On Chattering-Free Sliding-Mode Control

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***Abstract*—High-Order Sliding Mode (HOSM) control was originally proposed to overcome the dangerous chattering effect. The idea is to treat the time derivative of the actual control as a new control artificially raising the relative degree of the sliding variable. The resulting HOSM features finite-time stability, ultimate accuracy of sliding mode, and smooth control. Unfortunately, due to the interaction between the control and its derivative, the convergence to the HOSM is only ensured, if the initial values of the successive sliding-variable derivatives are small enough. It is proved in the paper that under mild conditions that restriction is removed. Output-feedback controllers are constructed. Computer simulation confirms the applicability of the approach.**

I. Introduction

CONTROL under heavy uncertainty conditions remains one of the main subjects of the modern control theory. One of the most popular approaches to the problem is based on the sliding-mode control. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Although very robust and accurate, the approach also features certain restrictions. The standard sliding mode may be directly implemented only if the relative degree of the constraint is 1, i.e. control has to explicitly appear already in the first total time derivative of the constraint function. Another problem is that the high-frequency control switching may cause dangerous vibrations (chattering effect) [2], [5]-[7], [10], [11], [27].

The issues can be settled in a few ways. High-gain control with saturation is used to overcome the chattering effect approximating the sign-function in a narrow boundary layer around the switching manifold [25], the sliding-sector method [12] avoids chattering in control of disturbed linear time-invariant systems. The sliding-mode order approach [2]-[4], [9], [15]-[25] is capable to treat both the chattering and the relative-degree restrictions, while preserving the sliding-mode features and improving the accuracy.

High order sliding mode (HOSM) [16], [17] actually is a movement on a discontinuity set of a dynamic system understood in Filippov's sense [8]. The sliding order characterizes the dynamics smoothness degree in the vicinity of the sliding mode.

Consider a smooth dynamic system with a smooth output function σ. The function σ can for example be a tracking error. Let the system be closed by some possibly-dynamical discontinuous feedback and be understood in the Filippov sense [8]. The task is to make σ vanish, keeping it at zero afterwards. Successively differentiating σ along trajectories, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma \equiv 0$ may be classified by the number r of the first successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason, like trajectory nonuniqueness. That number is called the sliding order. The rth order sliding modes are also called r-sliding modes.

Recall that, roughly speaking, the relative degree is the number of the total time derivative of the output, in which the control appears explicitly for the first time [13]. The sliding order r coincides with the relative degree, if the output relative degree is well-defined., and the control is discontinuous.

One of the main results of the HOSM theory is that a number of predefined standard controllers are developed, defined for each given relative degree r, which solve the problem of keeping $\sigma = 0$ in finite time. Such controllers are called r-sliding controllers [16], [17] and actually require only the knowledge of the system relative degree r. The produced control is a discontinuous function of σ and of its real-time-calculated successive derivatives $\dot{\sigma}$, ..., $\sigma^{(r-1)}$. Such controllers provide for the accuracy $\sigma = O(\tau^r)$ with the sampling interval τ [18]. This asymptotics is preserved, when a robust exact differentiator of the order r - 1 [17] is applied as a standard part of the output-feedback r-sliding controller.

The produced HOSM control features the high, theoretically infinite frequency of control switching, which can still be troublesome. In practice the control is inevitably based on sensor outputs. Also the control signal does not directly influence the system, either affects it via an actuator, being itself a dynamic system. The complicated interaction of frequent switching with various noises, delays, and fast dynamics of sensors and actuators produces dangerous high-frequency system vibrations called the chattering effect.

Nevertheless, HOSMs were historically created to remove the chattering effect. The idea is to consider the kth-order time derivative of the actual control as the new control input. As a result, the relative degree raises, and a new $(r + k)$-sliding controller is applied, corresponding to the new relative degree $r + k$. The real control is now the output of an integrator chain, i.e. is smooth of the needed order k. The input u and its derivatives $\dot{u}$, …, $u^{(k-1)}$ are considered now as system coordinates. The $(r + k)$-sliding-mode conditions $\sigma = \dot{\sigma} = ... = \sigma^{(r+k-1)} = 0$ define a manifold in that extended space.

Recent results [5], [6], [10], [11], [21] show that the produced $(r + k)$-sliding dynamics is robust with respect to the influence of unaccounted-for small noises, delays, fast stable actuators and sensors. Moreover, it was proved that

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the dangerous chattering effect is removed, and only negligibly small vibrations of infinitesimal energy persist [21]. Note that it is the specific combination of k integrators and the $(r + k)$-sliding control, which remove the chattering, and not just the integration chain itself.

Any $(r + k)$-sliding controller is based on the domination of the new control $u^{(k)}$ in the expression for $\sigma^{(r+k)}$. Unfortunately, in the general case u and its lower derivatives explicitly appear in $\sigma^{(r+k)}$. Some interaction of u and its derivatives during the convergence to the $(r + k)$-sliding mode $\sigma = \dot{\sigma} = ... = \sigma^{(r+k-1)} = 0$ is inevitable. Thus, generally speaking, such an $(r + k)$-sliding controller is for sure effective in some vicinity of the $(r + k)$-sliding mode only. Indeed, the conditions $\sigma^{(r)} = ... = \sigma^{(r+k-1)} = 0$ determine u, $\dot{u}$, …, $u^{(k-1)}$ in a unique way (the equivalent control and its derivatives [26]), which excludes the above interaction in some vicinity of the sliding mode.

The global convergence is so far assured only for the transfer from the relative degree 1 to 2, i.e. for $r = k = 1$ [16]. Semi-global convergence is provided in the general case, implementing $(r + k)$th order integral sliding mode [22]. Note that in the latter case one needs to calculate a predefined transient trajectory in the coordinates σ, $\dot{\sigma}$, ..., $\sigma^{(r+k-1)}$ connecting the initial point with the origin $\sigma = \dot{\sigma} = ... = \sigma^{(r+k-1)} = 0$.

The above integral sliding mode approach is significantly simplified in this paper. One does not need anymore to calculate a transient trajectory in advance. Thus, the method can be easily applied for the chattering attenuation procedure, or just to exterminate the system uncertainty from the very beginning.

The produced controller can be equipped with a robust finite-time convergent differentiator [3], [14], [17], producing output-feedback control. The asymptotical accuracy of the sliding mode is calculated in the presence of measurement noises and discrete sampling. The results are illustrated by computer simulation.

II. The Chattering Attenuation Problem

Consider a dynamic system of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \tag{1}$$

where $x \in \mathbf{R}^n$, a, b and σ: $\mathbf{R}^{n+1} \to \mathbf{R}$ are unknown smooth functions, $u \in \mathbf{R}$ is the control, n might be also uncertain. The task is to get $\sigma \equiv 0$.

All differential equations are understood in the Filippov sense [8], which allows discontinuous dynamics. The system relative degree r is assumed to be constant and known, which implies [13] that

$$\sigma^{(r)} = h(t,x) + g(t,x)u, \tag{2}$$

where $h(t,x) = \sigma^{(r)}|_{u=0}$, $g(t,x) = \frac{\partial}{\partial u}\sigma^{(r)} \neq 0$ are some unknown smooth functions. It is supposed that

$$0 < K_{\mathrm{m}} \le g(t,x) \le K_{\mathrm{M}}, \quad | h(t,x) | \le C \tag{3}$$

for some K_{m}, K_{M}, $C > 0$. Note that at least locally it is always true. The corresponding local or semiglobal convergence Theorems [17] also make sense due to the finite-time convergence. Note that (3) indeed holds in most engineering applications. For example, any aircraft can operate only with bounded velocities, accelerations, altitudes, etc.

Trajectories of (2) are assumed infinitely extendible in time for any Lebesgue-measurable bounded control $u(t, x)$. Though formally not needed, it is probably required in practice that the system feature bounded-input-bounded-state property.

The above problem statement is standard and is solved by a number of known r-sliding controllers [4], [17] - [19]

$$u = \alpha U_r(\sigma, \dot{\sigma}, ..., \sigma^{(r-1)}), \tag{4}$$

which actually solve the problem for the differential inclusion

$$\sigma^{(r)} \in [-C, C] + [K_{\mathrm{m}}, K_{\mathrm{M}}]u \tag{5}$$

instead of (2), (3). Here U_r is a bounded discontinuous function. Only the control gain $\alpha > 0$ needs to be adjusted for the concrete values of C, K_{m}, K_{M}, providing for the finite-time convergence of the inclusion trajectories to zero. Controllers considered in this paper in detail are quasi-continuous [19], and are defined further.

Control (4) is discontinuous and therefore produces considerable chattering, when the sliding mode $\sigma \equiv 0$ is kept. In order to remove the dangerous high-energy vibrations consider $\dot{u}$ as the new system input. Differentiating (2) obtain

$$\begin{aligned} \sigma^{(r+1)} &= h_1(t,x,u) + g(t,x)\,\dot{u}, \\ h_1 &= h'_t + h'_x a + (h'_x b + g'_t + g'_x a)u + g'_x b u^2. \end{aligned} \tag{6}$$

According to the standard procedure the control is defined as

$$\dot{u} = \alpha U_{r+1}(\sigma, \dot{\sigma}, ..., \sigma^{(r)}). \tag{7}$$

It is natural to suppose that

$$| h'_t + h'_x a | \le c_a, \ | h'_x b + g'_t + g'_x a | \le c_b, \ | g'_x b | \le c_d, \tag{8}$$

where c_a, c_b, c_d are some positive constants. Also this assumption is always true at least locally.

Unfortunately, restrictions (3), (8) cannot ensure the domination in (6) of the term with $\dot{u}$, even if α is large. Therefore, the control (7) might not provide for the convergence of σ, $\dot{\sigma}$, ..., $\sigma^{(r)}$ to zero. The problem is resolved, if the initial point is close to the $(r+1)$-sliding manifold

$$\sigma = \dot{\sigma} = ... = \sigma^{(r)} = 0. \tag{9}$$

Indeed, in that case $\sigma^{(r)} = 0$ implies that u is close to its *unknown* equivalent value

$$u_{eq} = -\, h(t,x)/g(t,x),$$

and h_1 in (6) remains bounded. Thus, only local convergence in the extended space t, x, u is guaranteed. Unlike the conditions (3), (8) this condition is indeed very restrictive, since the convergence region can be really small, and u_{eq} is uncertain. A simple effective solution of this problem is the main subject of this paper.

III. Quasi-continuous Controllers

While any r-sliding controller can be applied here, only one controller family is considered. Let $i = 0, 1, \ldots, r-1$. The following recursive procedure defines the family of quasi-continuous controllers [19] stabilizing (5) in finite time, and therefore solving the standard problem (1) - (3):

$$\begin{aligned}
&\varphi_{0,r} = \sigma,\ N_{0,r} = |\sigma|, \quad \Psi_{0,r} = \varphi_{0,r}/N_{0,r} = \operatorname{sign}\sigma,\\
&\varphi_{i,r} = \sigma^{(i)} + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}\, \Psi_{i-1,r},\\
&N_{i,r} = |\sigma^{(i)}| + \beta_i N_{i-1,r}^{(r-i)/(r-i+1)}, \quad \Psi_{i,r} = \varphi_{i,r}/N_{i,r},\\
&u = -\alpha \Psi_{r-1,r}(\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}).
\end{aligned} \tag{10}$$

Here $\beta_1, \ldots, \beta_{r-1} > 0$, $\alpha > 0$ are the controller parameters. Obviously, α is to be negative with $(\partial/\partial u)\sigma^{(r)} < 0$. Denote $\Sigma_i = (\sigma, \dot\sigma \ldots, \sigma^{(i)})$. It is easy to see that $N_{i,r}(\Sigma_i)$ is a positive-definite continuous function, $\Psi_{i,r}(\Sigma_i)$ is continuous everywhere except $\Sigma_i = 0$, and $|\Psi_{i,r}| \le 1$. These controllers feature specific homogeneity properties [18].

A function f: $\mathbf{R}^n \to \mathbf{R}$ (respectively a vector-set field $F(x) \subset \mathbf{R}^n$, $x \in \mathbf{R}^n$, or a vector field f: $\mathbf{R}^n \to \mathbf{R}^n$) is called *homogeneous of the degree $q \in \mathbf{R}$ with the dilation* [1], [18]

$$d_\kappa: (x_1, x_2, \ldots, x_n) \mapsto (\kappa^{m_1} x_1, \kappa^{m_2} x_2, \ldots, \kappa^{m_n} x_n),$$

where $m_1, \ldots, m_n$ are some positive numbers (*weights*), if for any $\kappa > 0$ the identity $f(x) = \kappa^{-q} f(d_\kappa x)$ holds (respectively $F(x) = \kappa^{-q} d_\kappa^{-1} F(d_\kappa x)$, or $f(x) = \kappa^{-q} d_\kappa^{-1} f(d_\kappa x)$). The non-zero homogeneity degree q of a vector field can always be scaled to ± 1 by an appropriate proportional change of the weights $m_1, \ldots, m_n$.

Note that the homogeneity of a vector field $f(x)$ (a vector-set field $F(x)$) can equivalently be defined as the invariance of the differential equation $\dot x = f(x)$ (differential inclusion $\dot x \in F(x)$) with respect to the homogeneity time-coordinate transformation

$$G_\kappa : (t, x) \mapsto (\kappa^p t, d_\kappa x),$$

where p, $p = -q$, might naturally be considered as the weight of t. Indeed, the homogeneity condition can be rewritten as $\dot x \in F(x) \Leftrightarrow \dfrac{d(d_\kappa x)}{d(\kappa^p t)} \in F(d_\kappa x)$.

Let the coordinates $\sigma, \dot\sigma, \ldots, \sigma^{(r-1)}$ have the homogeneity weights r, $r-1$, …, 1 respectively. Then the differential inclusion (5), (10) is homogeneous with the degree -1 and the homogeneity transformation

$$G_\kappa : (t, \Sigma_{r-1}) \mapsto (\kappa t, d_\kappa \Sigma_{r-1}),\ d_\kappa \Sigma_{r-1} = (\kappa^r \sigma, \kappa^{r-1}\dot\sigma, \ldots, \kappa \sigma^{(r-1)}).$$

The corresponding homogeneity is called r-sliding homogeneity, and the quasi-continuous controller (10) is respectively called r-sliding homogeneous [18].

Choice of the parameters β_i. The idea of the controller (10) is to keep $\Psi_{r-1,r} = \varphi_{r-1,r}/N_{r-1,r}$ close to zero. The equality

$$\varphi_{r-1,r} = \sigma^{(r-1)} + \beta_i N_{r-2,r}^{1/2}(\Sigma_{r-2})\, \Psi_{r-2,r}(\Sigma_{r-2}) = 0 \tag{11}$$

defines an r-sliding homogeneous differential equation of the order $r-1$. The sufficient condition for the convergence of (5) (10) to zero in finite time with sufficiently large α is that (11) be finite-time stable. It is shown that (11) is finite-time stable if $\beta_1, \ldots, \beta_{r-1}$ are chosen sufficiently large in the list order.

Note that while enlarging α increases the class (3) of systems, to which the controller is applicable, parameters β_i, are tuned to provide for the needed convergence rate. The following procedure provides for the approximately λ times reduction of the convergence time [20]. The new parameters $\tilde\beta_1, \ldots, \tilde\beta_{r-1}, \tilde\alpha$ are calculated according to the formulas $\tilde\beta_1 = \lambda\beta_1$, $\tilde\beta_2 = \lambda^{r/(r-1)}\beta_2$, ..., $\tilde\beta_{r-1} = \lambda^{r/2}\beta_{r-1}$, $\tilde\alpha = \lambda^r\alpha$. Following are the resulting quasi-continuous controllers with $r \le 4$ and simulation-tested β_i:

1. $u = -\alpha \operatorname{sign}\sigma$,
2. $u = -\alpha\, (\dot\sigma + \lambda |\sigma|^{1/2} \operatorname{sign}\sigma)/(|\dot\sigma| + \lambda|\sigma|^{1/2})$,
3. $u = -\alpha\, [\ddot\sigma + 2\lambda^{3/2} (|\dot\sigma| + \lambda|\sigma|^{2/3})^{-1/2} (\dot\sigma + \lambda|\sigma|^{2/3}\operatorname{sign}\sigma)] / [|\ddot\sigma| + 2\lambda^{3/2} (|\dot\sigma| + \lambda|\sigma|^{2/3})^{1/2}]$,
4. $\varphi_{3,4} = \dddot\sigma + 3\lambda^2 [\ddot\sigma + \lambda^{4/3} (|\dot\sigma| + 0.5\lambda|\sigma|^{3/4})^{-1/3} (\dot\sigma + 0.5\lambda\, |\sigma|^{3/4} \operatorname{sign}\sigma)]\, [|\ddot\sigma| + \lambda^{4/3} (|\dot\sigma| + 0.5\lambda\, |\sigma|^{3/4})^{2/3}]^{-1/2}$,
   $N_{3,4} = |\dddot\sigma| + 3\lambda^2\, [|\ddot\sigma| + \lambda^{4/3} (|\dot\sigma| + 0.5\lambda\, |\sigma|^{3/4})^{2/3}]^{1/2}$,
   $u = -\alpha\, \varphi_{3,4}/N_{3,4}$.

IV. Problem Solution

Return to the chattering attenuation problem (1) – (3), (8). Consider the controller

$$\dot u = -\alpha \Psi_{r,r+1}(S, \dot S, \ldots, S^{(r)}), \tag{12}$$

$$s^{(r+1)} = -\alpha_0 \underline{\Psi}_{r,r+1}(s, \dot s, \ldots, s^{(r)}), \quad S = \sigma - s, \tag{13}$$

$$s(t_0) = \sigma(t_0), \ldots, s^{(r)}(t_0) = \sigma^{(r)}(t_0). \tag{14}$$

Here the controller $-\alpha_0 \underline{\Psi}_{r,r+1}$ is any r-sliding homogeneous controller, providing for the desired convergence rate and the global finite-time stability of (13). The classical conception of the (first order) integral sliding mode [27] corresponds in the considered $(r+1)$th order case to

$$s^{(r+1)} = - \alpha_0 \underline{\Psi}_{r,r+1}(\sigma, \dot\sigma, ..., \sigma^{(r)}) \qquad (15)$$

instead of (13). In such a case the sliding manifold $S = 0$ motion is defined by the "pure" feedback principle. The term "integral sliding mode" is clarified after one notes that s is the $(r+1)$th order integral of (13) or (15).

Theorem 1. *Let the initial values of the system* (1) *belong to some compact region. Then with properly chosen parameters of the controllers* (12), (13) *and any sufficiently large* α *both controllers* (12) - (14) *and* (12), (14), (15) *provide for the establishment in finite time and keeping afterwards of the* $(r+1)$*-sliding mode* $\sigma \equiv 0$. *The transient dynamics is described by the finite-time stable equation*

$$\sigma^{(r+1)} = - \alpha_0 \underline{\Psi}_{r,r+1} (\sigma, \dot\sigma, ..., \sigma^{(r)}).$$

Proof. First take the controller (12) – (14). Consider the output function $S = \sigma - s$ for the system (1), (12), (13). As follows from (12), (13)

$$S^{(r+1)} = h_1(t,x,u) + g(t,x)\,\dot u + \alpha_0 \underline{\Psi}_{r,r+1}(s, \dot s, ..., s^{(r)}), \qquad (16)$$
$$S^{(r)} = h(t,x) + g(t,x)u - s^{(r)}.$$

Let $s(t)$ be the solution of (13), $t \geq t_0$. It vanishes in finite time and stays at zero afterwards. Thus, from the equation $S^{(r)} = 0$ obtain

$$u_{\rm eq} = (s^{(r)}(t) - h(t,x))/ g(t,x).$$

Hence, the $(r+1)$-sliding manifold $S \equiv 0$ is described by the equations

$$\sigma - s(t) = \dot\sigma - \dot s = \ldots = \sigma^{(r-1)} - s^{(r-1)} = 0, \; u - u_{\rm eq}(t,x) = 0.$$

The motion in the $(r+1)$-sliding mode $\sigma - s(t) \equiv 0$ is described by the equations

$$\dot x = a(t,x) + b(t,x)\,(s^{(r)}(t) - h(t,x))/ g(t,x), \qquad (17)$$
$$\sigma = s(t),\; \dot\sigma = \dot s(t),\; \ldots\; \sigma^{(r-1)} = s^{(r-1)}(t). \qquad (18)$$

There is a correspondence between trajectories of (17), (18) and trajectories of (13), (14) in the auxiliary coordinates s. The initial conditions of (13), (14) also automatically belong to some compact set. Take a time segment so large that all the trajectories of (13), (14) terminate at zero. The corresponding trajectory segments of (17), (18) with the initial conditions (14) comprise a compact point set Ω which is the projection of a high-dimensional compact set in the coordinates t, x, s. The continuous function $u_{\rm eq}(t,x)$ is obviously bounded on Ω. Let

$$C_1 > c_a + c_b \sup |u_{\rm eq}| + c_d \sup^2 |u_{\rm eq}|.$$

Due to (16) any trajectory of (12), (13), (14) satisfies the inclusion

$$S^{(r+1)} \in [-C_1 - \alpha_0, C_1 + \alpha_0] + [K_{\rm m}, K_{\rm M}]\,\dot u \qquad (19)$$

whenever $(t,x) \in \Omega$. Taking now α sufficiently large, provide for the finite-time stability of (19). Taking into account the initial conditions (14), obtain that $S \equiv 0$ is kept from the very beginning. Therefore, the resulting trajectories are exactly the trajectories of (17), (18), (13), (14).

The case of the controller (12), (14), (15) is similarly considered. The trajectories still satisfy inclusion (19) whenever $(t,x) \in \Omega$. Due to the initial conditions (14) the resulting trajectories are exactly the same.■

Implementing an rth order robust exact differentiator [17] obtain the output-feedback controller

$$\dot u = - \alpha\Psi_{r,r+1}(z_0 - s, z_1 - \dot s, ..., z_r - s^{(r)}), \qquad (20)$$
$$s^{(r+1)} = - \alpha_0 \underline{\Psi}_{r,r+1}(s, \dot s, ..., s^{(r)}), \qquad (21)$$
$$s(t_0) = z_0(t_0), \ldots, s^{(r)}(t_0) = z_r(t_0). \qquad (22)$$

$$\begin{aligned}
\dot z_0 &= v_0,\; v_0 = - \lambda_r L^{1/(r+1)} | z_0 - \sigma|^{r/(r+1)} \operatorname{sign}(z_0 - \sigma) + z_1,\\
\dot z_1 &= v_1,\; v_1 = - \lambda_{r-1} L^{1/r} | z_1 - v_0|^{(r-1)/r} \operatorname{sign}(z_1 - v_0) + z_2,\\
&\ldots\\
\dot z_{r-1} &= v_{r-1},\; v_{r-1} = -\lambda_1 L^{1/2} | z_{r-1} - v_{r-2}|^{1/2} \operatorname{sign}(z_{r-1} - v_{r-2}) + z_r,\\
\dot z_r &= -\lambda_0 L \operatorname{sign}(z_r - v_{r-1}),
\end{aligned} \qquad (23)$$

where $L > C_1 + \alpha K_{\rm M}$, and parameters λ_i of differentiator (12) are chosen in advance. In particular, the following values can be used for any $r \leq 5$: $\lambda_0 = 1.1$, $\lambda_1 = 1.5$, $\lambda_2 = 3$, $\lambda_3 = 5$, $\lambda_4 = 8$, $\lambda_5 = 12$ [19].

Theorem 2. *The discrete-measurement version of the controller* (20)-(23) *with the sampling interval* τ *provides in the absence of measurement noises for the inequalities*

$$|\sigma| < \gamma_0 \tau^{r+1}, |\dot\sigma| < \gamma_1 \tau^{r}, ..., \sigma^{(r)} < \gamma_r \tau$$

for some $\gamma_0, \gamma_1, ..., \gamma_r > 0$. *In the presence of any Lebesgue-measurable sampling noise of the magnitude* ε *the accuracies*

$$|\sigma| < \delta_0 \varepsilon, |\dot\sigma| < \delta_1 \varepsilon^{r/(r+1)}, ..., \sigma^{(r)} < \delta_{r-1} \varepsilon^{1/(r+1)}$$

are obtained for some $\delta_0, \delta_1, ..., \delta_{r-1} > 0$.

That asymptotic accuracy is the best possible with discontinuous $\sigma^{(r)}$ and discrete sampling [16].

Proof. Denote $\sigma_i = z_i - \sigma^{(i)}$. Then using $\sigma^{(r+1)} \in [-L, L]$ the control can be rewritten as

$$\dot u = - \alpha\Psi_{r,r+1}(\sigma_0 + S, \sigma_1 + \dot S, ..., \sigma_r + S^{(r)}), \qquad (24)$$

$$\begin{aligned}
\dot\sigma_0 &= - \lambda_r L^{1/(r+1)}|\sigma_0|^{r/(r+1)} \operatorname{sign}(\sigma_0) + \sigma_1,\\
\dot\sigma_1 &= - \lambda_{r-1} L^{1/(r-1)}| \sigma_1 - \dot\sigma_0 |^{(r-2)/(r-1)} \operatorname{sign}(\sigma_1 - \dot\sigma_0) + \sigma_2,\\
&\ldots\\
\dot\sigma_{r-1} &= - \lambda_1 L^{1/2}|\sigma_{r-1} - \dot\sigma_{r-2}|^{1/2} \operatorname{sign}(\sigma_{r-1} - \dot\sigma_{r-2}) + \sigma_r,
\end{aligned} \qquad (25)$$

$\dot{\sigma}_r \in -\lambda_0 L \,\text{sign}(\sigma_r - \dot{\sigma}_{r-1}) + [-L, L]$.

Solutions of (19) - (23) satisfy the Filippov differential inclusion (19), (24), (25). Assign the weights $r+1-i$ to σ_i, $S^{(i)}$, $i = 0, 1, ..., r$, and obtain a homogeneous differential inclusion (6), (13), (14) of the degree -1. The part (25) of this inclusion collapses in finite time [17] and $\sigma_i \equiv 0$ from that moment. The inclusion (19), (12) is also finite-time stable. Thus, (19), (24), (25) is finite-time stable, and its accuracy is readily given by [18].■

V. Chattering analysis

A few notions recently introduced in [21] are briefly reviewed here. Define the L_1-chattering of the signal $\xi(t)$ with respect to $\bar{\xi}(t)$ as

$$L_1\text{-chat}(\xi, \bar{\xi}; 0, T) = \int_0^T |\dot{\xi}(t) - \dot{\bar{\xi}}|\, dt.$$

In other words, L_1-chattering is the distance between $\dot{\xi}$ and $\dot{\bar{\xi}}$ in L_1-metric, or the variation of the signal difference $\Delta\xi$. It can be also interpreted as the work of the virtual dry friction force $f = -\,\text{sign}\, \Delta\dot{\xi}$. L_p-chattering, $p > 1$, is similarly defined. These notions are almost equivalent [21], and the prefix "L_p-" is further omitted.

Let $x(t) \in \mathbf{R}^n$, $t \in [0, T]$, be an absolutely continuous vector function, and $M(t, x)$ be some positive-definite continuous symmetric matrix with the determinant separated from 0. The chattering of a trajectory $x(t)$ with respect to $\bar{x}(t)$ is defined as

$$\text{chat}(x, \bar{x}; 0, T) = \int_0^T [(\dot{x}^t(t) - \dot{\bar{x}}^t) M(t,x) (\dot{x}(t) - \dot{\bar{x}})]^{1/2} dt .$$

Consider a *chattering family* of absolutely continuous trajectories (signals) $x(t,\varepsilon) \in \mathbf{R}^n$, $t \in [0, T]$, $\varepsilon \in \mathbf{R}^l$. The family parameters ε_i measure some imperfections and tend to zero. Define the nominal trajectory (signal) as the limit trajectory (signal) $\bar{x}(t) = \lim_{\varepsilon\to 0} x(t,\varepsilon)$, $t \in [0, T]$. The chattering is not defined in the case, when the limit trajectory $\bar{x}(t)$ does not exist or is not absolutely continuous.

The chattering is **infinitesimal** if $\lim_{\varepsilon\to 0} \text{chat}(x, \bar{x}; 0,T) = 0$.

The chattering is **bounded** if $\overline{\lim_{\varepsilon\to 0}} \text{chat}(x, \bar{x}; 0,T) > 0$.

The chattering is **unbounded** if $\overline{\lim_{\varepsilon\to 0}} \text{chat}(x, \bar{x}; 0, T) = \infty$.

In the case of a uniformly converging chattering family the notions of the chattering infinitesimality, boundedness and unboundedness are invariant with respect to smooth transformations of time and coordinates, and to the choice of a continuous positive-definite symmetric matrix M.

It is said that there is infinitesimal chattering in a closed-loop control system depending on a small vector chattering parameter, if any local chattering family of *plant state* trajectories features infinitesimal chattering. It is said that the chattering is bounded (unbounded) if there exists a local chattering family with bounded (unbounded) chattering.

Some coordinates of the closed-loop-system mathematical model can be excluded from the plant state. It happens, in particular, if a dynamic controller is used, based on the computer technique. The chattering of the internal computer variables can be ignored.

Note that the least possible chattering in this classification is the infinitesimal one. In other words the infinitesimal chattering is present in any control system as a result of infinitesimal disturbances of different nature. One can prove that under mild conditions infinitesimal-chattering mechanical systems have only infinitesimal heat emission.

It is proved [21] that sliding-mode control systems feature bounded chattering. High gain systems feature unbounded chattering, if infinitesimal sampling noises are accounted for. Systems obtained by means of the above chattering attenuation procedure have only infinitesimal chattering in the presence of small delays, sampling noises, fast stable actuators and sensors [21].

VI. Simulation

Consider a simple pendulum without friction. An engine transmits a torque u. The features of the engine uncontrollably change in time as a result of some additional load. The task is to track some function x_c given in real time by the angular coordinate x of the pendulum.

The system is described by the equation

$$\ddot{x} = -\sin x + R(t)\, u, \quad R(t) \in [0.5, 1.5].$$

Here $\dot{R}$ and $\dot{x}_c$ are assumed bounded, $\sigma = x - x_c$ is available. The relative degree of the system is 2, and straight-forward sliding-mode control leads to the hard chattering of the system coordinate $\dot{x}$. Artificially increase the relative degree to 3, considering $\dot{u}$ as a new control.

The initial conditions are $x(0) = -6$, $\dot{x}(0) = -15$, $u(0) = 3$. Following are the functions R and x_c considered in the simulation:

$$R = 2 + \sin 4t, \quad x_c = 0.5 \sin 0.5t + 0.5 \cos t .$$

The output-feedback controller takes on the form

$$\dot{u} = -40\, [S_2 + 2\,(|S_1| + |S_0|^{2/3})^{-1/2}(S_1 + |S_0|^{2/3} \text{sign}\, S_0)] \,/\, [|S_2| + 2\,(|S_1| + |S_0|^{2/3})^{1/2}];$$

$\sigma = x - x_c$, $S_0 = z_0 - s$, $S_1 = z_1 - \dot{s}$, $S_2 = z_2 - \ddot{s}$;

$$\dddot{s} = -\lambda^3 20[\ddot{s} + 2\lambda^{3/2} (|\dot{s}| + \lambda|s|^{2/3})^{-1/2}(\dot{s} + \lambda|s|^{2/3} \text{sign}\, s)] \,/\, [|\ddot{s}| + 2\lambda^{3/2} (|\dot{s}| + \lambda|s|^{2/3})^{1/2}],$$

$s(t_0) = z_0(t_0)$, $\dot{s}(t_0) = z_1(t_0)$, $\ddot{s}(t_0) = z_2(t_0)$;

$\dot{z}_0 = v_0$, $v_0 = -18.5664\, |z_0 - \sigma|^{2/3} \text{sign}(z_0 - \sigma) + z_1$,

$\dot{z}_1 = v_1$, $v_1 = -42.4264\, |z_1 - v_0|^{1/2} \text{sign}(z_1 - v_0) + z_2$,

$\dot{z}_2 = -880\, \text{sign}(z_2 - v_1)$.

Here $L = 800$. The time $t_0 = 0.5$ provides some time for the differentiator convergence; $z_0(0) = \sigma(0)$, $z_1(0) = z_2(0) = 0$.

Two values $\lambda = 0.8, 0.5$ were considered in order to demonstrate the transient dynamics rate change.

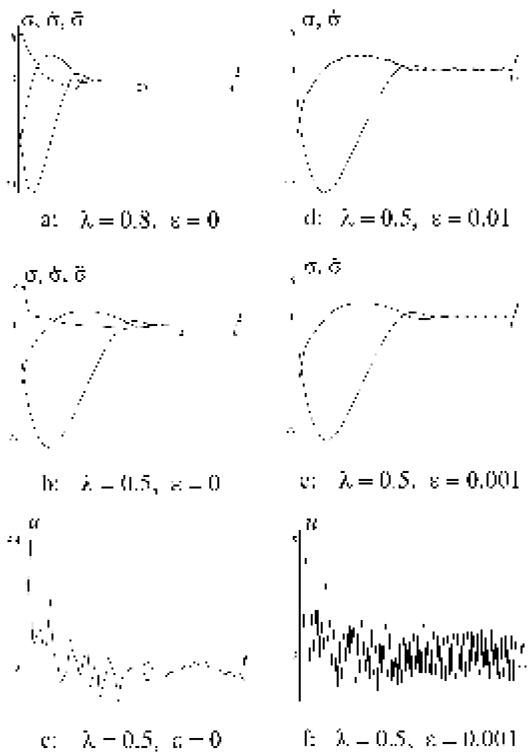

Fig. 1. Performance with different λ and noise magnitude ε

The integration was carried out according to the Euler method (the only reliable integration method with discontinuous dynamics). The sampling step was equal to the integration step $\tau = 10^{-5}$.

In the absence of output noises the tracking accuracies $|\sigma| \le 1.4{\cdot}10^{-10}$, $|\dot\sigma| \le 6.0{\cdot}10^{-7}$, $|\ddot\sigma| \le 0.007$ are obtained with $\lambda = 0.8, 0.5$ and $\tau = 10^{-5}$ (Fig. 1a,b). With the sampling noise of the magnitude $\varepsilon = 0.01$ the accuracies $|\sigma| \le 0.018$, $|\dot\sigma| \le 0.28$, $|\ddot\sigma| \le 7.2$ are obtained. They change to $|\sigma| \le 0.0015$, $|\dot\sigma| \le 0.062$, $|\ddot\sigma| \le 3.3$ with $\varepsilon = 0.001$. Coordinates σ, $\dot\sigma$ of the system demonstrate infinitesimal chattering only (Fig. 1d,e), while $\ddot\sigma$, and u feature bounded chattering (Fig. 1c,f), which corresponds to [21]. One needs to raise the relative degree once more if the bounded chattering of the control is not acceptable.

VII. Conclusions

A simple chattering attenuation procedure is proposed based on HOSMs, which is free of the interaction between the control and its derivatives. Also output feedback controllers are constructed, and asymptotic accuracies calculated.

Each procedure step increases the order of the plant coordinate derivatives with bounded chattering by one. As a result the hard chattering phenomena are removed from the plant to auxiliary control devices, usually computers.

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