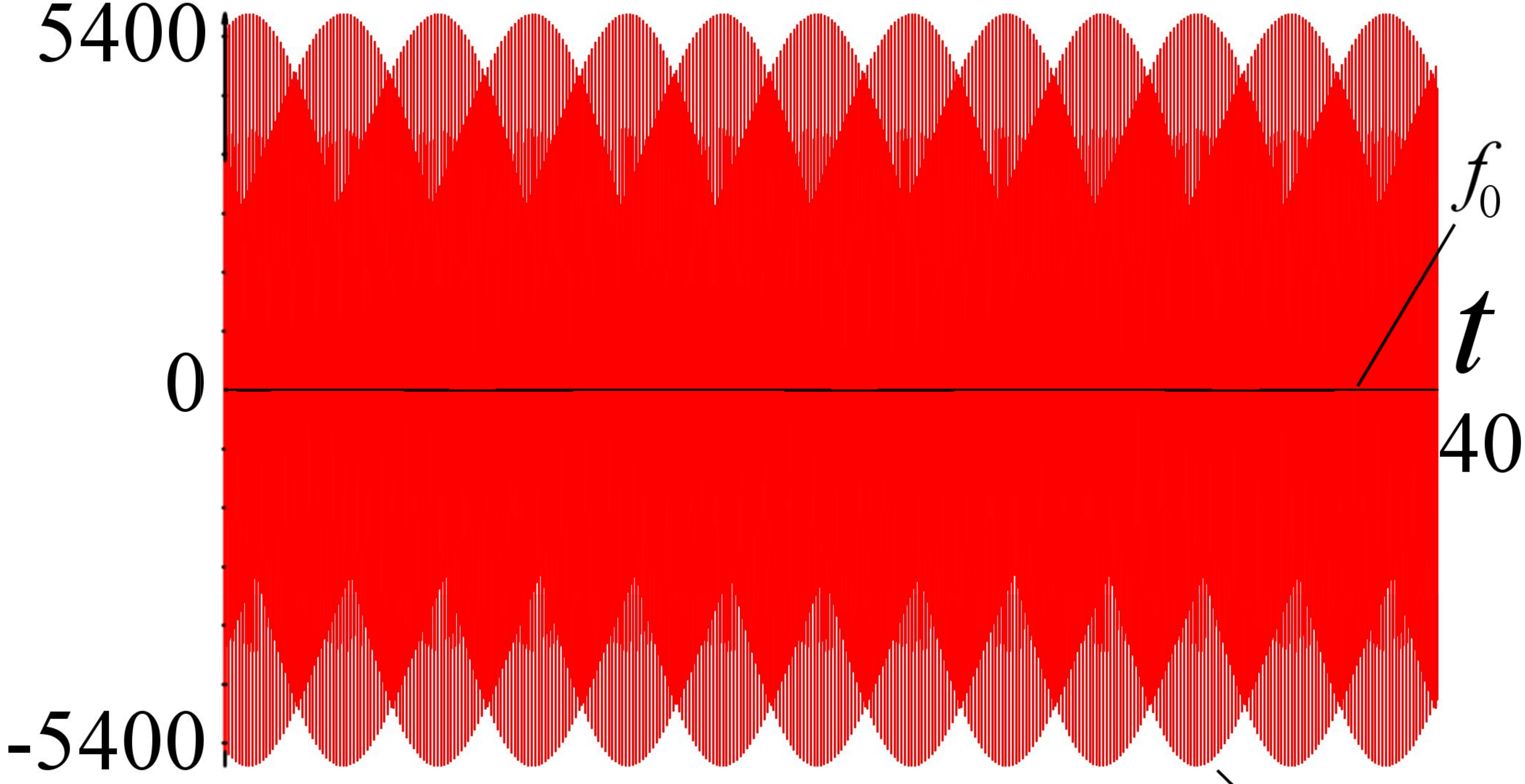
$$\eta(t) = \cos(10000t) - 0.5\sin(20000t) + 2\cos(70000t)$$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (1.5 \cdot 10^{-5}, 0.001, 0.034)$$

Filtering differentiation: $n_d = 2, n_f = 5, \tau = 10^{-5}$

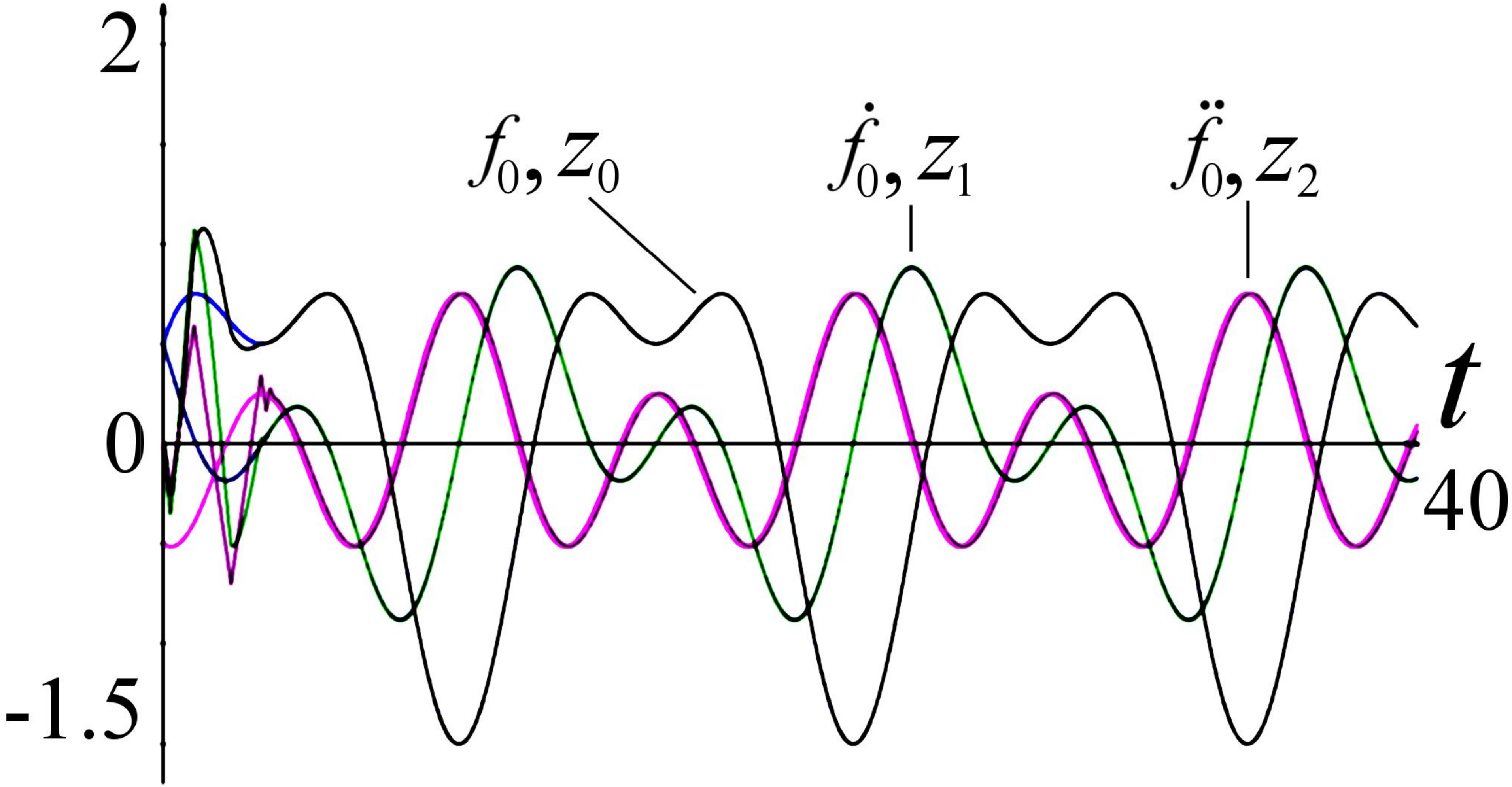
$\eta(t) = 1500\cos(10000t) + 3000\sin(20000t) + 2000\cos(70000t).$



The input

Filtering differentiation: $n_d = 2, n_f = 5, \tau = 10^{-5}$

$$\eta(t) = 1500\cos(10000t) + 3000\sin(20000t) + 2000\cos(70000t).$$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|) \le (9.0 \cdot 10^{-5}, 3.3 \cdot 10^{-3}, 0.063)$$

Explanation

Let $\eta = \gamma \cos(\omega t),\ \omega >> 1$, then

$$| z_i(t) - f_0^{(i)}(t) | \le \mu_i L \rho^{n_d + 1 - i},$$

$$\rho = \left(\frac{\gamma}{L}\right)^{\frac{1}{n_d + n_f + 1}} \omega^{-\frac{n_f}{n_d + n_f + 1}}$$

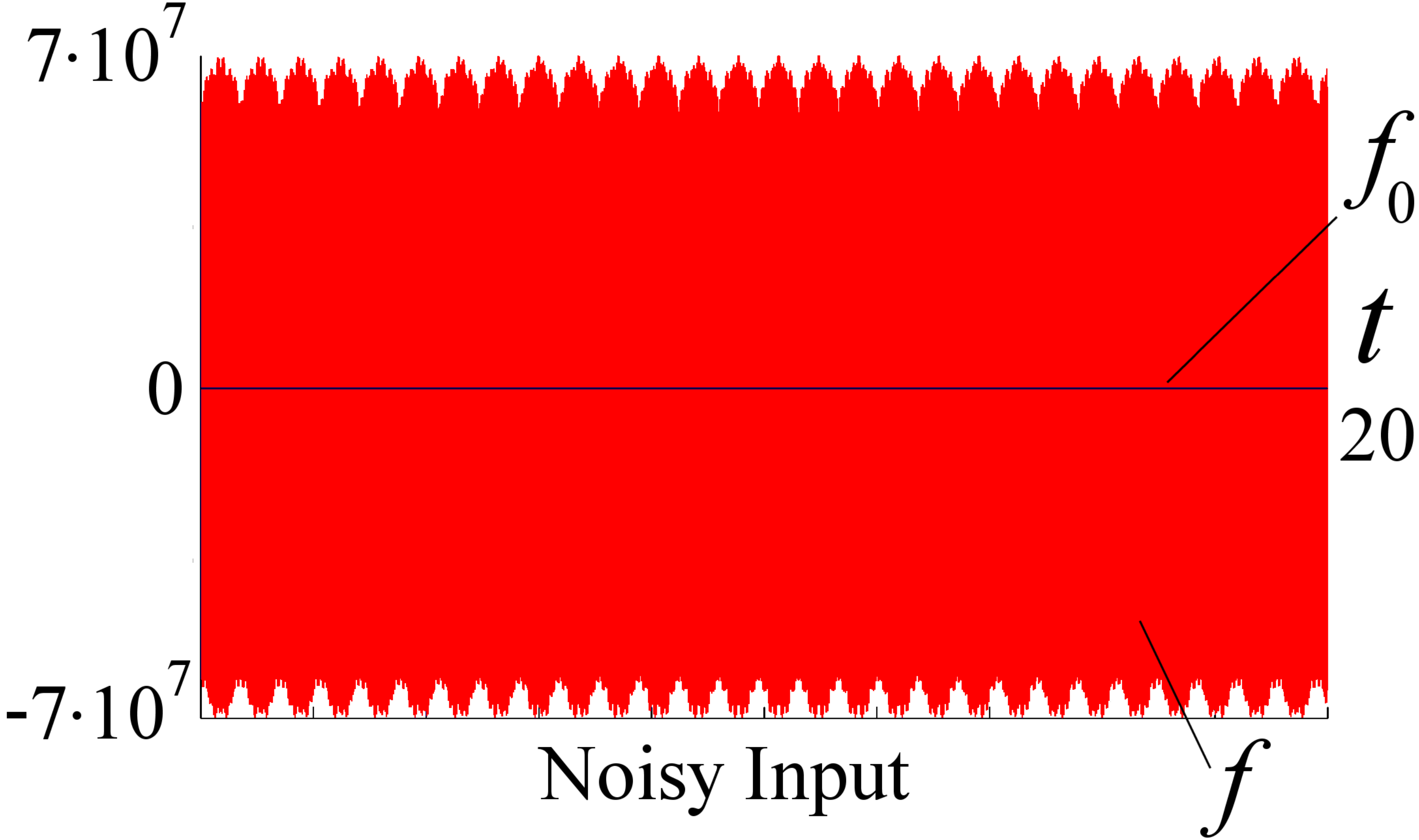
take $n_d = 3,\ n_f = 7,\ \gamma = 10^7,\quad \omega = 10^4,\quad L = 1$

then $\rho \approx 0.005$

+ digital error and discrete sampling

Filtering differentiation: $n_d = 3, n_f = 7, \tau = 10^{-5}$

$$\eta(t) = 10^7 \cos(10^4 t) - 2 \cdot 10^7 \cos(1.7 \cdot 10^5 t) + 5 \cdot 10^7 \cos(1.33 \cdot 10^5 t)$$

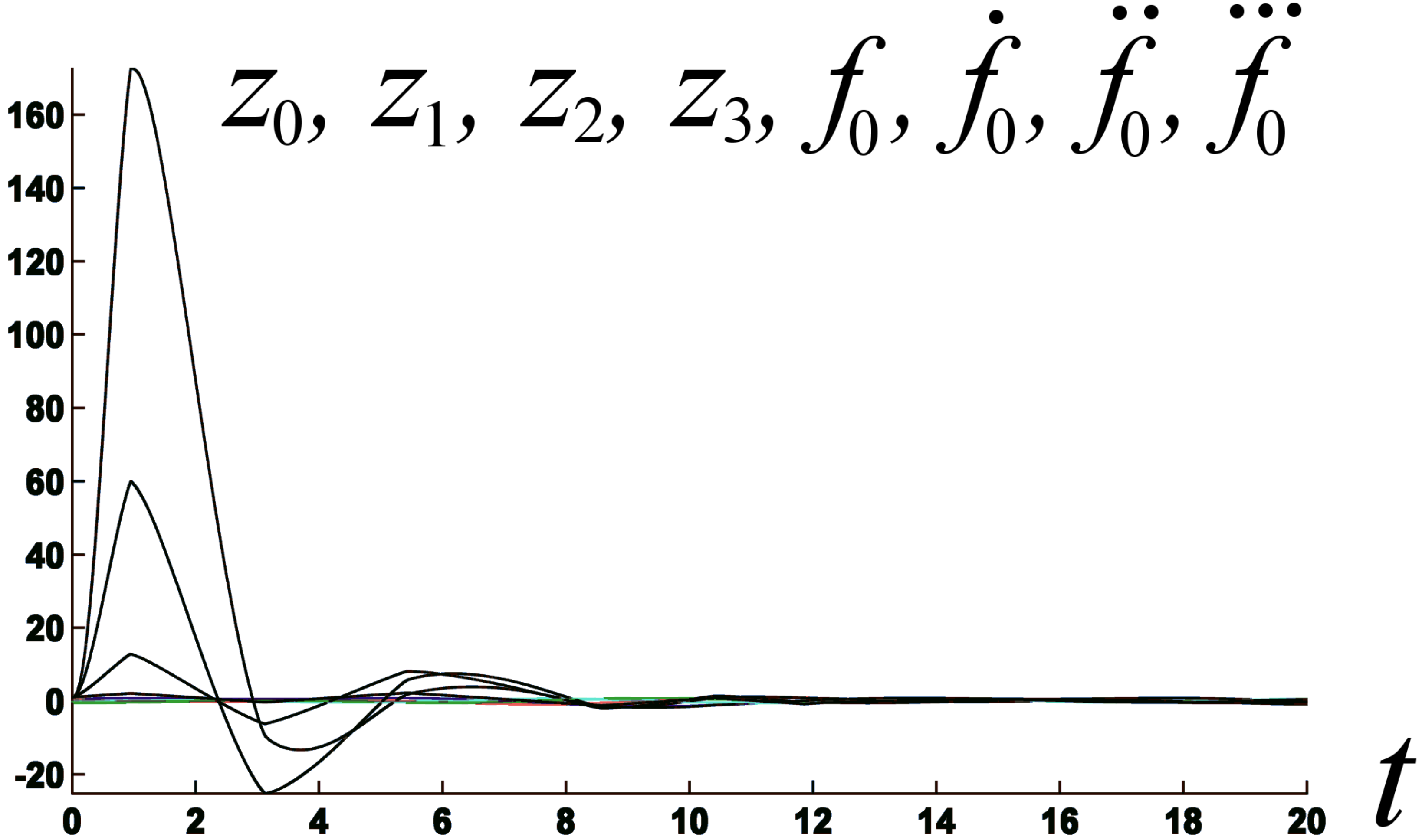


Noisy Input

$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \leq (0.00018, 0.0034, 0.031, 0.15)$$

Convergence, $n_d = 3, n_f = 7, L = 1, \tau = 10^{-5}$

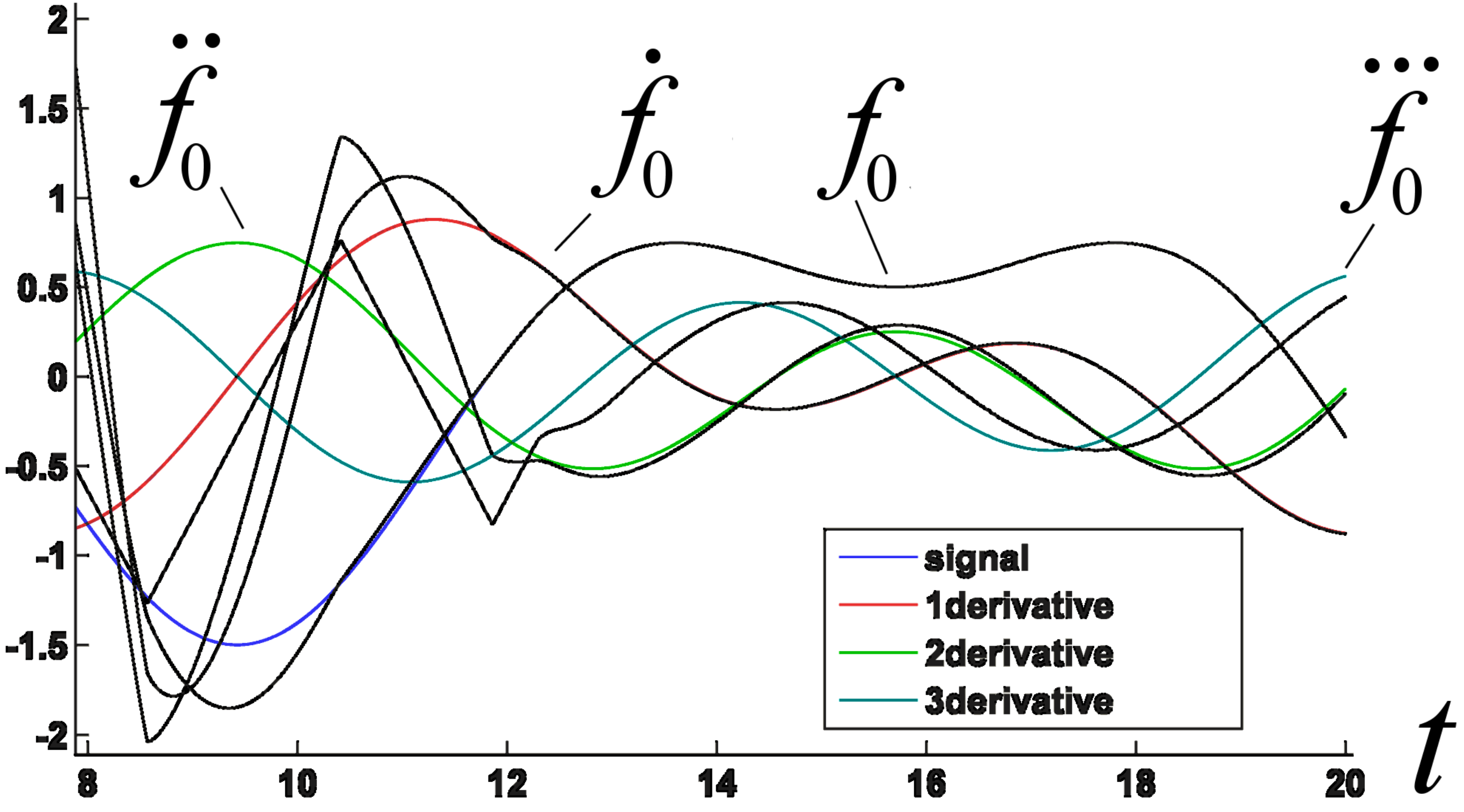
$$\eta(t) = 10^7 \cos(10^4 t) - 2\cdot 10^7 \cos(1.7\cdot 10^5 t) + 5\cdot 10^7 \cos(1.33\cdot 10^5 t)$$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \le (0.00018, 0.0034, 0.031, 0.15)$$

Zoom, $n_d = 3, n_f = 7, L = 1, \tau = 10^{-5}$

$$\eta(t) = 10^7 \cos(10^4 t) - 2\cdot 10^7 \cos(1.7\cdot 10^5 t) + 5\cdot 10^7 \cos(1.33\cdot 10^5 t)$$



$$(|z_0 - f_0|, |z_1 - \dot{f}_0|, |z_2 - \ddot{f}_0|, |z_3 - \dddot{f}_0|) \le (0.00018, 0.0034, 0.031, 0.15)$$