QUANTIFICATION AND MODALITY

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PART 1: QUANTIFICATION
INTRODUCTION

I. SEMANTIC MEANING/PRAGMATIC MEANING

Recommendation letter: I only write

*He has beautiful azure eyes*

**Pragmatic implication:** Don't take the guy.

Gricean reasoning, speechcontext, etc.

knowledge about what one is supposed to write in a recommendation letter.

**Semantic implications:**

- He has azure eyes
- He has eyes etc.

Depends only on the speaker/hearer knowledge of the language → semantic competence

So semantic facts are –it seems- much more boring than pragmatic facts.

But even stupid facts like the above are interesting because they are part of patterns that are interesting.

II ADJECTIVES

**Intersectivity:**

A. An azure eye is an eye

B. An azure eye is azure

Many adjectives are intersective.

Some adjectives do not quite look intersective, but are what is called **subsective**.

These are typically degree adjectives:

- A. A small elephant is an elephant
- X. B. A small elephant is small

It is not clear that subsective adjectives aren't really intersective.

**Assumption 1:**

Degree adjectives have an interpretation aspect which is not lexicalized , a comparison class.

**Assumption 2:** Pragmatics of comparison class

1. Prenominal/attributive adjectives:

Out of the blue the comparison class is the denotation of the noun:

small [rel. C] elephant \( \rightarrow \) small [rel elephant] elephant

C = elephant

2. Predicative adjectives:

Out of the blue the comparison class is contextual:

small [rel C] \( \rightarrow \) small [rel C]

C = set of contextual objects

Now look at:

* A small elephant is small

interpretation:

* A small [rel \(C_i\)] elephant is small [rel \(C_2\)]
Intersectivity only says that the following should be true:

A small [rel $C_1$] elephant is small [rel $C_1$]

And this is uncontroversial: if Jumbo is a small elephant, then Jumbo is small for an elephant.

But the pragmatics of comparison class gives you out of the blue:

A small [rel elephant] elephant is small [rel set of contextual objects].

This is, of course, not necessarily true, on anybody’s theory. This means that we can maintain intersectivity, despite the seeming evidence to the contrary.

Intersectivity claims that the inference in (1) is true:

(1) a. Jumbo is a small elephant.
    b. Jumbo is an elephant and small (in comparison with the other elephants).

Intersectivity does not claim that the inference in (2) is true:

(2) a. Jumbo is a small elephant
    b. Jumbo is an elephant and small (in comparison with the other animals in the zoo)

Evidence for comparison class:
Even for attributive adjectives, the comparison class can be contextually determined: Kamp & Partee

My three year old built a huge snowman

The college team

$C_{\text{huge}} \neq \text{Snowmen}$

$C_{1,\text{huge}} = \text{Snowmen built by 3 year olds}$

$C_{2,\text{huge}} = \text{Snowmen built by college teams}$

EXCURSUS
One could speculate –but this is more tentative – that an argument for intersectivity even applies to adjectives like dead and fake.

1 A A dead poet is a poet
    B A dead poet is dead

2 A A fake Rembrandt is a Rembrandt
    B A fake Rembrandt is fake

also a fake diamond
Here the inference to the adjective is valid, but the inference to the noun is not.
The idea would be that the pragmatics of deadfake allows for 'temporary widening of
the denotation of the noun.

2A would be ambiguous:
\[
\begin{align*}
A_{\text{wide}} & : \text{A fake Rembrandt}^{\text{wide}} \text{ is a Rembrandt}^{\text{wide}} \quad \text{True} \\
B_{\text{narrow}} & : \text{A fake Rembrandt}^{\text{wide}} \text{ is a Rembrandt}^{\text{narrow}} \quad \text{False}
\end{align*}
\]

cf: \textit{Most Rembrandts are fake.}

However, whether this line is ultimately fruitful is questionable.
The reason is that there is ultimately a big difference between (3a) and (3b):

(3) a. A small elephant is not small.
    b. A fake Rembrandt is not a Rembrandt.

While the truth of (3a) (with stress indicated) is dependent on the context, (3b) seems
to be true absolutely.

\textbf{END OF EXCURSUS}

We see that intersectivity applies to a wide class of adjectives.
But not to all: \textbf{Temporal and modal adjectives}.

Temporal:
\[
\begin{align*}
A & : \text{A former friend is a friend} \quad \text{FALSE} \\
B & : \text{A former friend is former} \quad \text{INFELICITOUS}
\end{align*}
\]

Similarly future wife, etc…
-Not subsective (A is false)
-Most intensional adjectives (= temporal or modal can not be use predicatively.

Modal:
\[
\begin{align*}
A & : \text{A potential counterexample is a counterexample} \quad \text{FALSE} \\
B & : \text{A potential counterexample is potential} \quad \text{INFELICITOUS}
\end{align*}
\]

So the 'stupid facts' actually form part of a semantic classification of adjectives in
terms of \textbf{intersective} versus \textbf{intensional}.

\textbf{And this generalizes.}
I. We find the same distribution for \textbf{adverbials}:

\textbf{Intersective adverbials}: for example manner adverbials:
\[
\begin{align*}
Elegantly, \text{ Dafna danced} \\
A: & \text{Dafna's dancing was dancing} \\
& \text{i.e. Dafna danced} \\
B: & \text{Dafna's dancing was elegant} \\
& \text{i.e. Something that happened was elegant}
\end{align*}
\]
Intensional adverbials:

Potentially, Dafna will dance
A: Dafna will dance                       FALSE
B: Something that will happen is potential  INFELICITOUS

III GENERALIZATIONS

There is a different kind of generalization that we are particularly interested in.

THREE KINDS OF SEMANTIC MEANING

1. WORD MEANING  [Lexicography]
2. SENTENCE MEANING  [Logic]
3. CONSTITUENT MEANING  [Semantics]

Sentence meaning: We use judgements of native speakers about inference and felicity as data. These judgements involve sentence meanings.

Constituent meaning: Semantic generalizations are most often best stated neither at the level of word meaning, nor at the level of sentence meaning, but at an intermediate level of constituent meaning.

Example.

He has beatiful azure eyes which shine in the dark with black eye lashes

Adjectives, relative clause, prepositional phrase.

Facts:
A Azure eyes are eyes
B Azure eyes are azure

A Eyes with black eye lashes are eyes
B Eyes with black eye lashes have black eye lashes

A Eyes which shine in the dark are eyes
B Eyes which shine in the dark shine in the dark

Observation: Intersectivity is a principle that concerns not just adjectives, but also prepositional phrases and relative clauses.

This means that intersectivity is not a lexical property of the meanings of certain words (like adjectives), but of the meanings of classes of PHRASES. More precisely, it is a meaning constraint on how the meanings of ADJUNCTS like APs, PPs, CPs combine with the meanings nouns, verbs.

But this means that we need a theory of constituent meanings and a theory of the meaning of adjunction in order to even state the generalization. This is what semantics is about.
Generalization:
Syntactic adjuncts come in two kinds:
A  Those derived from predicates
B  Those not derived from predicates (intensional)
The semantic interpretation of adjunction for class A is predicate \textit{intersection}.

IV ABOUTNESS AND SEMANTIC COMPETENCE.

A core part of what we call meaning concerns the relation between linguistic expressions and non-linguistic entities, or 'the world' as our semantic system assumes it to be, the world as structured by our semantic system.

Some think about semantics in a realist way: semantics concerns the relation between language and the world.
Others think about semantics in a more conceptual, or if you want idealistic way: semantics concerns the relation between language and an intersubjective level of shared information, a conceptualization of the world, the world as we jointly structure it. Both agree that semantics is a theory of \textit{interpretation} of linguistic expressions: semantics concerns the relation between linguistic expressions and what those expressions are about. Both agree that important semantic generalizations are to be captured by paying attention to what expressions are about, and important semantic generalizations are missed when we don't pay attention to that.

But semantics concerns semantic \textit{competence}. Semantic competence does not concern \textit{what} expressions happen to be about, but \textit{how} they happen to be about them.

Native speakers obviously do not have to know what, say, a name happens to stand for in a certain situation, or what the truth value of a sentence happens to be in a certain situation. That is not necessarily part of their semantic competence. What is part of their semantic competence is \textit{reference conditions, truth conditions}:

Take the Dutch sentence:
\textit{Er is geen pen onder de tafel}.

A Dutch speaker can use that sentence to distinguish situation one [pen under the table] from situation two [pen above the table]. In which do you think is the sentence true?
Well, what Dutch speakers know is that \textit{g-} is a negative morpheme in Dutch, so it is situation two. So, the Dutch speaker can use this sentence to distinguish these two types of situation, while you can't. This is not because the dutch are more intelligent than you are, but only because the Dutch speakers have something that you don't have: semantic competence in Dutch.

Note that it is not part of the Dutch speakers competence to know whether the sentence is true or false (that is the business of detectives and scientists). What is part of your semantic competence is that, in principle, you're able to distinguish situations where that sentence is true, from situations where it is false, i.e. that you know \textit{what it takes} for a possible situation to be the kind of situation in which that string of words, that sentence, is true, and what it takes for a situation to be the kind of situation where that sentence is false.
Note too that we are talking about linguistic competence: my cat too can classify situations in terms of situations where there is a cockroach in the house, and where there isn't. But she cannot use language to do that classification, and we can.

The first thing to stress is: semantics is not interested in truth; semantics is interested in truth conditions.

From this it follows too that we're not interested in truth conditions per se, but in truth conditions relative to contextual parameters.

Take the sentence: I was behind the table one minute ago. The truth of this sentence depends on who the speaker is, when it is said, what the facts in the particular situation are like. But we're not interested in the truth of this sentence, hence we're not interested in who is the speaker, when it was said, and what the facts are like.

What we're interested in is the following: given a certain situation (any situation) at a certain time where a certain speaker (any speaker) utters the above sentence, and certain facts obtain in that situation (any combination of facts): do we judge the sentence true or false under those circumstantial conditions?

A semantic theory assumes that when we have set such contextual parameters, native speakers have the capacity to judge the truth or falsity of a sentence in virtue of the meanings of the expressions involved, i.e. in virtue of their semantic competence. And that is what we're interested in.

Semantic competence involves recognizing how truth values of sentences of your native language change, when you vary aspects of evaluation situations.

- vary the facts: make my green t-shirt yellow.
- vary the time: go to a point where I am 23.
- vary the speaker: go to a speaker who now is 23.
- vary the person pointed at: she has azure eyes.

Some of these aspects are linguistically creative, they get linguistically encoded in many languages, and classes of expressions, often cross-linguistically, are sensitive to this aspect, others are not.

i.e. Facts are less linguistically creative than time is: Changing the color of my shirt is not going affect the truth value of sentence that are not about me, but varying the time is. Languages evaluate relative to time and have time-operations, but they do no evaluate relative to fred-shirt-color, and they do not have fred-shirt-color operations.

To summarize: a semantic theory contains a theory of aboutness and this will include a theory of truth conditions.

Given the above, when I say truth, I really mean, truth relative to settings of contextual parameters.
Furthermore, given what I said before about realistic vs. idealistic interpretations of the domain of non-linguistic entities that the expressions are about, you should not necessarily think of truth in an absolute or realistic way: that depends on your ontological assumptions. If you think that semantics is directly about the real world as it is in itself, then truth means **truth in a real situation**. If you think that what we're actually talking about is a level of shared information about the 'real' world, then situations are shared conceptualizations, structurings of the real world, and truth means **truth in a situation which is a structuring of reality**. This difference has very few practical consequences for most actual semantic work: it concerns the interpretation of the truth definition rather than its formulation.

This is a gross overstatement, but for all the phenomena that we will be concerned with in this course, this is true enough. Specifying a precise theory of truth conditions, makes our semantic theory **testable**. We have a general procedure for defining a notion of **entailment** in terms of truth conditions. Once we have formulated a theory of the truth conditions of sentences containing the linguistic expressions whose semantics we are studying, our semantic theory gives a theory of what entailments we should expect for such sentences. Those predictions we can compare with our judgments, the intuitions concerning the entailments that such sentences actually have.

This may sound trivial, but it isn’t really. We will mention later Aristotle’s theory of the Syllogism, a theory which dominated logical thought for 2000 years, but which patently fails to makes any predictions at all about large classes of data that it is concerned with. [i.e. it is easy for interesting theories to be nevertheless inadequate]

Even for good ideas, it is easy for interesting theories to go wrong [even if at heart they are good, fruitful theories], and we will need to think about how to make them go right.

**Example: Event Theory**  
Event theory proposes that verbs have an event argument. The theory allows for insightful analyses of the semantics of adverbials, tense and aspect. A simple minded version of the theory is based on the following paraphrases:

*John kissed Mary*
Analysis: There is a kissing event in the past with John as kisser and Mary as kissee.

*John kissed Mary quickly*
Analysis: There is a kissing event in the past with John as kisser and Mary as kissee and that event was done in a quick manner.

*Some boy kissed Mary*
Analysis: There is a kissing event in the past with some boy as kisser and Mary as kissee.

*Some boy kissed some girl*
Analysis: There is a kissing event in the past with some boy as kisser and some girl as kissee.
Based on this, we would expect the following analysis:

Some boy kissed no girl
Analysis: There is a kissing event in the past with some boy as kisser and no girl as kissee.

But this analysis derives the wrong meaning: it says that some boy kissed a non-girl, which is, of course not what the sentence means.

It is easy to see what the most natural reading that the sentence *does* have should be:

*Some boy kissed no girl*
Analysis: There is a boy for which there isn’t a kissing event in the past with that boy as kisser and a girl as kissee.

There are *bona fide* versions of grammars using event theory that block the wrong readings and get the right readings. But in order to get this and maintain the advantages that were the rationale for introducing event theory in the first place requires a subtle balance and requires subtle thinking about the syntax-semantics relation.

The task of formulating elegant semantic theories that get the facts right is highly non-trivial, challenging (and fun!).

**End of Example.**

**David Lewis' Practical Guide:**
Do not ask what a meaning is, but what a meaning does, and find something that does that.

\[ \rightarrow \text{Intension of } \varphi: \text{ function from situations to truthvalues} \]

Intension of \( \varphi \) does (by and large) what we want a meaning to do.

This is not yet a theory: we need to specify what we put in situations (which distinctions are linguistically relevant)

Facts, time, speaker, events,…

When we fix that we have a precise theory of objects that do what we want meanings to do, a theory that makes predictions about entailments which can be checked with the facts.

If you tell me: 'but that's not what meanings are", I will ask you: ' Well, what more do you want meanings to do?''

- If you want meanings to do the dishes, intensions won't
- Possibly you find the particular notion of intention used not finegrained enough. In that case, I will try to make my situations more finegrained.

But the fact it that practically speaking the theories that *have* been developed are successful in dealing with a large number of phenomena, and in stating important generalizations.
V. COMPOSITIONALITY.

The interpretation of a complex expression is a function of the interpretations of its parts and the way these parts are put together.

Semantic theories differ of course in what semantic entities are assumed to be the interpretations of syntactic expressions. They share the general format of a compositional interpretation theory.

Let us assume that we have certain syntactic structures, say, the following trees:

In a compositional theory of interpretation, we choose semantic entities as the interpretations, meanings of the parts. This means that we start with meanings for the lexical items:

1. \( m(a(n)) \) \( m(\text{American}) \) \( m(\text{girl}) \)

What these are will depend, of course, on your semantics theory.

We assume that corresponding to the build up rules in the syntax, there are corresponding semantic interpretation rules.

First we make the standard assumption that the little trees projected from the lexicon have as their meaning the meanings of the lexical items:

\[
\begin{align*}
m(D) &= m(a) \\
m(\text{ADJ}) &= m(\text{American}) \\
m(N) &= m(\text{girl})
\end{align*}
\]

Next, we assume that, corresponding to the syntactic operation of adjunction forming a noun phrase out of an adjective and a noun(phrase), there is a semantic operation forming the meaning of the complex noun phrase as a function of the meaning of the adjective and the meaning of the noun. And the same for the operation forming a determiner phrase out of a determiner and a noun phrase.

Moreover, we have argued above that the semantic operation that combines an adjective with a noun and the semantic operation that combines a relative clause with a noun should be the same intersective semantic operation. This is a generalisation that we want to express in the grammar:

\[
\text{DET} + \text{NP} \Rightarrow \text{DP} \\
m(\text{DP}) = \text{OP}[m(\text{D}), m(\text{NP})]
\]
NP₁ + ADJUNCT ⇒ NP₂
m(NP₂) = OP₂[ m(ADJUNCT), m(NP₁) ]

Our grammatical assumption is that the same semantic operation corresponds in
adjunction in both trees.

Let us now make a specific assumption to illustrate compositionality.
We assume that the adjective American and the relative clause Who is American have
the same meaning.

Note, we do not have to make that assumption, but let us assume here that we are
dealing with a notion of meaning for which that is reasonable.

Assumption:

\[ m([_{CP} \text{ who is American}] = m(\text{American}) \]

In that case, the principle of compositionality tells that in this grammar the two trees
derived have the same meaning: in both cases we derive for the whole tree:

\[ \text{OP₁[ m(α), OP₂[ m(\text{American}), m(\text{girl}) ] ]} \]

It is easy to see that the principle of Compositionality of Meaning entails a principle
of Substitution of Meaning:

Let \( \alpha \) be an expression and \( \beta \) be a syntactic part of \( \alpha \).
For our purposes here, we take this to mean that the grammar makes \( \beta \) a constituent of
\( \alpha \).
Let \( \gamma \) be an expression that can be grammatically successfully substituted for \( \beta \) in \( \alpha \)
Let \( \alpha[\gamma/\beta] \) be the result of substituting \( \gamma \) for \( \beta \) in \( \alpha \).
The grammatical analysis of \( \alpha[\gamma/\beta] \) has a tree corresponding to \( \gamma \) at the node where \( \alpha \)
has a tree corresponding to \( \beta \).

Compositionality entails Substitution:

**Substitution:** if \( m(\beta) = m(\gamma) \) then \( m(\alpha) = m(\alpha[\gamma/\beta]) \)

If you substitute in an expression a sub-expression \( \beta \) by an expression \( \gamma \) with the same
meaning, the meaning of the whole stays the same.
ARGUMENTS FOR COMPOSITIONALITY

1. *A priori* arguments.
Compositionality is semantic recursiveness. Frege 1918 *Der Gedanke* gives in essence the same argument for semantics as Chomsky for syntax later:
We understand sentences that we have never heard before. Sentence comprehension cannot be a creative exercise because we do it fast, on-line. It is not clear how this could possible work without assuming compositionality.

2. *Practical* arguments.
The meaning of a complex expression is a network of interacting factors: i.e. interesting phenomena on the intersection of aspect, quantification, mass-noun distinctions, plurality, etc. etc.
Compositionality is **analysis**, it separates the semantic contributions of the parts and the contribution of the semantic **glue**. So it helps you in telling in a complex of interacting factors which bits or meaning are contributed by what.

The compositional analysis in 2 allows you formulate your semantic generalizations at the appropriate level of constituent meaning.
For instance, **intersectivity** is a semantic correlate of the **adjunction operation**.
I. SET THEORY (Cantor, Boole)

Set Theory is based on the element-of relation $\in$.

The fundamental properties of sets and the element-of relation are given by the following principles:

**Separation**: Given a domain $D$ and a property $P$, we can form the set of all objects in $D$ that have property $P$: $\{x \in D : P(x)\}$.

-We write \{a,b,c\} for the set $\{x \in D : x = a \text{ or } x = b \text{ or } x = c\}$.

**Extensionality**: sets are only determined by their elements:

$$A = B \iff \text{for every } a \in D: a \in A \iff a \in B$$

-It follows from extensionality that \{c,b,a,c\} = \{a,b,c\}
-It follows from Separation that, if there is a domain $D$, there is an empty set, a set with no elements (because we can define the set of all elements of $D$ that have the property of being non-identical to itself).
-It follows from Extensionality that there is only one empty set (because any two empty sets have the same elements, and hence are identical):

**Empty set**: The empty set, $\emptyset = \{x \in D: x \neq x\}$

(\neq: 'is not identical to')

From now on we write $A,B,C$ for sets of objects in domain $D$.

**Subset relation**: $A$ is a subset of $B$, $A \subseteq B$, iff for every $a \in D$: if $a \in A$ then $a \in B$.

**FACTS about $\subseteq$**:

-For every set $A$: $\emptyset \subseteq A$ (reflexivity)
-For every sets $A,B,C$: if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (transitivity)
-For every sets $A,B$: if $A \subseteq B$ and $B \subseteq A$ then $A=B$ (anti-symmetry)

**Union**: The union of $A$ and $B$, $A \cup B = \{x \in D : x \in A \text{ or } x \in B\}$

**FACTS about $\cup$ and $\subseteq$**:

-For every $A$: $A \cup A = A$ (idempotency)
-For every $A,B$: $A \cup B = B \cup A$ (commutativity)
-For every $A,B,C$: $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)

-For every $A,B$: $A \cup B$ is the smallest set of elements of $D$ such that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ (the join of $A$ and $B$ in $D$)

**Intersection**: The intersection of $A$ and $B$, $A \cap B = \{x \in D : x \in A \text{ and } x \in B\}$

**FACTS about $\cap$ and $\subseteq$**:

-For every $A$: $A \cap A = A$ (idempotency)
-For every $A,B$: $A \cap B = B \cap A$ (commutativity)
-For every $A,B,C$: $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)

-For every $A,B$: $A \cap B$ is the biggest set of elements of $D$ such that $A \subseteq A \cap B$ and $B \subseteq A \cap B$ (the meet of $A$ and $B$ in $D$)
FACTS about $\cup$, $\cap$ and $\subseteq$:

- for every $A,B$: $A \cap (B \cup A) = A$  
  (absorption)
- for every $A,B$: $A \cup (B \cap A) = A$  
  (absorption)
- for every $A,B,C$: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributivity)
- for every $A,B,C$: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributivity)

**Complement:** The complement of $B$ in $A$, $A - B = \{a \in D : a \in A \text{ and } a \notin B\}$  
($\notin$ : 'is not an element of')

The complement of $B$, $-B = D - B$

FACTS about $-:$:

- $-\emptyset = D$  
  (laws of 0 and 1)
- $-D = \emptyset$  
  (  )
- for every $A$: $A \cup -A = D$  
  (  )
- for every $A$: $A \cap -A = \emptyset$  
  (  )
- for every $A$: $-A = A$  
  (double negation)
- for every $A,B$: $-(A \cap B) = (-A \cup -B)$ (de Morgan laws)
- for every $A,B$: $-(A \cup B) = (-A \cap -B)$ (de Morgan laws)

**Cardinality:** The cardinality of $A$, $|A|$ is the number of elements of $A$.

**Powerset:** The powerset of $A$, $\text{pow}(A) = \{B : B \subseteq A\}$

FACT about pow:

- If $A$ has $n$ elements, $\text{pow}(A)$ has $2^n$ elements.
- $\text{pow}(\emptyset) = \{\emptyset\}$
- $\text{pow}\{a\} = \{\emptyset, \{a\}\}$
- $\text{pow}\{a,b\} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$
- $\text{pow}\{\{a,b,c\}\} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

**Ordered pairs:**

A set with one element we call a **singleton** set.

A set with two elements we call an **unordered pair**.

Unordered means that $\{a,b\} = \{b,a\}$.

The **ordered pair** of $a$ and $b$, $\langle a,b \rangle$ differs from the unordered pair in that the order of the elements is fixed. Ordered pairs satisfy the following condition:

$\langle a_1,a_2 \rangle = \langle b_1,b_2 \rangle$ iff $a_1 = b_1$ and $a_2 = b_2$.

We understand the notion of ordered pair such that while $\{a,a\} = a$, $\langle a,a \rangle \neq a$.

FACT: if $a \neq b$, then $\langle a,b \rangle \neq \langle b,a \rangle$

Similarly, we call $\langle a,b,c \rangle$ an **ordered triple**. We use quadruple, quintuple, sextuple, etc. The general case we call an ordered $n$-tuple:

$\langle a_1,\ldots,a_n \rangle$ with $n$ a number is an **ordered $n$-tuple**.

**Cartesian product:** The cartesian product of $A$ and $B$,

$A \times B = \{\langle a,b \rangle : a \in A \text{ and } b \in B\}$

Similarly, the cartesian product of $A$, $B$ and $C$ is:

$A \times B \times C = \{\langle a,b,c \rangle : a \in A \text{ and } b \in B \text{ and } c \in C\}$
Given this, \( A \times A = \{<a,b>: a, b \in A}\). We also write \( A^2 \) for \( A \times A \)

Similarly, \( A^3 = A \times A \times A = \{<a,b,c>: a,b,c \in A}\)

**FACT:**
- If \( |A| = n \) and \( |B| = m \) then \( |A \times B| = n \times m \)
- Hence \( |A^2| = |A|^2, |A^3| = |A|^3 \), etc.

\[
\{-a,b\} \times \{c,d,e\} = \{<a,c>,<a,d>,<a,e>,<b,c>,<b,d>,<b,e>\}
\]

\[
\{-a,b\}^2 = \{a,b\} \times \{a,b\} = \{<a,a>,<a,b>,<b,a>,<b,b>\}
\]

**Relations:**

\( R \) is a (two-place) **relation** between \( A \) and \( B \) iff \( R \subseteq A \times B \)

Hence: the set of all (two-place) relations between \( A \) and \( B \) is \( \text{pow}(A \times B) \).

\( R \) is a (two place) relation on \( A \) iff \( R \subseteq A \times A \).

Hence \( \text{pow}(A^2) \) is the set of all (two-place) relations on \( A \).

Similarly, the set of all three-place relations on \( A, B \) and \( C \) is \( \text{pow}(A \times B \times C) \), the set of all three-place relations on \( A \) is \( \text{pow}(A^3) \), and the set of all \( n \)-place relations on \( A \) is the set: \( \text{pow}(A^n) \).

**Note:** We sometimes make the notational convention: \( <a> = a \). If we do that, we can write \( A^1 \) for \( A \). On this notation \( \text{pow}(A) = \text{pow}(A^1) \), the set of all one-place relations on \( A \), also called properties, is the set of all subsets of \( A \).

**Domain and range:**

Let \( R \) be a two-place relation between \( A \) and \( B \), \( R \subseteq A \times B \).

The **domain** of \( R \), \( \text{dom}(R) = \{a \in A: \text{for some } b \in B: <a,b> \in R\} \)

The **range** of \( R \), \( \text{ran}(R) = \{b \in B: \text{for some } a \in A: <a,b> \in R\} \)

Let \( A = \{a,b,c\}, B = \{a,c,d,e\}, R = \{<a,a>,<a,c>,<a,d>,<b,d>\}. \)

Then \( \text{dom}(R) = \{a,b\}, \text{ran}(R) = \{a,c,d\} \).

**Converse relation, total relation, empty relation:**

Let \( R \subseteq A \times B \) be a relation between \( A \) and \( B \).

The **converse relation** of \( R \), \( R^c = \{<b,a>: <a,b> \in R\} \)

\( A \times B \) is itself a relation between \( A \) and \( B \), we call it the **total** relation (everything relates to everything else).

\( \emptyset \) is also a relation between \( A \) and \( B \), we call it the **empty** relation (nothing relates to anything).

**Functions:**

\( f \) is a (one-place, total) **function** from \( A \) into \( B \), \( f: A \to B \) iff:

1. \( f \) is a relation between \( A \) and \( B \): \( f \subseteq A \times B \).
2. \( \text{dom}(f) = A \) and \( \text{ran}(f) \subseteq B \)
   i.e. for every \( a \in A \) there is a \( b \in B \) such that \( <a,b> \in f \).
3. for every \( a \in A, b_1, b_2 \in B \): if \( <a,b_1> \in f \) and \( <a,b_2> \in f \) then \( b_1 = b_2 \).

If \( \text{dom}(f) \subseteq A \) and the other conditions hold we call \( f \) a **partial** (one-place) function from \( A \) into \( B \).

When I say function, I mean total function unless I tell you differently explicitly.

The intuition is: a function from \( A \) into \( B \) takes each element of \( A \) and maps it onto an element of \( B \).
Arguments and values:
We call the elements of the domain of \( f \) the **arguments** of \( f \), and the elements of the range of \( f \) the **values** of \( f \).
A function maps each argument in its domain on one and only one value in its range.
So: each argument has a value, and no argument has more than one value.
(But note, different arguments may have the same value.)

We write: \( f(a)=b \) for \( \langle a,b \rangle \in f \).

Example: Let \( A = \{a,b,c\} \) and \( B = \{0,1\} \).
\[ f = \{ \langle a,1 \rangle, \langle b,1 \rangle, \langle c,0 \rangle \} \]
is a function from \( A \) into \( B \).

We also use the following notation for \( f \):
\[
\begin{align*}
f: & \quad a \rightarrow 1 \\
     & \quad b \rightarrow 1 \\
     & \quad c \rightarrow 0
\end{align*}
\]

\[ n \text{ place operations:} \]
If \( f: A \rightarrow A \) we call \( f \) a (one-place) **operation** on \( A \).

We call a function \( f: A \times B \rightarrow C \) a two-place function from \( A \) and \( B \) into \( C \).
If \( f: A \times A \rightarrow A \), we call \( f \) a two-place operation on \( A \).
Similarly, \( f: A^n \rightarrow A \) is an \( n \)-place operation on \( A \).

Function space: The **function space** of \( A \) and \( B \): \( (A \rightarrow B) = \{ f: f: A \rightarrow B \} \)
The function space of \( A \) and \( B \) is the set of all functions from \( A \) into \( B \).
This is also notated as \( B^A \).

**FACTS:**
- \( |(A \rightarrow B)| = |B|^{|A|} \)
- \( \{(a,b,c) \rightarrow \{0,1\}\} = \{f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8\} \) where:

\[
\begin{align*}
f_1: & \quad a \rightarrow 1 \\
     & \quad b \rightarrow 1 \\
     & \quad c \rightarrow 1 \\
\end{align*}
\[
\begin{align*}
f_2: & \quad a \rightarrow 1 \\
     & \quad b \rightarrow 1 \\
     & \quad c \rightarrow 0 \\
\end{align*}
\[
\begin{align*}
f_3: & \quad a \rightarrow 1 \\
     & \quad b \rightarrow 0 \\
     & \quad c \rightarrow 1 \\
\end{align*}
\[
\begin{align*}
f_4: & \quad a \rightarrow 0 \\
     & \quad b \rightarrow 1 \\
     & \quad c \rightarrow 0 \\
\end{align*}
\[
\begin{align*}
f_5: & \quad a \rightarrow 0 \\
     & \quad b \rightarrow 0 \\
     & \quad c \rightarrow 1 \\
\end{align*}
\[
\begin{align*}
f_6: & \quad a \rightarrow 0 \\
     & \quad b \rightarrow 0 \\
     & \quad c \rightarrow 0 \\
\end{align*}
\[
\begin{align*}
f_7: & \quad a \rightarrow 0 \\
     & \quad b \rightarrow 0 \\
     & \quad c \rightarrow 1 \\
\end{align*}
\[
\begin{align*}
f_8: & \quad a \rightarrow 0 \\
     & \quad b \rightarrow 0 \\
     & \quad c \rightarrow 0 \\
\end{align*}
\]

Note that indeed \( |\{(0,1)\}|^{\{a,b,c\}} = 2^3 = 8 \)
Injections, surjections, bijections:
Let f: A → B be a function from A into B.

f is a **injection** from A into B, a one-one function from A into B iff
for every a₁, a₂ ∈ A: if f(a₁) = f(a₂) then a₁ = a₂.
i.e. no two arguments have the same value.

f is a **surjection** from A into B, a function from A onto B iff
for every b ∈ B there is an a ∈ A such that f(a) = b.
i.e. every element of b is the value of some argument in A.

f is a **bijection** from A into B iff f is an injection and a surjection
from A into B.

Inverse function:
If f: A → B is an injection from A into B, f is a bijection from A into ran(f).
In this case, f⁻¹, the converse relation of f, is itself a function from ran(f) into A (and in fact, also a
bijection). We call this the **inverse function** and write f⁻¹ for f⁻¹.

Identity function on A:
The identity function on A, idₐ, is the function idₐ: A → A such that
for every a ∈ A: idₐ(a) = a.
(the function that maps every element onto itself).

Constant functions:
A function f: A → B is constant iff for every a₁, a₂ ∈ A: f(a₁) = f(a₂).

If f is a constant function and the value is b, we call f the constant function on b (and write cₐ).

Characteristic functions:
Let B ⊆ A
The characteristic function of B in A is the function:
ch₄: A → {0, 1} defined by:
for every a ∈ A: ch₄(a) = 1 if a ∈ B
ch₄(a) = 0 if a ∉ B

Let f: A → {0, 1} be a function from A into {0, 1}
The subset of A characterized by f, ch₄ = {a ∈ A: f(a) = 1}.

FACT: The elements of pow(A) (the subsets of A) and the elements of (A → {0, 1})
(the functions from A into {0, 1}) are in one-one correspondence:
- each function in (A → {0, 1}) uniquely characterizes a subset of A.
- each subset of A has a unique characteristic function in (A → {0, 1}).

We say that the domains pow(A) and (A → {0, 1}) are **isomorphic**, they have the
same structure. Mathematically, we do not distinguish between isomorphic domains.
This means that, mathematically, we do not distinguish between sets and
characteristic functions.
This means that if we assume that \textit{walk} is interpreted as a set, the set of walkers, this is for all purposes \textbf{the same} as saying that \textit{walk} is interpreted as the function mapping each individual onto 1 if that individual is a walker, and onto 0 if that individual isn't. It also means that if we identify the intension of a sentence as the function which maps each situation onto 1 if the sentence is true in it, and onto 0 otherwise, this is for all purposes \textbf{the same} as saying that the intension of that sentence is identical to the set of all situations where it is true.

\textbf{Composition of functions:}

Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be functions. Then the \textbf{composition} of $f$ and $g$, $g \circ f$, (g over f, or g after f), is the following function from $A$ into $C$:

$g \circ f: A \rightarrow C$ is the function such that:

for every $a \in A$: $g \circ f(a) = g(f(a))$

Intuitively, the composition takes you in one step where the functions $f$ and $g$ take you in two steps.

Let MOTHER: IND $\rightarrow$ IND be the function which maps every individual on its mother, and FATHER: IND $\rightarrow$ IND the function which maps every individual on its father. Then MOTHER $\circ$ FATHER is the paternal grandmother function, mapping every individual onto the mother of its father. Similarly, MOTHER $\circ$ MOTHER is the maternal grandmother function, mapping every individual onto the mother of its mother.

Similarly, if we take a function INT: LIVING-IND $\rightarrow$ TIME INTERVALS which maps every individual alive now onto the maximal time interval that it has been alive in up to now, and we take a function TIME: TIME INTERVALS $\rightarrow$ NUMBERS which assigns to every time interval a length measured in terms of years (so, intervals smaller than a year are assigned 0, etc.), then the function AGE: LIVING-IND $\rightarrow$ NUMBERS defined by:

$\text{AGE} = \text{TIME} \circ \text{INT}$

assigns to every living individual its current age measured in years.
II. L₁, A LANGUAGE WITHOUT VARIABLES (Frege, Boole)

SYNTAX OF L₁

1. Lexicon of L₁

NAME = {JOHN, MARY,...}  The set of names.
PRED₁ = {WALK, TALK, BOY, GIRL,...}  The set of one-place predicates.
PRED₂ = {LOVE, KISS,...}  The set of two-place predicates.
NEG = {¬}  "not"
CONN = {∧, ∨, →}  "and", "or", "if...then..."

LEX = NAME ⊔ PRED₁ ⊔ PRED₂ ⊔ NEG ⊔ CONN

2. Sentences of L₁

FORM, the set of all formulas of L₁ is the smallest set such that:
1. If P ∈ PRED₁ and α ∈ NAME, then P(α) ∈ FORM.
2. If R ∈ PRED₂ and α, β ∈ NAME, then R(α, β) ∈ FORM.
3. If φ ∈ FORM, then ¬φ ∈ FORM.
4. If φ, ψ ∈ FORM, then (φ ∧ ψ) ∈ FORM.
5. If φ, ψ ∈ FORM, then (φ ∨ ψ) ∈ FORM.
6. If φ, ψ ∈ FORM, then (φ → ψ) ∈ FORM.

SEMANTICS FOR L₁

1. Models for L₁  (evaluation situations)

A Model for L₁ is a pair M = <D_M, F_M>, where:
1. D_M is a (non-empty) set, the domain of M.
2. F_M, the interpretation function for the lexical items, is a function such that:
   a. F_M is a function from names to individuals in D_M.
      F_M: NAME → D_M
      i.e. for every α ∈ NAME: F_M(α) ∈ D_M.
   b. F_M is a function from one-place predicates to sets of individuals:
      F_M: PRED₁ → pow(D_M)
      i.e. for every P ∈ PRED₁: F_M(P) ⊆ D_M.
   c. F_M is a function from two-place predicates to sets of pairs of individuals (two-place relations):
      F_M: PRED₂ → pow(D_M × D_M)
      i.e. for every R ∈ PRED₂: F_M(R) ⊆ D_M × D_M.
   d. F_M(¬): {0, 1} → {0, 1}
      F_M(¬) =
      \[
      \begin{pmatrix}
      0 & 1 \\
      1 & 0
      \end{pmatrix}
      \]
      F_M(¬) is a one-place truth function: a function from truth values to truth values.
e. $F_M(\land): \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

\[
F_M(\land) = \begin{cases}
<1,1> & \rightarrow 1 \\
<1,0> & \rightarrow 0 \\
<0,1> & \rightarrow 0 \\
<0,0> & \rightarrow 0 
\end{cases}
\]

e. $F_M(\lor): \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

\[
F_M(\lor) = \begin{cases}
<1,1> & \rightarrow 1 \\
<1,0> & \rightarrow 1 \\
<0,1> & \rightarrow 1 \\
<0,0> & \rightarrow 0 
\end{cases}
\]

g. $F_M(\rightarrow): \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

\[
F_M(\rightarrow) = \begin{cases}
<1,1> & \rightarrow 1 \\
<1,0> & \rightarrow 0 \\
<0,1> & \rightarrow 1 \\
<0,0> & \rightarrow 1 
\end{cases}
\]

$F_M(\land)$, $F_M(\lor)$ and $F_M(\rightarrow)$ are two-place truth function.

2. Recursive semantics for $L_1$

We define for every expression $\alpha$ of $L_1$, $[\alpha]_M$, the interpretation of $\alpha$ in model $M$:

1. If $\alpha \in \text{LEX}$, then $[\alpha]_M = F_M(\alpha)$
2. If $P \in \text{PRED}^1$ and $\alpha \in \text{NAME}$ then:
   $[P(\alpha)]_M = 1$ iff $[\alpha]_M \in [P]_M$; 0 otherwise.
3. If $R \in \text{PRED}^2$ and $\alpha, \beta \in \text{NAME}$ then:
   $[R(\alpha,\beta)]_M = 1$ iff $<[\alpha]_M, [\beta]_M> \in [R]_M$; 0 otherwise.
4. If $\varphi \in \text{FORM}$ then:
   $[\neg \varphi]_M = [\neg]_M ([\varphi]_M)$
5. If $\varphi, \psi \in \text{FORM}$ then:
   $[\varphi \land \psi]_M = [\land]_M ([\varphi]_M, [\psi]_M)$
6. If $\varphi, \psi \in \text{FORM}$ then:
   $[\varphi \lor \psi]_M = [\lor]_M ([\varphi]_M, [\psi]_M)$
7. If $\varphi, \psi \in \text{FORM}$ then:
   $[\varphi \rightarrow \psi]_M = [\rightarrow]_M ([\varphi]_M, [\psi]_M)$
COMPOSITIONALITY AND SEMANTIC GLUE.

If you're interested in the lexical meanings of predicates and relations, the semantics of $L_1$ is disappointing. The semantics for $L_1$ has nothing interesting to say about that.

Let us assume that you already know how naming works and what the meanings of the predicates and relations in $L_1$.

So, you're a grown-up person, so you know what kissing is: you know how to distinguish situations where it is kissing from situations where it is not. And you know that KISS means when it is.

What else do you need to know in order to know the semantics of $L_1$?

Two things:
1. **The meaning of the semantic glue.**
2. The meanings of the connectives $\neg$, $\land$, $\lor$, $\rightarrow$.

The meaning of the semantic glue is the most universal bit. Remember, compositionality says:

\[
\begin{align*}
\llbracket P(\alpha) \rrbracket_M &= \text{OPERATION}_1 \llbracket P \rrbracket_M, \llbracket \alpha \rrbracket_M \\
\llbracket R(\alpha,\beta) \rrbracket_M &= \text{OPERATION}_2 \llbracket R \rrbracket_M, \llbracket \alpha \rrbracket_M, \llbracket \beta \rrbracket_M \\
\llbracket \neg \phi \rrbracket_M &= \text{OPERATION}_3 \llbracket \neg \rrbracket_M, \llbracket \phi \rrbracket_M \\
\llbracket (\phi \land \psi) \rrbracket_M &= \text{OPERATION}_4 \llbracket \land \rrbracket_M, \llbracket \phi \rrbracket_M, \llbracket \psi \rrbracket_M 
\end{align*}
\]

In order to master the semantics of $L_1$, you need to know what the operations $\text{OPERATION}_1$... $\text{OPERATION}_4$ are.

The idea of the semantics given is that there really is only one operation which is the interpretation of the semantic glue:

\[
\text{OPERATION}[ F, A_1, \ldots, A_n ] = F(A_1, \ldots, A_n)
\]

the result of applying function entity $F$ to argument entities $A_1$...$A_n$

So:

*In the semantics for $L_1$, the semantic glue is interpreted as function-argument application.*
This idea applies directly to OPERATION\textsubscript{3} and OPERATION\textsubscript{4}:
- we interpret \( \neg \) as a truth function \( \llbracket \neg \rrbracket_M: \{0,1\} \rightarrow \{0,1\} \) and any \( \phi \) as a truth value \( \llbracket \phi \rrbracket_M \in \{0,1\} \).

\[
\llbracket \neg \phi \rrbracket_M = \text{OPERATION}[\llbracket \neg \rrbracket_M, \llbracket \phi \rrbracket_M] = \\
\llbracket \neg \rrbracket_M (\llbracket \phi \rrbracket_M) \\
\llbracket \neg \rrbracket_M (\llbracket \phi \rrbracket_M) \in \{0,1\}
\]

- we interpret \( \land \) as a truth function \( \llbracket \land \rrbracket_M: \{0,1\} \times \{0,1\} \rightarrow \{0,1\} \) and any \( \phi \) and \( \psi \) as a truth values \( \llbracket \phi \rrbracket_M, \llbracket \psi \rrbracket_M \in \{0,1\} \).

\[
\llbracket \phi \land \psi \rrbracket_M = \text{OPERATION}[\llbracket \land \rrbracket_M, \llbracket \phi \rrbracket_M, \llbracket \psi \rrbracket_M] = \\
\llbracket \land \rrbracket_M (\llbracket \phi \rrbracket_M, \llbracket \psi \rrbracket_M) \\
\llbracket \land \rrbracket_M (\llbracket \phi \rrbracket_M, \llbracket \psi \rrbracket_M) \in \{0,1\}.
\]

The idea applies indirectly to OPERATION\textsubscript{1} and OPERATION\textsubscript{2}. The first argument of the operation is not a function, but a set (a set of individuals for OPERATION\textsubscript{1}, a set of ordered pairs of individuals for OPERATION\textsubscript{2}).

But we have learned that we can switch between sets and characteristic functions.
Instead of letting OPERATION operate on X, we can let OPERATION operate on \( \text{ch}_X \):
- If \( X \subseteq D_M \), then \( \text{ch}_X: D_M \rightarrow \{0,1\} \)
  for every \( d \in D_M \): \( \text{ch}_X(d) = 1 \) iff \( d \in X \)

So: \( \text{ch}_{P_M}: D_M \rightarrow \{0,1\} \)
for every \( d \in D_M \): \( \text{ch}_{P_M}(d) = 1 \) iff \( d \in \llbracket P \rrbracket_M \)

- If \( Y \subseteq D_M \times D_M \), then \( \text{ch}_Y: D_M \times D_M \rightarrow \{0,1\} \)
  for every \( \langle d_1, d_2 \rangle \in D_M \times D_M \): \( \text{ch}_Y(\langle d_1, d_2 \rangle) = 1 \) iff \( \langle d_1, d_2 \rangle \in Y \)

So: \( \text{ch}_{R_M}: D_M \times D_M \rightarrow \{0,1\} \)
for every \( \langle d_1, d_2 \rangle \in D_M \times D_M \): \( \text{ch}_{R_M}(\langle d_1, d_2 \rangle) = 1 \) iff \( \langle d_1, d_2 \rangle \in \llbracket R \rrbracket_M \)

Now we can assume that OPERATION\textsubscript{1} and OPERATION\textsubscript{2} are the very same operation OPERATION of functional application:

\[
\llbracket P(\alpha) \rrbracket_M = \text{OPERATION} [\text{ch}_{P_M}, \llbracket \alpha \rrbracket_M] = \\
\text{ch}_{P_M}(\llbracket \alpha \rrbracket_M) \\
\text{ch}_{P_M}(\llbracket \alpha \rrbracket_M) \in \{0,1\}
\]

This specifies exactly what we specified in the semantics for L\textsubscript{1}:

\[
\llbracket P(\alpha) \rrbracket_M = 1 \iff \llbracket \alpha \rrbracket_M \in \llbracket P \rrbracket_M; 0 \text{ otherwise.}
\]
\[ [R(\alpha,\beta)]_M = \text{OPERATION} [ \text{ch}_{RM}([\alpha]_M, [\beta]_M) = \\
\text{ch}_{RM}([\alpha]_M, [\beta]_M) \\
\text{ch}_{RM}([\alpha]_M, [\beta]_M) \in \{0,1\} \]

This specifies \textbf{exactly} what we specified in the semantics for \( L_1 \):

\[ [R(\alpha,\beta)]_M = 1 \text{ iff } <[\alpha]_M, [\beta]_M> \in [R]_M; \text{ 0 otherwise.} \]

Thus, the first thing we need to know to master the semantics of \( L_1 \) is the interpretation of the semantic glue:

\textbf{The semantic glue in} \( L_1 \) \textbf{is function-argument application.}

Function-argument application is one of the basic operations for building meanings. Later in this class, we will see (one instance of) a second basic operation for building meanings: \textbf{functional abstraction. General functional abstraction}, and also other operations, like \textbf{function composition} and \textbf{type shifting operations} we will not discuss in this class: they are discussed in Advanced Semantics.

So, if you have learned the meanings of the lexical items of \( L_1 \) (including those of the connectives), and, say, function-argument application is a universal cognitive capacity, then \textbf{the only thing} you need to learn to master the semantics of \( L_1 \) is the \textbf{syntax-semantics map}:

\begin{itemize}
  \item How to properly divide a complex expression into an expression denoting a funcion, and expressions denoting its arguments.
\end{itemize}

Arguably, this is eminently learnable: natural languages provide ample clues for this, in \( L_1 \) it is by and large written into the notation of the language.

This means that, we can prove for \( L_1 \) that if the meanings of the lexical items are learnable (and why shouldn't they), the semantics of the whole language is learnable.

The second thing we need to know is what the semantics of \( L_1 \) is \textbf{really} a theory about: the meanings of the connectives \( \neg, \wedge, \vee, \rightarrow \).

Really the only interesting \textbf{predictions} of the semantics given for \( L_1 \) concern the interrelations between those meanings:
3. Entailment for $L_1$

Let $\phi, \psi \in \text{FORM}, \Delta \subseteq \text{FORM}$

We write $\phi \Rightarrow \psi$ for $\phi$ entails $\psi$:

$$\phi \Rightarrow \psi \iff \text{for every } M: \text{if } \llbracket \phi \rrbracket_M = 1 \text{ then } \llbracket \psi \rrbracket_M = 1$$

on every model where $\phi$ is true, $\psi$ is true as well.

$$\Delta \Rightarrow \psi \iff \text{for every } M: \text{if for every } \phi \in \Delta: \llbracket \phi \rrbracket_M = 1 \text{ then } \llbracket \psi \rrbracket_M = 1$$

on every model where all the premises in $\Delta$ are true, $\psi$ is true as well.

$\phi$ and $\psi$ are equivalent, $\phi \iff \psi$ iff $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$.

So:

$$\phi \iff \psi \iff \text{for every } M: \llbracket \phi \rrbracket_M = 1 \text{ iff } \llbracket \psi \rrbracket_M = 1$$

$\phi$ and $\psi$ are true on exactly the same models.

**FACT:**

For any $\phi \in \text{FORM}$:

$$\neg\neg \phi \iff \phi$$

Namely:

For every $M$:

1. $\llbracket \neg \neg \phi \rrbracket_M = 1$ iff
2. $\neg \llbracket \phi \rrbracket_M = 1$ iff
3. $F_M(\neg) \llbracket \neg \phi \rrbracket_M = 1$ iff
4. $\begin{cases} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{cases}$ ( $\neg \phi$ ) $\llbracket \neg \phi \rrbracket_M = 1$ iff
5. $\llbracket \neg \phi \rrbracket_M = 0$ iff
6. $\neg \llbracket \phi \rrbracket_M = 0$ iff
7. $F_M(\neg) ( \llbracket \phi \rrbracket_M ) = 0$ iff
8. $\begin{cases} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{cases}$ ( $\phi$ ) $\llbracket \phi \rrbracket_M = 0$ iff
9. $\llbracket \phi \rrbracket_M = 1$
FACT:
Let $\phi, \psi \in \text{FORM}:
\{ (\phi \lor \psi), \neg \phi \} \Rightarrow \psi

Namely:
(1) Assume $\sem(\phi \lor \psi)_M = 1$ and $\sem(\neg \phi)_M = 1$.
(2) Then $\sem(\lor)_M (\langle \sem\phi_M, \sem\psi_M \rangle >) = 1$ and $\sem(\neg \phi)_M = 1$.
(3) Then $F_M(\lor) (\langle \sem\phi_M, \sem\psi_M \rangle >) = 1$ and $F_M(\neg \phi)_M = 1$.
(4) Then $\begin{bmatrix}
<1,1> \rightarrow 1 \\
<1,0> \rightarrow 1 \\
<0,1> \rightarrow 1 \\
<0,0> \rightarrow 0
\end{bmatrix}
(\sem\phi_M, \sem\psi_M >) = 1$ and $\begin{bmatrix}
1 \rightarrow 0 \\
0 \rightarrow 1
\end{bmatrix}
(\sem\phi_M) = 1$.

Then, since, the (a) and the (b) cases are impossible, the (c) case holds, so:
(5) Then $\sem\phi_M = 0$ and $\sem\psi_M = 1$.
(6) Then $\sem\psi_M = 1$.

Other facts:
$(\phi \land \psi) \Leftrightarrow \neg (\neg \phi \lor \neg \psi)$
$(\phi \lor \psi) \Leftrightarrow \neg (\neg \phi \land \neg \psi)$
III. QUANTIFIERS AND VARIABLES (Frege)

(1) a. Mary sings.
    b. SING(m)

\[\llbracket \text{SING}(m) \rrbracket_M = 1 \text{ iff } F_M(m) \in F_M(\text{SING})\]

(2) a. Everybody sings.
    b. SING(everybody)
(3) a. Somebody sings.
    b. SING(somebody)
(4) a. Nobody sings.
    b. Sing(nobody).

\[\llbracket \text{SING}(\alpha) \rrbracket_M = 1 \text{ iff } F_M(\alpha) \in F_M(\text{SING})\]

So: \(F_M(\text{everybody}), F_M(\text{somebody}), F_M(\text{nobody}) \in D_M\)

Problem 1: \(F_M(\text{nobody}) \in D_M\)?

Alice: I saw nobody on the road.
The white king: I wish I had your eyes.

Problem 2: No predictions about entailment patterns:

I. Every girl sings. \(\text{SING}(\alpha)\) 
   Mary is a girl. \(\text{GIRL}(\beta)\) 
   entails Mary sings. \(\text{SING}(\beta)\)

II. No girl sings. \(\text{SING}(\alpha)\) 
    Mary is a girl. \(\text{GIRL}(\beta)\) 
    entails Mary doesn’t sing. \(\neg \text{SING}(\beta)\)

III. Some boy kisses every girl. \(\text{KISS}(\alpha, \beta)\) 
    Mary is a girl. \(\text{GIRL}(\gamma)\) 
    entails Some boy kisses Mary. \(\text{KISS}(\alpha, \gamma)\)

Problem 3: Wrong predictions about entailment patterns.

\(F_M(\text{SING}) \cup (D_M - F_M(\text{SING})) = D_M\)
\(F_M(\text{SING}) \cap (D_M - F_M(\text{SING})) = \emptyset\)

Hence:
for every model \(M\) and every \(\alpha \in \text{name}\): \[\llbracket \text{SING}(\alpha) \lor \neg \text{SING}(\alpha) \rrbracket_M = 1\]
\(\text{SING}(\alpha) \lor \neg \text{SING}(\alpha)\) is a **tautology**.
for every model \(M\) and every \(\alpha \in \text{name}\): \[\llbracket \text{SING}(\alpha) \land \neg \text{SING}(\alpha) \rrbracket_M = 0\]
\(\text{SING}(\alpha) \land \neg \text{SING}(\alpha)\) is a **contradiction**.
Ok for names:

(5) a. Mary sings or Mary doesn't sing. Tautology
    b. Mary sings and Mary doesn't sing. Contradiction

But not for the others:

(6) Every girl sings or every girl doesn't sing. No tautology
(7) Some girl sings and some girl doesn't sing. No contradiction

Problem: (6) is predicted to be a tautology, (7) is predicted to be a contradiction.

Aristotle: partial account of the entailment problem:
Stipulation of a set of entailment rules (syllogisms).
Problems:
- Rules are stipulated, not explained by the meanings of the expressions involved.
- Only for noun phrases in subject position: 2000 years of logic failed to come up with a satisfactory set of rules for entailments like those in (III) or the following:

(8) a. Some boy gave every girl her favorite flower
    b. Mary is a girl and her favorite flower is a Lily.
    c. Some boy gives Mary a Lily.

All these problems were solved once and for all in 1879 in Gottlob Frege's *Begriffsschrift*.

Frege's solution: quantifiers and variables.
Frege: Do not analyse Everybody sings as SING(everybody), but analyse Everybody sings in two stages:

STAGE 1: Replace everybody in Everybody sings by a pronoun: he:

he sings \( SING(x) \)

This is a sentence whose truth value depends on what you are pointing at.

STAGE 2: Let everybody express a constraint on what you are pointing at:

For every pointing with he: he sings \( \forall x[SING(x)] \)

Note: this is not Frege's notation, and while Frege gave the idea of the semantics intuitively, he didn't give the semantics: he gave a set of inference rules fitting this semantics.

Everybody sings.
For every pointing with he: he sings \( \forall x[SING(x)] \)

Somebody sings.
For some pointing with he: he sings \( \exists x[SING(x)] \)

Nobody sings.
For no pointing with he: he sings \( \neg \exists x[SING(x)] \)
Every girl sings.
For every pointing with she: if she is a girl, then she sings
\( \forall x [\text{GIRL}(x) \rightarrow \text{SING}(x)] \)

Some girl sings.
For some pointing with she: she is a girl and she sings.
\( \exists x [\text{GIRL}(x) \land \text{SING}(x)] \)

No girl sings.
For no pointing with she: she is a girl and she sings.
\( \neg \exists x [\text{GIRL}(x) \land \text{SING}(x)] \)

-Frege's inference rules for these expressions predict the entailments in I and II.

I \( \forall x [\text{GIRL}(x) \rightarrow \text{SING}(x)] \) GIRL(m) entails SING(m)

II \( \neg \exists x [\text{GIRL}(x) \land \text{SING}(x)] \) GIRL(m) entails \( \neg \text{SING}(m) \)

-Frege's solves the problem of tautologies and contradictions:

(6) Every girl sings or every girl doesn't sing.

The trick is to analyse every girl in every girl doesn't sing after doesn't, the same for some girl in some girl doesn't sing:

\( \forall x [\text{GIRL}(x) \rightarrow \text{SING}(x)] \lor \forall x [\text{GIRL}(x) \rightarrow \neg \text{SING}(x)] \) No tautology.

(7) Some girl sings and some girl doesn't sing.

\( \exists x [\text{GIRL}(x) \land \text{SING}(x)] \land \exists x [\text{GIRL}(x) \land \neg \text{SING}(x)] \) No contradiction.
Frege solves the problem of entailments for noun phrases not in subject position. 
Frege's solution: apply the same analysis in stages:

Some boy kisses every girl.  
Stage 1a. Replace every girl in this by a pronoun she (her):  
Some boy kisses her.  
Truth value depends on pointings for she  
Some boy kiss y

Stage 1b: every girl constrains pointings for she:  
For every pointing with she: if she is a girl, some boy kisses her  
∀y[GIRL(y) → some boy kisses y]

Stage 2a. Now replace some boy by a pronoun he:  
For every pointing with she: if she is a girl, he kisses her  
Truth value depends on pointings for he  
∀y[GIRL(y) → KISS(x,y)]

Stage 2b. some boy constrains pointings for he:  
For some pointing with he: he is a boy and for every pointing with she:  
if she is a girl, he kisses her.  
∃x[BOY(x) ∧ ∀y[GIRL(y) → KISS(x,y)]]

With this analysis, Frege doesn't have to stipulate anything separate for entailments for sentences with quantifiers not in subject position: the same inference rules predict the entailment pattern in III:

III  ∃x[BOY(x) ∧ ∀y[GIRL(y) → KISS(x,y)]]  
GIRL(m)  
entails  ∃x[BOY(x) ∧ KISS(x,m)]

After 2000 years of failure, this is very impressive!

Alfred Tarski developed the semantics for Frege's analysis in The Concept of Truth in Formalized Languages, first published in Polish in 1932. He did this by precisely specifying the notions of 'truth relative to a pointing for pronoun (s)he' and the notion of quantifiers as 'constraints on pointings for pronoun (s)he.' Frege told us what the meanings of quantifiers and variables do. Tarski told us, given that, what the meanings of quantifiers and variables are.
Frege's notation:

\[ \begin{align*}
\text{\( \psi \)} & \quad (\varphi \rightarrow \psi) \\
\text{\( \varphi \)} & \quad \neg \varphi \\
\text{\( a \quad P(a) \)} & \quad \forall x P(x)
\end{align*} \]

Our linear notation:

\[ \forall x \forall y ((P(x) \rightarrow (Q(y) \rightarrow \neg \forall z - R(x,y,z))) \]

becomes two-dimensional:

And we would need definitions of connectives corresponding to \( \land \), \( \lor \) and \( \exists \) to write the equivalent and more legible:

\[ \forall x \forall y ((P(x) \land Q(y)) \rightarrow \exists z R(x,y,z)) \]

Clearly bookprinters preferred the linear notation (which stems by and large from Peano).
IV. L₂, A LANGUAGE WITH VARIABLES

SYNTAX OF L₂

1. Lexicon of L₁
NAME = {JOHN, MARY, ...}
VAR = {x₁, x₂, ..., x, y, z} An infinite set of variables ('pronouns')
PRED¹ = {WALK, TALK, BOY, GIRL, ...}
PRED² = {LOVE, KISS, ...}
NEG = {¬}
CONN = {∧, ∨, →}
LEX = NAME ∪ PRED¹ ∪ PRED² ∪ NEG ∪ CONN
TERM = NAME ∪ VAR Terms are names or variables.

2. Sentences of L₂
FORM, the set of all formulas of L₂ is the smallest set such that:
1. If P ∈ PRED¹ and α ∈ TERM, then P(α) ∈ FORM.
2. If R ∈ PRED² and α, β ∈ TERM, then R(α, β) ∈ FORM.
3. If φ ∈ FORM, then ¬φ ∈ FORM.
4. If φ, ψ ∈ FORM, then (φ ∧ ψ) ∈ FORM.
5. If φ, ψ ∈ FORM, then (φ ∨ ψ) ∈ FORM.
6. If φ, ψ ∈ FORM, then (φ → ψ) ∈ FORM.

SEMANTICS FOR L₂

1. Models for L₂
A Model for L₂ is a pair M = <Dₘ, Fₘ>, where:
1. Dₘ, the domain of M, is a (non-empty) set.
2. Fₘ, the interpretation function for the lexical items of L₂, is given by:
a. Fₘ: NAME → Dₘ
   i.e. for every α ∈ NAME: Fₘ(α) ∈ Dₘ.
b. Fₘ: PRED¹ → pow(Dₘ)
   i.e. for every P ∈ PRED¹: Fₘ(P) ⊆ Dₘ.
c. Fₘ: PRED² → pow(Dₘ × Dₘ)
   i.e. for every R ∈ PRED²: Fₘ(R) ⊆ Dₘ × Dₘ.
d. Fₘ(¬): {0, 1} → {0, 1}
   Fₘ(¬) = \begin{bmatrix}
   0 & 1 \\
   1 & 0
\end{bmatrix}
e. Fₘ(∧): \{0, 1\} × \{0, 1\} → {0, 1}
   Fₘ(∧) = \begin{bmatrix}
   0 & 1, 1 > → 1 \\
   <1, 0> → 0 \\
   <0, 1> → 0 \\
   <0, 0> → 0
\end{bmatrix}
f. $F_M(\lor): \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

$$F_M(\lor) = \begin{cases} <1,1> & \rightarrow 1 \\ <1,0> & \rightarrow 1 \\ <0,1> & \rightarrow 1 \\ <0,0> & \rightarrow 0 \end{cases}$$

g. $F_M(\rightarrow): \{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

$$F_M(\rightarrow) = \begin{cases} <1,1> & \rightarrow 1 \\ <1,0> & \rightarrow 0 \\ <0,1> & \rightarrow 1 \\ <0,0> & \rightarrow 1 \end{cases}$$

2. Variable assignments.
Variables are not yet interpreted. We introduce pointing devices and call them variable assignments:

A variable assignment for $L_2$ on $M$ is a function $g: \text{VAR} \rightarrow D_M$, a function from variables to individuals. i.e. for every $x \in \text{VAR}$: $g(x) \in D_M$.

3. Recursive semantics for $L_2$.

We define the interpretation of $\alpha$ in model $M$, relative to variable assignment $g$:

1. If $\alpha \in \text{LEX}$, then $[\alpha]_{M,g} = F_M(\alpha)$
2. If $\alpha \in \text{VAR}$, then $[\alpha]_{M,g} = g(\alpha)$
3. If $P \in \text{PRED}^1$ and $\alpha \in \text{TERM}$ then:
   $[P(\alpha)]_{M,g} = 1$ iff $[\alpha]_{M,g} \in [P]_{M,g}$; 0 otherwise.
4. If $R \in \text{PRED}^2$ and $\alpha, \beta \in \text{TERM}$ then:
   $[R(\alpha,\beta)]_{M,g} = 1$ iff $<[\alpha]_{M,g}, [\beta]_{M,g}> \in [R]_{M,g}$; 0 otherwise.
5. If $\varphi \in \text{FORM}$ then:
   $[\neg \varphi]_{M,g} = [\neg]_{M,g} ([\varphi]_{M,g})$
6. If $\varphi, \psi \in \text{FORM}$ then:
   $[[\varphi \land \psi]_{M,g} = [[\land]_{M,g} ([\varphi]_{M,g}, [\psi]_{M,g})]
7. If $\varphi, \psi \in \text{FORM}$ then:
   $[[\varphi \lor \psi]_{M,g} = [[\lor]_{M,g} ([\varphi]_{M,g}, [\psi]_{M,g})]
8. If $\varphi, \psi \in \text{FORM}$ then:
   $[[\varphi \rightarrow \psi]_{M,g} = [[\rightarrow]_{M,g} ([\varphi]_{M,g}, [\psi]_{M,g})]
4. Truth for L$_2$. (Independent of assignments)
We define, for formulas of L$_2$, in terms of the recursively defined notion of 'interpretation in M relative to g' ([$\mathcal{I}$]|$_Mg$), the notions of 'truth in M' ([$\mathcal{I}$]|$_M= 1$) and 'falsity in M' ([$\mathcal{I}$]|$_M= 0$).

Let $\varphi \in \text{FORM}$:

- $[\varphi]|_M = 1$ iff for every assignment $g$ for L$_2$: $[\varphi]|_{M,g} = 1$
- $[\varphi]|_M = 0$ iff for every assignment $g$ for L$_2$: $[\varphi]|_{M,g} = 0$

3. Entailment for L$_2$: Defined in terms of [ ]$_M$.
Let $\varphi, \psi \in \text{FORM}$

$\varphi$ entails $\psi$, $\varphi \Rightarrow \psi$ iff for every model $M$ for L$_2$: if $[\varphi]|_M = 1$ then $[\psi]|_M = 1$

For formulas without variables we have:

FACT: if $\varphi$ is a formula without variables, then:

- for every model $M$: either $[\varphi]|_M=1$ or $[\varphi]|_M=0$

Formulas with variables are often neither true, nor false on a model (but undefined), because their truth varies with assignment functions.
Example:
Let $F_M(P) \subseteq D_M$, $d_1, d_2 \in D_M$ and $d_1 \in F_M(P)$, $d_2 \not\in F_M(P)$.
Let $g_1(x)=d_1$, $g_2(x)=d_2$.

Then: $[P(x)]|_M \neq 1$, because $[P(x)]|_{M,g_2} = 0$
$[P(x)]|_M \neq 0$, because $[P(x)]|_{M,g_1} = 1$

Not all formulas with variables come out as undefined, though:

- $[P(x) \lor \neg P(x)]|_M = 1$ iff
- for every $g$: $[P(x) \lor \neg P(x)]|_{M,g} = 1$ iff
- for every $g$: $g(x) \in F_M(P)$ or $g(x) \not\in F_M(P)$ iff
- for every $g$: $g(x) \in F_M(P)$ or $g(x) \in D_M - F_M(P)$ iff
- for every $g$: $g(x) \in D_M$.
So: $[P(x) \lor \neg P(x)]|_M = 1$

Similarly:

- $[P(x) \land \neg P(x)]|_M = 1$ iff
- for every $g$: $g(x) \in F_M(P)$ and $g(x) \in D_M - F_M(P)$
So: $[P(x) \land \neg P(x)]|_M = 0$

Hence, $P(x) \lor \neg P(x)$ is a tautology, and $P(x) \land \neg P(x)$ is a contradiction.
Later we will follow the logical tradition in defining entailment only for formulas whose truth doesn't vary with assignments (formulas without free occurrences of variables). But it is important to note that the technique applies correctly to formulas with free variables as well.
The technique of defining truth in M as truth relative to all variation parameters, and falsity as falsity relative to all variation parameters plays an important role in semantics (for instance in the analysis of vagueness). It is called the technique of super valuations. (van Fraasen)

**Excursus: Vagueness as a problem for many-valued logic** (Kamp)

We assume that our domain consists only of humans (for simplicity)

(1) Bob is male
(2) Bob is a typical adolescent, borderline between grown-up and not-grown up
(3) A grown-up male is a man
   A non-grown-up male is a boy

\[
\begin{align*}
\text{male}(bob)_{M,g} &= 1 \\
grown-up(bob)_{M,g} &= \bot \text{ and } \neg\text{grown-up}(bob)_{M,g} = \bot \\
\text{(undefined, or any intermediate value between 1 and 0, this is allowed in many-valued logic)}
\end{align*}
\]

\[
\begin{align*}
\text{man}(bob)_{M,g} &= 1 \text{ iff } \text{male}(bob)_{M,g} = 1 \text{ and } \text{grown-up}(bob)_{M,g} = 1 \\
\text{boy}(bob)_{M,g} &= 1 \text{ iff } \text{male}(bob)_{M,g} = 1 \text{ and } \text{grown-up}(bob)_{M,g} = 0
\end{align*}
\]

So:

\[
\begin{align*}
\text{man}(bob)_{M,g} &= \bot \\
\text{boy}(bob)_{M,g} &= \bot
\end{align*}
\]

**Kamp’s problem of conditions:**
Intuitively (1a) and (1b) are true:

(1) a. If Bob is grown-up, he is a man.
   b. If Bob is not grown-up, he is boy.

We show the problem with material conditional, though the problem can be reconstructed for other analyses of the conditional as well.

\[
\begin{align*}
[1] & \text{grown-up}(bob) \rightarrow \text{man}(bob)
\end{align*}
\]

working out the definition of man, this is equivalent to:

\[
\begin{align*}
[2] & \text{grown-up}(bob) \rightarrow (\text{male}(bob) \land \text{grown-up}(bob))
\end{align*}
\]

using the truth-table for the material implication ( \( \varphi \rightarrow \psi \) is equivalent to \( \neg\varphi \lor \psi \)), this is equivalent to:

\[
\begin{align*}
[3] & \neg\text{grown-up}(bob) \lor (\text{male}(bob) \land \text{grown-up}(bob))
\end{align*}
\]

using the distributive law ( \( \varphi \lor (\psi \land \chi) \) is equivalent to \( ((\varphi \lor \psi) \land (\varphi \land \chi)) \)), this is equivalent to:
[4] \((-\text{grown-up}(bob) \lor \text{male}(bob)) \land (-\text{grown-up}(bob) \lor \text{grown-up}(bob))\)

**The first conjunct:** \(-\text{grown-up}(bob) \lor \text{male}(bob)\)

\([[-\text{grown-up}(bob)]_{M,g} = \bot\text{ and } [\text{male}(bob)]_{M,g} = 1\]

In the strong-systems of many-valued logic, this means that the disjunction is true (one true disjunct is enough). So, let's assume that: the first conjunct is true.

We want [4] itself to come out as true. By the same reasoning as for disjunction, we will need both conjuncts in [4] to come out as true: in many-valued logic, if one of the conjuncts has a value less than 1, the conjunction will itself have a value less than 1, which is not good enough for us, because, with Kamp, we want (1a) to come out as true. So we need the second conjunct to come out as true.

**The second conjunct:** \(-\text{grown-up}(bob) \lor \text{grown-up}(bob)\)

We are now back to the problem of tautologies. The value of \text{grown-up}(bob) is intermediate between 0 and 1 (in three-valued logic \(\bot\), but the problem is the same for theories with more values, like fuzzy logic). This means by necessity in many-valued logic that the value of \(-\text{grown-up}(bob)\) is less than 1 (namely, depending on your definition of \(-\), 0 or intermediate).

But in many-valued logic, the disjunction of two values that are less than one is never 1. This means that the tautology does not come out as 1, and with that (1a) does not come out as true.

**Sketch of the solution (Kamp, Fine):**

- Truth of sentences with vague predicates depends on a contextual standard of precision and the ways in which this standard of precision can be refined.
- Three-valued logic: Bob is a borderline case of a man (boy), because our standard of precision doesn't count him among the grown-ups, nor among the non-grown-ups.
- Supervaluations: This means that some refinements of our standard of precision make him a grown-up, and some refinements of our standard of precision make him a non-grown-up. A completion of \(s\) is a refinement of \(s\) that makes all the predicates involved completely precise.

- Truth is 'super-truth': if \(s\) is our standard of precision, we define:
  \(\varphi\) is supertrue in \(s\) iff every completion of \(s\) makes \(\varphi\) true
  \(\varphi\) is superfalse in \(s\) iff every completion of \(s\) makes \(\varphi\) false

Results:

- \text{male}(bob) is supertrue it is already true in \(s\)
- \text{grown-up}(bob) is not supertrue some completion makes \text{grown-up}(bob) false
- \(-\text{grown-up}(bob)\) is not supertrue some completion makes \text{grown-up}(bob) true

but:

- \text{grown-up}(bob) \lor \text{where grown-up}(bob) is supertrue,
  because tautologies are true in all completions

and:

\text{grown-up}(bob) \rightarrow \text{man}(bob) is supertrue because in every completion where bob is counted among the grown-ups he is a grown-up male, hence a man.

Notice the parallel with the definition of truth/falsity as truth/falsity relative to all assignments.
V. L₃, A LANGUAGE WITH QUANTIFIERS AND VARIABLES

Syntax of L₃:
L₃ has the same syntax as L₂, except that we add two more clauses to the definition of FORM:

7. If x ∈ VAR and φ ∈ FORM, then ∀xφ ∈ FORM
8. If x ∈ VAR and φ ∈ FORM, then ∃xφ ∈ FORM

Semantic for L₃:
The notion of model for L₃ and variable assignment for L₃ on a model are the same as for L₂.

Note on compositionality:
I introduce the symbols ∀ and ∃ in the formula definition and not in the lexicon (such symbols are called syncategorematic, meaning, not part of a lexical category).
Similarly, I will specify the truth conditions of sentences with these symbols, but not give an explicit interpretation for them, i.e. their interpretation will be specified implicitly.
This is solely for your convenience. Just as in L₂ I defined explicitly Fₘ(¬) as a function, I can explicitly define Fₘ(∀) and Fₘ(∃) as functions.
But doing this is technically more involved.
The reason is that, whereas the operations introduced so far (like ¬, ∧, ∨) are extensional with respect to assignment functions (meaning that the interpretation of a complex in M relative to g, depends on the interpretations of the parts in M relative to that same g), the quantifiers are intensional with respect to assignment functions (meaning that the interpretation of a quantificational complex in M relative to g, depends on the interpretations of the parts in M relative to other assignments g').
And this means that if we want to introduce the interpretations of quantifiers explicitly, we need to introduce for their interpretations complex functions from sets of assignment functions to sets of assignment functions.
Since this is too technical at this point of the exposition, we explain for your convenience what a quantifier does in the theory, rather than what a quantifier is in the theory.
Importantly: this doesn't mean that the semantics for L₃ given is not compositional; it only means that for your convenience I won't work out all compositional details.

Resetting values of variables.
Let g be a variable assignment for L₃ on M, g: VAR → Dₘ
We define: gₓᵈ, the result of resetting the value of variable x in assignment g to object d.

\[ gₓᵈ = \text{the assignment function such that:} \]
1. for every y ∈ VAR−{x}: \( gₓᵈ(y) = g(y) \)
2. \( gₓᵈ(x) = d \)
i.e. gₓᵈ assigns to all variables except for x the same value as g assigns, but it assigns to variable x object d, it varies the value for variable x.
Example:

\[
g = \begin{cases}
  x_1 \rightarrow d_1 \\
  x_2 \rightarrow d_2 \\
  x_3 \rightarrow d_1 \\
  x_4 \rightarrow d_2
\end{cases}
\]

\[
g \times_{x_1} d_1 = \begin{cases}
  x_1 \rightarrow d_1 \\
  x_2 \rightarrow d_1 \\
  x_3 \rightarrow d_1 \\
  x_4 \rightarrow d_2
\end{cases}
\]

\[
g \times_{x_2} d_1 d_2 = \begin{cases}
  x_1 \rightarrow d_1 \\
  x_2 \rightarrow d_2 \\
  x_3 \rightarrow d_1 \\
  x_4 \rightarrow d_2
\end{cases}
\]

Compositional semantics for \( \forall x[\text{P}(x)] \).

The truth value \( [\forall x\text{P}(x)]_{M,g} \) is not enough to define compositionally the truth value \( [\forall x\text{P}(x)]_{M,g} \).

What you need is not the extension of \( \text{P}(x) \) in \( M \) relative to \( g \), the truth value in \( M \) relative to \( g \), but the pattern of variation of the extension, the truth value, of \( \text{P}(x) \) in \( M \), when you vary the value of \( x \).

Given \( D_M \) and \( g(x)=d_1 \).

The pattern of variation of the value of \( x \) over domain \( D_M \) is the list:

\[
\begin{cases}
  g_x^{d_1} : x \rightarrow d_1 \\
  g_x^{d_2} : x \rightarrow d_2 \\
  g_x^{d_3} : x \rightarrow d_3 \\
  \ldots \text{ for all } d \in D_M
\end{cases}
\]

The pattern of variation of the truth value of \( \text{P}(x) \) over domain \( D_M \) is the list:

\[
\begin{cases}
  < g_x^{d_1} : [\text{P}(x)]_{M,g x}^{d_1} > \\
  < g_x^{d_2} : [\text{P}(x)]_{M,g x}^{d_2} > \\
  < g_x^{d_3} : [\text{P}(x)]_{M,g x}^{d_3} > \\
  \ldots \text{ for all } d \in D_M
\end{cases}
\]

\( [\forall x\text{P}(x)]_{M,g} = 1 \) iff you get truth value 1 everywhere in the list.

Equivalently: \( \text{iff for every } d \in D_M : [\phi]_{M,g x}^d = 1; 0 \text{ otherwise} \)

\( [\exists x\text{P}(x)]_{M,g} = 1 \) iff you get truth value 1 somewhere in the list.

Equivalently: \( \text{iff for some } d \in D_M : [\phi]_{M,g x}^d = 1; 0 \text{ otherwise} \)

Moral: The meanings of expressions in predicate logic are not extensions, but these lists of assignment-extension pairs.
Explanation:

\[ \forall x \phi ]_{M,g} = 1/0? \quad [\exists x \phi ]_{M,g} = 1/0? \]

1. Form the list which varies in g the value of x through the domain:

\[
\begin{array}{cccc}
[\phi ]_{Mg}^d & [\phi ]_{Mg}^{d2} & [\phi ]_{Mg}^{d3} & [\phi ]_{Mg}^{d4} \\
g_x^d & g_x^{d2} & g_x^{d3} & g_x^{d4}
\end{array}
\]

2. Add the truth value of \([\phi ]_{M,h}\) relative to all these assignmens h;

\[
\begin{array}{cccc}
[\phi ]_{Mg}^d & [\phi ]_{Mg}^{d2} & [\phi ]_{Mg}^{d3} & [\phi ]_{Mg}^{d4} \\
g_x^d & g_x^{d2} & g_x^{d3} & g_x^{d4}
\end{array}
\]

1 1 1 0 ...

3. This is the relevant pattern of variation for \(\phi\). PV(\(\phi\))

The truth conditions say the following:

\[ [\forall x \phi ]_{M,g} = 1 \text{ if you only get 1’s in } PV(\phi) \]
\[ [\forall x \phi ]_{M,g} = 0 \text{ if you get one or more 0 in } PV(\phi) \]
\[ [\exists x \phi ]_{M,g} = 1 \text{ if you get one or more 1 in } PV(\phi) \]
\[ [\exists x \phi ]_{M,g} = 0 \text{ if you only get 0’s in } PV(\phi) \]

This means:

\[ [\forall x \phi ]_{M,g} = 1 \text{ iff for every } d \in D_M: [\phi ]_{M,gx}^d = 1 \]
\[ 0 \text{ iff for some } d \in D_M: [\phi ]_{M,gx}^d = 0 \]
\[ [\exists x \phi ]_{M,g} = 1 \text{ iff for some } d \in D_M: [\phi ]_{M,gx}^d = 1 \]
\[ 0 \text{ iff for every } d \in D_M: [\phi ]_{M,gx}^d = 0 \]
Tarski’s formalization of Frege’s intuition:
A Frege-Tarski-quantifier like $\forall x$ is a function that does **two things simultaneously**:  
1. The quantifier binds all occurrences of variable $x$ that are free in the input. 
What corresponds to this semantically is: the quantifier sets up a pattern of variation **for the input**. The occurrences of the variable $x$ are bound in this pattern of variation. **This bit is the same for all quantifiers.** 
2. The quantifier expresses a quantificational constraint, its particular lexical meaning. 
What corresponds to this semantically is: the quantifier expresses a constraint on the pattern of variation for the input. (i.e. the meaning of $\forall x$ tells you that you need to get value 1 at **every** place in the list, the meaning of $\exists x$ that you need to get value 1 at **some** place in the list. 

I will argue later that natural language semantics took off in the 1960s, when this analysis of quantification and binding was rejected for a similar, but nevertheless different analysis. But to understand that, we need to understand the Frege-Tarski analysis first.

**Recursive semantics for $L_3$:** 
We define $[a]_{M,g}$ in exactly the same way as for $L_2$, except that we add two interpretation clauses:

8. If $x \in \text{VAR}$ and $\varphi \in \text{FORM}$ then: 
$$[\forall x \varphi]_{M,g} = 1 \text{ iff for every } d \in D_M: [\varphi]_{M,gx^d} = 1; 0 \text{ otherwise}$$

9. If $x \in \text{VAR}$ and $\varphi \in \text{FORM}$ then: 
$$[\exists x \varphi]_{M,g} = 1 \text{ iff for some } d \in D_M: [\varphi]_{M,gx^d} = 1; 0 \text{ otherwise}$$

**Truth and entailment**: See below.
VI. L₄, FULL PREDICATE LOGIC WITH IDENTITY

Syntax of L₄

CON = \{c₁, c₂, \ldots\} \quad \text{The set of individual constants (= names)}

For every n>0: PREDₙ = \{Pⁿ₁, Pⁿ₂, \ldots\} \quad \text{The set of n-place predicates.}

(For CON and each PREDₙ you choose which and how many elements these sets have in L₄.)

VAR = \{x₁, x₂, \ldots\} \quad \text{The set of variables.}

(VAR contains infinitely many variables.)

NEG = \{¬\}, CONN = \{∧, ∨, →\}

LEX = CON \cup PREDⁿ \cup NEG \cup CONN (for each n>0)

TERM = CON \cup VAR

FORM is the smallest set such that:

1. If P ∈ PREDⁿ and α₁,...,αₙ ∈ TERM, then P(α₁,...,αₙ) ∈ FORM
2. If α₁,α₂ ∈ TERM, then (α₁=α₂) ∈ FORM
3. If φ,ψ ∈ FORM, then ¬φ, (φ \land ψ), (φ \lor ψ), (φ \rightarrow ψ) ∈ FORM
4. If x ∈ VAR and φ ∈ FORM, then \(\forall xφ\), \(\exists xφ\) ∈ FORM

Semantics for L₄.

A model for L₄ is a pair \(M = <D_M, F_M>\), where:

1. \(D_M\), the domain of M, is a non-empty set.
2. \(F_M\), the interpretation function for M, is given by:
   a. for every c ∈ CON: \(F_M(c) ∈ D_M\)
   b. for every P ∈ PREDⁿ: \(F_M(P) ⊆ (D_M)^n\)

Here

\((D_M)^1 = D_M\)
\((D_M)^2 = D_M \times D_M\)
\((D_M)^3 = D_M \times D_M \times D_M\)

etc.

d. \(F_M(¬): \{0,1\} → \{0,1\}\)
\(F_M(¬) = \begin{pmatrix}
0 & → & 1 \\
1 & → & 0
\end{pmatrix}

e. \(F_M(∧): \{0,1\} \times \{0,1\} → \{0,1\}\)
\(F_M(∧) = \begin{pmatrix}
<1,1> & → & 1 \\
<1,0> & → & 0 \\
<0,1> & → & 0 \\
<0,0> & → & 0
\end{pmatrix}

f. \(F_M(∨): \{0,1\} \times \{0,1\} → \{0,1\}\)
\(F_M(∨) = \begin{pmatrix}
<1,1> & → & 1 \\
<1,0> & → & 1 \\
<0,1> & → & 1 \\
<0,0> & → & 0
\end{pmatrix}

g. \(F_M(→): \{0,1\} \times \{0,1\} → \{0,1\}\)
\(F_M(→) = \begin{pmatrix}
<1,1> & → & 1 \\
<1,0> & → & 0 \\
<0,1> & → & 0 \\
<0,0> & → & 1
\end{pmatrix}
A variable assignment for $L_4$ on $M$ is a function $g: \text{VAR} \rightarrow D_M$

Let $g$ be a variable assignment for $L_4$.

$g_x^d$ = the variable assignment such that:

1. for every $y \in \text{VAR} - \{x\}$: $g_x^d(y) = g(y)$
2. $g_x^d(x) = d$

Recursive specification of $[\alpha]_{M,g}$, the interpretation of $\alpha$ in model $M$, relative to assignment $g$, for every expression of $L_4$:

0. If $\alpha \in \text{LEX}$, then $[\alpha]_{M,g} = F_M(\alpha)$
   If $\alpha \in \text{VAR}$, then $[\alpha]_{M,g} = g(\alpha)$

1. If $P \in \text{PRED}^n$ and $\alpha_1, \ldots, \alpha_n \in \text{TERM}$ then:
   $[P(\alpha_1, \ldots, \alpha_n)]_{M,g} = 1$ iff $< [\alpha_1]_{M,g}, \ldots, [\alpha_n]_{M,g} > \in [P]_{M,g}$; 0 otherwise.

2. If $\alpha_1, \alpha_2 \in \text{TERM}$, then:
   $[(\alpha_1 = \alpha_2)]_{M,g} = 1$ iff $[\alpha_1]_{M,g} = [\alpha_2]_{M,g}$; 0 otherwise.

3. If $\phi, \psi \in \text{FORM}$ then:
   $[\neg \phi]_{M,g} = [\neg]_{M,g}([\phi]_{M,g})$
   $[(\phi \land \psi)]_{M,g} = [\land]_{M,g}([\phi]_{M,g}, [\psi]_{M,g})$
   $[(\phi \lor \psi)]_{M,g} = [\lor]_{M,g}([\phi]_{M,g}, [\psi]_{M,g})$
   $[(\phi \rightarrow \psi)]_{M,g} = [\rightarrow]_{M,g}([\phi]_{M,g}, [\psi]_{M,g})$

4. If $x \in \text{VAR}$ and $\phi \in \text{FORM}$ then:
   $[\forall x \phi]_{M,g} = 1$ iff for every $d \in D_M$: $[\phi]_{M,g^x_d} = 1$; 0 otherwise
   $[\exists x \phi]_{M,g} = 1$ iff for some $d \in D_M$: $[\phi]_{M,g^x_d} = 1$; 0 otherwise

Note, we have introduced $=$ syncategorematically. We could also assume that $= \in \text{PRED}^2$,

specify its semantics as: $F_M(=) = \{ <d,d>: d \in D_M \}$,

and introduce a notation convention: $(\alpha = \beta) := (=\alpha,\beta)$
( := means 'is by definition')

Truth and entailment: See below.
VII: QUANTIFIER SCOPE: BOUND AND FREE VARIABLES

The construction tree of a formula of $L_4$ is the tree showing how the formula is built from $L_4$-expressions.
Rather than defining this notion precisely, I indicate in the following example what the construction trees look like.

Let $x, y \in \text{VAR}$, $j \in \text{CON}$, $P, Q \in \text{PRED}^1$, $R \in \text{PRED}^2$

$$(\forall x (P(x) \rightarrow \exists y (R(x,y) \land \neg R(y,j))) \lor Q(x)) \in \text{FORM}$$

We usually change the notation a bit to make the formula more legible. This can involve not write some brackets where this doesn't lead to confusion, adding some brackets to bring out the structure more clearly, or change the form of the brackets, so that you see more clearly which brackets belong together. So I write the above formula as:

$$(\forall x [P(x) \rightarrow \exists y [R(x,y) \land \neg R(y,j)]] \lor Q(x))$$

Its construction tree is:

![Construction Tree Diagram]

Note that in this tree all nodes are labeled by expressions of $L_4$, except for the nodes with labels $\forall x$ and $\exists y$, which are not $L_4$-expressions. As remarked earlier, we set up $L_4$ in this way to make the semantics simpler to read and understand for you.

For the purpose of the construction tree, we will assume that $\forall x$ and $\exists y$ are $L_4$ expressions, we call them universal and existential quantifiers.
**Important:** for the purpose of the notions defined below, we will **not** decompose $\forall x$ into $\forall$ and $x$, the same for $\exists y$.

This means that, while we normally call $\forall$ the universal quantifier and $\exists$ the existential quantifier, we will **here** call $\forall x$ a universal quantifier and $\exists y$ an existential quantifier.

Thus, on this mode of speech, $L_4$ contains infinitely many different universal quantifiers, and infinitely many existential quantifiers:

$\forall x_1, \forall x_2, \forall x_3, ...$

$\exists x_1, \exists x_2, \exists x_3, ...$

**FACT about $L_4$:** each formula of $L_4$ has a unique construction tree.

We say: $L_4$ is **syntactically unambiguous**.

Let $\phi$ be an $L_4$ formula and $\alpha$ an $L_4$ expression.

$\alpha$ **occurs in** $\phi$ iff there is a node in the construction tree of $\phi$ labeled by $\alpha$.

If $\phi$ and $\psi$ are formulas and $\psi$ occurs in $\phi$, we call $\psi$ a **subformula** of $\phi$.

Let $\alpha$ be an $L_4$ expression and $\phi$ an $L_4$ formula.

an **occurrence of** $\alpha$ **in** $\phi$ is a node in the construction tree of $\phi$ labeled by $\alpha$.

So an expression $\alpha$ may occur more than once, say, twice, in a formula $\phi$. In that case there are two occurrences of $\alpha$ in $\phi$, and these two occurrences are nodes in the construction tree of $\phi$.

Let $\phi$ be an $L_4$ formula, $x \in \text{VAR}$.

Let $\alpha$ be an occurrence of a quantifier $\forall x$ or $\exists x$ in $\phi$ (that is, $\alpha$ is a node in the construction tree of $\phi$ labeled by $\forall x$ or by $\exists x$).

The **scope of** $\alpha$ **in** $\phi$ is the sister node of $\alpha$ in the construction tree of $\phi$.

Let $\beta$ be a node in the construction tree of $\phi$.

$\beta$ **is in the scope of** $\alpha$ iff $\beta$ is a daughternode of the scope of $\alpha$.

**Example:** In the above formula, there is an occurrence of quantifier $\forall x$. Its scope is the sister node which is boldfaced. In the formula, there are three occurrences of variable $x$, two of these occurrences of $x$ are in the scope of the occurrence of $\forall x$, one occurrence of $x$ is not in the scope of the occurrence of $\forall x$.

There is an occurrence of quantifier $\exists y$ in the formula. Its scope is its boldfaced sister node. There are two occurrences of variable $y$ in the formula. Both these occurrences are in the scope of the occurrence of $\exists y$. 

44
Let $\phi$ be an $L_4$ formula, let $\alpha$ be an occurrence of quantifier $\forall x$ or $\exists x$ in $\phi$, let $\beta$ be an occurrence of variable $x$ in $\phi$.

$\beta$ is **bound by** $\alpha$ in $\phi$ iff

1. $\beta$ is in the scope of $\alpha$.
2. There is no occurrence $\gamma$ of either $\forall x$ or $\exists x$ in $\phi$ such that both (a.) and (b.) hold:
   a. $\gamma$ is in the scope of $\alpha$.
   b. $\beta$ is in the scope of $\gamma$.

This means that an occurrence $\beta$ of a variable $x$ is bound by an occurrence $\alpha$ of a quantifier $\forall x$ or $\exists y$ in $\phi$ if $\beta$ is in the scope of $\alpha$, and there is no occurrence of a quantifier with the same variable $x$ (i.e. $\forall x$ or $\exists x$) in between $\alpha$ and $\beta$ in $\phi$. Thus an occurrence of $x$ is bound by the closest occurrence of $\forall x$ or $\exists x$ in $\phi$ that it is in the scope of.

Note that this means that an occurrence of a variable $x$ is never bound by an occurrence of a quantifier which is not in variable $x$ (i.e. never by $\forall y$ or $\exists y$).

Let $\phi$ be an $L_4$ formula.

Occurrence $\beta$ of variable $x$ in $\phi$ is **free for** occurrence $\alpha$ of quantifier $\forall x$ or $\exists x$ in $\phi$ iff $\beta$ is not bound by $\alpha$ in $\phi$.

Occurrence $\beta$ of variable $x$ in $\phi$ is **bound in** $\phi$ iff $\beta$ is bound by some occurrence of quantifier $\forall x$ or $\exists x$ in $\phi$.

Occurrence $\beta$ of variable $x$ in $\phi$ is **free in** $\phi$ iff $\beta$ is not bound in $\phi$.

Variable $x$ **occurs bound in** $\phi$ iff some occurrence of $x$ in $\phi$ is bound in $\phi$.
Variable $x$ **occurs free in** $\phi$ iff some occurrence of $x$ in $\phi$ is free in $\phi$.

Variable $x$ is **bound in** $\phi$ iff every occurrence of $x$ in $\phi$ is bound in $\phi$.
Variable $x$ is **free in** $\phi$ iff every occurrence of $x$ in $\phi$ is free in $\phi$.

Example:
Let $\phi$ be the following $L_4$ formula:

$$ ( \forall x[P(x) \rightarrow \exists x[Q(x) \land \exists y[R(x,y,z)]]]) \land S(x,y) $$

We write $\forall x$ for the occurrence of $\forall x$ in $\phi$, similarly for the other quantifiers. Let's indicate the occurrences of the variables in $\phi$ by superscripts:
\[(\forall x[P(x) \rightarrow \exists x[Q(x) \land \exists y[R(x, y, z)]]]) \land S(x, y)\]

 occurrence \(x^1\) is bound by occurrence \(\forall x\) in \(\phi\)
 occurrence \(x^2\) is bound by occurrence \(\exists x\) in \(\phi\)
 occurrence \(x^3\) is bound by occurrence \(\exists x\) in \(\phi\)
 occurrence \(y^1\) is bound by occurrence \(\exists y\) in \(\phi\)
 occurrence \(z^1\) is free in \(\phi\)
 occurrence \(x^4\) is free in \(\phi\)
 occurrence \(y^2\) is free in \(\phi\)

 variable \(z\) is free in \(\phi\)
 variables \(x, y\) are neither free, nor bound in \(\phi\), they occur both free and bound in \(\phi\).

A formula \(\phi\) of \(L_4\) is a sentence of \(L_4\) iff every variable occurring in \(\phi\) is bound in \(\phi\).

\[\text{SENT} = \{\phi \in \text{FORM}: \phi \text{ is a sentence of } L_4\}\]

**Truth for \(L_4\)**

Let \(\phi \in \text{FORM}:

\[\llbracket \phi \rrbracket_M = 1\] iff for every assignment \(g\) for \(L_2\):

\[\llbracket \phi \rrbracket_{M, g} = 1\]

\[\llbracket \phi \rrbracket_M = 0\] iff for every assignment \(g\) for \(L_2\):

\[\llbracket \phi \rrbracket_{M, g} = 0\]

**FACT:** If \(\phi \in \text{SENT}\) then for every model \(M\) for \(L_4\):

\[\llbracket \phi \rrbracket_M = 1\text{ or }\llbracket \phi \rrbracket_M = 0\]

i.e. formulas in which every variable occurring is bound are true or false independent of assignment functions.

Thus, even though the truth conditions of the formula \((P(x) \rightarrow Q(x))\) depend on assignment functions, the truth conditions of the sentence \(\forall x[P(x) \rightarrow Q(x)]\), built from it, do not depend on assignment functions.

**Entailment for \(L_4\)**

Let \(\varphi, \psi \in \text{SENT}\)

\(\varphi\) entails \(\psi\), \(\varphi \Rightarrow \psi\) iff for every model \(M\) for \(L_2\):

if \(\llbracket \varphi \rrbracket_M = 1\) then \(\llbracket \psi \rrbracket_M = 1\)

\(\varphi\) and \(\psi\) are equivalent, \(\varphi \iff \psi\) iff \(\varphi \Rightarrow \psi\) and \(\psi \Rightarrow \varphi\)

Let \(\Delta \subseteq \text{SENT}, \psi \in \text{SENT}\)

We write \(\Delta \backslash \psi\) for an argument with as premises the sentences in \(\Delta\), and as conclusion the sentence \(\psi\).

Argument \(\Delta \backslash \psi\) is valid, \(\Delta \Rightarrow \psi\) iff for every model \(M\) for \(L_3\):

if for every \(\varphi \in \Delta\): \(\llbracket \varphi \rrbracket_M = 1\), then \(\llbracket \psi \rrbracket_M = 1\)

i.e. \(\Delta \backslash \psi\) is valid iff in every model where all the premises in \(\Delta\) are true, the conclusion \(\psi\) is true.

\(\psi\) is valid, \(\Rightarrow \psi\), iff \(\emptyset \Rightarrow \psi\)

i.e. \(\psi\) is valid iff \(\psi\) is true in every model.
VIII. THE SEMANTICS OF BOUND AND FREE VARIABLES

\[\exists x[P(x) \land R(x,y)] \land Q(x)\]

bound bound free free

1. \[\exists x[P(x) \land R(x,y)] \land Q(x)\] in \(M,g\) = 1 iff
2. \[\land_{M,g} (\exists x[P(x) \land R(x,y)]\] in \(M,g\), \[Q(x)\] in \(M,g\) = 1 iff
3. \(F_M(\land_\langle \exists x[P(x) \land R(x,y)]\rangle_{M,g}, \langle Q(x)\rangle_{M,g} \rangle = 1\) iff
4. \(\langle \exists x[P(x) \land R(x,y)]\rangle_{M,g}, \langle Q(x)\rangle_{M,g} \rangle = 1\) iff
5. \(\exists x[P(x) \land R(x,y)]\] in \(M,g\) = 1 and \[Q(x)\] in \(M,g\) = 1 iff
6. \(\exists x[P(x) \land R(x,y)]\] in \(M,g\) = 1 and \[x\] in \(Q\] in \(M,g\) iff
7. \(\exists x[P(x) \land R(x,y)]\] in \(M,g\) = 1 and \(g(x)\) in \(Q\] in \(M,g\) iff
8. \(\exists x[P(x) \land R(x,y)]\] in \(M,g\) = 1 and \(g(x)\) in \(F\] in \(M,Q\) iff
9. for some \(d\) in \(D_M: [P(x) \land R(x,y)]_{M,g,d} = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
10. for some \(d\) in \(D_M: \land_{M,g} (\exists x[P(x) \land R(x,y)]_{M,g,d}, \langle R(x,y)\rangle_{M,g,d} \rangle = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
11. for some \(d\) in \(D_M: F_M(\land_\langle \exists x[P(x) \land R(x,y)]_{M,g,d}, \langle R(x,y)\rangle_{M,g,d} \rangle = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
12. for some \(d\) in \(D_M: \langle \exists x[P(x) \land R(x,y)]_{M,g,d}, \langle R(x,y)\rangle_{M,g,d} \rangle = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
13. for some \(d\) in \(D_M: [P(x) \land R(x,y)]_{M,g,d} = 1\) and \[R(x,y)\] in \(M,g,d = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
14. for some \(d\) in \(D_M: [x]_{M,g,d} \in [P]_{M,g,d} \land [R(x,y)]_{M,g,d} = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
15. for some \(d\) in \(D_M: [x]_{M,g,d} \in F_M(P) \land [R(x,y)]_{M,g,d} = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
16. for some \(d\) in \(D_M: g_{x,d}(x) \in F_M(P) \land [R(x,y)]_{M,g,d} = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
17. for some \(d\) in \(D_M: d \in F_M(P) \land [R(x,y)]_{M,g,d} = 1\) and \(g(x)\) in \(F\] in \(M,Q\) iff
18. for some \(d\) in \(D_M: d \in F_M(P) \land [x]_{M,g,d} \land [y]_{M,g,d} \land [R]_{M,g,d} \land [g(x) \in F\] in \(M,Q\) iff
19. for some \(d\) in \(D_M: d \in F_M(P) \land [x]_{M,g,d} \land [y]_{M,g,d} \land [F\] in \(M,R) \land g(x) \in F\] in \(M,Q)\) iff
20. for some \(d\) in \(D_M: d \in F_M(P) \land [g_{x,d}(x)]_{M,g,d} \land [y]_{M,g,d} \land [F\] in \(M,R) \land g(x) \in F\] in \(M,Q)\) iff
21. for some \(d\) in \(D_M: d \in F_M(P) \land [g_{x,d}(y)]_{M,g,d} \land [y]_{M,g,d} \land [F\] in \(M,R) \land g(x) \in F\] in \(M,Q)\) iff
22. for some \(d\) in \(D_M: d \in F_M(P) \land [d, g_{x,d}(y)]_{M,g,d} \land [y]_{M,g,d} \land [F\] in \(M,R) \land g(x) \in F\] in \(M,Q)\) iff
23. for some \(d\) in \(D_M: d \in F_M(P) \land [d, g_{x,d}(y)]_{M,g,d} \land [y]_{M,g,d} \land [F\] in \(M,R) \land g(x) \in F\] in \(M,Q)\) iff

Assume that for every \(M: F_M(P)\) is the set of boys in \(M, F_M(Q)\) is the set of girls in \(M, F_M(R)\) is the love relation in \(M, g(x) = \text{YOU THERE}\) and \(g(x) = \text{YOU OVER THERE}\).

Then \(\exists x[P(x) \land R(x,y)] \land Q(x)\) is true in any situation \(M, relative to g, where some boy loves you there and you over there are a girl.\)
IX. ENTAILMENT FOR SENTENCES

(1) \((P(m) \rightarrow Q(m))\)

(2) \(P(m)\)

(3) \(Q(m)\)

1. Assume \(\|P(m)\|_M = 1\)
   Then:
   For every \(g\): \(\|P(m)\|_{M,g} = 1\)
   Then
   For every \(g\): \(F_{M}(m) \in F_{M}(P)\)
   Then:
   \(F_{M}(m) \in F_{M}(P)\)

2. Assume \(\|(P(m) \rightarrow Q(m))\|_M = 1\)
   Then:
   For every \(g\): \(\|(P(m) \rightarrow Q(m))\|_{M,g} = 1\)
   Then:
   For every \(g\): \(\|P(m)\|_{M,g} = 0\) or \(\|Q(m)\|_{M,g} = 1\)
   Then:
   For every \(g\): \(F_{M}(m) \notin F_{M}(P)\) or \(F_{M}(m) \in F_{M}(Q)\)
   Then: \(F_{M}(m) \notin F_{M}(P)\) or \(F_{M}(m) \in F_{M}(Q)\)

3. Combining (1) and (2), it follows that:
   \(F_{M}(m) \in F_{M}(Q)\)

   Hence:
   For every \(g\): \(F_{M}(m) \in F_{M}(Q)\)
   Hence:
   For every \(g\): \(\|Q(m)\|_{M,g} = 1\)
   Hence \(\|Q(m)\|_M = 1\)

This means, by definition of entailment that (1) and (2) entail (3).
\{(1),(2)\}\ 3
(1) \exists x [B(x) \land \forall y [G(y) \rightarrow L(x,y)]]
(2) G(m)
(3) \exists x [B(x) \land L(x,m)]

\{(1),(2)\} \Rightarrow 3 \text{ iff for every } M:\text{ if } \llbracket (1) \rrbracket_M = 1 \text{ and } \llbracket (2) \rrbracket_M = 1, \text{ then } \llbracket (3) \rrbracket_M = 1

1. \llbracket (1) \rrbracket_M = 1 \text{ iff }
2. \llbracket \exists x [B(x) \land \forall y [G(y) \rightarrow L(x,y)]] \rrbracket_M = 1 \text{ iff }
3. \text{for every } g: \llbracket \exists x [B(x) \land \forall y [G(y) \rightarrow L(x,y)]] \rrbracket_{M,g} = 1 \text{ iff }
4. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
5. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
6. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
7. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
8. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
9. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
10. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
11. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
12. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
13. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
14. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
15. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }
16. \text{for every } g: \text{ for some } d \in D_M: \llbracket B(x) \land \forall y [G(y) \rightarrow L(x,y)] \rrbracket_{M,g,x} = 1 \text{ iff }

(1) \llbracket (2) \rrbracket_M = 1 \text{ iff }
(2) \llbracket G(m) \rrbracket_M = 1 \text{ iff }
(3) \text{for every } g: \llbracket G(m) \rrbracket_{M,g} = 1 \text{ iff }
(4) \text{for every } g: \llbracket m \rrbracket_{M,g} \in \llbracket G \rrbracket_{M,g} \text{ iff }
(5) \text{for every } g: \llbracket F(M) \rrbracket_{M,g} \text{ iff }
(6) \llbracket F_M(m) \rrbracket \in \llbracket F_M (G) \rrbracket
(1) \( [3]_M = 1 \) iff \\
(2) \( \exists x[B(x) \land L(x,m)]_M = 1 \) iff \\
(3) for every g: \( \exists x[B(x) \land L(x,m)]_{M,g} = 1 \) iff \\
(4) for every g: for some d \( \in D_M: [B(x) \land L(x,m)]_{M,g}^d = 1 \) iff \\
(5) for every g: for some d \( \in D_M: [B(x)]_{M,g}^d = 1 \) and \( [L(x,m)]_{M,g}^d = 1 \) iff \\
(6) for every g: for some d \( \in D_M: g^d(x) \in F_M(B) \) and \( <g^d(x),F_M(m)> \in F_M(L) \) iff \\
(7) for every g: for some d \( \in D_M: d \in F_M(B) \) and \( <d,F_M(m)> \in F_M(L) \) iff \\
(8) for some d \( \in D_M: d \in F_M(B) \) and \( <d,F_M(m)> \in F_M(L) \) iff \\
(9) for some d \( \in D_M: <d,F_M(m)> \in F_M(L) \).

In sum:

\([1]_M = 1\) iff for some \( d \in F_M(B) \) for every \( b \in F_M(G): <d,b> \in F_M(L)\).

\([2]_M = 1\) iff \( F_M(m) \in F_M(G)\).

\([3]_M = 1\) iff for some \( d \in F_M(B): <d,F_M(m)> \in F_M(L)\).

Now let \( M \) be any model such that \([1]_M = 1\) and \([2]_M = 1\).

This means that:

for some \( d \in F_M(B) \) for every \( b \in F_M(G): <d,b> \in F_M(L) \) and \( F_M(m) \in F_M(G)\).

Then for some \( d \in F_M(B): <d,F_M(m)> \in F_M(L) \), hence \([3]_M = 1\).

We have shown that the semantics predicts that \{(1),(2)\} \( \Rightarrow 3\).
Arguing in a picture.

Some boy loves every girl
(1) $\exists x [\text{BOY}(a) \land \forall y [\text{GIRL}(y) \rightarrow \text{LOVE}(x, y)]]$
\[\llbracket(1)\rrbracket_M = 1 \text{ iff whatever else holds in } M, \text{ you find the following:}

\[\begin{array}{ccc}
\text{F}_M(\text{BOY}) & \text{F}_M(\text{GIRL}) & M \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}\]

The arrows indicate part of $F_M(\text{LOVE})$

Mary is a girl
(2) $\text{GIRL}(\text{Mary})$
\[\llbracket(1)\rrbracket_M = 1 \text{ iff whatever else holds in } M, \text{ you find the following:}

\[\begin{array}{ccc}
\text{F}_M(\text{GIRL}) & & M \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}\]

A model that satisfies both (1) and (2) hence look like this:

\[\begin{array}{ccc}
\text{F}_M(\text{BOY}) & \text{F}_M(\text{GIRL}) & M \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array}\]

You can read off this picture that in any such model $M$ (3) is true:

(3) Some boy loves Mary.
$\exists x [\text{BOY}(x) \land \text{LOVE}(\text{Mary}, x)]$
X. ALPHABETIC VARIANTS

Let $\phi$ be an $L_4$ formula, $x$ and $y$ variables.
Let $q_x$ be an occurrence of $\forall x (\exists x)$ in $\phi$

$\text{BOUND}[q_x, \phi]$ is the set of all occurrences of variable $x$ in $\phi$ that are bound by $q_x$.

We define $\phi[q_y, y/x]$, a formula with the same construction tree as $\phi$ but different labels on the nodes:

$\phi[q_y, y/x]$ is the result of:
1. Replace occurrence $q_x$ of $\forall x$ in $\phi$ by $q_y$ of $\forall y$ (i.e. change on node $q_x$ in the construction tree label $\forall x$ into $\forall y$)
2. Replace every occurrence $n$ of $x$ in $\phi$ which is in $\text{BOUND}[q_x, \phi]$ by occurrence $n$ of $y$ (i.e. change on node $n$ in the construction tree label $x$ into $y$ for all these occurrences of $x$).
3. Adjust the nodes of the construction tree upwards accordingly.

The resulting tree is the construction tree for $\phi[q_y, y/x]$.

Let $T[\phi]$ be the construction tree of $\phi$ and $X$ a set of nodes in $T[\phi]$.

$\text{UL}[T[\phi]]$ is the unlabeled tree that results from deleting all the labels from the nodes in $T[\phi]$.
$\text{UL}[X]$ is the set of nodes in $\text{UL}[T[\phi]]$ that result from removing all the labels from the nodes in $X$.

Clearly, $\text{UL}([T[\phi]]) = \text{UL}(\phi[q_y, y/x])$.

We define:

1. $\phi$ and $\phi[q_y, y/x]$ are basic alphabetic variants iff
   \[ \text{UL}(\text{BOUND}[q_y, \phi[q_y, y/x]]) = \text{UL}(\text{BOUND}[q_x, \phi]) \]
   This means that occurrence $q_y$ of $\forall y (\exists y)$ in $\phi[q_y, y/x]$ binds exactly the occurrences of $y$ that were occurrences of $x$ in bound by occurrence $q_x$ of $\forall x (\exists x)$ in $\phi$.
   (i.e. that occur at exactly the same (unlabeled) nodes as the occurrences of $x$ bound by $q_x$ in $\phi$).
2. $\phi$ and $\psi$ are alphabetic variants iff there is a sequence of formulas $<\phi_1, \ldots, \phi_n>$ such that $\phi_1 = \phi$ and $\phi_n = \psi$ and for every $i < n$: $\phi_i$ and $\phi_{i+1}$ are basic alphabetic variants.

THEOREM: if $\phi$ and $\psi$ are alphabetic variants then $\phi \Leftrightarrow \psi$. 
EXAMPLE:

\[ \forall x P(x) \land Q(x) \]

\[ \exists y P(y) \land Q(x) \]

\( \varphi \)

\( \varphi[q, y/x] \)

\( q \) binds \( n_1 \) but not \( n_2 \)

\( q \) binds \( n_1 \)

\( \varphi \) and \( \varphi[q, y/x] \) are alphabetic variants.

EXAMPLE:

\[ \forall x R(y, x) \]

\[ \exists y R(y, y) \]

\( \varphi \)

\( \varphi[q, y/x] \)

\( q \) binds \( m \)

\( q \) binds \( n \) and \( m \)

Not alphabetic variants.

Hence the theorem does **not** say that \( \forall x R(y, x) \) and \( \exists y R(y, y) \) are equivalent (which indeed they are not)

\[ \exists x[ \forall x[P(x)] \land Q(x)] \land S(x) \]
\[ \exists x[ \forall y[P(y)] \land Q(x)] \land S(x) \]
\[ \exists z[ \forall y[P(y)] \land Q(z)] \land S(x) \]

More examples:

\( \forall x \exists y[R(x, y)] \) and \( \forall u \exists z[R(u, z)] \) are alphabetic variants.

\( \forall x \exists y[R(x, y)] \land P(x) \) and \( \forall u \exists z[R(u, z)] \land P(x) \) are alphabetic variants.

(only the first occurrence of \( x \) is bound by \( \forall x \), so only the first occurrence of \( x \) gets changed.)
∀x∃y[R(x,y)] ∧ P(x) and ∀u∃z[R(u,z)] ∧ P(u) are not alphabetic variants. (because you have changed also an occurrence of x which wasn't bound by ∀x). So, P(x) and P(y) are not alphabetic variants.

∀x∃y[R(x,y)] and ∀x∃x[R(x,x)] are not alphabetic variants. You change ∃y to ∃x and y to x. But after the change, ∃x binds not only the occurrence of x where we changed the label, but also the occurrence of x which was an occurrence of x to start with. This means that we do not satisfy the constraint of basic alphabetic variants.

Hence: ∀x∃y[R(x,y)] ↔ ∀u∃z[R(u,z)].
Note: ∀x∃y[R(x,y)] and ∀x∃x[R(x,x)] are not equivalent. (and if we extend the notion of equivalence to formulas in general, P(x) and P(y) are not equivalent.)
Showing this semantically:

(1) $\forall x \exists y[R(x,y)]_M = 1$ iff 
(2) for every $g$: $\forall x \exists y[R(x,y)]_{M,g} = 1$ iff 
(3) for every $g$: for every $d \in D_M$: $\exists y[R(x,y)]_{M,g_d} = 1$ iff 
(4) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $[R(x,y)]_{M,g_d,b} = 1$ iff 
(5) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $<g_x^d b, g_x^d b> \in F_M(R)$ iff 
(6) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $<g_u^d z, g_u^d z> \in F_M(R)$ iff 
(7) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $[R(u,z)]_{M,g_d,b} = 1$ iff 
(8) for every $g$: for every $d \in D_M$: $\exists z[R(u,z)]_{M,g_d} = 1$ iff 
(9) for every $g$: $\forall u \exists z[R(u,z)]_{M,g} = 1$ iff 
(10) $\forall u \exists z[R(u,z)]_M = 1$

Hence for every $M$: $\forall x \exists y[R(x,y)]_M = 1$ iff $\forall u \exists z[R(u,z)]_M = 1$, which means, indeed, that: $\forall x \exists y[R(x,y)] \Leftrightarrow \forall u \exists z[R(u,z)]$.

(1) $\forall x \exists y[R(x,y)]_M = 1$ iff 
(2) for every $d \in D_M$ there is a $b \in D_M$: $<d,b> \in F_M(R)$

(1) $\forall x \exists x[R(x,x)]_M = 1$ iff 
(2) for every $g$: $\forall x \exists x[R(x,x)]_{M,g} = 1$ iff 
(3) for every $g$: for every $d \in D_M$: $\exists x[R(x,x)]_{M,g_d} = 1$ iff 
(4) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $[R(x,x)]_{M,g_d,b} = 1$ iff 
(5) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $<g_x^d b, g_x^d b> \in F_M(R)$ iff 
(6) for every $g$: for every $d \in D_M$ there is a $b \in D_M$: $<b,b> \in F_M(R)$ iff 
(7) there is a $b \in D_M$: $<b,b> \in D_M(R)$.

Let $M$ be a model with $D_M = \{d_1, d_2\}$ and $F_M(R) = \{<d_1,d_2>,<d_2,d_1>\}$
Then $\forall x \exists y[R(x,y)]_M = 1$ but $\forall x \exists x[R(x,x)]_M = 0$.
Hence the two are not equivalent.
This model shows that $\forall x \exists y[R(x,y)]$ does not entail $\forall x \exists x[R(x,x)]$.
A model $M'$ with $D_M = \{d_1, d_2\}$ and $F_M(R) = \{<d_1,d_1>\}$ shows that also $\forall x \exists x[R(x,x)]$ does not entail $\forall x \exists y[R(x,y)]$.
In fact, it is easy to show that: $\forall x \exists x[R(x,x)] \Leftrightarrow \exists x[R(x,x)]$.
We call the quantifier $\forall x$ in $\forall x \exists x[R(x,x)]$ vacuous, since it binds no variable.
And we see that semantically the vacuous quantifier doesn't contribute to the meaning of the whole.
THE GAME OF LOVE

\(D_M = \{a,b,c,d\}\)
\(F_{M(LOVE)} = \{<a,a>,<b,c>,<c,d>,<d,c>\}\)

Game: You win of
\(\forall x \exists y \text{LOVE}(x,y)\) \(M,g = 1\)

iff for every \(d \in D_M; \exists y \text{LOVE}(x,y)\) \(M,gx^d = 1\)

iff
a: \(\exists y \text{LOVE}(x,y)\) \(M,gx^a = 1\)
and b: \(\exists y \text{LOVE}(x,y)\) \(M,gx^b = 1\)
and c: \(\exists y \text{LOVE}(x,y)\) \(M,gx^c = 1\)
and d: \(\exists y \text{LOVE}(x,y)\) \(M,gx^d = 1\)

CASE a: To stay in the game you must show that
for some \(f \in D_M; \exists y \text{LOVE}(x,y)\) \(M,gx^af = 1\)

This means you must get 1 for one of:

\[a_1: \text{LOVE}(x,y)_{M,g} \text{xy}^a \text{a} \iff <a,a> \in F_{M(LOVE)}\]
\[a_2: \text{LOVE}(x,y)_{M,g} \text{xy}^a \text{b} \iff <a,b> \in F_{M(LOVE)}\]
\[a_3: \text{LOVE}(x,y)_{M,g} \text{xy}^a \text{c} \iff <a,c> \in F_{M(LOVE)}\]
\[a_4: \text{LOVE}(x,y)_{M,g} \text{xy}^a \text{d} \iff <a,d> \in F_{M(LOVE)}\]

You get 1 at \(a_1\), hence at a, so you stay in the game.

CASE b: To stay in the game you must show that
for some \(f \in D_M; \exists y \text{LOVE}(x,y)\) \(M,gx^bf = 1\)

This means you must get 1 for one of:

\[b_1: \text{LOVE}(x,y)_{M,g} \text{xy}^b \text{a} \iff <b,a> \in F_{M(LOVE)}\]
\[b_2: \text{LOVE}(x,y)_{M,g} \text{xy}^b \text{b} \iff <b,b> \in F_{M(LOVE)}\]
\[b_3: \text{LOVE}(x,y)_{M,g} \text{xy}^b \text{c} \iff <b,c> \in F_{M(LOVE)}\]
\[b_4: \text{LOVE}(x,y)_{M,g} \text{xy}^b \text{d} \iff <b,d> \in F_{M(LOVE)}\]

You get 1 at \(b_3\), hence at b, so you stay in the game.

CASE c: To stay in the game you must show that
for some \(f \in D_M; \exists y \text{LOVE}(x,y)\) \(M,gx^cf = 1\)

This means you must get 1 for one of:

\[c_1: \text{LOVE}(x,y)_{M,g} \text{xy}^c \text{a} \iff <c,a> \in F_{M(LOVE)}\]
\[c_2: \text{LOVE}(x,y)_{M,g} \text{xy}^c \text{b} \iff <c,b> \in F_{M(LOVE)}\]
\[c_3: \text{LOVE}(x,y)_{M,g} \text{xy}^c \text{c} \iff <c,c> \in F_{M(LOVE)}\]
\[c_4: \text{LOVE}(x,y)_{M,g} \text{xy}^c \text{d} \iff <c,d> \in F_{M(LOVE)}\]

You get 1 at \(c_4\), hence at c, so you stay in the game.
CASE d: To stay in the game you must show that for some \( f \in D_M: \| \text{LOVE}(x,y) \|_{M.g}^d f = 1 \)

This means you must get 1 for one of:

\[d_1: \| \text{LOVE}(x,y) \|_{M.g}^d a \quad \text{iff} \quad <d,a> \in F_M(\text{LOVE})\]
\[d_2: \| \text{LOVE}(x,y) \|_{M.g}^d b \quad \text{iff} \quad <d,b> \in F_M(\text{LOVE})\]
\[d_3: \| \text{LOVE}(x,y) \|_{M.g}^d c \quad \text{iff} \quad <d,c> \in F_M(\text{LOVE})\]
\[d_4: \| \text{LOVE}(x,y) \|_{M.g}^d d \quad \text{iff} \quad <d,d> \in F_M(\text{LOVE})\]

You get 1 at \( d_3 \), hence at \( d \), so you stay in the game.

You have gotten 1 at \( a,b,c,d \): YOU WIN!

Change the model to:
\( D_M = \{a,b,c,d\} \)
\( F_M(\text{LOVE}) = \{<a,a>,<b,c>,<c,d>\} \)
The cases \( a,b,c \) stay the same, but now on case \( d \) you get 0 overwhere in the list, cases \( d_1,d_2,d_3,d_4 \). This means you get 0 on \( d \), and you lose!.

When you get more experienced, you may do without writing out all the cases and work out the semantics directly:

\[\forall x \exists y \text{LOVE}(x,y) \|_{M.g} = 1 \quad \text{iff} \quad \text{for every } d \in D_M \text{ there is an } f \in D_M: \| \text{LOVE}(x,y) \|_{M.g}^d f = 1 \quad \text{iff} \quad \text{for every } d \in D_M \text{ there is an } f \in D_M: <g_x^d f(x), g_y^d f(y)> \in F_M(\text{LOVE}) \quad \text{iff} \quad \text{for every } d \in D_M \text{ there is an } f \in D_M: <f,g> \in F_M(\text{LOVE}) \quad \text{iff} \quad \text{DOM}(F_M(\text{LOVE})) = D_M\]

You check:
In the first example:
\( F_M(\text{LOVE}) = \{<a,a>,<b,c>,<c,d>,<d,c>\} \)
So \( \text{DOM}(F_M(\text{LOVE})) = \{a,b,c,d\} = D_M \quad \text{TRUE} \)

In the second example:
\( F_M(\text{LOVE}) = \{<a,a>,<b,c>,<c,d>\} \)
So \( \text{DOM}(F_M(\text{LOVE})) = \{a,b,c\} \neq D_M \quad \text{FALSE} \)
Similarly

$$\forall y \exists x \text{LOVE}(x,y)_{M,g} = 1 \text{ iff }$$

for every \( f \in D_M \) there is a \( d \in D_M \): \( <d,f> \in F_M(\text{LOVE}) \) iff

RAN\((F_M(\text{LOVE})) = D_M \)

In our example:

\( F_M(\text{LOVE}) = \{<a,a>,<b,c>,<c,d>,<d,c>\} \)

RAN\((F_M(\text{LOVE})) = \{a,c,d\} \neq D_M \) FALSE

$$\exists x \forall y \text{LOVE}(x,y)_{M,g} = 1 \text{ iff }$$

for some \( d \in D_M \) for every \( f \in D_m \): \( <d,f> \in F_M(\text{LOVE}) \)

Let \( L_d = \{f \in D_M: <d,f> \in F_M(\text{LOVE})\} \)

Hence:

$$\exists x \forall y \text{LOVE}(x,y)_{M,g} = 1 \text{ iff }$$

for some \( d \in D_M \): \( L_d = D_M \)

In our example:

\( L_a = \{a\}, L_b = \{c\}, L_c = \{d\}, L_d = \{c\} \) FALSE

$$\exists y \forall x \text{LOVE}(x,y)_{M,g} = 1 \text{ iff }$$

for some \( f \in D_M \) for every \( d \in D_M \): \( <d,f> \in F_M(\text{LOVE}) \)

Let \( BL_d = \{f \in D_M: <d,f> \in F_M(\text{LOVE})\} \)

Hence:

$$\exists y \forall x \text{LOVE}(x,y)_{M,g} = 1 \text{ iff }$$

for some \( f \in D_M \): \( BL_f = D_M \)

\( BL_a = \{a\}, BL_b = \emptyset, BL_c = \{b,d\}, BL_d = \{c\} \) FALSE

We see already here that \( \forall x \exists y \text{LOVE}(x,y) \) does not entail \( \exists y \forall x \text{LOVE}(x,y) \)

However, assume a model \( M' \) where \( \exists y \forall x \text{LOVE}(x,y) \) is true.

then for some \( f \in D_{M'} \): \( BL_f = D_{M'} \),

i.e. for some \( f \in D_{M'} \): \( \{d \in D_M: <d,f> \in F_M(\text{LOVE})\} = D_{M'} \)

But, obviously, then DOM\((F_M(\text{LOVE})) = D_M \)

and this means that \( \forall x \exists y \text{LOVE}(x,y) \) is true in \( M' \)

This means, that \( \exists y \forall x \text{LOVE}(x,y) \) entails \( \forall x \exists y \text{LOVE}(x,y) \).
XI. EXTENSIONALITY

We define:

\[(\varphi \leftrightarrow \psi) := (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)\]

Let \(\varphi, \psi, \chi\) be sentences of \(L_4\) and let \(a'\) be an occurrence of \(\psi\) in \(\varphi\) (so \(\psi\) is a subformula of \(\varphi\)). Let \(T(\varphi)\) be the construction tree of \(\varphi\). Let \(a'\) be the result of changing the label \(\psi\) on \(a'\) to \(\chi\) in \(T(\varphi)\), let \(T(\varphi)[a'/a']\) be the resulting construction tree and let \(\varphi[a'/a']\) be the \(L_4\) formula of which \(T(\varphi)[a'/a']\) is the construction tree.

**EXTENSIONALITY OF \(L_4\) (for subsentences of sentences):**

\[(\psi \leftrightarrow \chi) \Rightarrow (\varphi \leftrightarrow \varphi[a'/a'])\]

Thus, in every model where \(\psi\) and \(\chi\) have the same truth value, \(\varphi\) and the result of substituting \(\chi\) for \(\psi\) in \(\varphi\) have the same truth value.

It follows from this that if \(\varphi \Leftrightarrow \chi\), then \(\varphi \Leftrightarrow \varphi[a'/a']\).

Let \(\varphi\) be a sentence of \(L_4\) and \(t, s \in CON\) and let \(a'\) be an occurrence of \(t\) in \(\varphi\). Let \(a'\) be the result of changing the label on \(a'\) in \(T(\varphi)\) from \(t\) to \(s\), and let \(T(\varphi)[a'/a']\) and \(\varphi[a'/a']\) be construction tree and formula resulting from this change.

**EXTENSIONALITY OF \(L_4\) (for constants in sentences):**

\[(t = s) \Rightarrow (\varphi \leftrightarrow \varphi[a'/a'])\]

In every model where \(t\) and \(s\) have the same interpretation, \(\varphi\) and the result of substituting \(s\) for \(t\) in \(\varphi\) have the same truth value.

It follows from this that if \(t\) and \(s\) have the same interpretation in **every model** (i.e. \(t=s\)), then \(\varphi \leftrightarrow \varphi[a'/a']\).

There are also more general versions of these principles for formulas and terms in general:

Let \(x_1, \ldots, x_n\) be exactly the variables occurring free in \(\psi\) or \(\chi\).

**EXTENSIONALITY OF \(L_4\) (for subformulas of formulas):**

\[\forall x_1 \ldots \forall x_n (\psi \leftrightarrow \chi) \Rightarrow (\varphi \leftrightarrow \varphi[a'/a'])\]

In every model where for every assignment \(g\) every resetting of the values of \(x_1, \ldots, x_n\) in \(g\) gives the same truth value to \(\psi\) and \(\chi\) in every such model, every assignment \(g\) gives \(\varphi\) and \(\varphi[a'/a']\) the same truth value.

Let \(t, s\) be terms, \(x_1, x_2\) the variables occurring in \(t, s\) (since in our version of predicate logic we don’t have complex terms, \(x_1, x_2\) can only possibly occur in \(t, s\) if \(t\) or \(s\) is \(x_1\) or \(x_2\)).

**EXTENSIONALITY OF \(L_4\) (for terms in formulas):**

\[\forall x_1 \forall x_2 (t = s) \Rightarrow (\varphi \leftrightarrow \varphi[a'/a'])\]

In every model where for every assignment \(g\) every resetting of the values of \(x_1, x_2\) in \(g\) assigns \(t\) and \(s\) the same interpretation, in every such model, every assignment \(g\) gives \(\varphi\) and \(\varphi[a'/a']\) the same truth value.
XII. CONNECTIONS BETWEEN CONNECTIVES AND QUANTIFIERS

∀xP(x) ⇔ ¬∃x¬P(x).

(1) [∀xP(x)]_M = 1 iff
(2) for every g: [∀xP(x)]_{M,g} = 1 iff
(3) for every g: for every d ∈ D_M: [P(x)]_{M,gx}^d = 1 iff
(4) for every g: for every d ∈ D_M: \begin{cases} 1 → 0 \\ 0 → 1 \end{cases} ([P(x)]_{M,gx}^d) = 0
(5) for every g: for every d ∈ D_M: F_M(¬) ([P(x)]_{M,gx}^d) = 0 iff
(6) for every g: for every d ∈ D_M: [¬P(x)]_{M,gx}^d = 0 iff
(7) for every g: for every d ∈ D_M: [P(x)]_{M,gx}^d = 0 iff
(8) for every g: for no d ∈ D_M: [¬P(x)]_{M,gx}^d = 1 iff \text{[semantics of ∃]}
(9) for every g: [∃x¬P(x)]_{M,g} = 0 iff
(10) for every g: \begin{cases} 1 → 0 \\ 0 → 1 \end{cases} ([∃x¬P(x)]_{M,g}) = 1 iff
(11) for every g: F_M(¬) ([∃x¬P(x)]_{M,g}) = 1 iff
(12) for every g: [¬P(x)]_{M,g} = 1 iff
(13) for every g: [¬P(x)]_{M,g} = 1 iff
(14) [¬P(x)]_{M} = 1

Similarly ∃xP(x) ⇔ ¬∀x¬P(x).

Note that ∀ generalizes ∧ and ∃ generalizes ∨:
Let D_M = \{d_1, \ldots, d_n\} and F_M(c_i) = d_1, \ldots, F_M(c_n) = d_n
Then:
\[ [∀xP(x)]_{M,g} = 1 \text{ iff } [P(c_1) ∧ \ldots ∧ P(c_n)]_{M,g} = 1 \]
\[ [∃xP(x)]_{M,g} = 1 \text{ iff } [P(c_1) ∨ \ldots ∨ P(c_n)]_{M,g} = 1 \]

This explains the similarity between ∀xP(x) ⇔ ¬∃x¬P(x) and the de Morgan law which says that: \((φ ∧ ψ) ⇔ ¬(¬φ ∨ ¬ψ)\).
(1) a. Every cat is smart.
   b. $\forall x [\text{CAT}(x) \to \text{SMART}(x)]$

(2) a. Some cat is smart.
   b. $\exists x [\text{CAT}(x) \land \text{SMART}(x)]$

**Question:** Does (1) entail (2)?

**Answer:** (1b) does not entail (2b).

Namely, assume: $F_M(\text{CAT}) = \emptyset$.
Then $\llbracket \exists x [\text{CAT}(x) \land \text{SMART}(x)] \rrbracket_M = 0$
But, $\llbracket \forall x [\text{CAT}(x) \to \text{SMART}(x)] \rrbracket_M = 1$ iff for every $d \in F_M(\text{CAT})$: $d \in F_M(\text{SMART})$,
and this is trivially the case:
$\llbracket \forall x [\text{CAT}(x) \to \text{SMART}(x)] \rrbracket_M = 1$
Hence (1b) does not entail (2b).

**FACT:** \{ $\forall x [\text{CAT}(x) \to \text{SMART}(x)]$, $\exists x [\text{CAT}(x)]$ \} $\implies \exists x [\text{CAT}(x) \land \text{SMART}(x)]$

So: on every model where there are cats and every cat is smart, there is indeed a smart cat.

**Question:** Why don't we make this part of the meaning?
   Why don't we change the semantics of every to:

(1) c. $\exists x [\text{CAT}(x)] \land \forall x [\text{CAT}(x) \to \text{SMART}(x)]$

**Answer:** Because we think that *It is not the case that every cat is smart* should be equivalent to *some cat isn't smart.*

**FACT:** $\neg \forall x [\text{CAT}(x) \to \text{SMART}(x)] \iff \exists x [\text{CAT}(x) \land \neg \text{SMART}(x)]$

Namely:
$\neg \forall x [\text{CAT}(x) \to \text{SMART}(x)] \iff$ (as we saw above)
$\exists x [\neg [\text{CAT}(x) \to \text{SMART}(x)] \iff (\neg (\varphi \to \psi) \iff (\varphi \land \neg \psi) )$
$\exists x [\text{CAT}(x) \land \neg \text{SMART}(x)]$

**FACT:** $\neg (\exists x [\text{CAT}(x) \land \forall x [\text{CAT}(x) \to \text{SMART}(x)]) \iff$
$\neg (\exists x [\text{CAT}(x)] \lor \exists x [\text{CAT}(x) \land \neg \text{SMART}(x)])$

So: *It is not the case that every cat is smart* would mean: either there are no cats, or some cat is not smart. And this seems too weak: if anything, you would want it to mean:
$\exists x [\text{CAT}(x)] \land \neg \forall x [\text{CAT}(x) \to \text{SMART}(x)]$
But this is just equivalent to: $\neg \forall x [\text{CAT}(x) \to \text{SMART}(x)]$.
(1) a. Not every picture ascribed to Rembrandt is by Rembrandt.
    b. Some picture ascribed to Rembrandt is not by Rembrandt.

Not a question of knowledge, but of fact:

(2) We have no techniques available to tell of any picture ascribed to Rembrandt that it is not by Rembrandt, but still I claim (1a)/but still I claim (1b).

(1a) and (1b) make the same claim in this context.

**Question:** why don't we make it a presupposition?

*Every cat is smart* presupposes that there are cats.

**Answer:** Some people do. But the more standard view is that that is too strong.

We all agree that there is an effect: *normally*, when we assert (1a), we commit ourselves to (2a) as well.

But the effect can be canceled:

(3) [I run a crackpot lottery, and solemnly swear in court:]  
a. Every person who has come to me over the last year, has gotten a prize.  
[aside:] Fortunately, I was away on a polar expedition all year.

My statement of (3a) may be insincere, but it is not infelicitous or false.  
It would be false, if *every entails some*, it would be infelicitous, if *every presupposes some*.  
But it is neither, it is only insincere, because I am well aware that my statement of (3a) is *trivially true*.

**With the semantics given, we can explain the effect pragmatically as an implicature:**

1. My semantics is the standard semantics for *every* which does not entail *some*.  
2. I obey Grice's Maxim of Quality: "Do not say what you know to be false."
   So I do claim (3a) to be true.  
3. But I violate part of Grice's Maxim of Quantity: "Do not give less information than is required."
   I violate this, because, in fact, I knowingly give *no information at all*, because I well know that the content of my statement is *trivial*.  
   Since I violate the maxim of quantity to mislead the judge and jury, I am insincere.  
4. But this explains directly, why, in normal contexts, *every conversationally implicates some*:
   The maxim of Quantity entails a maxim of:  
   **Avoid Triviality:** make your statement non-trivial.
We go back to (1) and (2). Since (1b) is trivial if there are no cats, the assumption that (1b) is asserted in accordance with Grice's maxims entails that there are cats, and this means that:

Even though (1b) does not entail (2b), (1b) conversationally implicates (2b).

And this is enough to explain the effect.

MORE EXTENDED: ENTAILMENT, PRESUPPOSITION, IMPLICATURE

ENTAILMENT?
Let p be a contingent sentence. If φ 'implies' p and ¬φ 'implies' p then p cannot be an entailment of φ:

| φ ⇒ p | every model where φ is true p is true |
| ¬φ ⇒ p | every model where ¬φ is true p is true | p is true in every model (hence not contingent)

Every cat is smart 'implies' there are cats
Not every cat is smart 'implies' there are cats
So: there are cats is not an entailment.

PRESUPPOSITION OF IMPLICATURE?
Let ψ entail ¬p. If p is a presupposition of φ, then φ is only felicitous in a context that already contains p.
This means that I cannot felicitously assert: φ ∧ ψ, because ψ entails ¬p, and φ requires p, this gives, p ∧ ¬p.
The conjunction test is a test for presuppositions:

Example:
I knew that John was rich 'implies' John was rich
I didn't know that John was rich 'implies' John was rich

John was poor entails John was not rich.

Check:
I knew that John was rich, even though he was poor.

If this feels inconsistent (a contradiction), the implication relation is presupposition (given that it is not entailment).
If it is consistent, the implication relation is implicature (and can, apparently, be canceled).
We check:

\(\varphi_1\) The person who presented me with a winning lottery ticket last year got a prize.

\(\varphi_2\) The three persons who presented me with a winning lottery ticket last year got a prize.

\(\varphi_3\) The persons who presented me with a winning lottery ticket last year got a prize.

\(\varphi_4\) Every person who presented me with a winning lottery ticket last year got a prize.

\(\psi\): Fortunately, I was away all year on a polar expedition.
(We assume that in the relevant context \(\psi\) entails that nobody could have presented me with a winning lottery ticket last year.)

Now we check the intuitions:

\[
\begin{array}{ccc}
\varphi_1 \land \psi & \text{inconsistent} & \text{the } N & \text{presupposes } N \neq \emptyset \\
\varphi_1 \land \psi & \text{inconsistent} & \text{the three } N & \text{presupposes } N \neq \emptyset \\
\varphi_1 \land \psi' & \text{consistent} & \text{the } Ns & \text{implicates } N \neq \emptyset \\
\varphi_1 \land \psi & \text{consistent} & \text{every } N & \text{implicates } N \neq \emptyset
\end{array}
\]

The standard theory of every and the Boolean theory of plurality and definites (in the version of Landman 2004) predicts these facts.

**Confirmation of the facts:**

\(\varphi_1\) in every family, the boy goes into the army.
\(\varphi_2\) in every family, the three boys go into the army.
\(\varphi_3\) in every family, the boys go into the army.
\(\varphi_4\) in every family, every boy goes into the army.

Data: \(\varphi_1\) presupposes: *In every family, there is a boy*
\(\varphi_2\) presupposes: *In every family, there are three boys*

\(\varphi_3,\ \varphi_4\) do not presuppose *In every family there are boys*, they only quantify over families in which there are boys, i.e. they mean:

*In every family where there are boys, the boys go into the army.*

Explanation:
Existence Presupposition failure leads to undefinedness, infelicity
Existence Implicature failure leads to triviality.
The universal quantification over families can be seen as a long conjunction:

\[ \varphi_1 \]

The boy in family 1 goes into the army \( \land \ldots \land \) the boy in family \( n \) goes into the army

If in family \( i \) there are no boys, the statement *The boy in family \( i \) goes into the army* is, as we have seen above, infelicitous.

But then the whole conjunction is infelicitous, and hence \( \varphi_1 \) is infelicitous. hence \( \varphi_1 \) presupposes *In every family there is a boy.*

\[ \varphi_3 \]

Every boy in family 1 goes into the army \( \land \ldots \land \) Every boy in family \( n \) goed into the army

If in family \( i \) there are no boys, the statement *Every boy in family \( i \) goes into the army* is, as we have seen above, trivially true.

But if \( \varphi_i \) is trivially true, \( \varphi \land \varphi_i \) is equivalent to \( \varphi \). Thus, the cases of families where there are no boys are truth conditionally irrelevant and drop out of the conjunction. hence \( \varphi_3 \) indeed only quantifies over families where there are boys.

This means that the standard theory of *every* and the boolean theory of plurality and definiteness needs to add nothing to make the right predictions here.

**AVOID TRIVIALITY**

1. under quantification the triviality of \( \forall x \varphi \) over boys on an empty domain guarantees, as it should, that the quantification over families is restricted in the right way.
2. In some cases we use triviality to stay within the law (tell the truth): violating quantity is not as bad as violating quality.
3. What do we get in normal cases?
   I say *Every cat is smart.*
   -You and I assume that I adhere to quality, so I am assumed to make a true statement.
   -You and I assume that I adhere to quantity. Trivial statements give no information, hence violate quantity. This brings in an existence implicature; *There are cats.*
Connections between $\forall$, $\land$, $\lor$ and $\exists$, $\land$, $\lor$

(1) $\forall x[P(x) \lor Q(x)]$
(2) $\forall xP(x) \lor \forall xQ(x)$
(3) $\forall xP(x) \land \forall xQ(x)$
(4) $\forall x[P(x) \land Q(x)]$

Entailment Pattern for Every(body):

(1) 

(2)

(3) \iff (4)

(3) \iff (4):
If everybody sings and dances, then everybody sings.
If everybody sings and dances, then everybody dances.
If everybody sings and everybody dances, then everybody sings and dances.

(3) \implies (2)
This is just: $\varphi \land \psi \implies \varphi \lor \psi$
(2) does not entail (3): again, $\varphi \lor \psi$ does not entail $\varphi \land \psi$

(2) \implies (1)
$\forall xP(x) \implies \forall x[P(x) \lor Q(x)]$
$\forall xQ(x) \implies \forall x[P(x) \lor Q(x)]$
If $\varphi \implies \chi$ and $\psi \implies \chi$ then $(\varphi \land \psi) \implies \chi$

(1) does not entail (2).
Let $D_M = \{a,b\}$, $F_M(P) = \{a\}$, $F_M(Q) = \{b\}$. 
$\llbracket \forall x[P(x) \lor Q(x)] \rrbracket_M = 1$
$\llbracket \forall xP(x) \lor \forall xQ(x) \rrbracket_M = 0$. 
(1) \( \exists x [P(x) \lor Q(x)] \)
(2) \( \exists x P(x) \lor \exists x Q(x) \)
(3) \( \exists x P(x) \land \exists x Q(x) \)
(4) \( \exists x [P(x) \land Q(x)] \)

**Entailment Pattern for Some(body):**

(1) \( \iff \) (2)

(3) \( \implies \)

(4)

(1) \( \iff \) (2)
If somebody sings or dances then somebody sings or somebody dances.
If somebody sings or somebody dances then somebody sings or dances.

(3) \( \implies \) (2)
same as above \( \phi \land \psi \implies \phi \lor \psi \)

(4) \( \implies \) (3)
If somebody sings and dances, somebody sings.
If somebody sings and dances, somebody dances.
If \( \phi \implies \psi \) and \( \psi \implies \chi \), then \( \phi \implies \psi \land \chi \)

(3) does not entail (4).
The same model as above.
\[ [\exists x P(x) \land \exists x Q(x)]_M = 1 \]
\[ [\exists x [P(x) \land Q(x)]]_M = 0 \]
(1) \( \neg \exists x [P(x) \lor Q(x)] \)
(2) \( \neg \exists x P(x) \lor \neg \exists x Q(x) \)
(3) \( \neg \exists x P(x) \land \neg \exists x Q(x) \)
(4) \( \neg \exists x [P(x) \land Q(x)] \)

Entailment Pattern for No(body):

(4) \uparrow
(2) \uparrow
(3) \iff (1)

(3) \iff (1)
If nobody sings and nobody dances, nobody sings or dances.
If nobody sings or dances, nobody sings.
If nobody sings or dances, nobody dances.

(3) \implies (2)
Same as above.

(2) \implies (4)
If nobody sings or nobody dances, nobody sings and dances.
Assume that nobody sings or nobody dances.
There are three cases:
- nobody sings. In that case obviously nobody sings and dances.
- nobody dances. Also nobody sings and dances.
- nobody sings and nobody dances. The same.

(4) does not entail (2)
The same model.
(4) is true, since a sings (P) but doesn't dance (Q) and b dances but doesn't sing.
(2) is false: it is not the case that nobody sings (since a sings) and it is not the case that nobody dances (since b dances). Hence it is not the case that nobody sings or nobody dances.

Generalize:

(1) NP sing or dance.
(2) NP sing or NP dance.
(3) NP sing and NP dance.
(4) NP sing and dance.

We saw above that everybody, somebody, nobody have different characteristic patterns. If you try other noun phrases you find that their patterns differ:
most boys
(1) Most boys sing or dance.
(2) Most boys sing or most boys dance.
(3) Most boys sing and most boys dance.
(4) Most boys sing and dance.

(1) \[\uparrow\]
(2) \[\uparrow\]
(3) \[\uparrow\]
(4)

(4) \[\Rightarrow\] (3)
If most boys sing and dance, more than half of the boys sing and dance.
Then more than half of the boys sing and more than half of the boys dance.

(3) does not entail (4)
Let \(F_M(BOY) = \{a,b,c,d,e\}\)
Let \(F_M(SING) = \{a,b,c\}\) and \(F_M(DANCE) = \{c,d,e\}\)
Then more than half of the boys sing, since \(\{a,b,c\}\) is more than half of \(\{a,b,c,d,e\}\)
Also more than half of the boys dance, since \(\{c,d,e\}\) is more than half of \(\{a,b,c,d,e\}\).
But less than half of the boys sing and dance, since \(\{c\}\) is less than half of \(\{a,b,c,d,e\}\).

As usual (3) \[\Rightarrow\] (2).

(2) \[\Rightarrow\] (1)
Assume (2) is true.
There are again three cases:
- More than half of the boys sing. Since everybody who sings sings or dances, it follows that more than half of the boys sing or dance.
- More than half of the boys dance. A similar argument.
- More than half of the boys sing and more than half of the boys dance. The same argument.

(1) does not entail (2)
Let \(F_M(BOY) = \{a,b,c,d,e\}\)
Let \(F_M(SING) = \{a,b\}\) and \(F_M(DANCE) = \{d,e\}\)
(1) is true, since the set of singers together with the set of dancers \(\{a,b,d,e\}\) is more than half of \(\{a,b,c,d,e\}\).
(2) is false, since the set of singers \(\{a,b\}\) is less than half of the boys, and the set of dancers \(\{d,e\}\) is less than half of the boys.
Exactly three boys
(1) Exactly three boys sing or dance.
(2) Exactly three boys sing or exactly three boys dance.
(3) Exactly three boys sing and exactly three boys dance.
(4) Exactly three boys sing and dance.

(1) (4) (2)    (3)

Here we find only the obvious entailment from (3) to (2), all the others are logically independent.

(4) does not entail (1), (4) doesn't entail (2), (4) doesn't entail (3):
F_M(BOY) = {a,b,c,d,e}
F_M(SING) = {a,b,c,d}, F_M(DANCE) = {b,c,d,e}
(4) is true, (1) is false, (2) is false, (3) is false.

(3) does not entail (1), (3) doesn't entail (4)
F_M(SING) = {a,b,c}, F_M(dance) = {c,d,e}.
(3) is true, (1) is false, (4) is false.

(1) doesn't entail (2), (1) doesn't entail (3), (1) doesn't entail (4):
F_M(SING) = {a}, F_M(DANCE) = {b,c} 
(1) is true, (2) is false, (3) is false, (4) is false.

(2) doesn't entail (1), (2) doesn't entail (3), (2) doesn't entail (4)
F_M(SING) = {a,b,c}, F_M(DANCE) = {d,e}.
(2) is true, (1) is false, (3) is false, (4) is false.

Inverse logic: if you're not sure whether an expression in a language means, say, every or most, check how that expression interacts with ∧ and ∨. The characteristic pattern will tell you.
XIII. NUMERICALS AND THE DEFINITE ARTICLE

Expressing numericals in predicate logic
(1) At least one cat is smart.
   $\exists x [C(x) \land S(x)]$
(2) At least two cats are smart.
   $\exists x \exists y [C(x) \land S(x) \land C(y) \land S(y) \land \neg(x=y)]$
(3) At least three cats are smart.
   $\exists x \exists y \exists z [C(x) \land S(x) \land C(y) \land S(y) \land C(z) \land S(z) \land \neg(x=y) \land \neg(x=z) \land \neg(y=z)]$
(4) At most one cat is smart.
   $\forall x \forall y [C(x) \land S(x) \land C(y) \land S(y) \rightarrow (x=y)]$
(5) At most two cats are smart.
   $\forall x \forall y \forall z [B(x) \land W(x) \land B(y) \land W(y) \land B(z) \land W(z) \rightarrow [(x=y) \lor (x=z) \lor (y=z)]]$
(6) At most three boys walk.
   $\forall x \forall y \forall z \forall u [C(x) \land S(x) \land C(y) \land S(y) \land C(z) \land S(z) \land C(u) \land S(u) \rightarrow$
   $[(x=y) \lor (x=z) \lor (x=u) \lor (y=z) \lor (y=u) \lor (z=u)]]$
(7) Exactly n cats are smart = at least n cats are smart \& at most n cats are smart.

Russell:
(7) The cat is smart.
   $\exists x [C(x) \land \forall y [C(y) \rightarrow (x=y)] \land S(x)]$
There is exactly one cat and that cat is smart.
Frege, Strawson: The existence and uniqueness are not asserted but presupposed.

Add to \(L_3\):
If \(P \in \text{PRED}^1\), then \(\sigma(P) \in \text{TERM}\)
Semantics:

$$\llbracket \sigma(P) \rrbracket_{M,g} = \begin{cases} 
  d & \text{if } \llbracket P \rrbracket_{M,g} = \{d\} \\
  \bot & \text{otherwise}
\end{cases}$$

\(\bot\) stands for undefined

This requires a three valued semantics which allows the truth value of expression to be undefined. Example:

(8) The cat is smart.
SMART(\(\sigma(CAT)\))

$$\llbracket \text{SMART}(\sigma(CAT)) \rrbracket_{M,g} = \begin{cases} 
  1 & \text{if } \llbracket \sigma(CAT) \rrbracket_{M,g} \in F_M(\text{SMART}) \\
  0 & \text{if } \llbracket \sigma(CAT) \rrbracket_{M,g} \in D_M - F_M(\text{SMART}) \\
  \bot & \text{otherwise}
\end{cases}$$

(8) is undefined if there is no cat, and also if there is more than one cat.
The use of an expression to talk about a situation \(M\) presupposes that it is defined in \(M\). Hence the use of (8) to talk about \(M\), presupposes that \(F_M(CAT)\) is a set with exactly one element, a singleton set.
Similar modifications are needed for sentences involving n-place relations and identity statements.

Also the connectives need to be modified.

Three valued negation:

\[
\begin{align*}
\neg & (0 \rightarrow 1) \\
& (1 \rightarrow 0) \\
& (\bot \rightarrow \bot)
\end{align*}
\]

Now it follows that both \( \text{WHITE}(\sigma(\text{CAT})) \) and \( \neg \text{WHITE}(\sigma(\text{CAT})) \) presuppose that there is a unique cat (in the context).

Context: You walk through an alley. There are two cats sitting on a garbage can.

You say: The cat is white.  # infelicitous.

Presuppositions of speech acts (Stalnaker): assertion, denial, questioning, supposition.

Further modifications of the semantics: strong Kleene three values semantics for connectives and quantifiers.

Strong Kleene three valued truth tables:

<table>
<thead>
<tr>
<th>( \land \varphi )</th>
<th>1</th>
<th>0</th>
<th>( \bot )</th>
<th>( \lor \varphi )</th>
<th>1</th>
<th>0</th>
<th>( \bot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \psi )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \bot )</td>
<td>1</td>
<td>( \bot )</td>
<td>0</td>
<td>( \bot )</td>
<td>1</td>
<td>( \bot )</td>
<td>( \bot )</td>
</tr>
</tbody>
</table>

Generalization to quantifiers:

\[
\llbracket \forall x \varphi \rrbracket_{M,g} =
\begin{align*}
1 & \text{ iff } \text{ for every } d \in D_M: \llbracket \varphi \rrbracket_{M,gx^d} = 1 \\
0 & \text{ iff } \text{ for some } d \in D_M: \llbracket \varphi \rrbracket_{M,gx^d} = 0 \\
\bot & \text{ otherwise}
\end{align*}
\]

\[
\llbracket \exists x \varphi \rrbracket_{M,g} =
\begin{align*}
1 & \text{ iff } \text{ for some } d \in D_M: \llbracket \varphi \rrbracket_{M,gx^d} = 1 \\
0 & \text{ iff } \text{ for every } d \in D_M: \llbracket \varphi \rrbracket_{M,gx^d} = 0 \\
\bot & \text{ otherwise}
\end{align*}
\]

These clauses generalize the clauses for \( \land \) and \( \lor \).
What about the following counterexample:

(1) A letter to the Times:

Sir. In contradiction to what was written in the Times yesterday, the president of Belgium was not sent to prison, because, as you ought to know, Belgium is a monarchy.

No contradiction.
Metalinguistic negation.

In Dutch –n after schwa is not pronounced, except in Groningen, despite what the crazy new spelling reform rules try to make you believe.
You are ordering pancakes in an Amsterdam pancake restaurant, and you read your order aloud from the menu (in new spelling) to the waiter. The waiter says to you with a sneer:

(2) Sir, we do not have “pannekoek” on the menu, we only have “pannekoek” on the menu (we are in Amsterdam here).

(3) Larry Horn’s example:

No Johnny, Phideau didn’t shit on the rug. he defecated on the carpet.

These cases are instances of metalinguistic negation (Horn):

Metalinguistic negation: Use negation to object to some aspect of the utterance other than truth value.

Like pronunciation, register, or indeed: presupposition.
WHY MOST IS NOT FIRST ORDER DEFINABLE

Let $\varphi = \text{Every Cat is Smart}$

If I start with a domain $D$ and I add an object, I can let the truth value of $\varphi$ *flip*:

i.e

- $D = \emptyset$ $\varphi$ is true
- $D = \{d_1\}$ $\varphi$ is true if $F(\text{CAT}) \cap F(\text{SMART}) = \{d_1\}$
- $D = \{d_1,d_2\}$ I can make $\varphi$ false, by setting, $F(\text{CAT}) = \{d_1,d_2\}$, and $F(\text{SMART}) = \{d_1\}$ (I keep the settings I set before for $d_1$).

So I can flip the truth value by extending the domain, but, it is easy to see, that I can let the truth value flip *at most once*, if I keep all the settings I set before:

If *every cat is smart* is true on domain $D$, it can become false by adding a cat, but as soon as it is false on a domain, no matter how many individuals I add to the domain, *every cat is smart* stays false (i.e. one non-smart cat is enough).

The same holds for a sentence like *Some cat is smart* for inverse reasons: it starts out false; in letting the domain grow, we can keep it false, or make it true, but once it is true on a domain, adding new individuals is not going to make a difference.

Other sentences can flip *more than once*.
Take *Exactly three cats are smart*.
On a domain of less than three individuals, the sentence is false. I can make it true once I have three individuals (*flip one*), and I can make it false again when I have four individuals (by making four cats smart) (*flip two*). Again, once we have done two flips, we cannot let it flip more.

A sentence like *exactly 3 cats or exactly 10 cats are smart* can flip four times.

This leads to the question:
For an arbitrary predicate logical sentence, how many times can it flip?

The answer is given in a theorem:

**Theorem**: Every predicate logical sentence can flip maximally a *finite number of times*, meaning: for each predicate logical sentence $\varphi$ there is a boundary number $n_\varphi$, which is the number of times that $\varphi$ can flip (this number can actually be computed for each sentence)

Now look at $\varphi = \text{Most cats are smart}$.
The truth conditions say: $|\text{CAT} \cap \text{SMART}| > |\text{CAT} - \text{SMART}|$

We start out with a domain on which $\varphi$ is true.
- Add non-smart cats to make the numbers equal: $\varphi$ flips: $\varphi$ is false.
- Add a smart cat: $\varphi$ flips: $\varphi$ is true
- Add a non-smart cat: $\varphi$ flips: $\varphi$ is false.
- Add a smart cat: $\varphi$ flips: $\varphi$ is true

etc…
Hence, for \( \varphi = \text{most cats are smart} \) the truth value of \( \varphi \) can continue to flip: there is no number \( n_0 \) where \( n \) is the maximal number of flips that \( \varphi \) makes. This means, by the theorem, that there is no predicate logical formula which is equivalent to \( \text{Most cats are smart} \), because for all predicate logical formulas there is such a number. This means that \textit{most} is not first order definable.

The proof of the above theorem is nasty, it involves keeping track of quantifier depth, quantifiers embedded into other quantifiers.

The following theorem is a straightforward consequence of the basic completeness theorem for predicate logic, the theorem that says that every valid inference can be derived in the proof theory of predicate logic.

If \( \Delta \) is a set of sentences we say that \( M \) is a model for \( \Delta \) if all the sentences in \( \Delta \) are true on \( M \).
We say that that a model \( M \) has cardinality \( n \) iff \( |D_M| = n \)
Thus, a finite model is a model with a finite domain.

\textbf{Theorem}: If a set of sentences has arbitrarily large finite models, it has an infinite model.

We use this theorem to prove that, while we can express all sorts of cardinality statements in predicate logic, we cannot express that the domain is finite or that the domain is infinite.

We look at the following three sentences:

(a) \( \forall x \text{[Angel}(x)\text{]} \)
(b) \( \forall x \text{[Angel}(x) \rightarrow \text{Stand-on-this-pin}(x)] \)
(c) Only finitely many angels stand on this pin.

Our set of sentences is \( \Delta = \{a,b,c\} \)
Hence in any model for \( \Delta \) there are only angels (by a), and they all stand on this pin by (b).
Obviously, if I take a domain with one object, specify that it is an angel and that it is standing on this pin, I have a model for \( \Delta \), because it is also true that only finitely many angels stand on this pin in this model, namely one.
Now, obviously, I can do this for any finite domain: if I interpret all the objects as angels standing on this pin, I have a model for \( \Delta \).
This means that \( \Delta \) has arbitrarily large finite models.
By the theorem, it means that \textbf{if (c) is definable in predicate logic, then} \( \Delta \) has an infinite model \( M^{\text{inf}} \), whose domain is the infinite set \( D^{\text{inf}} \).
Since this model is a model for \( \Delta \), everything in it is an angel standing on this pin and since \( D^{\text{inf}} \) is infinite, infinitely many angels stand on this pin.
But that means that it is false that only finitely many angels stand on this pin, so \( M^{\text{inf}} \) isn’t a model for \( \Delta \) after all. This can only be, if there is no predicate logical sentence defining (c), and that means that the notions finiteness/infinite are not definable.
XIV. ORDER RELATIONS

Let R be a two-place relation.

R is **reflexive**: \( \forall x [R(x,x)] \)
R is **irreflexive**: \( \forall x [\neg R(x,x)] \)

R is **transitive**: \( \forall x \forall y \forall z [R(x,y) \land R(y,z) \rightarrow R(x,z)] \)
R is **intransitive**: \( \forall x \forall y \forall z [R(x,y) \land R(y,z) \rightarrow \neg R(x,z)] \)

R is **symmetric**: \( \forall x \forall y [R(x,y) \rightarrow R(y,x)] \)
R is **asymmetric**: \( \forall x \forall y [R(x,y) \rightarrow \neg R(y,x)] \)
R is **antisymmetric**: \( \forall x \forall y [R(x,y) \land R(y,x) \rightarrow (x=y)] \)

R is **connected**: \( \forall x \forall y [R(x,y) \lor R(y,x)] \)
R is **s-connected**: \( \forall x \forall y [R(x,y) \lor R(y,x) \lor (x=y)] \)

R is a **pre-order**: R is reflexive and transitive
R is a **partial order**: R is reflexive and transitive and antisymmetric.
R is a **strict partial order**: R is irreflexive and transitive and asymmetric.

R is a **total (or linear) order**: R is a connected partial order.
R is a **strict total order**: R is an s-connected partial order.

R is an **equivalence relation**: R is reflexive and transitive and symmetric.
Partial order:

(ir)reflexivity understood:

Transitivity understood:

Direction of the graph understood:

Equivalence relations and partitions:
XV. AMBIGUITY

1 Lexical ambiguity.

(1) I took my money to the bank.  
    Reading 1: And deposited it there.  
    Reading 2: And buried it there.  

Ambiguity of the lexical meaning of bank.

Lexical drift:

**Flemish:** *knap* = 1. Intelligent  
                  2. Admirable  
                  3. Skilful  

**Dutch:** *knap* = 0. Pretty  
    1. Intelligent  
    2. Admirable  
    3. Skilful  

In certain idioms 4. Narrow  

**German:** *knapp* = 4 By a narrow margin, Narrow, Short, Brief,  
    *Knapp verfehlt* - Just failed (exam result)

-Systematic Lexical Ambiguity

MASS AND COUNT NOUNS  
(using recent work by Susan Rothstein, by me and others)

1. Count nouns and mass nouns

<table>
<thead>
<tr>
<th>Count Nouns</th>
<th>singular</th>
<th>plural</th>
<th>Counting</th>
</tr>
</thead>
<tbody>
<tr>
<td>boy</td>
<td>boys</td>
<td>one boy</td>
<td>three boys</td>
</tr>
<tr>
<td>rabbit</td>
<td>rabbits</td>
<td>one rabbit</td>
<td>three rabbits</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mass Nouns</th>
<th>Counting</th>
</tr>
</thead>
<tbody>
<tr>
<td>mud</td>
<td>one mud</td>
</tr>
<tr>
<td>furniture</td>
<td>one furniture</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Ambiguous</th>
</tr>
</thead>
<tbody>
<tr>
<td>hair</td>
</tr>
<tr>
<td>My hairs are getting grey</td>
</tr>
</tbody>
</table>
2. Bare nouns

Bare nouns are nouns that occur in argument position without determiners.

ENGLISH

English has Bare Plural Count nouns:

(1) a. Dogs were running around.
    b. Dogs play with each other when they are cheerful
    c. There were dogs running around all afternoon.

English has Bare Mass nouns:

(2) a. Mud was thrown at the prime minister.
    b. There was mud on my shoe.

English does not have Bare Singular Count nouns:

(3) a. #Dog was running around.
    b. #Dog likes each other.
    c. #There was dog running around all afternoon.

3. Grinding

(4) a. After the accident with the fan, there was rabbit all over the wall.
    b. When Fred stopped trying to repair the watch, there was watch all over the table.

David Lewis: Universal grinder.
The meaning can shift from count to mass by grinding.

-A shift from the singular count meaning of the noun rabbit to a ground mass reading: rabbit stuff

In English: grinding is a Last Resort Mechanism (Rothstein)

In (4) we find the bare singular count noun rabbit. English doesn have bare singular count nouns.
We get a grammar conflict.
The conflict can be resolved in some positions in English by grinding.
Not in normal argument position, but in the position that the subject is in in there-insertion contexts:

Compare (5a) with (5b):

(5) a. After the accident with the fan there was cockroach all over the wall.
    b. #After the accident with the fan there were cockroaches all over the wall.
(5a) has the grinding reading. (5b) does not have a grinding reading. This is explained if grinding is indeed a last resort mechanism: English does have bare plurals, so we get a perfectly felicitous plural interpretation in (5b): no conflict, no resolution of a conflict, no grinding.

**Mandarin Chinese**

Mandarin Chinese:
1. No number; no lexical distinction between mass and count nouns. 
2. Number expressions cannot directly modify nouns.

```
#Liăng nîu    #Liăng ròu
two    cow    two    meat
```

3. Classifiers are used to mediate this relation.

Classifiers exist and are normal in many languages:

English: collection classifiers: *a flight of* birds, *a school of* fish
        container classifiers: *a bottle of* milk

Classifiers are packagers (the opposite of grinders)

Classifier languages: classifiers are obligatory for counting.

Mandarin does distinguish count and mass nouns at this level.
The general individual classifier *ge* goes with nouns that are prototypically count, but not with nouns that are prototypically mass:

```
✓Liăng ge nîu    #Liăng ge ròu
two    CL cow    two    CL meat
```

4. Mandarin Chinese allows all nouns as bare nouns.

What about the grinding context?

```
(6) Qiang-shang dou shì gou
    wall-topic    all    copula    dog
- This means: There are dogs all over the wall (doggy wallpaper)
- This does not mean: There is dog all over the wall (grinding reading)
- To express the ground reading, you need to use *gou-rou/dog meat*
```

**Conclusion:**

With respect to grinding, Chinese works like English:
- grinding is a last resort device which comes into play in the case of a conflict.
- since in Chinese all nouns occur bare anyway, a plural interpretation is in principle available in (6).
- No grammatical conflict, no grinding.
BRASILIAN PORTUGUESE

Brasilian Portuguese has number, and bare plural count nouns (like English), but also allows bare singular count nouns.

(7) Eu vi criança na sala. ✓ E ela estava ouvindo / E elas estavam ouvindo. I saw child in the room. And she was listening/ And they were listening

boy was playing with each other

In Brasilian Portuguese, what readings you get is dependent on aspect:

In the perfective aspect, the facts are the same as in Chinese: no grinding.
In the imperfective aspect, you get an ambiguity between a plural and a grinding reading:

(8) a. Depois do acidente, teve cachorro na parede in teira
   After the accident was[perfective] dog in the wall whole
   Only reading: There were dogs on the wall
No grinder reading
   b. Depois do acidente, tinha cachorro na parede in teira
   After the accident was[imperfective] dog in the wall whole
Ambiguous: There was dog-stuff on the wall/ there were dogs on the wall.

Three languages, three different patterns to do with how bare nouns work in the grammar of these languages.

FOODSTUFF NOUNS

English:

(9) a. There is apple in the salad.
   b. There is pig in the salad   Grinding
cf: In the restaurant with three Michelin stars I take a hair out of my soup and say:

(10) Yeagh, there is Cordonbleu cook in this soup.

   Where has the big apple gone that was lying here
(11) #There is big apple in the salad

The contrast:

(12) a. There is big dog in the salad (Labrador)
   = grinding: there is stuff in the salad derived from big dog
   b. There is big banana in the salad
   ≠ grinding: there is stuff in the salad derived from big banana

Why is (11) weird and doesn't it have reading (12b) in analogy to (12a)?
Answer: (Landman)
Foodstuff nouns like *apple* are ambiguous between a count and a mass interpretation
- *big* does not modify prototypical mass nouns (cf. *#big mud*)
- The head noun of the bare noun (i.e. *apple*) determines whether the grinding strategy is available or not.

So:
(1) The foodstuff noun *apple* is lexically ambiguous between a count and a mass noun interpretation.
   The complex *big apple* is not ambiguous in the same way: there is no food stuff: *big apple*.
(2) In the bare noun *big apple*, *apple* is the head, and it allows a mass interpretation, hence no conflict, no grinding (because bare mass nouns are allowed in English).
(3) However, *big* can not modify the mass interpretation of *apple*, only the count interpretation.
(4) We get a conflict after all.
   But, by assumption, we can no longer resolve this conflict by grinding (because it is the head that determines whether you will grind or not).

In other words: the last resort nature of grinding predicts that since the headnoun *banana/apple* has a felicitous mass reading in this context, there is no grinding. But then *big* must apply to the mass interpretation, which is weird.
*Dog* does not have a mass interpretation, so *dog* can be ground in this context. If you can grind *dog*, you can grind *big dog*, hence the felicity of (12a).

Does this make sense?
Yes, when we look cross-linguistically.

**Mandarin Chinese**

(12) a. Shala li you *pinggui* foodstuff
    Salad inside have *apple*

b. Shala li you *zhu* conceptually count
    Salad inside have *pig*

(12a) is ambiguous:
- There are *apples* in the salad
- There is *apple-stuff* in the salad

(12b) is not ambiguous:
(12b) means: there is a whole pig in the salad
(12b) does not mean: there is *pig-meat* in the salad
Brasilian Portuguese

Change *dog* to *apple*:
Both in the imperfective and in the perfective do you get two readings:
- There are *apples* in the salad
- There is *apple-stuff* in the salad

This suggests indeed that foodstuff nouns are systematically ambiguous between mass and count readings.

**Conclusion:**
Cross linguistic variation, but systematic ambiguities and systematic connections:

For singular count nouns we find two readings:
- a lexical count reading
- a derived ground reading, derivable in contexts of conflict

For foodstuff nouns in English we find two readings:
- a lexical count reading
- a lexical mass reading

And there is reason to think that there are two distinct mass readings:
- lexical mass vs. ground mass

We see that Chinese and Brasilian Portuguese in essence bring out the same distinctions, with some differences:
- Chinese does not have grinding (at least not in the examples studied here), because there isn’t a conflict to be resolved
- In Brasilian Portuguese, grinding is not a last resort option, but generally available (in the imperfective).

Thus, the ambiguities form regular patterns: there is method in this madness.
2. Syntactic ambiguity.

(1) Old men and women danced. Reading 1 entails: Old women danced. Reading 2 doesn’t entail: Old women danced.

This is an ambiguity of the scope of *old*.

Usual assumption: represented in syntactic constituent structure (at surface structure): 

\[
[\text{NP} \text{old} \text{[NP men and women]]} \text{ vs. } [\text{NP} \text{old men }] \text{ and } [\text{NP women}]
\]

**RELATIVE CLAUSES**

(2) a. In this opera, the prince is in love with \underbrace{\begin{array}{c} \text{a girl} \\ \text{the girl} \\ \text{every girl} \end{array}}_{\text{every girl}} 

b. In this opera, the prince is in love with \underbrace{\begin{array}{c} \text{a girl} \\ \text{the girl} \end{array}}_{\text{every girl}}, who doesn’t love him

(2a) is a restrictive relative, (2b) a non-restrictive relative, an appositive.

The data shows a similarity between *non-restrictive* relatives and discourse anaphora:

(3) a. In this opera, if \underbrace{\begin{array}{c} \text{a girl} \\ \text{the girl} \\ \text{every girl} \end{array}}_{\text{every girl}} hides in the cupboard, it is because \underbrace{\begin{array}{c} \text{she} \\ \#\text{she} \end{array}}_{\text{she}} doesn’t want to meet the prince.

Syntactic ambiguity of the relatives:

Restricted relative:

\[
\text{DP} \quad \text{D} \quad \text{NP} \\
\quad \text{a/the/every} \quad \text{NP} \quad \text{CP} \\
\quad \text{girl} \\
\quad \text{who doesn’t love him}
\]

Non-restricted relative:

\[
\text{DP} \quad \text{PRED[CP]} \\
\text{D} \quad \text{NP} \\
\text{a/the/every} \quad \text{girl} \\
\text{who doesn’t love him}
\]

The syntactic ambiguity accounts for the discourse anaphora facts:

- in the restrictive relatives there is normal binding
- the non-restrictive relative patterns with discourse anaphora. It adjoines to a full DP, which functions as its discourse anaphora antecedent.
3. Scope ambiguity: quantifiers and negation (English)

(3) Everybody isn't smart.

Reading 1: \( \forall x[\neg \text{SMART}(x)] \) \( \neg \) in the scope of \( \forall \)
Reading 2: \( \neg \forall x[\text{SMART}(x)] \) \( \forall x \) in the scope of \( \neg \)

Usual assumption: not represented in syntactic constituent structure (at surface structure).
Alternative approaches:
I. Ambiguity is represented in constituent structure at a different level: Logical Form.
   - build one surface structure.
   - Give rules for deriving two logical representations from this.
   - Interpret these two logical representations.
Theoretical Claim: There is a level of Logical Form ordered after the surface syntax:
Semantic interpretation is interpretation of fully derived surface structures.

II. Ambiguity is represented in semantic derivation: the same syntactic constituent structure at surface structure is derived in two different ways:
   - the semantic operations for building the meaning of one surface structure for (1) can be applied in two different orders. This gives two meanings.
Theoretical Claim:
You don't need to wait with interpreting till you have derived surface structure, there is no independent level of logical form.

Elke verandering is geen verbetering

-Scope ambiguity: multiple quantifiers.

(4) Every man admires a woman.

Reading 1: His mother. (or a list…)
Reading 2: Madonna.

\( \forall x[\text{MAN}(x) \rightarrow \exists y[\text{WOMAN}(y) \land \text{ADMIRE}(x,y)]] \)

\( \exists y[\text{WOMAN}(y) \land \forall x[\text{MAN}(x) \rightarrow \text{ADMIRE}(x,y)]] \)

(5) Some man admires every woman.

Inverse reading is a bit harder to get (but try intonation: no stress on some man + stress on every woman).
But the inverse reading is easy to get in other cases:

(6) A flag hung in front of every window.

A flag spanned every window from left to right

cf. the contrast in (7):

(7) a. At the finish, a bus is waiting for every participant from Ramala.

Preferred reading: \( \exists x[\text{BUS}(x) \land \forall y[\text{P}(y) \rightarrow \text{AWAIT}(x,y)]] \)

b. At the finish, a medal is waiting for every participant from Ramala.

Preferred reading: \( \forall y[\text{P}(y) \rightarrow \exists x[\text{M}(x) \land \text{AWAIT}(x,y)]] \)

Inverse scope: easy to get because medal naturally has a relational interpretation ('his medal'), and the implicit argument is easily bound by the other quantifier: but that requires inverse scope:

\( \forall y[\text{P}(y) \rightarrow \exists x[\text{M}(x,y) \land \text{WAIT}(x,y)]] \)
(8) a. In New York City a pedestrian is run over by a car every 3 minutes.
   b. In Soviet Russia a tour guide accompanied every foreign visitor.

The readings in scope ambiguities with multiple quantifiers seen so far are not independent: one reading entails the other: i.e. \( \exists y \forall x [R(x,y)] \) entails \( \forall x \exists y [R(x,y)] \), but not vice versa.

In general, if one reading \( \alpha \) entails the other \( \beta \), you have to take into account the possibility that the grammar generates only the weaker reading \( \beta \), and derives in context the interpretation as a special case. This is what we assume for three cats are smart: most of us let the grammar generate an at least interpretation and see the stronger exactly interpretation as a special case: we don't assume an ambiguity. This strategy has been attempted for scope ambiguities as well (by Tanya Reinhart in the seventies), but not successfully. If the readings are logically independent, such a pragmatic strategy will not work. In that case you either have to argue that the grammar derives a third weaker reading \( \gamma \) that is entailed both by \( \alpha \) and by \( \beta \) and treat both \( \alpha \) and \( \beta \) as special cases (this has been attempted by Kempson and Cormack in the early eighties, also not successfully, I think), or you have to accept that there is indeed an ambiguity that the grammar must derive.


**Predicates of individuals:** have blue eyes:
Distributive interpretation:
   (8) a. John and Bill have blue eyes iff John has blue eyes and Bill has blue eyes iff each of John and Bill has blue eyes.
   b. Three boys have blue eyes iff there is a group of three boys and each of those three boys has blue eyes.

**Predicates of groups of individuals:** meet in the park:
In simple cases: collective interpretation:
   (9) a. John and Bill met in the park.
     does not mean: John met in the park and Bill met in the park.
     does not mean: each of John and Bill met in the park.
The intransitive predicate meet in the park is not a predicate of individuals.
   b. Three boys met in the park.
     means: there is a group of three boys and that group met in the park.
     does not mean: there is a group of three boys and each of those three boys met in the park.

**Predicates of individuals or groups of individuals:** carry the piano upstairs:
Collective/distributive ambiguity:
   (10) a. John and Bill carried the piano upstairs.
     **Reading 1**: John and Bill together carried the piano upstairs,
     John and Bill carried the piano upstairs as a group. (Collective)

Diagnostics of collective reading: weak involvement of the group members:
the boys carried the piano upstairs allows a boy that doesn't do any carrying but walks in front with a flag.
**Reading 2**: John carried the piano upstairs and (after that) Bill carried the piano upstairs. (Distributive)
b. Three boys carried the piano upstairs.

**Reading 1**: There is a group of three boys, and as a group, they carried the piano upstairs. (Collective)

**Reading 2**: There is a group of three boys, and each of those three boys carried the piano upstairs (Distributive).

**FACT**: For sentences with multiple noun phrases we find **scopal** and **non-scopal** interpretations.

**Example:**

(11) Two flags hung in front of three windows.

```
<table>
<thead>
<tr>
<th>Flag</th>
<th>Window</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

**Non-scopal reading**: Representation something like the following:

\[
\exists X[\text{FLAG}(X) \land |X|=2 \land \exists Y[\text{WINDOW}(Y) \land |Y|=3 \land \text{HIFO}(X,Y)]]
\]

\[f_1+f_2 \rightarrow w_1+w_2+w_3\]

Two flags hung in front of three windows.

We went into town, and saw two flags sown together spanning three windows.

Theories of plurality discuss whether there is one non-scopal reading or several (the question is: do we need to distinguish: group \(f_1+f_2\) spans \(w_1+w_2+w_3\) from say: \(f_1\) spans \(w_1+w_2+w_3\) and \(f_2\) spans \(w_1+w_2+w_3\)?)

**Models for non-scopal readings** involve maximally two flags and three windows.

**Cumulative readings (total-total)**

20 Chickens laid 140 eggs last week.

20 CH + 140 eggs + every one of these chickens laid some of these eggs + every one of these eggs was laid by one of these chickens

These readings are not collective: *laying, give birth to* are non-collective relations.)

(argument from Landman 1994, 2000)

**Collective:**

(1) a. Five women *met* with ten children 5 – 10

b. Ten women *met* with five children 10 – 5

**Cumulative:**

(2) a. Five women *gave birth to* ten children 5 – 10

b. #Ten women *gave birth to* five children #10 - 5

Why the infelicity of (2b)? Because *give birth to* does not allow a collective interpretation.
But then the felicitous (2a) is not collective either.
So cumulative readings and collective readings are not the same thing.
Note: I say *infelicity* of (2b), but I am not saying that (2b) is strictly speaking infelicitous. Rather it is uncomfortable. Why? Because it seems to treat *giving birth* as something that can be treated as the responsibility of the whole group of *ten women*. The point is: that is a collectivity effect and often uncomfortable (as group responsibility often is).

Why do we get this effect in (2b)? Because (2b) cannot have a cumulative reading (because the numbers don't allow a cumulative reading).

Why don't we get this effect in (2a)? Because (2a) *does* allow a cumulative reading. If there were only a collective reading, then (2a) should be as uncomfortable as (2b), but it is not. The existence of cumulative readings explains the contrast.

**Scopal readings**

Every theory needs to distinguish non-*scopal* readings from *scopal* readings, which associate with distributive interpretations.

**Models for scopal readings** involve a maximum of two flags and six windows, or six flags and three windows.

The most natural *scopal* interpretations of (12) are:

**Distributive-flag takes scope over collective-window):**

\[
\begin{align*}
\exists X & [\text{FLAG}(X) \land |X|=2 \land \\
\forall x & \in X: \exists Y [\text{WINDOW}(Y) \land |Y|=3 \land \text{HIFO}(x,Y)]
\end{align*}
\]

\[
\begin{align*}
& f_1 \rightarrow w_1 + w_2 + w_3 \\
& f_2 \rightarrow w_4 + w_5 + w_6
\end{align*}
\]

Two flags hung in front of three windows:
We found two three-window spanning flags.
Distributive-widow takes scope over collective-flag: INVERSE SCOPE

\[ \exists Y[\text{WINDOW}(Y) \land |Y|=3 \land \forall y \in Y: \exists X[\text{FLAG}(X) \land |X|=2 \land \text{HIFO}(X,y)] \]

\[ f_1 + f_2 \rightarrow w_1 \]
\[ f_3 + f_4 \rightarrow w_2 \]
\[ f_5 + f_6 \rightarrow w_3 \]

Two flags hung in front of three windows. Of windows with two flags, we found three.

We’ve seen many windows with one flag. What about two flags? Well, two flags...hm...two flags....let me count.... Ok, two flags hung in front of three windows.

In this case, the recto-scope reading and the inverse scope reading are logically independent, neither entails the other. This is evidence that a mechanism for recto and inverse scope must be part of the grammar.

- Scope islands

A medal was given to every girl

A medal *that was given to every girl* was put in the museum.

Unavailable reading: (wide scope of *every girl*)

\[ \forall x[\text{Girl}(x) \rightarrow \exists y[\text{Medal}(y) \land \exists z[\text{Give}(z,y,x) \land \text{PIM}(y)]]] \]

For every girl, there is a medal that was given to her and put in the museum.

Available reading: (narrow scope of *every girl*)

\[ \exists y[\text{Medal}(y) \land \forall x[\text{Girl}(x) \rightarrow \exists z[\text{Give}(z,y,x)] \land \text{PIM}(y)]] \]

Some medal was put in the museum and each girl was given that medal (say, in turn).
5. De dicto-de re- ambiguity.
Intensional contexts have scope.

-Modals: may

(12) As far as I know, everybody may have done it.
   a. \(\forall x[\textit{may}(x, \text{it})]\)
   b. \(\textit{may}(\exists x[\text{DONE}(x, \text{it})])\)

Reading a.: Beginning of a detective novel.
Reading b.: Towards the end in a famous detective novel by Agatha Christy.

-Intensional verbs: try

(13) John tries to find a unicorn

Representation, something like the following:
   a. \(\texttt{TRY}(j, \exists y[\textit{UNICORN}(y) \land \textit{FIND}(j, y)])\) \([\textit{de dicto}]\)
   b. \(\exists y[\textit{UNICORN}(y) \land \texttt{TRY}(j, \textit{FIND}(j, y))]\) \([\textit{de re}]\)

The \textit{de dicto} reading does not entail that there \textit{is} a unicorn:
TRY-TO-FIND is not a relation between John and an actual unicorn, but a relation between John and the unicorn-property:
John tries to bring himself in a situation where he has found an instance of the unicorn-property.

The \textit{de re} reading \textit{does} entail that there \textit{is} a unicorn:
The sentence expresses that there is an actual unicorn, say, Fido, and John tries to bring himself in a situation where he has found Fido.

Also \textit{de dicto} and \textit{de re} are generally logically independent.

-Propositional attitude verbs: know, believe:

(14) John believes that a former soccer player was elected Governor.
   a. \(\texttt{BELIEVE}(j, \exists y[\textit{FSP}(y) \land \textit{EG}(y)])\) \([\textit{de dicto}]\)
   b. \(\exists y[\textit{FSP}(y) \land \texttt{BELIEVE}(j, \textit{EG}(y))]\) \([\textit{de re}]\)

Reading a:
John reads in the newspaper: "The newly elected governor used to play Rambo." He thinks Rambo is a soccer team, and he tells me: "A former soccer player got elected governor." I report what he told me to you: I say (14). I report a belief of John about the property \textit{former soccer player}: in the world according to John, the newly elected governor is a former soccer player.
(14) is true, even though John has no belief \textit{about} any actual individual that that individual got elected governor.
Reading b:
John watched the Governor election, and saw there Arnold getting elected. But he wasn't wearing his glasses, and he thought it was Johan Cruyff. He thinks that Johan Cruyff got elected governor. Not knowing any Dutch, but having seen Johan Cruyff on Dutch television a lot while zapping, he thinks that Johan Cruyff is the Dutch prime minister.
John says to me: "Johan Cruyff got elected governor."
Now, I know very well who Johan Cruyff is, and that he is a famous former soccer player, but I don't know that John doesn't know that, and I do know that you don't know who Johan Cruyff is. For the latter reason, I report what John said to me to you by saying (14).
In this case, John would not himself accept: "A former soccer player got elected governor." (He would accept: "The Dutch prime minister got elected governor."). What I report to you by saying (14) is a belief of John about Johan Cruyff, about someone who actually is a former soccer player.

The situations were chosen in such a way that in the first one the de dicto reading is true, but the de re reading false, while in the second situation the de re reading true, but the de dicto reading false. So indeed, the two readings are logically independent (neither entails the other).
This means that if we agree that (14) can be truthfully said in those two types of situations, there is an ambiguity that the grammar must account for.
XVI. GENERALIZED QUANTIFIERS

Frege/Tarski:
Quantifier \( \forall x \) or \( \exists x \) does **two things simultaneously**:
1. Frege: It binds the occurrences of variable \( x \) free in its scope.
   Tarski: It sets up a variation range for the truth value of its scope along the variation of the value for variable \( x \).
2. Frege: It expresses its lexical meaning.
   Tarski: It expresses a constraint according to its lexical meaning on this variation range.


Montague-Lewis: Successful compositional semantic analysis of **natural language quantification** becomes possible only when we realize that for natural language quantification the Frege/Tarski theory is **wrong**.
(Note: Montague and Lewis do not say this explicitly, but it follows from their work)
And what is wrong, is part one of the Frege/Tarski analysis of quantification:

Montague-Lewis: **Natural language quantifiers do not bind variables.**
(Montague doesn't say this explicitly, but it follows from the theory in Montague 1973. Lewis is explicit about this.)

For quantification in natural language, we must separate the setting up of Tarski's variable range from the lexical restriction on the variable range: quantifiers only do the latter.

As it turns out, this separation is linguistically motivated **both** from the perspective of variable binding, and **from** the perspective of quantification.

**Variable Binding:** quantifiers do not bind variables, because variables are already bound inside the scope of the quantifier.

Some linguistic evidence.

1. **Evidence from variables: reflexives.**

1a. Reflexives without quantificational binders.
   (1) Every boy admires himself.
   \( \forall x[\text{BOY}(x) \rightarrow \text{ADMIRE}(x,x)] \)
   Frege/Tarski: The quantifier \( \forall x \) binds the interpretation of the reflexive, the third occurrence of \( x \).
Problems:
- Non-quantificational subjects.

(2) John admires himself.
   \text{ADMIRE}(j,j)

Intuitively, the interpretation of the reflexive is bound in (2) in the same way as it is in (1). (i.e. we do have something of the form \text{ADMIRE}(a,a)) in the semantics). But there is no quantifier in (2), and hence no binding operator.

- No subjects.

(3) a. \text{To admire oneself too much} is regarded as vanity.

   b. \text{Excessive admiration of oneself} is regarded as vanity.

Intuitively, the reflexive is bound in the infinitive and in the noun phrase in the same way as it is in (1) and (2). (We need, in the semantics, something of the form \text{ADMIRE}(a,a)). But there is no subject, let alone a quantificational subject binding the reflexive.

1b. Reflexives in VP-ellipsis.

VP-ellipsis:

(4) John is smart and Mary is too.

\text{be smart} \longrightarrow \text{be smart}

John is smart and Mary is smart

(5) John kissed Ronya and Mary did too.

\text{kiss} \text{Ronya} \longrightarrow \text{kiss} \text{Ronya}

John kissed Ronya and Mary kissed Ronya.

(6) John likes himself and Mary does too. (sloppy identity reading)

\text{like yourself} \longrightarrow \text{like yourself}

John likes himself and Mary likes herself.

(6) Every boy likes himself and Every girl does too.

\text{like yourself} \longrightarrow \text{like yourself}

Every boy likes himself and Every girl likes herself.

Think about (6):

\forall x [\text{BOY}(x) \rightarrow \text{LIKE}(x,x)] \wedge \forall y [\text{GIRL}(y) \rightarrow ?]

What we want is a one-place predicate, the interpretation of \text{like yourself} in which the variable is bound:

\forall x [\text{BOY}(x) \rightarrow \text{like yourself}(x)] \wedge \forall y [\text{GIRL}(y) \rightarrow ?(y)]
\forall x [\text{BOY}(x) \rightarrow \text{like yourself}(x)] \wedge \forall y [\text{GIRL}(y) \rightarrow \text{like yourself}(y)]

\text{Lambda Notation} (\text{to be defined shortly}):
\[ \lambda x. \varphi(x) \]
The property that you have if \( \varphi \) is true of you.

\textit{like yourself}: \( \lambda z. \text{LIKE}(z,z) \)
The property that you have if you like yourself.

Equivalences:

\[ \forall x [\text{BOY}(x) \rightarrow \text{LIKE}(x,x)] \land \forall y [\text{GIRL}(y) \rightarrow ?(y)] \]

Equivalent in property form:

\[ \forall x [\text{BOY}(x) \rightarrow \lambda z. \text{LIKE}(z,z) (x)] \land \forall y [\text{GIRL}(y) \rightarrow \lambda z. \text{LIKE}(z,z) (y)] \]
For every \( x \) if \( x \) is a boy then \( x \) has the like-yourself property

\[ \forall x [\text{BOY}(x) \rightarrow \lambda z. \text{LIKE}(z,z) (x)] \land \forall y [\text{GIRL}(y) \rightarrow \lambda z. \text{LIKE}(z,z) (y)] \]
For every \( x \) if \( x \) is a boy then \( x \) has the like-yourself property and
For every \( y \) if \( y \) is a girl then \( y \) has the like-yourself property.

Every boy likes himself:
\[ \forall x [\text{BOY}(x) \rightarrow \text{LIKE}(x,x)] \]
\[ \forall x [\text{BOY}(x) \rightarrow \lambda z. \text{LIKE}(z,z) (x)] \]
every:
\[ \forall x [------(x) \rightarrow -----------\equiv--(x)] \]
\text{BOY} \quad \lambda z. \text{LIKE}(z,z)

\textit{every} relates two one-place predicates:

\[ \text{EVERY} (\text{BOY}, \lambda z. \text{LIKE}(z,z)) \]

But then, the crucial observation is:

\textbf{The quantifier doesn’t bind any variables, because variables like reflexives are already bound (in the predicate, by the \( \lambda \)-operator).}

1c. Pronouns ‘bound’ by quantifiers that cannot bind them: Functional readings.

(7) The woman that every Englishman adores most is his mother.

Meaning of (7):

\[ \forall x [\text{ENGLISHMAN}(x) \rightarrow \]
\text{For every englishman}
\[ \sigma(\lambda y. \text{WOMAN}(y) \land \text{ADORE-MOST}(x,y)) = \sigma(\lambda y. \text{Mother-of}(y,x)) ] \]
\text{The woman he adores most} is his mother
To read this formula, read the predicates first:

$$\lambda y.\text{WOMAN}(y) \land \text{ADORE-MOST}(x,y)$$  the property you have if you are a woman and $x$ adores you most

$$\lambda y. \text{Mother-of}(y,x)$$  The property that you have if you are $x$ 's mother

$\sigma$ is the definiteness operator, so:

$$\sigma(\lambda y.\text{WOMAN}(y) \land \text{ADORE-MOST}(x,y)))$$  The woman that $x$ adores most

$$\sigma(\lambda y. \text{Mother-of}(y,x)))$$  The mother of $x$

$$\sigma(\lambda y.\text{WOMAN}(y) \land \text{ADORE-MOST}(x,y))) = \sigma(\lambda y. \text{Mother-of}(y,x)))$$  The woman that $x$ adores most is the mother of $x$.

Problem: this involves scoping every Englishman out of the relative clause that every Englishman adores, which violates the scope island.

Alternative analysis:  **Functional readings.**

(7) is analyzed as an equation of two functions $f$ and $g$, both of which are functions from individuals to individuals

$$f, g: D_M \rightarrow D_M$$

*The woman that every Englishman adores most*

Interpretation:  The function $f$ that maps every Englishman onto the woman that he adores most.

*his mother*

Interpretation:  The function that maps every individual onto his/her mother:

We can represent these readings also with help of the $\lambda$-operator.  We interpret the expression *his mother (one's mother)* as:

$$g \quad \lambda x. \sigma(\lambda y. \text{MOTHER}(y,x))$$

We read this as:  the function that maps every individual $x$ onto $\sigma(\lambda y. \text{MOTHER}(y,x))$, the mother of $x$.

$$f \quad \lambda x \in \text{ENGLISHMEN}: \sigma(\lambda y.\text{WOMAN}(y) \land \text{ADORE-MOST}(x,y)))$$

We read this as:  the function that maps every englishman $x$ onto the woman that $x$ adores most.

(7)  The woman that *every Englishman* adores most is *his* mother.

Semantics:  when restricted to their common domain:

$$f = g \upharpoonright \text{ENGLISHMEN}$$

$$\lambda x \in \text{ENGLISHMEN}: \sigma(\lambda y.\text{WOMAN}(y) \land \text{ADORE-MOST}(x,y)))$$
\[ \lambda x \in \text{ENGLISHMEN}: \sigma(\lambda y. \text{MOTHER}(y, x)) \]

(7) then expresses that the function that maps every Englishman onto the woman he adores is the mother function (with the domain restricted to Englishmen).

It turns out that an elegant compositional semantics can be given that derives for the woman that every Englishman adores this functional interpretation \( f \), without giving every Englishman wide scope out of the relative clause.

An analysis in terms of functional readings along those lines is generally assumed to be the correct way of analyzing cases like (7).

But this means, again, that the pronoun his in his mother in (7) is not bound by the quantifier every Englishman at all. It is bound inside the expression his mother:

The interpretation of his mother is the function denoted by the expression:

\[ \lambda x . \sigma(\lambda y. \text{MOTHER}(y, x)) \]

and the pronoun his is bound by the \( \lambda \)-operator in this expression.

So, by introducing the \( \lambda \)-operator, we can separate quantification and variable binding.

The facts about variables suggest that we should.
2. Evidence from quantification.
Applying the Frege/Tarski's analysis of quantifiers to natural language quantifiers has well known problems.
- There is no good theory of the restricting effect of the noun:
  Every cat is smart.
  \( \forall x [\text{CAT}(x) \rightarrow \text{SMART}(x)] \)
  Some cat sings.
  \( \exists x [\text{CAT}(x) \land \text{SMART}(x)] \)
Sometimes you use \( \rightarrow \), sometimes you use \( \land \). There is no theory of when you use the one and when the other.
For \( \forall \) and \( \exists \), this is not a very serious problem, since can introduce restricted quantifiers:

If \( x \) is a variable and \( \varphi \) a formula, \( P \) a one-place predicate, then
\( \forall x \in P : \varphi \) and \( \exists x \in P : \varphi \) are formulas.

\[
\begin{align*}
\llbracket \forall x \in P : \varphi \rrbracket_{M,g} & = 1 \text{ iff for every } d \in \llbracket P \rrbracket_{M,g} : \llbracket \varphi \rrbracket_{M,g,d} = 1; 0 \text{ otherwise} \\
\llbracket \exists x \in P : \varphi \rrbracket_{M,g} & = 1 \text{ iff for some } d \in \llbracket P \rrbracket_{M,g} : \llbracket \varphi \rrbracket_{M,g,d} = 1; 0 \text{ otherwise}
\end{align*}
\]

But what about other quantifiers?

Most cats are smart
\( Mx[\text{CAT}(x) \land \text{SMART}(x)] \)
\( Mx \in \text{CAT} : \text{SMART}(x) \)
Try: \( Mx[\text{CAT}(x) \rightarrow \text{SMART}(x)] \)
\( Mx[\text{CAT}(x) \land \text{SMART}(x)] \)

You can prove that there is no Frege/Tarski quantifier \( Mx \) and connective \( \land \) that get the truth conditions of \textit{Most cats are smart} right.
You can prove that there is no restricted Frege/Tarski quantifier over individuals \( Mx \in \text{CAT} \) that gets the truth conditions of \textit{Most cat are smart} right.
This requires, of course, a proper definition of a 'quantifier over individuals', but it reflects the intuition about the semantics of \textit{most}: most compares the cardinalities of two sets of individuals.

Montague and Lewis solve these problems by using a different perspective on quantifiers introduced in logic in the 1950s by Andrej Mostowski, that of generalized quantifiers.
I will introduce the theory here as a theory of generalized quantificational determiners, by which we mean expressions like \textit{every}, \textit{some}, \textit{no}, \textit{most}, \textit{at least three}, etc.
The idea is very simple:

Determiners like \textit{every} do not express Frege/Tarski quantifiers at all, they express relations between sets of individuals.
Analogy:

V is a 2 place relation between individuals
D is a 2 place relation between sets of individuals

This idea combines in the following way with the analysis of predicates discussed above. We argued that in *every boy admires himself*, the noun phrase *every boy* or the determiner *every* does not bind the reflexive variable at all, that variable is already bound in the predicate, *admires himself*.

We analyzed that with the variable binding operation \( \lambda x: \)

*admires himself* is interpreted as \( \lambda x.\text{ADMIRE}(x,x) \).

We are not doing without the Frege/Tarski analysis of variable binding: the semantic interpretation of \( \lambda x.\text{ADMIRE}(x,x) \) is **built**, semantically, from Tarski's variable range.

Mathematically, we can identify this with a function from objects \( d \in D_M \) to truth values:

\[
\{ d \in D_M \mid \text{\text{ADMIRE}(x,x)}_{M,gx}^d = 1 \}
\]

But this is precisely the interpretation of \( \lambda x.\text{ADMIRE}(x,x) \).
From this we derive the all important conclusion:

**Tarski's value ranges can be identified with sets of individuals.**

Now the two theories come together:

- Predication formation on $\text{ADMIRE}(x)$ binds variable $x$ to abstraction operator $\lambda x$. This forms a set of individuals, equivalent to the Tarski value range of $\text{ADMIRE}(x,x)$: the set of individuals that admire themselves.
- The determiner meaning *every* in *every boy* expresses a restriction on this set, a restriction which relates it to the set which is the noun interpretation, the set of boys.

In sum, then: we get:

$$\text{EVERY}(\text{BOY}, \lambda x. \text{ADMIRE}(x,x))$$

The semantics of determiner *every* expresses a constraint on the relation between the set of boys and the set of self-admirers.

**We have now separated variable binding from quantification:**
- variable binding is what Tarski assumed it was, except that it is done by operation $\lambda x$, and not by quantifiers.
- quantificational determiners express relations between sets of individuals.

The advantage of this perspective for quantificational determiners is that it provides a unified theory of natural language quantification: in this perspective we can study the semantic contribution of any determiner element, and, importantly, we can formulate semantic generalizations about the meanings of classes of determiners.

While developed by Montague and Lewis, the theory was first formulated as a theory of semantic generalizations about classes of determiners by Jon Barwise and Robin Cooper in 1981 in a paper called 'Generalized quantifiers and natural language'.
THE LANGUAGE L₅: PREDICATE LOGIC EXTENDED WITH GENERALIZED QUANTIFIERS

For comparison reasons, we don't redefine quantification along the lines indicated here, but add the new approach to predicate logic. Our language L₅ has the same syntax as L₄, but with the following additions:

DET = {EVERY, SOME, NO, n, AT MOST n, AT LEAST n, EXACTLY n, MOST} where n>0.

DET ⊆ LEX

ABSTRACTION:
If x ∈ VAR and φ ∈ FORM, then \( \lambda x \phi \) ∈ PRED¹

QUANTIFICATION:
If D ∈ DET and P,Q ∈ PRED¹, then D(P,Q) ∈ FORM

EXIST:
EXIST ∈ PRED¹

The semantics for L₅ is exactly the same as for L₄ with the following additions:

For every D ∈ DET: \( [D]_{M,g} = F_M(D) \)

If x ∈ VAR and φ ∈ FORM, then:
\[ [\lambda x \phi]_{M,g} = \{ d ∈ D_M: [\phi]_{M,gx}^d = 1 \} \]

If D ∈ DET and P,Q ∈ PRED¹, then:
\[ [D(P,Q)]_{M,g} = 1 \text{ iff } [P]_{M,g}, [Q]_{M,g} \geq [D]_{M,g} \]

\[ [\text{EXIST}]_{M,g} = D_M \]

The existence predicate will be useful in some of the technical discussions below. This leaves the specification of the new lexical items, the determiners:

For every D ∈ DET: \( F_M(D) \subseteq \text{pow}(D_M) \times \text{pow}(D_M) \)

Every determiner is interpreted as a relation between sets of individuals.

\[
\begin{align*}
F_M(\text{EVERY}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } X \subseteq Y \} \\
F_M(\text{SOME}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } X \cap Y \neq \emptyset \} \\
F_M(\text{NO}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } X \cap Y = \emptyset \} \\
F_M(\text{AT LEAST n}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } |X \cap Y| \geq n \} \\
F_M(\text{AT MOST n}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } |X \cap Y| \leq n \} \\
F_M(\text{n}) &= F_M(\text{AT LEAST n}) \\
F_M(\text{EXACTLY n}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } |X \cap Y| = n \} \\
F_M(\text{MOST}) &= \{ <X,Y>: X,Y \subseteq D_M \text{ and } |X \cap Y| > |X - Y| \}
\end{align*}
\]
We can now prove useful things:

\[
\text{EVERY[CAT, SMART]} \iff \forall x [\text{CAT}(x) \rightarrow \text{SMART}(x)]
\]
\[
\text{EVERY[CAT, } \lambda x \text{ADMIRE}(x, x)] \iff \forall x [\text{CAT}(x) \rightarrow \text{ADMIRE}(x, x)]
\]
\[
\text{SOME[CAT, SMART]} \iff \exists x [\text{CAT}(x) \wedge \text{SMART}(x)]
\]
\[
\text{NO(CAT, SMART)} \iff \neg \exists x [\text{CAT}(x) \wedge \text{SMART}(x)]
\]

\[
\text{AT LEAST 2(CAT, SMART)} \iff \exists x \exists y [\text{CAT}(x) \wedge \text{CAT}(y) \wedge \text{SMART}(x) \wedge \text{SMART}(y) \wedge (x \neq y)]
\]

\[\text{MOST(CAT, SMART)}\text{ is not equivalent to any L}_4\text{ sentence.}\]

\[
\text{EVERY}[\text{BOY}, \lambda x \text{SOME}[\text{GIRL}, \lambda y \text{KISS}(x, y)]] \iff \forall x [\text{BOY}(x) \rightarrow \exists y [\text{GIRL}(y) \wedge \text{KISS}(x, y)]]
\]
\[
\text{SOME}[\text{GIRL}, \lambda y \text{EVERY}[\text{BOY}, \lambda x . \text{KISS}(x, y)]] \iff \exists y [\text{GIRL}(y) \wedge \forall x [\text{BOY}(x) \rightarrow \text{KISS}(x, y)]]
\]

We show:

\[
\text{EVERY}[\text{BOY}, \lambda x \text{ADMIRE}(x, x)] \iff \forall x [\text{BOY}(x) \rightarrow \text{ADMIRE}(x, x)]
\]

(1.) \[\llbracket \text{EVERY}[\text{BOY}, \lambda x \text{ADMIRE}(x, x)] \rrbracket_M = 1\text{ iff}
\]
(2) for every g: \[\llbracket \text{BOY} \rrbracket_{M,g} \subseteq \llbracket \lambda x \text{ADMIRE}(x, x) \rrbracket_{M,g}\text{ iff}
\]
(3) for every g: \[\llbracket \text{BOY} \rrbracket_{M,g} \subseteq \llbracket \lambda x \text{ADMIRE}(x, x) \rrbracket_{M,g}\text{ iff}
\]
(4) for every g: \[\text{F}_M(\text{BOY}) \subseteq \{ d \in D_M : \llbracket \text{ADMIRE}(x, x) \rrbracket_{M,g}^d = 1 \} \text{ iff}
\]
(5) for every g: \[\text{F}_M(\text{BOY}) \subseteq \{ d \in D_M : <d, d> \in \text{F}_M(\text{ADMIRE}) \} \text{ iff}
\]
(6) for every g: for every d \in D_M: if d \in \text{F}_M(\text{BOY}) then <d, d> \in \text{F}_M(\text{ADMIRE}) \text{ iff}
\]
(7) for every g: for every d \in D_M:

\[
\text{if } \llbracket \text{BOY}(x) \rrbracket_{M,g}^d = 1 \text{ then } \llbracket \text{ADMIRE}(x, x) \rrbracket_{M,g}^d = 1 \text{ iff}
\]
(8) for every g: for every d \in D_M: \[\llbracket \text{BOY}(x) \rightarrow \text{ADMIRE}(x, x) \rrbracket_{M,g}^d = 1 \text{ iff}
\]
(9) for every g: \[\llbracket \forall x [\text{BOY}(x) \rightarrow \text{ADMIRE}(x, x)] \rrbracket_{M,g} = 1 \text{ iff}
\]
(10) \[\llbracket \forall x [\text{BOY}(x) \rightarrow \text{ADMIRE}(x, x)] \rrbracket_M = 1\]

**A note on most**

Our semantics:

\[\text{MOST}(A, B): |A \cap B| > |A - B|\]

*most A's are B's* is true if there are more A's that are B's than A's that are not B.

An obvious alternative:

\[\text{MOST}(A, B): |A \cap B| > \frac{1}{2}|A|\]

*most A's are B's* is true if more than half of the A's are B's

Is there a difference? Not on finite domains, obviously.

But do we native speakers have intuitions about infinite domains?

Cantor told us that there are as many even natural numbers as there are natural numbers, but do we have an *intuition* that (1) below is false (as it is according to our semantics) rather than infelicitous (as it is, if we assume that \(\frac{1}{2}|A|\) is not defined, if |A| is infinite).
(1) Most natural numbers are even.

I don't think we do, but – interestingly enough – we do have intuitions about comparison between finite and infinite sets, as in (2):

(2) Most primenumbers are odd.

In (2) we are comparing the cardinality of the set of odd primenumbers (infinite) and the cardinality of the set of even primenumbers (one). We have no problem counting (2) as true.

This is predicted by our semantics of most, but interestingly enough, not by an analysis that assume that $\frac{1}{2}|A|$ is infelicitous if $|A|$ is infinite.

With this, we can now analyze many new inference patterns, like:

\[
\{ (1),(2),(3) \} \Rightarrow (4)
\]

(1) There are exactly 10 apples
(2) Every apple is either green or red, not both
(3) Most apples are green
(4) At most 4 apples are red

\[
\begin{align*}
(1) & \text{EXACTLY } 10 & \text{[APPLE, EXIST]} \\
(2) & \text{EVERY} & \text{[APPLE, } \lambda x. (\text{GREEN}(x) \lor \text{RED}(x)) \land \neg (\text{GREEN}(x) \land \text{RED}(x)) \\
(3) & \text{MOST} & \text{[APPLE, GREEN]} \\
(4) & \text{AT MOST} & \text{4[APPLE, RED]}
\end{align*}
\]
**SKETCH OF THE SEMANTICS FOR PARTIAL DETERMINERS.**

We add to the lexicon a special set of determiners:

\[ \text{DET}^p = \{ \text{THE}_{\text{sing}}, \text{THE}_{\text{plur}}, \text{THE } n, \text{BOTH}, \text{NEITHER} \} \text{ for } n > 0 \]

We have the same syntactic rule for \( \text{DET}^p \) as for \( \text{DET} \):

\[ \text{If } D \in \text{DET}^p \text{ and } P, Q \in \text{PRED}^1, \text{ then } D(P, Q) \in \text{FORM} \]

We add to the models an interpretation function pair \( <F^+_M, F^-_M> \), where \( F^+_M \) and \( F^-_M \) are functions from \( \text{DET}^p \) to \( \text{pow} \{\text{D}_M\} \times \text{pow} \{\text{D}_M\} \), specified below.

We add the following interpretation rules:

\[ [D(P, Q)]_{M,g} = \begin{cases} 
1 \text{ iff } <[P]_{M,g}, [Q]_{M,g}> \in F^+_M(D) \\
0 \text{ iff } <[P]_{M,g}, [Q]_{M,g}> \in F^-_M(D) \\
\text{undefined otherwise} 
\end{cases} \]

Now we specify the lexical meanings of the partial determiners. In fact, we give here a schema for their interpretation:

\[ F^+_M(\text{THE}_a) = \{ <X, Y>: X, Y \subseteq \text{D}_M \text{ and } X \subseteq Y \text{ and pres}_a \} \]
\[ F^-_M(\text{THE}_a) = \{ <X, Y>: X, Y \subseteq \text{D}_M \text{ and not } X \subseteq Y \text{ and pres}_a \} \]

and we specify:

\[ \text{THE}_{\text{sing}}: \text{pres}_\text{sing}: |X|=1 \]
\[ \text{THE}_{\text{plur}}: \text{pres}_\text{plur}: |X|\neq0 \]

A stricter variant of the latter would be the strict plural:

\[ \text{THE}_{\text{plur}}: \text{pres}_\text{plur}: |X|>1 \]

\[ \text{THE}_n: \text{pres}_n: |X|>n \]
\[ \text{THE}_{\text{at least } n}: \text{pres}_\text{at least } n: |X|\geq n \]

\[ \text{BOTH} = \text{THE } 2 \]

\[ \text{NEITHER}: \text{presuppositional form of } \text{no}_2 \]

\[ F^+_M(\text{NEITHER}) = \{ <X, Y>: X, Y \subseteq \text{D} \text{ and } |X \cap Y|=0 \text{ and } |X|=2 \} \]
\[ F^-_M(\text{NEITHER}) = \{ <X, Y>: X, Y \subseteq \text{D} \text{ and } |X \cap Y|\neq0 \text{ and } |X|=2 \} \]
THE\text{sing}(\text{CAT,SMART}) is true if every cat is smart and there is exactly one cat. THE\text{sing}(\text{CAT,SMART}) is false if not every cat is smart and there is exactly one cat. (meaning: that cat isn't smart) THE\text{sing}(\text{CAT,SMART}) is undefined if there isn't exactly one cat.

THE n(\text{CAT,SMART}) is true if every cat is smart and there are exactly n cats. THE n(\text{CAT,SMART}) is false if not every cat is smart and there are exactly n cats. THE n(\text{CAT,SMART}) is undefined if there aren't exactly n cats.

NEITHER(\text{CAT,SMART}) is true if no cat is smart and there are two cats. NEITHER(\text{CAT,SMART}) is false if some cat is smart and there are two cats. NEITHER(\text{CAT,SMART}) is undefined if there aren't two cats.

We can show:
THE\text{sing}(\text{CAT,SMART}) and THE 1(\text{CAT,SMART}) are strongly equivalent.

φ and ψ are \textbf{strongly equivalent} iff they are true in the same models and false in the same models.

**FEW AND MANY.**
Lots of literature. Here, analysis of the simplest cases.

\[
\begin{align*}
F_M(\text{FEW}) &= \{ <X,Y>: X,Y \subseteq D_M \land |X \cap Y| < f(X,Y,C) \} \\
F_M(\text{MANY}) &= \{ <X,Y>: X,Y \subseteq D_M \land |X \cap Y| > m(X,Y,C) \}
\end{align*}
\]

Here f is a \textbf{contextual function} that determines, in context, a number that \textbf{counts as few}. Which number this is is contextually determined, and can depend on X, on Y, on both, or even on a comparison set C distinct from X and Y. Similarly, m is a contextual function that determines, in context, a number that \textbf{counts as many}.

Given this semantics, we expect FEW(\text{CAT,SMART}) to \textbf{pattern semantically} in some ways like AT MOST n(\text{CAT,SMART}), and we expect MANY(\text{CAT,SMART}) to pattern semantically in some ways like AT LEAST n(\text{CAT,SMART}).

There is much more to be said and done about the semantics of \textit{few} and \textit{many}. The semantics given here is introduced here mainly for comparison reasons later.
GENERAL CONSTRAINTS ON DETERMINER INTERPRETATION.

With some notorious problematic cases, discussed in the literature (eg. few, many),
natural language determiners all satisfy the following principles of extension, conservativity and quantity (van Benthem 1983).

EXTENSION

Determiner \( \alpha \) satisfies extension iff for all models \( M_1, M_2 \) and
for all sets \( X,Y \) such that \( X,Y \subseteq D_{M_1} \) and \( X,Y \subseteq D_{M_2} \):
\[
< X,Y > \in F_{M_1}(\alpha) \iff < X,Y > \in F_{M_2}(\alpha)
\]

If you assign CAT and SMART the same interpretation in models \( M_1 \) and \( M_2 \),
then \( DET(CAT,SMART) \) has the same truth value in \( M_1 \) and \( M_2 \).

Let \( F_{M_1}(P) = F_{M_2}(P) = X \) and \( F_{M_1}(Q) = F_{M_2}(Q) = Y \).
If \( \alpha \) satisfies extension, then the truth value of \( \alpha(P,Q) \) depends only on what is in
\( X \cup Y \), not on what is in \( D_{M_1} - (X \cup Y) \) or in \( D_{M_2} - (X \cup Y) \).

The intuition is the following:
If \( \alpha \) satisfies extension then, if we only specify of a model \( F_M(CAT) \) and
\( F_M(SMART) \), the truth value of \( \alpha(CAT,SMART) \) in \( M \) is already determined.

This is a natural constraint on natural language determiners:
The truth value of every cat/some cat/no cat/most cats...isare smart does not depend
on the presence or absence of objects that are neither cats nor smart.

In a picture:

\[
\begin{array}{c}
\text{D} \\
\text{CAT} \quad \text{SMART}
\end{array}
\]

If \( D \) satisfies extension then only what is inside \( CAT \cup SMART \) is relevant for the
truth of \( D(CAT,SMART) \)

\[
\begin{array}{c}
\text{CAT} \quad \text{SMART}
\end{array}
\]

Extension: if we extend the domain with stupid dogs, the truth value of
\( D(CAT,SMART) \) is unaffected.
CONSERVATIVITY

Determiner $\alpha$ is **conservative** iff for every model $M$ and for all sets $X,Y \subseteq D_M$:

$$<X,Y> \in F_M(\alpha) \text{ iff } <X \cap Y> \in F_M(\alpha)$$

There is another formulation of conservativity and extension, which is useful:

Determiner $\alpha$ satisfies **extension** and **conservativity** iff for all models $M_1,M_2$, and all sets $X_1,Y_1,X_2,Y_2$ such that $X_1, Y_1 \subseteq D_{M_1}$ and $X_2, Y_2 \subseteq D_{M_2}$:

If $X_1 \cap Y_1 = X_2 \cap Y_2$ and $X_1 - Y_1 = X_2 - Y_2$ then

$$<X_1,Y_1> \in F_{M_1}(\alpha) \text{ iff } <X_2,Y_2> \in F_{M_2}(\alpha).$$

If you let $\lambda x. \text{CAT}(x) \land \text{SMART}(x)$ and $\lambda x. \text{CAT}(x) \land \neg \text{SMART}(x)$ have the same interpretation in $M_1$ and $M_2$ then $\text{DET}((\text{CAT},\text{SMART}))$ has the same truth value in $M_1$ and $M_2$.

Let $F_{M_1}(P) = X_1$ and $F_{M_2}(P) = Y_1$ and $F_{M_1}(Q) = X_2$ and $F_{M_2}(Q) = Y_2$.

If $\alpha$ satisfies extension, and conservativity, then the truthvalue of $\alpha(P,Q)$ depends only on what is in $X_1 \cap Y_1 (= X_2 \cap Y_2)$ and in $X_1 - Y_1 (= X_2 - Y_2)$.

The intuition is the following:

If $\alpha$ satisfies extension and conservativity, then if we specify of a model $M$, not even what $F_M(\text{CAT})$ and $F_M(\text{SMART})$ are, but only what $F_M(\text{CAT}) \cap F_M(\text{SMART})$ and $F_M(\text{CAT}) - F_M(\text{SMART})$ are, then still the truth value of $\alpha(\text{CAT,SMART})$ in $M$ is already determined.

This is a natural constraint on natural language determiners:

The truth value of every cat/some cat/no cat/most cat... is/are smart does not depend on the presence or absence of objects that are neither cats nor smart, and also not on the presence or absence of smarties that are not cats: it only depends on what is in the set of cats that are smart, and what is in the set of cats that are not smart.

In a picture:

```
CAT−SMART  CAT ∩ SMART  SMART − CAT
```

If $D$ satisfies extension and monotonicity, then only $\text{CAT} \cap \text{SMART}$ and $\text{CAT} - \text{SMART}$ are relevant for the truth of $D(\text{CAT,SMART})$
Conservativity can be checked in the following pattern:

\[ \alpha \text{ is conservative iff } \alpha(\text{CAT}, \text{SMART}) \text{ is equivalent to } \alpha(\text{CAT}, \lambda x \text{CAT}(x) \land \text{SMART}(x)) \]

\[ \alpha \text{ cat is smart iff } \alpha \text{ cat is a cat that is smart} \]
\[ \alpha \text{ cats are smart iff } \alpha \text{ cats are cats that are smart} \]

cf:

Every cat is smart iff Every cat is a smart cat
Most cats are smart iff Most cats are smart cats

**QUANTITY** (Independent definition technically complex, see literature)

Determiner \( \alpha \) satisfies extension and conservativity and quantity iff for all models \( M_1, M_2 \), and all sets \( X_1, Y_1, X_2, Y_2 \) such that

\[ X_1, Y_1 \subseteq D_{M_1} \text{ and } X_2, Y_2 \subseteq D_{M_2} : \]

If \[ |X_1 \cap Y_1| = |X_2 \cap Y_2| \text{ and } |X_1 - Y_1| = |X_2 - Y_2| \text{ then} \]
\[ <X_1,Y_1> \in F_{M_1}(\alpha) \text{ iff } <X_2,Y_2> \in F_{M_2}(\alpha). \]

If you let each of \( \lambda x. \text{CAT}(x) \land \text{SMART}(x) \) and \( \lambda x. \text{CAT}(x) \land \neg \text{SMART}(x) \) have the same cardinality in \( M_1 \) as it has in \( M_2 \), then \( \text{DET(\text{CAT,SMART})} \) has the same truth value in \( M_1 \) and \( M_2 \).

Let \( F_{M_1}(P) = X_1 \) and \( F_{M_2}(P) = Y_1 \) and \( F_{M_1}(Q) = X_2 \) and \( F_{M_2}(Q) = Y_2 \).

If \( \alpha \) satisfies extension, and conservativity and extension, then the truth value of \( \alpha(P,Q) \) depends only on the cardinality of \( X_1 \cap Y_1 \) (= \( |X_2 \cap Y_2| \)) and the cardinality of \( X_1 - Y_1 \) (= \( |X_2 - Y_2| \)).

The intuition is the following:

If \( \alpha \) satisfies extension and conservativity and quantity, then if we specify of a model \( M \), not even what \( F_M(\text{CAT}) \) and \( F_M(\text{SMART}) \) are, and not even what \( F_M(\text{CAT}) \cap F_M(\text{SMART}) \) and \( F_M(\text{CAT}) - F_M(\text{SMART}) \) are, but only what \( |F_M(\text{CAT}) \cap F_M(\text{SMART})| \) and \( |F_M(\text{CAT}) - F_M(\text{SMART})| \) are then still the truth value of \( \alpha(\text{CAT,SMART}) \) in \( M \) is already determined.

This is a natural constraint on natural language determiners:

The truth value of every cat/some cat/no cat/most cats...is/are smart does not depend on the presence or absence of objects that are neither cats nor smart, and also not on the presence or absence of smarties that are not cats; it doesn't even depend on what is in the set of smart cats, and what is in the set of non-smart cats, but only on how many things there are in the set of smart cats and on how many things there are in the set of non-smart cats.

In a picture:

\[
\begin{align*}
\text{where } n &= |\text{CAT} \cap \text{SMART}| \\
\text{where } m &= |\text{CAT} - \text{SMART}|
\end{align*}
\]
For determiners that satisfy extension, conservativity and quantity we can set up the semantics in the following more general way.

We let the model $M$ associate with every determiner $\alpha$ that satisfies extension, conservativity and quantity a relation $r_\alpha$ between numbers. We associate for every model the same relation $r_\alpha$ with $\alpha$.

In terms of this, we define $F_M(\alpha)$:

$$F_M(\alpha) = \{ <X,Y>: X,Y \in D_M \text{ and } <|X \cap Y|,|X \setminus Y|> \in r_\alpha \}$$

Given this, the meaning of the determiner $\alpha$ is now reduced to the relation $r_\alpha$ between numbers. These meanings we specify as follows:

$$r_{\text{EVERY}} = \{ <n,0>: n \in \mathbb{N} \}$$
$$r_{\text{SOME}} = \{ <n,m>: n,m \in \mathbb{N} \text{ and } n \neq 0 \}$$
$$r_{\text{NO}} = \{ <0,m>: m \in \mathbb{N} \}$$
$$r_{\text{AT LEAST } k} = \{ <n,m>: n,m \in \mathbb{N} \text{ and } n \geq k \} \text{ for } k \in \mathbb{N}$$
$$r_{\text{AT MOST } k} = \{ <n,m>: n,m \in \mathbb{N} \text{ and } n \leq k \} \text{ for } k \in \mathbb{N}$$
$$r_{\text{EXACTLY } k} = \{ <k,m>: m \in \mathbb{N} \} \text{ for } k \in \mathbb{N}$$
$$r_{\text{MOST}} = \{ <n,m>: n,m \in \mathbb{N} \text{ and } n > m \}$$

$$1 + \ldots + k = \frac{k \times (k+1)}{2}$$

If $|D| = n$
Then $|\text{pow}(D)| = 2^n$ 2$^n$ distinct properties

Then $|\text{pow}(D) \times \text{pow}(D)| = 2^{2n}$ $2^{2n}$
Then $|\text{pow}^2(D)| = 2$

So:

$|D| = 1$ \hspace{1cm} $|\text{REL}| = 16$ distinct relations between sets on a domain of 1 ind.
$|D| = 2$ \hspace{1cm} $|\text{REL}| = 65,536$ \hspace{1cm} 2 ind
$|D| = 3$ \hspace{1cm} $|\text{REL}| = 2^{64}$ (Famous from the Chinese chessboard)

Let DET be the set of all relations satisfying extension, conservativity and quantity

If $|D| = n$
Then $|\text{DET}| = 2^{1+\ldots+n+1} = 2^{\frac{(n+1)(n+2)}{2}}$

So:

$|D| = 1$ \hspace{1cm} $|\text{DET}| = 8$
$|D| = 2$ \hspace{1cm} $|\text{DET}| = 64$
$|D| = 3$ \hspace{1cm} $|\text{DET}| = 1024$
If $|\text{CAT}| = 3$, then there are four possibilities for the cardinalities in $<|\text{CAT} \cap \text{PURR}|, |\text{CAT} - \text{PURR} |>:$

- $<0,3>$ means: $|\text{CAT} \cap \text{PURR}| = 0$ and $|\text{CAT} - \text{PURR}| = 3$
- $<1,2>$ means: $|\text{CAT} \cap \text{PURR}| = 1$ and $|\text{CAT} - \text{PURR}| = 2$
- $<2,1>$ means: $|\text{CAT} \cap \text{PURR}| = 2$ and $|\text{CAT} - \text{PURR}| = 1$
- $<3,0>$ means: $|\text{CAT} \cap \text{PURR}| = 3$ and $|\text{CAT} - \text{PURR}| = 0$

We can write down a **tree of numbers** which shows for each cardinality of CAT, all the possibilities for the cardinalities of $<|\text{CAT} \cap \text{PURR}|, |\text{CAT} - \text{PURR} |>$:

We can now study the **pattern** that each determiner meaning $r_\alpha$ makes on the tree of numbers, by **highlighting** (bold italic) the extension of $r_\alpha$:

$r_{\text{EVERY}}$

- $<0,0>$
- $<0,1> <1,0>$
- $<0,2> <1,1> <2,0>$
- $<0,3> <1,2> <2,1> <3,0>$
- $<0,4> <1,3> <2,2> <3,1> <4,0>$
- $<0,5> <1,4> <2,3> <3,2> <4,1> <5,0>$
- $<0,6> <1,5> <2,4> <3,3> <4,2> <5,1> <6,0>$
- $<0,7> <1,6> <2,5> <3,4> <4,3> <5,2> <6,1> <7,0>$
- $<0,8> <1,7> <2,6> <3,5> <4,4> <5,3> <6,2> <7,1> <8,0>$
- $<0,9> <1,8> <2,7> <3,6> <4,5> <5,4> <6,3> <7,2> <8,1> <9,0>$
- $|\text{CAT}|=9$

...
Problematic cases: vague, context-dependent determiners:

many and few seem to allow readings which are not conservative.

only is not conservative:

Only cats purr is not equivalent to Only cats are cats that walk
(it entails it, but isn't entailed by it).

But then, only probably just isn't a determiner (it is cross-categorial).
SYMMETRY

Determiner $\alpha$ is symmetric iff for every model $M$ and all sets $X, Y \in D_M$:

$<X,Y> \in F_M(\alpha)$  iff  $<Y,X> \in F_M(\alpha)$

Pattern:  $\alpha$(CATS,PURR) is equivalent to $\alpha$(PURR,CATS)

$\alpha$ cat purrs iff $\alpha$ purrer is a cat
$\alpha$ cats purr iff $\alpha$ purrers are cats

Technically:  $F_M(\alpha)$ only depends on $|A \cap B|$: Symmetry follows from commutativity of $\cap$.

every  NO
some  YES
no  YES
at least n  YES
at most n  YES
exactly n  YES
many  YES  (on the analysis given, keeping $m$ constant)
few  YES  (on the analysis given, keeping $f$ constant)
most  NO
the$_{\text{sing}}$  NO
the$_{\text{plur}}$  NO
the n  NO
both  NO
neither  NO

Felicity in *there*-insertion contexts (Milsark 1974), definiteness effects:

(1)  a. #There is every cat in the garden.
b. There is some cat in the garden.
c. There is no cat in the garden.
d. There are at least three cats in the garden.
e. There are at most three cats in the garden.
f. There are exactly three cats in the garden.
g. There are many cats in the garden.
h. There are few cats in the garden.
i. #There are most catss in the garden.
j. #There is the cat in the garden.
k. #There are the cats in the garden.
l. #There are the three cats in the garden.
m. #There are both boys in the garden.
n. #There is neither cat in the garden.
There are some zoologists who don't know what a platypus is.

There are not many zoologists who don't know what a platypus is.

There are no australian zoologists who don't know what a platypus is.

#There are all islandic zoologists who don't know what a platypus is.

#There are most czech zoologists who don't know what a platypus is.

The same pattern with relational nouns like sister in existential have sentences:

John has DET sister(s) in the army.

John has a sister in the army/ #John has the sister in the army.

Note: exceptions:
(1) a. Who should we ask to sing Auld lang Syne at the party.
   Well, there’s always Fred.
   b. What is there in the fridge? Well, there’s the milk and the wine and the cheese.
   c. There’s every reason to distrust him (= there is good reason to distrust him.
      (1c) does not mean:
      For every reason to distrust him, there is it.)

Milsark:

[NP D NOUN] is felicitous in there-insertion contexts iff D is an indefinite determiner

But Milsark doesn't define what an indefinite determiner is.

Observation: Keenan 1987, varying Barwise and Cooper 1981:
(Keenan’s actual statement is a bit more subtle, since it applies also to complex noun phrases.)

[NP D NOUN] is felicitous in there-insertion contexts iff D is symmetric.

D is indefinite iff D is symmetric.

Technically:
We defined above $[\exists\text{EXIST}]_{M,g} = D_M$

Then: $\text{DET}(A,B) \iff_{\text{conservativity}} \text{DET}(A,A \cap B) \iff_{\text{symmetry}} \text{DET}(A \cap B,A)$

$\text{DET}(A \cap B,A) \iff_{\text{conservativity}} \text{DET}(A \cap B,A \cap B)$

$\iff \text{r}_\text{DET}(|A \cap B|,0)$

$\iff \text{r}_\text{DET}(|A \cap B| \cap \exists\text{EXIST}, |(A \cap B) - \exists\text{EXIST}|)$

$\iff \text{DET}(A \cap B,\exists\text{EXIST})$

Thus:

D is symmetric iff $\text{DET}(A,B) \iff \text{DET}(A \cap B,\exists\text{EXIST})$

This means that the truth conditions of $\text{DET}(A,B)$ only depend on the cardinality of $A \cap B$, ie. are completely determined by that.
A determiner D is **weak** iff D is symmetric; otherwise D is **strong**.

**Generalization:** Weak determiners are determiners D for which the truthvalue of D(A,B) only depends on |A ∩ B|.

It is then the **commutativity** of ∩ (i.e. the fact that A ∩ B = B ∩ A), which brings in symmetry.
For strong determiners, the semantics of D(A,B) depends not on |A ∩ B| or on more than |A ∩ B|.
Thus, the semantics of EVERY and MOST depends on |A – B|, and the semantics of presuppositional noun phrases like the, both, neither have presuppositions that interfere with symmetry: the semantics of D(A,B) associates a presupposition with A, that of D(B,A) associates a presupposition with B. Obviously, this is a failure of symmetry.

**A remark about there-insertion:**
- **there** is is **not** an existential quantifier ∃: there is nothing to that theory (no existential quantifier takes scope over nothing).
- **there** is is **not** a locative. Cross-linguistic variation.

The there-insertion construction is a construction in which the subject does not occur in the normal external subject position but in some lower position. The definiteness effects are presumably related to the special properties of that lower position (at least that is what I argue in my 2004 book Indefinites and the Type of Sets).
Instead what appears in the external subject position is, what we call a **pleonastic** element (not a great term, because the element may be null). What appears there is open to variation. We find the effects not just with there be but also with unaccusative verbs like arrive.

**English:** There have just now arrived three girls from Paris.
Pleonastic: **there**

**Dutch:** Er zijn net drie meisjes aangekomen uit Parijs.
Misschien zijn (er) net drie meisjes aangekomen uit Parijs.
The finite verb (zijn) is in second position.
Pleonastic: If the subject is first position, then optionally **er** [there]
   - If the subject is not first position, it is third position and either **er** or – [null] (i.e. optionally filled with **er**)

**German:** Es sind - gerade drei Mädchen angekommen aus Paris.
Vielleicht sind - gerade drei Mädchen angekommen aus Paris.
The finite verb (sind) is in second position.
Pleonastic: If the subject is first position, then obligatorily **es** [it]
   - If the subject is not first position, it is third position and obligatorily – [null]

**French:** Il sont arrivé trois filles de Paris.
Pleonastic: **Il** [he]
Idiomatic: **Il y a** un chat dans le jardin
   - he there has a cat in the garden
   - **there** is a cat in the garden
**MONOTONICITY.**

Let \(\alpha\) be a determiner.

In \(\alpha(P,Q)\) we call \(P\) the **first argument** of \(\alpha\) and \(Q\) the **second argument** of \(\alpha\).

**Terminology:**
- \(\alpha\) is \(\uparrow_1\): \(\alpha\) is **upward monotonic**, upward entailing, on its first argument
- \(\alpha\) is \(\downarrow_1\): \(\alpha\) is **downward monotonic**, downward entailing, on its first argument
- \(\alpha\) is \(\not\uparrow_1\): \(\alpha\) is neither upward nor downward monotonic on its first argument
- \(\alpha\) is \(\uparrow_2\): \(\alpha\) is **upward monotonic**, upward entailing, on its second argument
- \(\alpha\) is \(\downarrow_2\): \(\alpha\) is **downward monotonic**, downward entailing, on its second argument
- \(\alpha\) is \(\not\uparrow_2\): \(\alpha\) is neither upward nor downward monotonic on its second argument

\(\alpha\) is \(\uparrow_1\) iff for every model \(M\) and all sets \(X_1, X_2, Y \subseteq D_M\):

\[
\text{if } <X_1, Y> \in F_M(\alpha) \text{ and } X_1 \subseteq X_2 \text{ then } <X_2, Y> \in F_M(\alpha)
\]

\(\alpha\) is \(\downarrow_1\) iff for every model \(M\) and all sets \(X_1, X_2, Y \subseteq D_M\):

\[
\text{if } <X_2, Y> \in F_M(\alpha) \text{ and } X_1 \subseteq X_2 \text{ then } <X_1, Y> \in F_M(\alpha)
\]

\(\alpha\) is \(\not\uparrow_1\) iff \(\alpha\) is not \(\uparrow_1\) and \(\alpha\) is not \(\downarrow_1\)

\(\alpha\) is \(\uparrow_2\) iff for every model \(M\) and all sets \(X, Y_1, Y_2 \subseteq D_M\):

\[
\text{if } <X, Y_1> \in F_M(\alpha) \text{ and } Y_1 \subseteq Y_2 \text{ then } <X, Y_2> \in F_M(\alpha)
\]

\(\alpha\) is \(\downarrow_2\) iff for every model \(M\) and all sets \(X, Y_1, Y_2 \subseteq D_M\):

\[
\text{if } <X, Y_2> \in F_M(\alpha) \text{ and } Y_1 \subseteq Y_2 \text{ then } <X, Y_1> \in F_M(\alpha)
\]

\(\alpha\) is \(\not\uparrow_2\) iff \(\alpha\) is not \(\uparrow_2\) and \(\alpha\) is not \(\downarrow_2\)

**Diagnostic Tests:**

For every model \(M\) for English and \(g\): \([\text{BLUE-EYED BOY}]_{M,g} \subseteq [\text{BOY}]_{M,g}\)

For every model \(M\) for English and \(g\): \([\text{WALK}]_{M,g} \subseteq [\text{MOVE}]_{M,g}\)

\(\alpha\) is \(\uparrow_1\) iff \(\alpha(\text{BLUE-EYED BOY}, \text{WALK}) \Rightarrow \alpha(\text{BOY}, \text{WALK})\)

\(\alpha\) is \(\downarrow_1\) iff \(\alpha(\text{BOY}, \text{WALK}) \Rightarrow \alpha(\text{BLUE-EYED BOY}, \text{WALK})\)

\(\alpha\) is \(\uparrow_2\) iff \(\alpha(\text{BOY}, \text{WALK}) \Rightarrow \alpha(\text{BOY}, \text{MOVE})\)

\(\alpha\) is \(\downarrow_2\) iff \(\alpha(\text{BOY}, \text{MOVE}) \Rightarrow \alpha(\text{BOY}, \text{WALK})\)

115
ARGUMENT 1           ARGUMENT 2
  every ↓ ↑
  some ↑ ↑
  no ↓ ↓
  at least n ↑ ↑
  at most n ↓ ↓
  exactly n – –
  most – ↑
  many ↑ ↑ (on the analysis given)
  few ↓ ↓ (on the analysis given)

(we ignore the partial determiners here)

Example: MOST(BOY,WALK) ⇒ MOST(BOY,MOVE)
Reason:
|BOY ∩ MOVE| ≥ |BOY ∩ WALK|
|BOY – MOVE| ≤ |BOY – WALK|

If MOST(BOY,WALK) then |BOY ∩ WALK| > |BOY – WALK|
Then |BOY ∩ MOVE| > |BOY – MOVE|
Hence MOST(BOY,MOVE)
So most is ↑2.

But MOST(BLUE-EYED BOY,WALK) doesn't entail MOST(BOY,WALK)
and MOST(BOY,WALK) doesn't entail MOST(BLUE-EYED BOY,WALK)
Hence MOST is −1.

**Polarity sensitivity items:** any, ever, a red cent, budge an inch, a damn,…

(1) a. I don't see anything
       b. #I see anything.
(2) a. I haven't ever visited him.
       b. #I have ever visited him.
(3) a. I don't give a damn.
       b. #I give a damn.

Polarity sensitivity items are licensed in the scope of negation.
But not just negation, also other contexts:
-Questions: Did you ever love me?
-Antecedents of conditionals: If Fred reads anything, it is Harry Potter.
-and more…

116
We use *ever*.

We check: \(\alpha\) boy(s) *ever* visited Paris  \(\text{ever}\) in the second argument of \(\alpha\)
\(\alpha\) boy(s) who *ever* visited Paris was/were happy  
\(\text{ever}\) in the first argument of \(\alpha\)

(1) a. \#*Every* boy ever visited Paris.
    b. *Every* boy who ever visited Paris was happy.

(2) a. \#*Some* boy ever visited Paris.
    b. \#*Some* boy who ever visited Paris was happy.

    b. *No* boy who ever visited Paris was happy.

(4) a. \#*At least three* boys ever visited Paris.
    b. \#*At least three* boys who ever visited Paris were happy.

(5) a. *At most three* boys ever visited Paris.
    b. *At mostt three* boys who ever visited Paris were happy.

(6) a. \#*Exactly three* boys ever visited Paris.
    b. \#*Exactly three* boys who ever visited Paris were happy.

(7) a. \#*Most* boys ever visited Paris.
    b. ?*Most* boys who ever visited Paris were happy.

(8) a. \#*Many* boys ever visited Paris.
    b. \#*Many* boys who ever visited Paris were happy.

    b. *Few* boys who ever visited Paris were happy.

Results: *ever* felicitous inside:

<table>
<thead>
<tr>
<th></th>
<th>ARGUMENT 1</th>
<th>ARGUMENT 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>every</td>
<td>YES</td>
<td>NO</td>
</tr>
<tr>
<td>some</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>no</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>at least n</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>at most n</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>exactly n</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>most</td>
<td>NO(?)</td>
<td>NO</td>
</tr>
<tr>
<td>many</td>
<td>NO</td>
<td>NO</td>
</tr>
<tr>
<td>few</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>

**Correlation:** (Ladusaw 1979) Polarity sensitivity item \(\alpha\) is felicitous iff \(\alpha\) occurs in a downward monotonic environment.
What is it about *any* that makes it occur in DE contexts? (Kadmon and Landman 1993)

**Intensifiers:**

John is a fool

John is a *damn* fool

1. What does *damn* do?
   Answer: it creates a **stronger** expression

2. What does **stronger** mean?
   Answer: The expression *damn* entails the expression without *damn* (Kadmon and Landman allow also pragmatic implication here)

3. How does it create a stronger meaning?
   Answer: By being a subsective/intersective adjective
   (a damn fool is a fool, but not every fool is a damn fool)

4. When will it work?
   Answer: In upward entailing contexts.

Cf. John isn't a damn fool, he is only a bit of a fool (only metalinguistic negation)
Cf. a. I have always told you Jane, your husband is a DAMN fool.
    b.# I have alsways told you Jane, your husband isn't a DAMN fool.

* A damn fool* is an indefinite which is stronger and more restricted than
  a fool.

5. How do you intensify in downward entailing contexts?
   Answer: By finding an expression that creates a **stronger** expression in downward entailing contexts.

Adjectives **restrict** the noun interpretation: this is **weaker** in DE contexts.
So what we want is an **anti-adjective**: an expression that doesn't **restrict** the noun interpretation but **liberates it, widenes it**.

6. Polarity sensitivity items are anti-adjectives

   *We don't have potatoes* = We don't have potatoes\textsuperscript{NARROW}
   *We don't have any potatoes* = We don't have potatoes\textsuperscript{WIDE}

*Any fool* is an indefinite which is **stronger and less restricted** than *a fool.*

But, of course, anti-adjectives only create a stronger expression in DE contexts.

So the restriction on DE contexts can be explained through the interaction of the two properties: widening and strengthening.
ANOTHER CHARACTERIZATION OF $\uparrow_2$ AND $\downarrow_2$

D is $\uparrow_2$ iff if D(A,B) and you move objects from A – B to A \cap B then D(A,B)
D is $\downarrow_2$ iff if D(A,B) and you move objects from A \cap B to A – B then D(A,B)
D is $\uparrow_1$ iff if D(A,B) and you add new objects to A \cup B then D(A,B)
D is $\downarrow_1$ iff if D(A,B) and you take away objects from A \cup B then D(A,B)

Technically:
We define for A,B \subseteq D_M:

$-(A,B) = \{<X,Y> : X \cup Y = A \cup B \text{ and } X \cap Y \subseteq A \cap B\}$

$+(A,B) = \{<X,Y> : X \cup Y = A \cup B \text{ and } A \cap B \subseteq X \cap Y\}$

Intuitively, +(A,B) stands for all the possible ways of moving objects from A \cap B to A \cap B.

We define:

DET is $\uparrow_2^*$ iff for every M for every A,B \subseteq D_M:
If DET(A,B) and <X,Y> \in +(A,B), then DET(X,Y)

If DET(A,B) and you move objects from A – B to A \cap B, getting from A to X and from B to Y, then DET(X,Y).

DET is $\downarrow_2^*$ iff for every M for every A,B \subseteq D_M:
If DET(A,B) and <X,Y> \in -(A,B), then DET(X,Y)

**Fact:** (assuming conservativity, quantitativity, extension)
For every determiner DET: DET is $\uparrow_2^*$ iff DET is $\uparrow_2$
DET is $\downarrow_2^*$ iff DET is $\downarrow_2$

$\uparrow_2$ says that DET(A,B) is preserved if you extend B.
$\uparrow_2^*$ says that DET(A,B) is preserved if you extend B inside A.
But, given extension and conservativity, the truthvalue of DET(A,B) is insensitive to any extension of B that you do outside A.

Formally:
A: If DET is $\uparrow_2$ then DET is $\uparrow_2^*$ (because that is a special case).
B: Assume that DET is $\uparrow_2^*$, and assume DET(A,B) and let B \subseteq B'.
Then A \cap B' \in +(A,B)
Hence, by $\uparrow_2^*$, DET(A,A \cap B')
Now, by conservativity: DET(A,B') iff DET(A,A \cap B')
Hence: DET(A,B')
So DET satisfies $\uparrow_2^*$
The argument with $\downarrow_2^*$ goes in the same way.
These characterisations allow us to define the patterns that monotonicity make on the tree of numbers:

$\uparrow^*_2$ if $(n,m) \in r_{DET}$, then every number to the right is in DET

|CAT|=0
|CAT|=1
|CAT|=2
|CAT|=3
|CAT|=4
|CAT|=5
|CAT|=6
|CAT|=7
|CAT|=8
|CAT|=9

Downwards:

$\downarrow^*_2$ if $(n,m) \in r_{DET}$ then every number to the left is in DET

|CAT|=0
|CAT|=1
|CAT|=2
|CAT|=3
|CAT|=4
|CAT|=5
|CAT|=6
|CAT|=7
|CAT|=8
|CAT|=9

We see:

$r_{EVERY}$ is $\uparrow^*_2$
We can give similar definitions on the tree for \( \uparrow_1 \) and \( \downarrow_1 \).

I will here state the facts about the trees:

\( r_{\text{DET}} \) is \( \uparrow_1 \) iff if \( <n,m> \in r_{\text{DET}} \) then \( <n+1,m>, <n,m+1> \in r_{\text{DET}} \).

This means that \( r_{\text{DET}} \) is \( \uparrow_1 \) iff if \( <n,m> \in r_{\text{DET}} \) then the whole triangle with top \( <n,m> \) is in \( r_{\alpha} \).
Example: $r_{\geq 4}$ is $\uparrow_1$:

\[ r_{\text{AT LEAST 4}} \]

$\begin{array}{cccccccc}
0,0 & 0,1 & 0,2 & 0,3 & 0,4 & 0,5 & 0,6 & 0,7 \\
0,1 & 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & 1,7 \\
0,2 & 1,2 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 \\
0,3 & 1,3 & 2,3 & 3,3 & 3,4 & 3,5 & 3,6 & 3,7 \\
0,4 & 1,4 & 2,4 & 3,4 & 4,4 & 4,5 & 4,6 & 4,7 \\
0,5 & 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 5,6 & 5,7 \\
0,6 & 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 & 6,7 \\
0,7 & 1,7 & 2,7 & 3,7 & 4,7 & 5,7 & 6,7 & 7,7 \\
0,8 & 1,8 & 2,8 & 3,8 & 4,8 & 5,8 & 6,8 & 7,8 \\
0,9 & 1,9 & 2,9 & 3,9 & 4,9 & 5,9 & 6,9 & 7,9 \\
\end{array} \]

\[ \vdots \]

\[ r_{\text{DET is } \downarrow_1} \text{ iff if } <n,m> \in r_{\text{DET}} \text{ then } <n-1,m>, <n,m-1> \in r_{\text{DET}} \]

(when $n$ or $m$ is 0, set $n-1, m-1$ to 0 as well)

This means that $r_a$ is $\downarrow_1$ iff if $<n,m> \in r_a$ then the whole inverted triangle with bottom $<n,m>$ is in $r_a$.

Example: $r_{\geq 4}$ is $\downarrow_1$:

\[ r_{\text{AT MOST 4}} \]

$\begin{array}{cccccccc}
0,0 & 0,1 & 0,2 & 0,3 & 0,4 & 0,5 & 0,6 & 0,7 \\
0,1 & 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & 1,7 \\
0,2 & 1,2 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 \\
0,3 & 1,3 & 2,3 & 3,3 & 3,4 & 3,5 & 3,6 & 3,7 \\
0,4 & 1,4 & 2,4 & 3,4 & 4,4 & 4,5 & 4,6 & 4,7 \\
0,5 & 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 5,6 & 5,7 \\
0,6 & 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 & 6,7 \\
0,7 & 1,7 & 2,7 & 3,7 & 4,7 & 5,7 & 6,7 & 7,7 \\
0,8 & 1,8 & 2,8 & 3,8 & 4,8 & 5,8 & 6,8 & 7,8 \\
0,9 & 1,9 & 2,9 & 3,9 & 4,9 & 5,9 & 6,9 & 7,9 \\
\end{array} \]

\[ \vdots \]

It is easy to check that $r_{\leq 3}$ is none of the above:

\[ r_{\text{EXACTLY 4}} \]

$\begin{array}{cccccccc}
0,0 & 0,1 & 0,2 & 0,3 & 0,4 & 0,5 & 0,6 & 0,7 \\
0,1 & 1,1 & 1,2 & 1,3 & 1,4 & 1,5 & 1,6 & 1,7 \\
0,2 & 1,2 & 2,2 & 2,3 & 2,4 & 2,5 & 2,6 & 2,7 \\
0,3 & 1,3 & 2,3 & 3,3 & 3,4 & 3,5 & 3,6 & 3,7 \\
0,4 & 1,4 & 2,4 & 3,4 & 4,4 & 4,5 & 4,6 & 4,7 \\
0,5 & 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 5,6 & 5,7 \\
0,6 & 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 & 6,7 \\
0,7 & 1,7 & 2,7 & 3,7 & 4,7 & 5,7 & 6,7 & 7,7 \\
0,8 & 1,8 & 2,8 & 3,8 & 4,8 & 5,8 & 6,8 & 7,8 \\
0,9 & 1,9 & 2,9 & 3,9 & 4,9 & 5,9 & 6,9 & 7,9 \\
\end{array} \]

\[ \vdots \]
\( r_{\text{every}} \) is clearly not \( \uparrow_1 \), since the downward triangles are not preserved. \( r_{\text{every}} \) is \( \downarrow_1 \), since the upward inverted triangle is just the right edge.

\[
\begin{array}{ccc}
\text{EVERY} & \text{CAT} = 0 & \text{CAT} = 1 \\
<0,0> & <1,0> & <2,0> \\
<0,1> & <1,1> & <2,1> \\
\vdots & \vdots & \vdots \\
<0,9> & <1,9> & <2,9> \\
\end{array}
\]

\( r_{\text{no}} \) is again clearly not \( \uparrow_1 \), but it is \( \downarrow_1 \), because, again, the upward inverted triangle is just the left edge.

\[
\begin{array}{ccc}
\text{NO} & \text{CAT} = 0 & \text{CAT} = 1 \\
<0,0> & <1,0> & <2,0> \\
<0,1> & <1,1> & <2,1> \\
\vdots & \vdots & \vdots \\
<0,9> & <1,9> & <2,9> \\
\end{array}
\]
$r_{\text{most}}$ is $\uparrow_2$, but neither $\uparrow_1$ nor $\downarrow_1$: for no point in $r_{\text{most}}$ is the downward triangle completely in $r_{\text{most}}$ and for no point is the upward triangle completely in $r_{\text{most}}$ (because $\langle 0,0 \rangle$ is not).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{triangle_diagram}
\end{figure}
SYMMETRY AS A PATTERN ON THE TREE OF NUMBERS

$\alpha$ is **symmetric** iff for every $M$, for every $X,Y$: $\langle X,Y \rangle \in F_M(\alpha)$ iff $\langle Y,X \rangle \in F_M(\alpha)$

We showed above that:

If $\alpha$ is symmetric then $\alpha(X,Y)$ iff $r_\alpha([X \cap Y],0)$

With this, we can define:

$\alpha$ is **symmetric** iff $r_\alpha$ is symmetric.

$r_\alpha$ is symmetric iff for every $n,m \geq 0$: $\langle n,m \rangle \in r_\alpha$ iff $\langle n,0 \rangle \in r_\alpha$

i.e.
FACT: if $\alpha$ satisfies EXT, CONS, QUANT, then $\alpha$ is symmetric iff for every $M$ for every $X,Y$: whether $\langle X,Y \rangle$ is in $F_M(\alpha)$ or not depends only on $|X \cap Y|$.

In terms of the tree of numbers this means that:

$r_\alpha$ is symmetric iff for every $n$: **either** for every $m$: $\langle n,m \rangle \in r_\alpha$

or for every $m$: $\langle n,m \rangle \not\in r_\alpha$

In terms of the tree of numbers this means the following.

For number $n$, $\{\langle n,k \rangle: k \in \mathbb{N}\}$ is a **diagonal line** in the tree going from left below to right up:

Like, for $n = 3$:

```
<0,0> |CAT|=0
<0,1> <1,0> |CAT|=1
<0,2> <1,1> <2,0> |CAT|=2
<0,3> <1,2> <2,1> <3,0> |CAT|=3
<0,4> <1,3> <2,2> <3,1> <4,0> |CAT|=4
<0,5> <1,4> <2,3> <3,2> <4,1> <5,0> |CAT|=5
<0,6> <1,5> <2,4> <3,3> <4,2> <5,1> <6,0> |CAT|=6
<0,7> <1,6> <2,5> <3,4> <4,3> <5,2> <6,1> <7,0> |CAT|=7
<0,8> <1,7> <2,6> <3,5> <4,4> <5,3> <6,2> <7,1> <8,0> |CAT|=8
<0,9> <1,8> <2,7> <3,6> <4,5> <5,4> <6,3> <7,2> <8,1> <9,0> |CAT|=9
... ...
```

$r_\alpha$ is symmetric iff every such diagonal line is either **completely inside** $r_\alpha$ or **completely outside** $r_\alpha$. 

126
With this we can check straightforwardly in the trees which $r_\alpha$'s are symmetric:

$r_{\text{every}}$ is **not** symmetric:

\[
\begin{array}{c}
\text{EVERY}
\\
\begin{array}{cccccccc}
& & & & & & & <0,0> \\
& & & & & & <0,1> & <1,0> \\
& & & & & <0,2> & <1,1> & <2,0> \\
& & & & <0,3> & <1,2> & <2,1> & <3,0> \\
& & & <0,4> & <1,3> & <2,2> & <3,1> & <4,0> \\
& & <0,5> & <1,4> & <2,3> & <3,2> & <4,1> & <5,0> \\
& <0,6> & <1,5> & <2,4> & <3,3> & <4,2> & <5,1> & <6,0> \\
<0,7> & <1,6> & <2,5> & <3,4> & <4,3> & <5,2> & <6,1> & <7,0> \\
<0,8> & <1,7> & <2,6> & <3,5> & <4,4> & <5,3> & <6,2> & <7,1> & <8,0> \\
<0,9> & <1,8> & <2,7> & <3,6> & <4,5> & <5,4> & <6,3> & <7,2> & <8,1> <9,0> \\
\end{array}
\end{array}
\]

$\text{CAT}|=0$  $\text{CAT}|=1$  $\text{CAT}|=2$  $\text{CAT}|=3$  $\text{CAT}|=4$  $\text{CAT}|=5$  $\text{CAT}|=6$  $\text{CAT}|=7$  $\text{CAT}|=8$  $\text{CAT}|=9$

It is easy to check that $r_{\leq n}$, $r_{\geq n}$, $r=n$ are symmetric, but that $r_{\text{most}}$ is not symmetric.
Computationally

CAT \quad CAT \cap \text{SMART} \quad CAT - \text{SMART}

Give every individual in CAT a collar with the letter \(i\) (for intersection) or \(d\) (for difference):

ronya has label \(i\) if and only if \(ronya \in CAT \cap \text{SMART}\)
ronya has label \(d\) if and only if \(ronya \in CAT - \text{SMART}\)

In going through the set of cats, we can write a sequence

\(i \ i \ d \ i \ d \ i \ i\)

a string of labels indicating that \(|CAT \cap \text{SMART}| = 6\) and \(|CAT - \text{SMART}| = 2\)

The every language is
the set of strings \(\alpha\) in alphabet \(\{i, d\}\) such that \(d\) doesn’t occur in \(\alpha\)
\(\{e, i, ii, iii, iiii, \ldots\}\) (\(e\) is the empty string)

The some language is
the set of strings \(\alpha\) in alphabet \(\{i, d\}\) such that \(i\) does occur in \(\alpha\)
\(\{i, id, di, idd, did, ddi, iid, idi, dii, iii, \ldots\}\)

The most language is the set of strings with more \(d\)'s and \(i\)'s.
\(ddiiidi\) is in the most language, \(dddddiid\) is not.

The every automaton (smiley’s are accepting states):

\[
\begin{array}{c}
\begin{array}{c}
\text{Accepted: } \text{iiiiii} \\
\text{Rejected: } \text{iiidi} \\
\text{(one cat isn't smart)}
\end{array}
\end{array}
\]
The *some* automaton:

![Diagram of the 'some' automaton]

**Fact:** For every determiner definable in predicate logic, there is a finite state automaton accepting its language (regular)
- Some ‘determiners’ that are not definable in predicate logic have a language accepted by a finite state automaton (*an even number of*)
- The most language is not accepted by a finite state automaton.
  - The most language is accepted by a pushdown storage automaton (context free).

Push down storage automaton: while reading you can push symbols onto a memory store or pop symbols from the store, where the store is a first-in last-out memory.

The *most* automaton:

**Start:** store the first symbol read
**Move:** - if what you read is the same as what is on top of the store, then push onto store what you read.
  - if what you read is different from what is on top of the store, then pop the topsymbol off the store.
**End:** Go to an accepting state if at the end of the string there are i’s in the store.

```
i d i i d d i d d i i e
↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑
S S S S S S S S S S S S
↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓
e → i → e → i → i → i → e → i → e → d → e → i
push pop push push pop pop push pop push pop push
```

You end up with i in store, which means that the string is accepted, so MOST(CAT,SMART) is true, which is good, since there are 6 CATS that are smart, and 5 cats that are not.